Behavior of cosmological models with varying G

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We provide a detailed analysis of Friedmann-Robertson-Walker universes in a wide range of scalar-tensor theories of gravity. We apply solution-generating methods to three parametrized classes of scalar-tensor theory which lead naturally to general relativity in the weak-field limit. We restrict the parameters which specify these theories by the requirements imposed by the weak-field tests of gravitation theories in the solar system and by the requirement that viable cosmological solutions be obtained. We construct a range of exact solutions for open, closed, and flat isotropic universes containing matter with equation of state $p \leq \frac{1}{3}\rho$ and in vacuum. We study the range of early- and late-time behaviors displayed, examine when there is a ''bounce'' at early times, and expansion maxima in closed models. [S0556-2821(97)00704-2]

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I. INTRODUCTION

Cosmological models arising from theories of gravity in which the Newtonian gravitational ''constant'' G varies with time have a long history. They were first studied in detail in response to Dirac's claims that a coincidence between the values of 'large numbers' arising in dimensionless combinations of physical and cosmological constants could emerge naturally if one of the constants involved possessed a time variation that was significant over cosmological time scales [1]. Dirac ascribed that time variation to G and simply wrote the time variation into the Newtonian expressions which held for constant G. Subsequently, mathematically well-posed gravitation theories were developed in which Einstein's theory of general relativity (GR) was generalized to include a varying G by deriving it from a scalar field satisfying a conservation equation. These scalar-tensor gravity theories, first formulated by Jordan [2], were most fully exploited by Brans and Dicke in 1961 [3]. Motivated by claims that the observations of light-bending by the sun were in significant disagreement with the predictions of GR, Brans and Dicke explored the possibility that the simplest scalar-tensor theory could provide predictions of the weak-field solar-system tests in agreement with light-bending and other geological and paleontological observations [4]. Cosmological models could be found in Brans-Dicke (BD) theory, but astronomical observations were unable to impose stronger limits upon them than had been found from solar system experiments. Subsequently, the doubts regarding the compatibility between observations of solar light-bending and the predictions of GR were removed by a fuller understanding of the uncertainties surrounding measurements of the solar diameter at times of high solar surface activity [5]. This removed the one observation that called for the replacement of GR by a scalar-tensor theory displaying varying G in the weak-field limit. Since that time there have been two observations which have led to renewed interest in gravity theories with non-Newtonian variation in G: one was the claim that a ''fifth force'' of Nature existed on laboratory length scales and influenced Eötvös experiments as if there existed a deviation from the inverse-square form of the law of gravitation in the nonrelativistic Newtonian regime [6]; the other has been the occasional claim that the flatness of the rotation curves displayed by spiral galaxies might be a signal of non-Newtonian gravitational attraction [7]. The evidence for the ''fifth force'' variations has not been supported by other experiments and flat rotation curves appear as natural outcomes of protogalaxy formation in conventional gravitation theories. In both cases, the deviations from conventional gravitation theory, with constant G, would introduce a new characteristic length scale into the law of gravity with no fundamental basis other than to explain a particular set of observations. Some attempts were made by Gibbons and Whiting [8] to see how simple scaling arguments might relate preferred high-energy physics scales to those observed in fifth force experiments. More recently, the study of scalar-tensor gravity theories has been rejuvenated by theoretical developments in the study of the evolutionary possibilities open to the early universe.

In a metric scalar-tensor theory of gravity the gravitational coupling is derived from some scalar field, ϕ, so $G=G(\phi)$. Historically, most interest has been focused upon the first and simplest theory of this type, presented by Brans and Dicke [3], in which the coupling function $\omega(\phi)$ is a constant. In general, the consideration of scalar-tensor theories with nonconstant $\omega(\phi)$ [5,9], greatly enlarges the range of possible G variations and weakens the impact of observational limits accordingly. In the weak-field limit Nordtvedt found an expression for the observed value of the gravitation ''constant'' in these theories, to leading order, as [10]

$$G(t)=\phi^{-1}\left(\frac{4+2\omega(\phi)}{3+2\omega(\phi)}\right),$$

so that

$$\frac{\dot{G}}{G}=-\dot{\phi}\left(\frac{3+2\omega}{4+2\omega}\right)\left(G+\frac{2\omega'(\phi)}{(3+2\omega)^2}\right).$$

We see that in Brans-Dicke theory, with constant ω the variation of $G(t)$ is just inversely proportional to $\phi(t)$. Moreover, for one particular choice of $\omega(\phi)$,

$$\omega(\phi)=\frac{4-3\phi}{2(\phi-1)},$$

it is possible to have $\dot{G}=0$ to first order in the weak-field limit [11].

Our observational limits are a mixture of cosmological limits, studies of astrophysical objects, and weak-field tests of gravitation in the solar system. In Brans-Dicke theory the scope for significant deviations from constant G is very small because of the constancy of ω. However, if ω varies then it can increase with cosmic time in such a way that $\omega\to\infty$ and $\omega'\omega^{-3}\to 0$ as $t\to\infty$ so that weak-field observations at the present time accord well with the predictions of GR even though the theory may possess significant deviations from GR predictions at very early cosmological times [12].

The principal observational upper limits on $\dot{G}/G$ come from many different observations. Passive radar data on the distances to Mercury and Venus [13,14] give the limit $\dot{G}/G<4\times10^{-10}\ \mathrm{yr}^{-1}$. This was improved using the Mariner 9 Mars orbiter by Anderson *et al.* [15] to $\dot{G}/G<1.5\times10^{-10}\ \mathrm{yr}^{-1}$. Anderson *et al.* [16] then used Mariner 10 data and radar ranging to Mercury and Venus to obtain $\dot{G}/G<0.0\pm2.0\times10^{-12}\ \mathrm{yr}^{-1}$. Viking landers, Mars orbiters and transponders gave upper limits of $\dot{G}/G<3\times10^{-11}\ \mathrm{yr}^{-1}$ [17], $\dot{G}/G<2\pm4\times10^{-12}\ \mathrm{yr}^{-1}$ [18], and $\dot{G}/G<-2\pm10\times10^{-12}\ \mathrm{yr}^{-1}$ [19]. These are made uncertain by our incomplete knowledge of the asteroid distribution. Studies of the Binary pulsar PSR 1913+16 by Damour, Gibbons, and Taylor [20] give a limit of $\dot{G}/G=-(1.10\pm1.07)\times10^{-11}\ \mathrm{yr}^{-1}$ with uncertainties due to the pulsar's proper motion. Gravitational lensing promises to yield new tests, but depends upon other uncertain cosmological parameters [21]. Krauss and White argue that a variation in G of $\Delta G/G\leqslant 20$ would influence lenses at $z\sim1.5$ and these studies might eventually achieve constraints of order $\dot{G}/G\leqslant10^{-11}$ in Brans-Dicke theories. White dwarf cooling will be affected by variations in G [22], and recently, García-Berro *et al.* [23] gave limits, depending upon the chemical composition of the white dwarf of $\dot{G}/G\leqslant-(1\pm1)\times10^{-11}\ \mathrm{yr}^{-1}$ for chemically stratified models, and $\dot{G}/G\leqslant-(3^{+1}_{-3})\times10^{-11}\ \mathrm{yr}^{-1}$ for models with constant C/O composition. However, these studies have theoretical uncertainties introduced by the quasi-Newtonian modeling of the variation of G [24]. This is also true of attempts to constrain $\dot{G}/G$ by measurements of the time-variation of neutron star masses [25]. Cosmological nucleosynthesis gives limits that are of order $\dot{G}/G\leqslant0.01H$ in Brans-Dicke theory [26]. Nucleosynthesis limits have also been obtained in Bekenstein's variable mass theory [27], the Schmidt-Greiner-Heinz-Müller theory of gravity [28], Kaluza-Klein and superstring theories [29,30], scale-covariant gravity [31], and theories with $2\omega+3\propto\phi^{\alpha}$, which display a slow logarithmic decrease in $G(t)$ [32,33]. Accetta, Krauss, and Romanelli have shown that in general $\dot{G}/G<9\times10^{-13}\ \mathrm{yr}^{-1}$ [34]. Other limits include those from solar evolution [35] of $\dot{G}/G\leqslant10^{-10}\ \mathrm{yr}^{-1}$; lunar occultations and eclipses [36] of $\dot{G}/G\leqslant0.4\times10^{-10}\ \mathrm{yr}^{-1}$; lunar laser-ranging studies of the Moon's orbit around the Earth [37] of $\dot{G}/G\leqslant0.3\times10^{-10}\ \mathrm{yr}^{-1}$, and Müller *et al.* [38] derive $\dot{G}/G<(0.1\pm10)\times10^{-12}\ \mathrm{yr}^{-1}$. There are proposals to measure via radar the position of a transponder in orbit around Mercury [39]. The ranging sensitivity is claimed to be able to reach just a few centimeters which would translate into limits of order $\dot{G}/G<3\times10^{-13}\ \mathrm{yr}^{-1}$. Most of these limits (especially cosmological ones from nucleosynthesis) are uncertain if varying G is coupled to the variation of other constants [30].

The coupling function of scalar tensor theories $\omega(\phi)$ is related to the parametrized post-Newtonian (PPN) parameters by [10]

$$\beta=1+\frac{\omega'}{(3+2\omega)^2(4+2\omega)}\to1+O\left(\frac{\omega'}{8\omega^3}\right)\quad\text{as}\quad\omega\to\infty,$$

$$\gamma=1-\frac{1}{\omega+2}\to1\quad\text{as}\quad\omega\to\infty.$$

Observational limits on the weak-field PPN parameter γ are $\gamma\simeq1\pm0.002$ from radio timing delays [40]; $\gamma\simeq1.0002\pm0.002$ from light deflection using very long baseline interferometry (VLBI) observations of quasars [41]; $\gamma\simeq1\pm0.02$ from lunar laser ranging [38]. Future experiments, GPB, POINTS, and Mercury Relativity Satellite, hope to reach sensitivities of $|\gamma-1|\sim3\times10^{-7}$ [42]. For combinations of two PPN parameters determining ω, limits of $|4\beta-\gamma-3|<5\times10^{-3}(1\sigma)$ have been found [19]. If we take the observational limits as $|\gamma-1|<0.002$ and $|4\beta-\gamma-3|<0.001$, then we have $\omega>498$ and

$$\left|\frac{\omega'}{(3+2\omega)^2(4+2\omega)}\right|<0.001,$$

where ω' is evaluated at the asymptotic ϕ value ϕ_0 where $\omega\to\omega(\phi_0)$. Hence, we have only a rather weak limit of $|\omega'|<10^{-6}\times999^2\sim1$.

Planetary data [43] also provides a limit on the spatial gradient of G over solar system scales of $\nabla G/G<3\times10^{-10}\mathrm{AU}^{-1}$. A limit of $\delta G/G<10^{-13}$ on possible spatial anisotropy of G in the solar system has been derived by studying the alignment of the Sun's rotation axis with the direction of the solar system's angular momentum vector [44], and of $\delta G/G<2\times10^{-12}$ from satellite and Laser Geodynamics Satellite (LAGEOS) laser ranging data [45].

A brief period of interest in extended Kaluza-Klein cosmological theories [46], which culminated in their replacement by superstring theories [47], revealed how time variations in any extra (>3) dimensions of space would manifest itself through the time variation of the ''constants'' defined in the three dimensions we observe. Over the same period, cosmologists showed growing interest in the behavior of all scalar fields during the early stages of the universe. The time variation of a field energy source in the early universe at a rate slower than the universe is expanding is a general possibility only for *scalar* fields and has become known as ''slow-rolling'' of the field. Typically, it produces an acceleration of the expansion scale factor of the universe with time: a phenomenon known as ''inflation'' [48]. This has led to the investigation of general-relativistic cosmological models containing a wide range of self-interacting scalar-field sources [49], the classification of different varieties of inflation that can result from their slow-rolling evolution, and the extraction of detailed predictions concerning the fluctuations imprinted in the cosmic microwave background radiation by the spectra of gravitational waves and density perturbations that emerge from a period of primordial inflation. However, it has also been recognized that natural scalar fields might be provided by the scalar component of a scalar-tensor theory of gravity [50]. Such theories possess close conformal relationships with GR plus explicit scalar fields, and although the scalar field determining the strength of the gravitational coupling does not easily drive inflation it does have a significant effect upon the pace of inflation that arises when self-interacting scalar fields are included in the universal energy-momentum tensor. Salgado, Sudarsky, and Quevedo have suggested a scalar-tensor description of gravity to explain the periodicity of galaxy counts [51], finding that the mass of their gravitational scalar field can also contribute significantly to cosmological dark matter. Following the long initial study of BD cosmological models, Barrow [12] showed how to generate cosmological solutions of vacuum or radiation-dominated isotropic cosmological models in any scalar-tensor gravity theory. The method used works for any isotropic cosmological model with an energy momentum tensor possessing a vanishing trace. Barrow and Mimoso [33] then devised a more complicated procedure which allows solutions to be generated for isotropic, zero-curvature universes with the pressure p and density ρ related by a perfect-fluid equation of state $p=(\gamma-1)\rho$, with the constant γ lying in the range $0\leqslant\gamma\leqslant 4/3$. In particular, they found the first dust solutions for scalar-tensor theories more general than BD together with a wide range of new inflationary solutions. Anisotropic cosmological models arising from scalar-tensor theories of gravitation have been studied by Mimoso and Wands [52]. Recently, the techniques introduced by Barrow and Mimoso have been used to study the asymptotic behavior of both the isotropic and anisotropic cases [53,54]. In this paper we are going to extend these studies to arrive at a more general and systematic understanding of the behavior of isotropic cosmological models containing matter with equation of state $p\leqslant\rho/3$. We will be interested in studying scalar-tensor gravity theories which can approach general relativity in the weak-field limit at late cosmic epochs. By studying the classes of gravity theory which allow this approach to occur we shall use our solution-generating techniques to build up a detailed picture of the behavior of scalar-tensor cosmological models. We shall be particularly interested in models containing radiation $(p=\rho/3)$, dust $(p=0)$, and inflationary stresses $(p=-\rho)$.

The plan of the paper is as follows. In Sec. II we introduce the Lagrangians and field equations which define scalar-tensor gravity theories in terms of their coupling function $\omega(\phi)$ and dictate their evolution. We shall specialize our study to the cases of Friedmann universes containing perfect fluid sources. In Sec. III we describe the two techniques for finding complete and asymptotic solutions of these equations for arbitrary choices of $\omega(\phi)$. The ''direct'' method works for universes of all curvatures but is restricted to the vacuum and radiation-dominated cases. The ''indirect'' method works only for flat universes but for any equation of state. We shall be especially interested in the vacuum, dust $(p=0)$, radiation and inflationary $(p=-\rho)$ cases. These techniques allow us to draw important conclusions about the early and late-time evolution of cosmological models in scalar-tensor theories. In Sec. IV we introduce three broad classes of gravity theory, defined by the form of $\omega(\phi)$, which allow distinct forms of cosmological evolution. The free parameters can be restricted to allow the theories to reproduce the successful weak-field predictions of general relativity and to give cosmological solutions. The cosmological consequences of these three classes of theory are explored systematically in Secs. V, VI, and VII. In each case we are interested in determining the early and late time behaviors, and finding exact solutions which describe the dust, radiation, and inflationary phases of expansion. The results are discussed in Sec. VIII.

II. SCALAR-TENSOR COSMOLOGIES

A. Field equations

We shall consider scalar-tensor theories of gravity defined by the action,

$$S_G=\int d^4x\sqrt{-g}\left[-\phi\mathcal{R}+\frac{\omega(\phi)}{\phi}g^{\mathrm{ab}}\partial_{\mathrm{a}}\phi\partial_{\mathrm{b}}\phi\right]. \tag{1}$$

where $\mathcal{R}$ is the curvature scalar arising from the spacetime metric g_{ab}, g is the determinant of g_{ab}, ϕ is a scalar field, and $\omega(\phi)$ is a function determining the strength of the coupling between the scalar field and gravity. We are working in units such that Newton's constant G_N is equal to unity. The field ϕ is the analogue of G_N^{-1} in GR except that here, in contrast to Einstein's theory, ϕ is a dynamical quantity. Scalar-tensor gravitational theories therefore permit histories in which the value of the gravitational ''constant'' varies. The simplest such case is that explored by Brans and Dicke, where ω is a constant. The action in Eq. (1) offers more general $\omega(\phi)$ theories as natural extensions to BD gravity. It is these theories that will be of primary interest in this paper. Demanding that the first-order variations of Eq. (1) with respect to ϕ and g_{ab} vanish, we derive the field equations

$$\mathcal{R}_{\mathrm{ab}}-\frac{1}{2}g_{\mathrm{ab}}\mathcal{R}=-8\pi\frac{T_{\mathrm{ab}}}{\phi}-\frac{\omega(\phi)}{\phi^2}\left(\phi_{\mathrm{a}}\phi_{\mathrm{b}}-\frac{1}{2}g_{\mathrm{ab}}\phi_{\mathrm{c}}\phi^{\mathrm{c}}\right)-\frac{1}{\phi}(\phi_{\mathrm{a;b}}-g_{\mathrm{ab}}\Box\phi), \tag{2}$$

$$\Box\phi=\frac{1}{2\omega(\phi)+3}[8\pi T-\omega'(\phi)\phi_{\mathrm{c}}\phi^{\mathrm{c}}], \tag{3}$$

and the conservation law,

$$T^{\mathrm{ab}}{}_{;\mathrm{b}}=0, \tag{4}$$

where $\mathcal{R}_{\mathrm{ab}}$ is the Ricci curvature tensor, T_{ab} is the energy-momentum tensor specifying the properties of the matter occupying the universe. Primes indicate derivatives with respect to ϕ. The first of these three equations is the scalar-tensor analogue of Einstein's equations, the second is the wave equation for ϕ, and the final expression is the energy-momentum conservation law for the matter, which ensures that each theory is consistent with the equivalence principle.

B. Friedmann universes

We shall examine solutions to these equations that describe homogeneous, isotropic Friedmann-Robertson-Walker (FRW) cosmological models, with time-varying G. The FRW metric line element in spherical polar coordinates (t,r,θ,ψ) is given by

$$ds^2=-dt^2+a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\psi^2)\right], \quad (5)$$

where the curvature parameter is $k=-1, 0, +1$ for open, flat or closed cosmologies respectively, and the scale-factor $a(t)$ characterizes the expansion history of the universe. We shall assume the universe contains a simple perfect-fluid which may be accurately described by a perfect-fluid equation of state,

$$p=(\gamma-1)\rho\ ; \qquad 0\leqslant\gamma\leqslant 2, \quad \gamma \ \text{const}\ ; \quad (6)$$

with these prescriptions, the equations of motion become

$$H^2+H\frac{\dot{\phi}}{\phi}-\frac{\omega(\phi)}{6}\frac{\dot{\phi}^2}{\phi^2}+\frac{k}{a^2}=\frac{8\pi}{3}\frac{\rho}{\phi}, \quad (7)$$

$$\ddot{\phi}+\left[3H+\frac{\omega(\dot{\phi})}{2\omega(\phi)+3}\right]\dot{\phi}=\frac{8\pi\rho}{2\omega(\phi)+3}(4-3\gamma), \quad (8)$$

$$\dot{H}+H^2+\frac{\omega(\phi)}{3}\frac{\dot{\phi}^2}{\phi^2}-H\frac{\dot{\phi}}{\phi} =-\frac{8\pi\rho}{3\phi}\frac{(3\gamma-2)\omega(\phi)+3}{2\omega(\phi)+3}+\frac{1}{2}\frac{\omega(\dot{\phi})}{2\omega(\phi)+3}\frac{\dot{\phi}}{\phi}, \quad (9)$$

$$\dot{\rho}+3\gamma H\rho=0, \quad (10)$$

where $H\equiv\dot{a}/a$ is the Hubble expansion parameter and overdots denote derivatives with respect to comoving proper time, t.

The structure of the solutions to these equations is sensitive to the form of the coupling function $\omega(\phi)$, which defines the scalar-tensor theory of gravity. As Nordvedt first showed [10], it is possible to place bounds on the parameter-space of many models, prior to extracting a full solution from the field equations, simply by inspecting the explicit form of $\omega(\phi)$. Nordvedt's constraints demand that the theories tend to GR in the weak-field limit, so that they concur with the observational limits on light-bending and perihelion precession. This requirement manifests itself explicitly in the conditions $\omega\to\infty$ and $\omega'\omega^{-3}\to 0$ as $t\to\infty$. Typically, scalar-tensor theories add a term proportional to $\omega'\omega^{-3}$ to the weak-field predictions of general relativity. While the first condition ($\omega\to\infty$) is well known, the second ($\omega'\omega^{-3}\to 0$) is not. As we shall see in Sec. IV there are many theories which satisfy the first condition but not the second, for example, those defined by $\omega(\phi)\sim(1-\phi/\phi_0)^{-\alpha}$, with $0<\alpha<1/2$, as $\phi\to\phi_0$ from below.

III. METHODS OF SOLUTION

Solutions to Eqs. (7)–(10) for Brans-Dicke-FRW models have existed for some years [55,56]. The Brans-Dicke case has also been studied qualitatively by Kolitch and Eardley [57]. Recently, Barrow [12,59] and later Barrow and Mimoso [33] have extended these treatments to general $\omega(\phi)$ theories, providing a method for obtaining zero-curvature cosmological solutions when $\gamma<4/3$, and solutions for vacuum and radiation-dominated universes of any curvature.

We now recapitulate the two methods of solution introduced in Refs. [12] and [33] to solve the cosmological field equations for theories specified by any $\omega(\phi)$.

A. Vacuum and radiation models

1. Exact solutions

The general solutions to Eqs. (7)–(10) contain four arbitrary integration constants, one more than their GR counterparts, the extra degree of freedom being attached to the value of $\dot{\phi}$. When the energy-momentum tensor is trace-free there exists a conformal equivalence between the theory and GR, the right-hand side of Eq. (3) vanishes and $\dot{\phi}=0$ is always a particular solution, corresponding to a special choice of the additional constant possessed by the model over GR. Consequently, the exact general solution of Einstein's equations when $T_{\rm ab}$ is trace-free is also a particular solution to Eqs. (7)–(10) with ϕ, and hence $\omega(\phi)$, constant.

It will seldom be the case that the particular solution obtained in this way will form the general solution for that particular choice of $\omega(\phi)$. Usually, however, it will be the late or early time attractor of the general solution. For example, in the case of Brans-Dicke theory the special GR solution is the late-time attractor for flat and open universes but not the early-time attractor. However, one of these authors [12] developed a method for integrating the field equations for models containing trace-free matter. The procedure is as follows.

Equation (10) integrates immediately to yield

$$8\pi\rho=3\Gamma a^{-3\gamma}, \quad (11)$$

where $\Gamma\geqslant 0$ is a constant of integration; $\Gamma=0$ describes vacuum models. Making the choice $\gamma=4/3$, corresponding to blackbody radiation, and introducing the conformal time coordinate η, defined by

$$a\,d\eta=dt, \quad (12)$$

Eq. (8) becomes

$$\phi_{\eta\eta}+\frac{2}{a}a_{\eta}\phi_{\eta}=-\frac{\omega'(\phi)}{2\omega(\phi)+3}(\phi_{\eta})^2, \quad (13)$$

where subscript η denotes a derivative with respect to conformal time. This integrates exactly to give,

$$\phi_{\eta}a^2=3^{1/2}A[2\omega(\phi)+3]^{-1/2}\ ; \quad A\ \text{const}. \quad (14)$$

We now employ the variable used by Lorenz-Petzold to study Brans-Dicke models [56],

$$y=\phi a^2\ , \tag{15}$$

to rewrite the scalar-tensor version of the Friedmann equation, Eq. (7), as

$$(y_\eta)^2=-4ky^2+4\Gamma y+\frac{1}{3}(\phi_\eta)^2a^4[2\omega(\phi)+3]\ . \tag{16}$$

Dividing Eq. (14) by Eq. (15), and using Eq. (14), we obtain the coupled pair of differential equations

$$\frac{\phi_\eta}{\phi}=3^{1/2}Ay^{-1}[2\omega(\phi)+3]^{-1/2}\ , \tag{17}$$

$$(y_\eta)^2=-4ky^2+4\Gamma y+A^2\ . \tag{18}$$

We may now obtain the general solution for a particular choice of $\omega(\phi)$, given k. Integrating Eq. (18) yields $y(\eta)$ which, in conjunction with $\omega(\phi)$, implies $\phi(\eta)$ from Eq. (17) and, without further integration, $a(\eta)$ from Eq. (15). If Eq. (12) is both integratable and invertible we may compute $\phi(t)$ and $a(t)$, so completing the solution.

The vacuum models are obtained by setting $\Gamma=0$ and in this case Eq. (18) has three possible solutions, according to the value of k,

$$y(\eta)=\begin{cases} A(\eta+\eta_0)\ , & k=0\ ,\\ \dfrac{1}{2}A\sinh[2(\eta+\eta_0)]\ , & k=-1\ ,\\ \dfrac{1}{2}A\sin[2(\eta+\eta_0)]\ , & k=+1\ ,\end{cases} \tag{19}$$

where η_0 is an arbitrary constant fixing the origin of η time. Combining these results with Eq. (17) yields the set of integral relations,

$$\int\frac{(2\omega(\phi)+3)^{1/2}}{\phi}d\phi=\begin{cases}\sqrt{3}\ln|\eta+\eta_0|, & k=0\ ,\\ \sqrt{3}\ln|\tanh(\eta+\eta_0)|, & k=-1\ ,\\ \sqrt{3}\ln|\tan(\eta+\eta_0)|, & k=+1\ .\end{cases} \tag{20}$$

Specification of $\omega(\phi)$ allows the full solutions to be completed. When $k=-1$ the negativity of the right-hand side, arising because $0<|\tanh(\eta+\eta_0)|<1$, $\forall\eta$, places strong constraints on the allowed form of the integral on the left.

Similarly, one may perform this treatment on the radiation models, i.e., those cases for which $\Gamma>0$. Again, the results are classified by the value of k,

$$y(\eta)=\begin{cases}\Gamma(\eta+\eta_0)^2-A^2/4\Gamma\ , & k=0\ ,\\ -\dfrac{1}{2}\Gamma+\dfrac{1}{2}(A^2-\Gamma^2)^{1/2}\sinh[2(\eta+\eta_0)]\ , & k=-1\\ \dfrac{1}{2}\Gamma+\dfrac{1}{2}(\Gamma^2+A^2)^{1/2}\sin[2(\eta+\eta_0)]\ , & k=+1\ .\end{cases} \tag{21}$$

Integrating Eq. (17) with the solutions above leads to

$$\int\frac{[2\omega(\phi)+3]^{1/2}}{\phi}d\phi=\begin{cases}\sqrt{3}\ln|(2\Gamma\eta+2\Gamma\eta_0-A)/(2\Gamma\eta+2\Gamma\eta_0+A)|\ , & k=0\ ,\\ \sqrt{3}\ln|[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]|\ , & k=-1\ ,\\ \sqrt{3}\ln|[\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A]/[\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A]|\ , & k=+1\ .\end{cases} \tag{22}$$

These expressions can be simplified by choosing the arbitrary integration constant η_0 such that $2\Gamma\eta_0=A$.

The domain on which each right-hand side exists strongly constrains the integral on the left. For instance, if we require $\phi\in(0,\phi_0)$ then the corresponding range of the function of ϕ resulting from the integral on the left must be compatible with the allowed range on the right. In general, ϕ must tend to its general relativistic form, i.e., a constant, at large η. We shall find in Secs. V–VII that this behavior is not generic and often requires the integration constant associated with the left-hand integral to assume a particular value. This integration constant can be interpreted as an initial boundary condition on ϕ or ω at, say, $\eta=0$. Further restrictions can be found by studying the evolution at early times.

In Secs. V–VII we shall exploit these relations to derive general $\omega(\phi)$ solutions for all values of the curvature parameter.

2. Approximation techniques

Many of the models we shall present are insoluble in terms of t. The reason for this is the noninvertibility of $t(\eta)$, arising from integrating $a(\eta)$. In these cases we shall invert $t(\eta)$ approximately at early and late times to obtain series solutions for the behavior of $\phi(t)$ and $a(t)$, indicating their limiting forms and their approach to these forms to leading order. We shall use the inversion technique of Olver [60]. If we have an expression

$$y(x)=f(x)g(x)\ , \tag{23}$$

and we require an approximation to $x(y)$, valid in a region where g dominates over f, we neglect f and write

$$x(y)\simeq g^{-1}(y)\ . \tag{24}$$

The next-order approximation is easily obtained by substituting this result into f, yielding $f\simeq f[g^{-1}(y)]$. Using this in Eq. (23) we have

$$x(y)\simeq g^{-1}\left(\frac{y}{f[g^{-1}(y)]}\right)\ . \tag{25}$$

This procedure can be iterated indefinitely but we shall use the 2nd iteration form as given in Eq. (25).

To analyze the asymptotic behavior we need to establish the value of η as $t\rightarrow\infty$ and our primary interest is in models which display GR, i.e., $\phi\rightarrow\phi_0$, in this limit. Radiation-dominated, $k=0$, universes in GR evolve like $a\propto t^{1/2}$ at late times and hence $t\propto\eta^2$ and $\eta\rightarrow\infty$ as $t\rightarrow\infty$. There does not exist a spatially flat vacuum model in GR, however, the negatively curved general relativistic models are asymptotically vacuum-dominated (as the matter density redshifts to zero) tending to the Milne solution, $a\propto t$, as $t\rightarrow\infty$, and $\eta\propto\ln t$ which diverges with t. We therefore study the asymptotic behavior of vacuum and radiation models in the limit $\eta\rightarrow\infty$ when $k\leqslant 0$. We will not consider the late-time limit of the $k>0$ models. Typically, they recollapse to a final singularity if $\rho>0$ and $\rho+3p>0$, although a bounce occurs for many choices of $\omega(\phi)$. The behavior in their recollapse phase is similar to the time-reverse of the early expansion of $k=0$ models.

We now define the early-time limit. We examine first models with $0<\phi<\phi_0$, in this range $\sqrt{2\omega+3}$ does not change sign, as guaranteed by the choices of the coupling function, $\omega(\phi)$, detailed in Sec. IV and Eq. (14) ensures that ϕ is monotonically increasing. We thus extrapolate the evolution back to $\phi=0$ and treat this as the early-time limit in vacuum and radiation models. In more general perfect-fluid models we find that we are easily able to examine the behavior when $\phi>\phi_0$ by exploiting the conformal invariance of the theory (this procedure is explained in detail in the Appendix) and we so examine early-time behavior in the neighborhood of the last zero, or nonzero minimum, of a.

3. Maxima and minima

It is easy to study the structure of nonsingular vacuum and radiation-dominated cosmological models within the framework presented in this section. Differentiating Eq. (15) and substituting from Eqs. (14) and (15) leads to

$$(a^2)_\eta=\frac{y_\eta}{\phi}-\frac{\sqrt{3}A}{\phi\sqrt{2\omega+3}}\ . \tag{26}$$

The condition for the scale-factor to contain a stationary point, $a_\eta=0$, is equivalent to $(a^2)_\eta=0$ when $a\neq 0$, which leads to the simple relation

TABLE I. $y_{\eta\eta}$ for vacuum and radiation models of all curvatures.

Matter source	Vacuum			Radiation		
k	-1	0	$+1$	-1	0	$+1$
$y_{\eta\eta}$	>0	$=0$	<0	$>2\Gamma>0$	$=2\Gamma>0$	$<2\Gamma$

$$y_\eta=\frac{\sqrt{3}A}{\sqrt{2\omega+3}}\ , \tag{27}$$

when ϕ is finite. This relation will prove useful for locating minima when we lack an exact solution for $a(\eta)$.

Equation (27) is a simple test for the existence of stationary points in the evolution of a. Stationary points of the type $\substack{\text{minimum}\\\text{maximum}}$ will occur when $a_{\eta\eta}\gtrless 0$, respectively but, because $a>0$, it is sufficient to replace $a_{\eta\eta}$ with $(a^2)_{\eta\eta}$ in this condition. Differentiating Eq. (15) twice and evaluating at a stationary point, i.e., where $(a^2)_\eta$ vanishes, we obtain

$$(a^2)_{\eta\eta}=\frac{y_{\eta\eta}}{\phi}-\frac{\phi_{\eta\eta}a^2}{\phi}\ . \tag{28}$$

A condition for the stationary point to be a $\substack{\text{minimum}\\\text{maximum}}$ is then

$$\frac{y_{\eta\eta}}{\phi}\gtrless\frac{\phi_{\eta\eta}a^2}{\phi}\ . \tag{29}$$

Differentiating Eq. (14) and discarding $(a^2)_\eta$ we have

$$\phi_{\eta\eta}a^2=-\frac{3A^2\omega'(\phi)}{a^2(2\omega+3)^2}\ . \tag{30}$$

Substituting this into Eq. (29), and remembering $\phi>0$ yields

$$y_{\eta\eta}\gtrless-\frac{3A^2\omega'(\phi)}{a^2(2\omega+3)^2}\ , \tag{31}$$

as the condition for the stationary point to be a $\substack{\text{minimum}\\\text{maximum}}$. When $0<\phi<\phi_0$ we have $\omega'(\phi)>0$ for the choices in Sec. IV and the right-hand side of Eq. (31) is negative definite. Thus whenever we can prove $y_{\eta\eta}\geqslant 0$ we may exclude the possibility of $a(\eta)$ possessing maxima, and then by continuity we can limit the number of minima to one. We test this condition for the forms of $y(\eta)$ given in Eqs. (19) and (21), the results are summarized in Table I. From the table it can be seen that all spatially flat and negatively curved models may only contain a single minimum. The $k=+1$ models are not bounded in this way and may contain an undetermined number of minima and maxima.

B. General perfect-fluid cosmologies

When T is nonvanishing the situation is substantially more complicated. In this instance, $\dot{\phi}=0$ is no longer a particular solution of the field equations, forcing us to resort to more elaborate methods to obtain solutions. Barrow and Mimoso [33] have done this, for the $k=0$ models, by generalizing the method of Gurevich *et al.* [55] for BD models to the case of varying ω. We now outline this procedure.

Introducing the new time coordinate ξ, and the two new variables x and v such that

$$dt = a^{3(\gamma-1)} \sqrt{\frac{2\omega+3}{3}}\ d\xi \ , \tag{32}$$

$$x \equiv [\phi a^{3(1-\gamma)}(a^3)_\xi] \ , \tag{33}$$

$$v \equiv [a^{3(2-\gamma)}\phi_\xi] \ , \tag{34}$$

and confining attention to the $k=0$ models, Eqs. (7), (8), and (9) transform to

$$\left(\frac{2}{3}x+v\right)^2 = \left(\frac{2\omega+3}{3}\right)[v^2+4\Gamma\ \phi\ a^{3(2-\gamma)}] \ , \tag{35}$$

$$v_\xi = \Gamma(4-3\gamma) \ , \tag{36}$$

and

$$x_\xi = 3\Gamma\ [(2-\gamma)\omega+1] + \frac{3}{2}\left(\frac{2}{3(2\omega+3)}x+v\right)\ \omega_\xi \ , \tag{37}$$

where subscript ξ represents a derivative with respect to ξ time. Equations (36) and (37) integrate easily to yield

$$v = \Gamma(4-3\gamma)\ (\xi-\xi_1) \ , \tag{38}$$

$$x = \frac{3}{2}\left[-v+\sqrt{2\omega+3}\left(C+\Gamma(2-\gamma)\int_{\xi_1}^{\xi}\sqrt{2\omega+3}\ d\overline{\xi}\right)\right] \ , \tag{39}$$

C is an integration constant and ξ_1 fixes the origin of ξ time. Noting the relations

$$\frac{3}{a\phi}a_\xi\phi_\xi = \frac{1}{\phi^2}(\phi_\xi)^2 3\frac{\phi}{a}\frac{a_\xi}{\phi_\xi} = \frac{1}{\phi^2}(\phi_\xi)^2\frac{x}{v} \ , \tag{40}$$

and differentiating y with respect to ξ, yields

$$\left(\frac{\phi_\xi}{\phi}\right)_\xi + \left[\frac{3\gamma-4}{2}+\frac{1}{\xi-\xi_1}f_\xi(\xi)\right]\left(\frac{\phi_\xi}{\phi}\right)^2 = \frac{1}{\xi-\xi_1}\frac{\phi_\xi}{\phi} \ , \tag{41}$$

where a new function $f(\xi)$, is defined by

$$f(\xi) \equiv \int_{\xi_1}^{\xi}\frac{3(2-\gamma)}{2\Gamma(4-3\gamma)}\sqrt{2\omega(\phi)+3}\ \left[C+\Gamma(2-\gamma)\times\int_{\xi_1}^{\xi}\sqrt{2\omega(\phi)+3}\ d\tilde{\xi}\right] d\overline{\xi} \ . \tag{42}$$

Solving Eq. (41), we have the solution

$$\ln\left(\frac{\phi}{\phi_0}\right) = \int_{\xi_1}^{\xi}\frac{\xi-\xi_1}{g(\xi)}d\xi \ , \tag{43}$$

with $g(\xi)$ simply related to $f(\xi)$ by

$$g(\xi) \equiv f(\xi) + \frac{3\gamma-4}{4}(\xi-\xi_1)^2 + D \ , \tag{44}$$

where D is a constant of integration. Equation (40) immediately reveals a simple formula for the scale factor:

$$a^3 = a_0^3\left(\frac{g}{\phi}\right)^{1/2-\gamma}\ \ a_0\ \ \text{const} \ . \tag{45}$$

Finally, the scalar-tensor coupling function $\omega(\phi)$ is given as a function of f by

$$2\omega[\phi(\xi)]+3 = \frac{(4-3\gamma)}{3(2-\gamma)^2}\ \frac{(f')^2}{\left[f+\frac{4-3\gamma}{3(2-\gamma)^2}f_0\right]} \ , \tag{46}$$

where f_0 is another arbitrary constant.

In the calculations to follow, we shall exploit the arbitrariness of ξ_1 and set it to zero. In addition, we note some relations between the other constants in the solution, arising from the scalar-tensor Friedmann constraint, Eq. (35)

$$D = \frac{4-3\gamma}{3(2-\gamma)^2}f_0 \ , \tag{47}$$

$$\phi_0\ a_0^{3(2-\gamma)} = \Gamma\ (4-3\gamma) \ , \tag{48}$$

and from the requirement that the BD theory be recovered when $\omega=\omega_0$ is a constant we obtain the further condition

$$C^2 = \left(\frac{2\Gamma(4-3\gamma)}{3(2-\gamma)}\right)^2 f_0 . \tag{49}$$

We shall require ϕ_0, i.e., G^{-1} today, to be positive and from Eq. (48) we see that this requires $\gamma<4/3$. We shall also confine our attention to theories with $2\omega+3>0$. Using Eq. (43) we can rewrite Eq. (46), as a function of ϕ and its derivatives, as

$$2\omega(\xi)+3 = \frac{4-3\gamma}{3(2-\gamma)^2} \times\frac{((\phi/\phi_\xi)-\xi[\phi\phi_{\xi\xi}/(\phi_\xi)^2]+(6-3\gamma/2)\xi)^2}{(\xi(\phi/\phi_\xi)-(3\gamma-4/4)\xi^2)} . \tag{50}$$

Asymptotically, $2\omega+3\rightarrow(g_\xi)^2/g$ as $\xi\rightarrow\infty$ and the above relation becomes

$$2\omega+3 \rightarrow \frac{\phi}{\xi\phi_\xi}\left(1+\xi\frac{\phi_\xi}{\phi}-\xi\frac{\phi_{\xi\xi}}{\phi_\xi}\right)^2 . \tag{51}$$

The choice of $\phi(\xi)$ is thus equally fundamental as that of $g(\xi)$, and amounts to specifying $G(\xi)$. We shall use the former. Once $\phi(\xi)$ has been specified, we may infer $g(\xi)$ from Eq. (43), $a(\xi)$ from Eq. (45), and $\omega(\xi)$ from Eq. (50). If Eq. (32) can be integrated and inverted to yield $\xi(t)$ these variables may be expressed in terms of cosmic time, t.

An important benchmark is provided by the behavior of the BD theory, where $\omega(\phi)=\omega_0=$ constant. In this case, the generating function, $f(\xi)$, is given by a quadratic in ξ:

$$f_{\mathrm{BD}}(\xi)=\frac{3(2-\gamma)}{2\Gamma(4-3\gamma)}\sqrt{2\omega_0+3}\times\left[C\xi+\frac{\Gamma(2-\gamma)}{2}\xi^2\sqrt{2\omega_0+3}\right]. \tag{52}$$

Hence, in general ($C\neq 0\neq\Gamma$) when $\gamma\neq 4/3,\,2$, we see that $f_{\mathrm{BD}}\propto\xi^2$ as $\xi\to\infty$ and $f_{\mathrm{BD}}\propto\xi$ as $\xi\to 0$, where $dt\propto a^{3(\gamma-1)}d\xi$. If we choose $C=0$ then $f_{\mathrm{BD}}\propto\Gamma\xi^2$ as $\xi\to 0$. The choice $C=0$ restricts the solution to the special ''matter-dominated'' solutions (termed ''Machian'' by Dicke [3,4], see also Weinberg [58]) which were first found for all perfect-fluids by Nariai [55]. If $C\neq 0$ then the early-time behavior is dominated by the dynamics of the ϕ field; such solutions are termed ''ϕ dominated'' (or ''non-Machian'' by Dicke).

Therefore if we choose a generating function $g(\xi)$ that grows slower than ξ^2 as $\xi\to\infty$ it will produce a theory that approaches BD at late times [$\phi\to$const, $\omega(\phi)\to$ const], while if $g(\xi)$ decreases slower than ξ as $\xi\to 0$ then the theory will approach the behavior of ϕ-dominated BD theory at early times. This means that we will find new (non-BD) late-time behaviors by studying generating functions which increase faster than $g(\xi)=\xi^2$ as $\xi\to\infty$ and new (non-BD) early-time behavior by picking generating functions which decrease slower than $g(\xi)=\xi$ as $\xi\to 0$ or $\xi\to\xi_{\min}$ [if there is no zero of ξ at the minimum of $a(t)$].

IV. THE COUPLING FUNCTION

We are interested in ascertaining the general behaviors displayed by cosmological models in the range of scalar-tensor gravity theories that approach GR in the weak-field, late-time limit. This requires $2\omega+3\to\infty$ as $t\to\infty$ and also $\omega'\omega^{-3}\to 0$ if the solar system tests are to accord with observation. The specific form of the leading-order corrections to the general relativistic predictions of light-bending, perihelion precession, and radar echo delay are all almost equal to $\omega'\omega^{-3}$ in the large ω limit. However, the rate at which $\omega(\phi)$ tends to infinity will determine the form of the cosmological models. In an earlier paper [33] we explored the behavior of simple power-law forms for $\omega(\phi)$ which, although growing with time, only attain the GR limit when $\phi=\infty$, although at any finite time ω can be made as large as we wish by the choice of the constants defining $\omega(\phi)$. Here, we turn our attention to a potentially more interesting situation in which $\omega\to\infty$ as $\phi\to\phi_0$ where ϕ_0 may be taken as the present value of $\phi(t)$, which determines the observed value of the Newtonian gravitation constant, $G=\phi_0^{-1}$.

We shall study the three general classes of theory, some examples of which are displayed in Figs. 1–3.

Theory 1. $2\omega(\phi)+3=2B_1|1-\phi/\phi_0|^{-\alpha}$; $\alpha>0$, $B_1>0$ const.

Theory 2. $2\omega(\phi)+3=B_2|\ln(\phi/\phi_0)|^{-2\delta}$; $\delta>0$, $B_2>0$ const.

Theory 3. $2\omega(\phi)+3=B_3|1-(\phi/\phi_0)^{\beta}|^{-1}$; $\beta>0$, $B_3>0$ const.

Theory 1 has been studied previously by Serna and Alimi [53], Comer *et al.* [54], Barrow [59], and Garcia-Bellido and Quiros [61]. Serna and Alimi [53] paid particular attention to the radiation eras of these models, extending their treatment

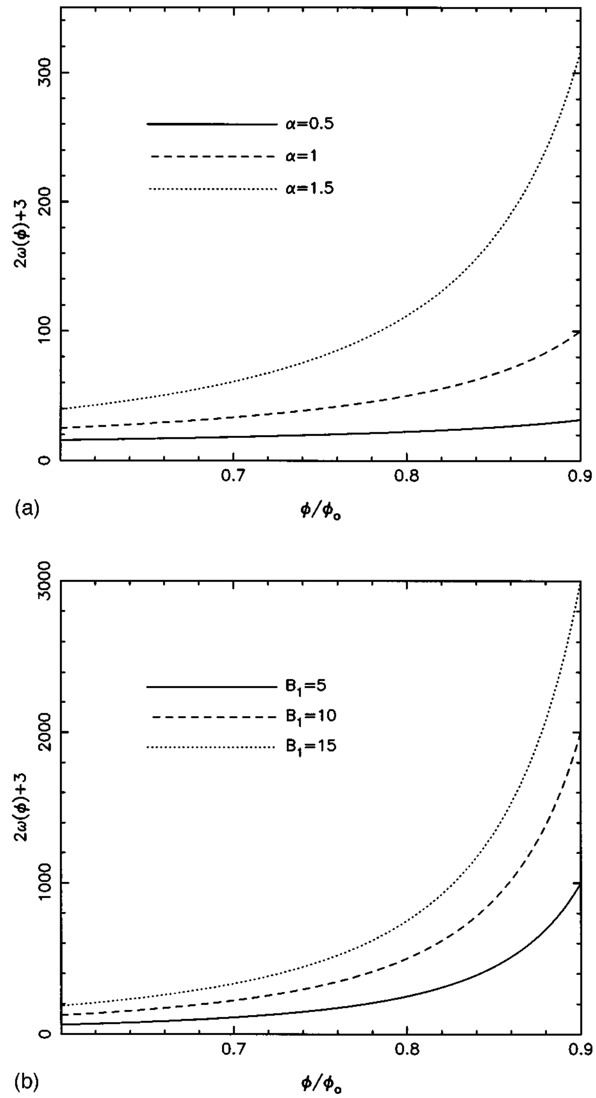

FIG. 1. Theory 1 with (a) $B_1=5$ and $\alpha=0.5,1,1.5$; and (b) $\alpha=2$ and $B_1=5,10,15$.

to allow $2\omega+3<0$, leading to ϕ-dominated initial conditions. Their analysis dealt mainly with the early-time behavior of the models, we shall examine in detail the late-time approach to GR. Theories 1–3 will all permit $\omega\to\infty$ if $\phi\to\phi_0$ at late times and span a wide range of different rates of approach to GR in the weak-field limit. They can all be reduced to Barker's constant-G theory [11] as $\phi\to\phi_0$ for special parameter choices. Theories 1 and 3 approach Brans-Dicke theory [3] as $\phi\to 0$ and the power law form studied in [12] as $\phi\to\infty$. General functional forms for $2\omega(\phi)+3$ can be expanded in a series of functions of this form and their asymptotic behaviors at small and large times will be dominated by one term of the above type. In fact, further restrictions can be placed upon the allowed theories within these three classes by the weak-field limit requirements, as follows:

Theory 1. We see that $\omega\to\infty$ as $\phi\to\phi_0$ is guaranteed if $\alpha>0$ and that

$$\frac{\omega'}{\omega^3}\propto\left(1-\frac{\phi}{\phi_0}\right)^{2\alpha-1}. \tag{53}$$

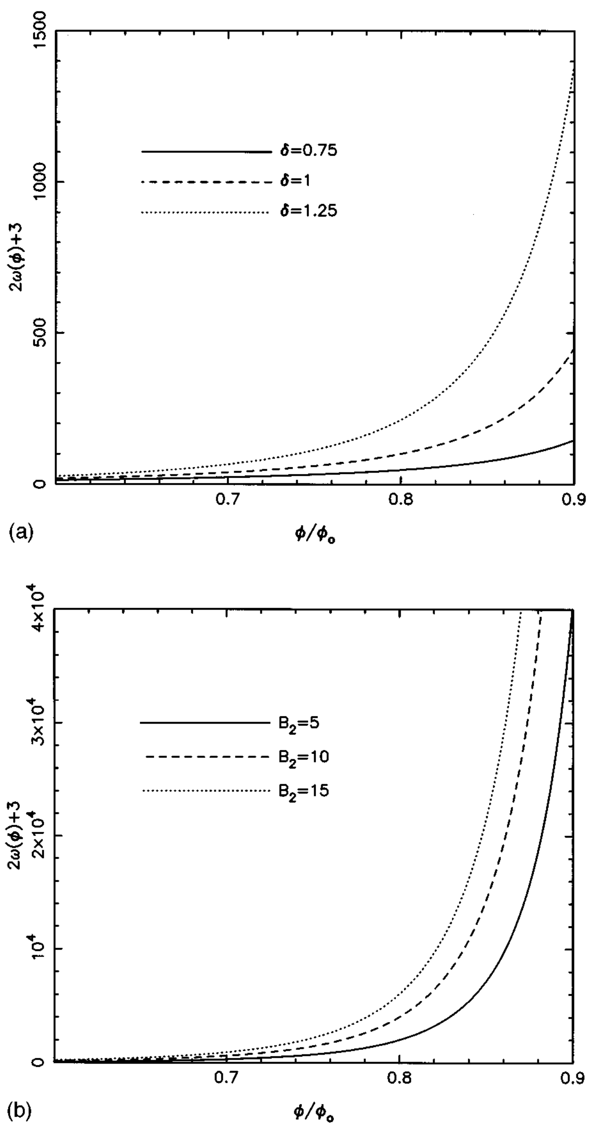

FIG. 2. Theory 2 with (a) $B_2=5$ and $\delta=0.75,1,1.25$ and (b) $\delta=2$ and $B_2=5,10,15$.

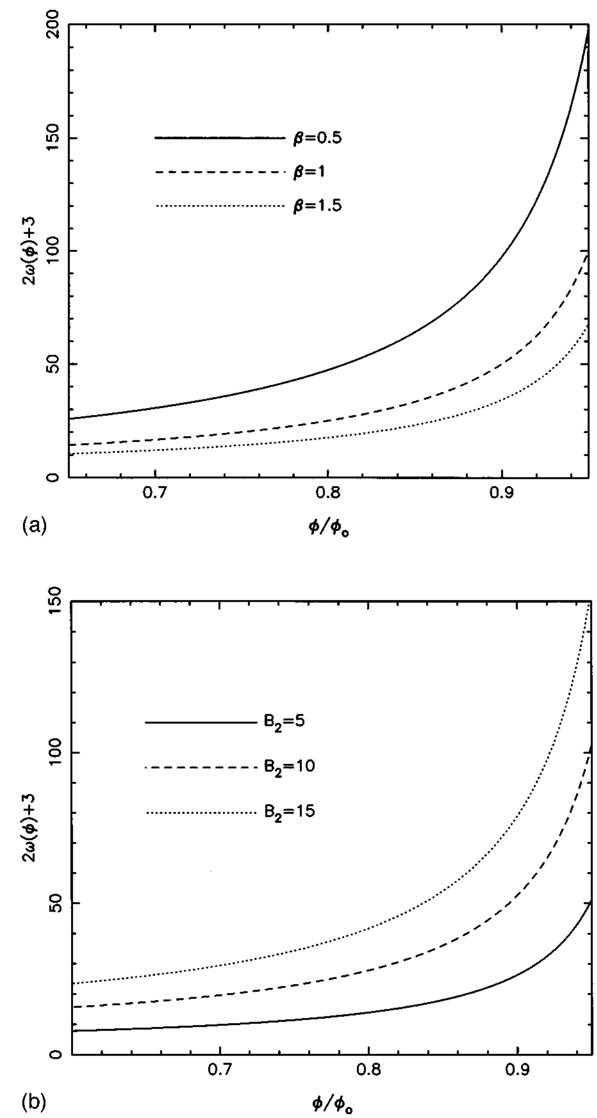

FIG. 3. Theory 3 with (a) $B_3=5$ and $\beta=0.5,1,1.5$; and (b) $\beta=2$ and $B_3=5,10,15$.

Hence $\omega'\omega^{-3}\to 0$ and the weak field limit will be compatible with observation as $\phi\to\phi_0$ so long as $\alpha>1/2$.

Theory 2. Here, $\omega\to\infty$ as $\phi\to\phi_0$ for $\delta>0$, but

$$\frac{\omega'}{\omega^3}\propto\left(\frac{\phi_0}{\phi}\right)\ln^{4\delta-1}\left(\frac{\phi}{\phi_0}\right), \tag{54}$$

and this tends to zero as $\phi\to\phi_0$ so long as $\delta>1/4$.

Theory 3. Here, $\omega\to\infty$ as $\phi\to\phi_0$, but

$$\frac{\omega'}{\omega^3}\propto\left[1-\left(\frac{\phi}{\phi_0}\right)^{\beta}\right]\left(\frac{\phi}{\phi_0}\right)^{\beta-1}, \tag{55}$$

and this tends to zero for all β as $\phi\to\phi_0$.

These constraints on the parameter ranges of theories 1–3 are independent of the form of the cosmological solutions so long as $\phi\to\phi_0$ at late cosmological times. The latter will introduce further restrictions on the allowed values of α, β and δ. With the exception of the $k\neq-1$ vacuum solutions, where there exists no FRW model within GR, we shall focus our attention on those particular parameter choices which reproduce Einstein's theory at late times and which we have just delineated in Eqs. (53)–(55). When $k=-1$ the Milne model, $a\propto t$, supplies the general relativistic solution at late times.

V. THEORY 1: $2\omega(\phi)+3=2B_1|1-\phi/\phi_0|^{-\alpha}$; $\alpha>1/2$, $B_1>0$ const

We study the evolution in the interval $\phi\in(0,\phi_0)$, allowing us to drop the modulus signs from $2\omega+3$. Making the substitution $u_1=(1-\phi/\phi_0)$, this bound becomes $u_1\in(0,1)$ and Eq. (20) is

$$\int\frac{\sqrt{2\omega+3}}{\phi}d\phi=\sqrt{2B_1}\int\frac{du_1}{u_1^{\alpha/2}(1-u_1)}-\sqrt{3}\ln K_1,$$
$$\alpha\neq 1,2, \tag{56}$$

where K_1 is an integration constant. In the models which asymptote to GR, $\omega\to\infty$ as $\phi\to\phi_0$.

A. Vacuum solutions ($k=0$)

1. Late-time behavior

For the theory to reduce to GR at late times we require $u_1\to 0$ as $\eta\to\infty$. Choosing $\eta_0=0$ with $k=0$ in vacuum we then have

$$-\frac{1}{\lambda_1}\frac{{u_1}^{1-\alpha/2}}{(1-\alpha/2)}=\ln(K_1\eta)\ , \tag{57}$$

as $u_1\to 0$. $\lambda_1=\sqrt{3/2B_1}$ and its sign determines the sign of $(2\omega+3)^{1/2}$. The requirement that $u_1\to 0$ as $\eta\to\infty$ demands we must have $\alpha>2$ and hence $\lambda_1>0$ to ensure the right-hand side of Eq. (57) is positive. Picking the appropriate right-hand side from Eq. (19), we thus obtain

$$\phi(\eta)\to\phi_0\left\{1-\left[\lambda_1\left(\frac{\alpha-2}{\alpha}\right)\right]^{2/(2-\alpha)}\ln^{2/(2-\alpha)}(K_1\eta)\right\}\ , \tag{58}$$

$$a^2(\eta)\to\frac{A}{\phi_0}\eta\left\{1+\left[\lambda_1\left(\frac{\alpha-2}{\alpha}\right)\right]^{2/(2-\alpha)}\ln^{2/(2-\alpha)}(K_1\eta)\right\}\ , \tag{59}$$

at large η. This leads to the asymptotic form for $t(\eta)$:

$$t(\eta)\simeq\frac{2}{3}\sqrt{\frac{A}{\phi_0}}\eta^{3/2}\left\{1+\frac{1}{2}\left[\lambda_1\left(\frac{\alpha-2}{2}\right)\right]^{2/(2-\alpha)}\times\ln^{2/(2-\alpha)}(K_1\eta)\right\}\ . \tag{60}$$

The first-order inversion of this at large η is

$$\eta(t)\simeq\left(\frac{3}{2}\right)^{2/3}\left(\frac{\phi_0}{A}\right)^{1/3}t^{2/3}\ . \tag{61}$$

This result is obtained by ignoring the factor in curly brackets on the right-hand side of Eq. (60). To obtain the next-order expression we substitute this first-order result into the weakly-varying $\ln(K_1\eta)$ term on the right-hand side of Eq. (60), take the curly brackets onto the left-hand side, and solve for η. This yields

$$\eta(t)\simeq\left(\frac{3}{2}\right)^{2/3}\left(\frac{\phi_0}{A}\right)^{1/3}t^{2/3}\left\{1-\frac{1}{3}\left(\frac{2}{3}\right)^{2/(2-\alpha)}\times\left[\lambda_1\left(\frac{\alpha-2}{2}\right)\right]^{2/(2-\alpha)}\ln^{2/(2-\alpha)}t\right\}\ , \tag{62}$$

and $\eta\to\infty$ as $t\to\infty$. At late times we then obtain the solution

$$\phi(t)\to\phi_0\left\{1-\left[\lambda_1\left(\frac{\alpha-2}{3}\right)\right]^{2/(2-\alpha)}\ln^{2/(2-\alpha)}t\right\}\ , \tag{63}$$

$$a(t)\to\left(\frac{3A}{2\phi_0}\right)^{1/3}t^{1/3}\left\{1+\frac{1}{3}\left[\lambda_1\left(\frac{\alpha-2}{3}\right)\right]^{2/(2-\alpha)}\ln^{2/(2-\alpha)}t\right\}\ . \tag{64}$$

2. Early-time behavior

At early times we see from Eqs. (20) and (56) that $u_1\to 1$ as $\eta\to 0$ is the only possibility consistent with $u_1\in(0,1)$ and $\lambda_1>0$. Treating ${u_1}^{\alpha/2}\simeq 1$ in Eq. (56) leads to the approximate relations

$$\ln(1-u_1)=\lambda_1\ln(K_1\eta) \tag{65}$$

and

$$\phi(\eta)\simeq\frac{\phi_0}{{K_1}^{-\lambda_1}}\eta^{\lambda_1}\ , \tag{66}$$

$$a^2(\eta)\simeq\frac{A{K_1}^{-\lambda_1}}{\phi_0}\eta^{1-\lambda_1}\ . \tag{67}$$

Taking the square root of the latter, integrating and inverting leads to

$$\eta(t)\simeq\left(\frac{\phi_0}{A{K_1}^{-\lambda_1}}\right)^{1/(3-\lambda_1)}\left(\frac{3-\lambda_1}{2}\right)^{2/(3-\lambda_1)}t^{(2/3-\lambda_1)}\ ,\quad \lambda_1\neq 3\ . \tag{68}$$

When $0<\lambda_1<3$ we see from the above that $t\to 0$ as $\eta\to 0$ and when $\lambda_1>3$ we have $t\to-\infty$ as $\eta\to 0$. In both cases we obtain the early-time behavior

$$\phi(t)\to A^{\lambda_1/(\lambda_1-3)}\left(\frac{\phi_0}{{K_1}^{-\lambda_1}}\right)^{3/(3-\lambda_1)}\times\left(\frac{2}{3-\lambda_1}\right)^{2\lambda_1/(\lambda_1-3)}t^{2\lambda_1/(\lambda_1-3)}\ , \tag{69}$$

$$a(t)\to\left(\frac{A{K_1}^{-\lambda_1}}{\phi_0}\right)^{1/(3-\lambda_1)}\times\left(\frac{3-\lambda_1}{2}\right)^{(1-\lambda_1)/(3-\lambda_1)}t^{(1-\lambda_1)/(3-\lambda_1)}\ . \tag{70}$$

When $\lambda_1=3$, Eq. (68) becomes

$$\eta(t)=\exp\left(\sqrt{\frac{\phi_0{K_1}^3}{A}}t\right)\ , \tag{71}$$

and $t\to-\infty$ as $\eta\to 0$. The early-time evolution is

$$\phi(t)\to\phi_0{K_1}^3\exp\left(3\sqrt{\frac{\phi_0{K_1}^3}{A}}t\right)\ , \tag{72}$$

$$a(t)\to\sqrt{\frac{A}{\phi_0{K_1}^3}}\exp\left(-\sqrt{\frac{\phi_0{K_1}^3}{A}}t\right)\ , \tag{73}$$

as $t\to-\infty$. Differentiating Eq. (70), we deduce that the early-time models will be expanding as long as $\lambda_1>3$ or $\lambda_1<1$.

3. Minima

We now probe the $k=0$ vacuum models generated by this choice of coupling for the existence of expansion minima. Equation (27) for these universes becomes

$$u_{1*}=\left(\frac{1}{\lambda_1}\right)^{2/\alpha}, \tag{74}$$

at a minimum. For u_{1*} to be real for all α requires $\lambda_1>0$ which is guaranteed when $u\in(0,1)$. In general, however, we find

$$0<\lambda_1<1\leftrightarrow u_{1*}>1\,, \tag{75}$$

$$\lambda_1=1\leftrightarrow u_{1*}=1\,, \tag{76}$$

$$\lambda_1>1\leftrightarrow u_{1*}<1\,, \tag{77}$$

and when $u_1\in(0,1)$, minima are only present when $\lambda_1>1$. We have the early-time solution for this model [Eqs. (72) and (73)] in which $a(0)=0$ for $\lambda_1>1$ and the universe experiences a phase of contraction before bouncing and approaching late time general relativistic expansion. When $\lambda_1=1$, then $a(0)$ itself becomes a nonzero minimum $(a^2\rightarrow A/K_1\phi_0)$.

4. Exact solution

The models with $\alpha=1$ are not described by the solutions presented above. These models tend (as $\phi\rightarrow\phi_0$) to the theory of gravity proposed by Barker [11], for their evolution we find the exact results

$$\phi(\eta)=\frac{4\phi_0 K_1{}^{-\lambda_1}\eta^{-\lambda_1}}{(K_1{}^{-\lambda_1}\eta^{-\lambda_1}+1)^2}\,, \tag{78}$$

$$a^2(\eta)=\frac{A}{4\phi_0 K_1{}^{-\lambda_1}}\,\eta^{1+\lambda_1}(K_1{}^{-\lambda_1}\eta^{-\lambda_1}+1)^2\,. \tag{79}$$

When $\eta\rightarrow\infty$ however, we observe that there is no combination of parameters allowing $\phi\rightarrow\phi_0$. Consequently, we shall not pursue this model any further. Theories with $\alpha=2$ have been solved exactly and studied earlier by Barrow [12,59] and in Ref. [53].

B. Vacuum solutions ($k=-1$)

1. Late-time behavior

When $k=-1$ the integral equation for the field evolution becomes (setting $\eta_0=0$)

$$-\frac{1}{\lambda_1}\int\frac{du_1}{u_1{}^{\alpha/2}(1-u_1)}=\ln[K_1\tanh\eta]\,. \tag{80}$$

at late times. As $\eta\rightarrow\infty$, $\tanh\eta\rightarrow1-2e^{-2\eta}$. Demanding $u_1\rightarrow0$ in this extreme, Eq. (80) may be approximated by

$$-\frac{1}{\lambda_1}\,\frac{u_1{}^{1-\alpha/2}}{1-\alpha/2}\simeq\ln K_1-2\exp(-2\eta)\,. \tag{81}$$

As $\eta\rightarrow\infty$ the right-hand side is finite and we require $1-\alpha/2>0$ (i.e., $\alpha<2$) in order that the left-hand side does not diverge as $u_1\rightarrow0$. We also require $K_1=1$ to ensure that $\eta\rightarrow\infty$ and $u_1\rightarrow0$ correspond to the same limit. The right-hand side approaches zero from below as η increases and $u_1>0$; thus to keep the left-hand side negative we need $\lambda_1>0$. The form of $y(\eta)$ for the negatively curved vacuum models, selected from Eq. (19), approaches $Ae^{2\eta}/4$ as $\eta\rightarrow\infty$ and the late-time solutions are

$$\phi(\eta)\rightarrow\phi_0\left[1-\{\lambda_1(2-\alpha)\}^{2/(2-\alpha)}\exp\left(\frac{4\eta}{\alpha-2}\right)\right], \tag{82}$$

$$a^2(\eta)\rightarrow\frac{A}{4\phi_0}e^{2\eta}\left[1+\{\lambda_1(2-\alpha)\}^{2/(2-\alpha)}\exp\left(\frac{4\eta}{\alpha-2}\right)\right], \tag{83}$$

as $\eta\rightarrow\infty$. Integrating and asymptotically inverting the second of these expressions we obtain the $\eta(t)$ relation

$$\eta(t)\rightarrow\ln\left\{2\sqrt{\frac{\phi_0}{A}}t\left[1-\frac{1}{2}\{\lambda_1(2-\alpha)\}^{2/(2-\alpha)}\left(\frac{\alpha-2}{\alpha+2}\right)\times\left(2\sqrt{\frac{\phi_0}{A}}\right)^{4/(\alpha-2)}t^{4/(\alpha-2)}\right]\right\}. \tag{84}$$

Substituting this back into Eqs. (82) and (83) yields the asymptotic forms

$$\phi(t)\rightarrow\phi_0\left[1-\{\lambda_1(2-\alpha)\}^{2/(2-\alpha)}\times\left(2\sqrt{\frac{\phi_0}{A}}\right)^{4/(\alpha-2)}t^{4/(\alpha-2)}\right], \tag{85}$$

$$a(t)\rightarrow t\left[1+\frac{2}{\alpha+2}\{\lambda_1(2-\alpha)\}^{2/(\alpha-2)}\times\left(2\sqrt{\frac{\phi_0}{A}}\right)^{4/(\alpha-2)}t^{4/(\alpha-2)}\right], \tag{86}$$

as $t\rightarrow\infty$. We note the asymptotic approach of this model to the general-relativistic Milne universe at late times.

2. Early-time behavior

At early times $u_1\rightarrow1$ as $\eta\rightarrow0$ and

$$\ln(1-u_1)\simeq\lambda_1\ln[\tanh\eta]\,, \tag{87}$$

which leads to

$$\phi(\eta)\rightarrow\phi_0\eta^{\lambda_1}\,, \tag{88}$$

$$a^2(\eta)\rightarrow\frac{A}{\phi_0}\eta^{1-\lambda_1}\,. \tag{89}$$

The functional form of this early-time behavior is identical to that detailed in Eqs. (68)–(73), after enforcing the choice $K_1=1$.

3. Minima

Searching for stationary points in the scale-factor evolution for this theory, Eq. (27) identifies

$$\cosh(2\eta_*)=\lambda_1 u_{1*}^{\alpha/2}. \tag{90}$$

The cosh function is bounded below by unity, as is the value of $u_1{}^{-\alpha/2}$ when $u_1\in(0,1)$, which gives $\lambda_1\geqslant1$ as the condition for the existence of a stationary point. When $0<\lambda_1<1$ the universe expands monotonically away from an initial singularity at $\eta=0$. When $1\leqslant\lambda_1\leqslant 3$, $a\to\infty$ as $\eta\to 0$, the subsequent evolution will contain a minimum, allowing the initial contraction to ''bounce'' back into late-time expansion. When $\lambda_1>3$ the evolution is expanding at all times. The special case $\alpha=1$ is treated by transforming $\eta+\eta_0\to\tanh(\eta+\eta_0)$ in Eqs. (78) and (79). Solutions for the $\alpha=2$ case may be obtained from the solutions in Ref. [59] after applying the same transformation.

C. Vacuum solutions $(k=+1)$

1. Late-time behavior

The positively curved $(k=+1)$ vacuum solutions are governed by

$$-\frac{1}{\lambda_1}\int\frac{du_1}{u_1{}^{\alpha/2}(1-u_1)}=\ln[K_1\tan\eta]\ , \tag{91}$$

taking $\eta_0=0$. In the GR limit $(u_1\to 0)$ this is approximated by

$$-\frac{1}{\lambda_1}\frac{u_1{}^{1-\alpha/2}}{(1-\alpha/2)}\simeq\ln[K_1\tan\eta]\ . \tag{92}$$

In conjunction with Eq. (19) we obtain

$$\phi(\eta)\to\phi_0\left[1-\left\{\lambda_1\left(\frac{\alpha-2}{2}\right)\right\}^{2/(2-\alpha)}\ln^{2/(2-\alpha)}(K_1\tan\eta)\right], \tag{93}$$

$$a^2(\eta)\to\frac{A}{2\phi_0}\frac{\sin 2\eta}{\left[1-\left\{\lambda_1\left(\frac{\alpha-2}{2}\right)\right\}^{2/(2-\alpha)}\ln^{2/(2-\alpha)}(K_1\tan\eta)\right]}\ . \tag{94}$$

These expressions approach GR when $\alpha<2$ as $\eta\to\tan^{-1}(K_1{}^{-1})$ and when $\alpha>2$ as $\eta\to n\pi/2$ where n is an integer. The behavior when $\alpha=2$ can be extracted from the $k=0$ treatment in Ref. [59] by substituting $2(\eta+\eta_0)\to\tanh(\eta+\eta_0)$.

D. Radiation solutions $(k=0)$

1. Late-time behavior

When $k=0$ Eq. (22) leads to the asymptotic behavior

$$-\frac{1}{\lambda_1}\frac{u_1{}^{1-\alpha/2}}{(1-\alpha/2)}=\ln K_1-\frac{2\eta_0}{\eta}\ , \tag{95}$$

where we have picked $2\Gamma\eta_0=A$ to fix the arbitrary integration η_0. The requirement that $u_1\to 0$ as $\eta\to\infty$ then demands $K_1=1$ and $\alpha<2$. This upper bound on α, in conjunction with the lower limit implied by the requirement that $\omega'\omega^{-3}\to 0$, yields the powerful constraint $1/2<\alpha<2$. We examine models for which $u_1\in(0,1)$, corresponding to $\phi\in(0,\phi_0)$. In this case we have the constraint $\lambda_1\eta_0>0$. Using $y(\eta)$ from Eq. (21), the late-time evolution of ϕ and a as functions of η is

$$\phi(\eta)\to\phi_0\left[1-\left(\frac{(2-\alpha)\lambda_1\eta_0}{\eta}\right)^{2/(2-\alpha)}\right], \tag{96}$$

$$a^2(\eta)\to\frac{\Gamma\eta^2}{\phi_0}\left[1+\left(\frac{(2-\alpha)\lambda_1\eta_0}{\eta}\right)^{2/(2-\alpha)}\right]. \tag{97}$$

The latter of these allows us to deduce

$$\eta(t)\to\left(\frac{4\phi_0}{\Gamma}\right)^{1/4}t^{1/2}\left[1-\frac{1}{4}\left(\frac{\alpha-2}{\alpha-1}\right)\{\lambda_1(2-\alpha)\eta_0\}^{2/(2-\alpha)}\times\left(\frac{4\phi_0}{\Gamma}\right)^{1/[2(\alpha-2)]}t^{1/(\alpha-2)}\right], \tag{98}$$

and hence, the evolution as a function of cosmic time:

$$\phi(t)\to\phi_0\left[1-\left(\lambda_1^2\eta_0{}^2(\alpha-2)^2\sqrt{\frac{\Gamma}{4\phi_0}}\right)^{1/(2-\alpha)}t^{1/(\alpha-2)}\right], \tag{99}$$

$$a(t)\to\left(\frac{4\Gamma}{\phi_0}\right)t^{1/2}\left[1+\frac{\alpha}{4(\alpha-1)}\times\left\{\lambda_1^2\eta_0^2(\alpha-2)^2\sqrt{\frac{\Gamma}{4\phi_0}}\right\}^{1/(2-\alpha)}t^{1/(\alpha-2)}\right], \tag{100}$$

as $t\to\infty$, so there is a power-law approach to the GR solution at late times.

2. Early-time behavior

At early times, we analyze the behavior in the neighborhood of $u_1\to 1$: i.e.,

$$\ln(1-u_1)\simeq\lambda_1\ln\left[\frac{\eta}{\eta+2\eta_0}\right]. \tag{101}$$

The left-hand side of Eq. (101) tends to $-\infty$ as GR is approached. In order that the right-hand side approach the same limit we see that $\eta\to 0$ when $\lambda_1>0$ and $\eta\to-2\eta_0$ when $\lambda_1<0$. Analyzing the former case leads to

$$\phi(\eta)\to(2\eta_0)^{-\lambda_1}\phi_0\eta^{\lambda_1}\ , \tag{102}$$

$$a^2(\eta)\to\frac{(2\eta_0)^{1+\lambda_1}\Gamma}{\phi_0}\eta^{1-\lambda_1}\ , \tag{103}$$

after setting $2\Gamma\eta_0=A$. The behavior of this system as a function of cosmic time t mirrors that described in Eqs. (68)–(73) with $K_1{}^{\lambda_1}$ replaced by $(2\eta_0)^{-\lambda_1}$ and A replaced by $2\Gamma\eta_0$. The bound $\lambda_1\eta_0>0$ implies $\eta_0>0$ when $\lambda_1>0$. When $\lambda_1<0$, $\eta\to-2\eta_0$ at early times. The inequality $\lambda_1\eta_0>0$ implies $\eta_0<0$ and hence $-2\eta_0>0$. The evolution is approximately

$$\phi(\eta)\simeq\phi_0(-2\eta_0)^{\lambda_1}(\eta+2\eta_0)^{-\lambda_1}\ , \tag{104}$$

$$a^2(\eta)\simeq\frac{\Gamma}{\phi_0}(-2\eta_0)^{1-\lambda_1}(\eta+2\eta_0)^{1+\lambda_1}\ . \tag{105}$$

The t-parametrized behavior of these equations is given by Eqs. (68)–(73), after applying the transformations

$$K_1{}^{\lambda_1} \to (-2\eta_0)^{\lambda_1}, \tag{106}$$

$$\lambda_1 \to -\lambda_1, \tag{107}$$

$$A \to -2\Gamma\eta_0. \tag{108}$$

3. Minima

As with the vacuum solutions, we can search for turning points of a^2 when $\lambda_1 > 0$. Equation (27) is

$$\frac{\eta_*}{\eta_0} = -(1-\lambda_1 u_{1*}^{\alpha/2}), \tag{109}$$

where subscript * denotes the value of a quantity at the stationary point. When $0<\lambda_1<1$, the scale-factor expands away from $a=0$ at $\eta=0$ by Eq. (67). Monotonicity of η implies $\eta_* \geqslant 0$ and we know $\eta_0>0$ from the sign of λ_1, which together give rise to the inequality

$$u_{1*}^{\alpha/2} \geqslant \frac{1}{\lambda^1}. \tag{110}$$

The range of $0 \leqslant u_1 \leqslant 1$ together with the condition $\alpha>0$ confirm $\lambda_1 \geqslant 1$ as a necessary condition for the existence of minima. When $\lambda_1=1$, $\eta_*=0$, and the universe expands from a nonsingular state of size $a \simeq (2\eta_0)^{1+\lambda_1}\Gamma/\phi_0$. When $\lambda_1>1$ the solution is initially contracting, bounces at η_* and tends to general-relativistic behavior at late times. Equation (110) still applies when $\lambda_1<0$, and asserts that none of them can contain stationary points in the evolution of the scale-factor. This does not effect the models in which $-1<\lambda_1<0$, which begin from $a=0$ at $\eta=-2\eta_0$ and monotonically expand. When $\lambda_1 \leqslant -1$ they are initially contracting and, due to the absence of minima, will always contract and never approach late-time general-relativistic expansion.

4. Exact solution

The case $\alpha=1$ possesses a simple exact form, which is instructive. Solving Eqs. (22) and (21) leads to

$$\phi(\eta) = \frac{4\phi_0 K_1{}^{\lambda_1}\eta^{\lambda_1}(\eta+2\eta_0)^{\lambda_1}}{[(\eta+2\eta_0)^{\lambda_1}+K_1{}^{\lambda_1}\eta^{\lambda_1}]^2}, \tag{111}$$

$$a^2(\eta) = \frac{\Gamma\eta(\eta+\eta_0)[(\eta+2\eta_0)^{\lambda_1}+K_1{}^{\lambda_1}\eta^{\lambda_1}]^2}{4\phi_0 K_1{}^{\lambda_1}\eta^{\lambda_1}(\eta+2\eta_0)^{\lambda_1}}, \tag{112}$$

where in these expressions we have again made the special choice $2\Gamma\eta_0=A$. Examining the large η limit reveals that $\phi \to \phi_0$ if and only if $K_1=1$. In this case we integrate the asymptotic $a(\eta)$ to obtain

$$\eta(t) \to \left(\frac{4\phi_0}{\Gamma}\right)^{1/4} t^{1/2}\left[1-\frac{\eta_0}{2}\left(\frac{\Gamma}{4\phi_0}\right)^{1/4} t^{-1/2} + \frac{1}{4}\left(\lambda_1^2-\frac{1}{4}\right)\eta_0^2\left(\frac{\Gamma}{4\phi_0}\right)^{1/2}\frac{\ln t}{t}\right], \tag{113}$$

and hence

$$\phi(t) \to \phi_0\left(1-\lambda_1^2\eta_0^2\sqrt{\frac{\Gamma}{4\phi_0}}t^{-1}\right), \tag{114}$$

$$a(t) \to \left(\frac{4\Gamma}{\phi_0}\right)^{1/4} t^{1/2}\left[1+\frac{1}{4}\left(\lambda_1^2-\frac{1}{4}\right)\eta_0^2\left(\frac{\Gamma}{4\phi_0}\right)^{1/2}\frac{\ln t}{t}\right], \tag{115}$$

valid as $t\to\infty$. At early times the solution becomes

$$\phi(\eta) \to 2^{2-|\lambda_1|}\eta_0^{-|\lambda_1|}\phi_0\eta^{|\lambda_1|}, \tag{116}$$

$$a^2(\eta) \to \frac{2^{|\lambda_1|-2}\Gamma\eta_0^{1+|\lambda_1|}}{\phi_0}\eta^{1-|\lambda_1|}, \tag{117}$$

as $\eta\to 0$. The behavior of the t-parametrized version of this model is given by Eqs. (68)–(73) after applying the transformations

$$\phi_0 \to 2^{2-|\lambda_1|}\eta_0^{-|\lambda_1|}\phi_0, \tag{118}$$

$$A \to \Gamma\eta_0, \tag{119}$$

and remembering $K_1=1$. Minima are a feature of this model when $|\lambda_1|>1$.

E. Radiation solution ($k=-1$)

When $k=-1$, Eq. (56) becomes

$$-\frac{1}{\lambda_1}\int\frac{du_1}{u_1{}^{\alpha/2}(1-u_1)} - \ln K_1 = \ln\left|\frac{(A^2-\Gamma^2)^{1/2}e^{2(\eta+\eta_0)}-\Gamma-A}{(A^2-\Gamma^2)^{1/2}e^{2(\eta+\eta_0)}-\Gamma+A}\right|. \tag{120}$$

1. Late-time behavior

Expanding the right-hand side at large η and integrating on the left as $u_1\to 0$ leads to the approximate formula

$$\frac{1}{\lambda_1}\frac{u_1{}^{1-\alpha/2}}{(1-\alpha/2)} \simeq \frac{2A}{(A^2-\Gamma^2)^{1/2}}e^{-2\eta}, \tag{121}$$

where we have chosen $K_1=1$ to ensure $\phi\to\phi_0$ at late times. The right-hand side of this expression tends to zero at late times and consistency on the left as $u_1\to 0$ requires $1-\alpha/2>0$, or $\alpha<2$. If $\phi_\eta>0$, as it must be when $\phi\to\phi_0$ and $0\leqslant\phi\leqslant\phi_0$, we recover the inequality $\lambda_1 A>0$. Selecting the $k=-1$ form for $y(\eta)$ from Eq. (19) we have the approximate solutions

$$\phi(\eta) \to \phi_0\left[1-\left\{\frac{A\lambda_1(2-\alpha)}{(A^2-\Gamma^2)^{1/2}}\right\}^{2/(2-\alpha)} e^{-[4\eta/(2-\alpha)]}\right], \tag{122}$$

$$a^2(\eta) \to \frac{(A^2-\Gamma^2)^{1/2}}{4\phi_0}e^{2\eta}\left[1+\left\{\frac{A\lambda_1(2-\alpha)}{(A^2-\Gamma^2)^{1/2}}\right\}^{2/(2-\alpha)} \times e^{-[4\eta/(2-\alpha)]}\right], \tag{123}$$

as $\eta\to\infty$. Integrating and inverting Eq. (123) to next-to-lowest-order yields

$$\eta(t)\to\ln\left[\frac{2\phi_0^{1/2}t}{(A^2-\Gamma^2)^{1/4}}\left\{1+\frac{1}{2}\left(\frac{2+\alpha}{2-\alpha}\right)\left[\frac{A\lambda_1(2-\alpha)}{(A^2-\Gamma^2)^{1/2}}\right]^{2/(2-\alpha)}\right.\right.$$
$$\left.\left.\times\left[\frac{2\phi_0^{1/2}}{(A^2-\Gamma^2)^{1/4}}\right]^{-[4/(2-\alpha)]}t^{[-4/(2-\alpha)]}\right\}\right], \qquad (124)$$

$t\to\infty$ as $\eta\to\infty$ and hence

$$\phi(t)\to\phi_0\left[1-\left\{\frac{A\lambda_1(2-\alpha)}{4\phi_0}\right\}^{2/(2-\alpha)}t^{-[4/(2-\alpha)]}\right], \qquad (125)$$

$$a(t)\to t\left[1+\frac{2}{(2-\alpha)}\left\{\frac{A\lambda_1(2-\alpha)}{4\phi_0}\right\}^{2/(2-\alpha)}t^{-[4/(2-\alpha)]}\right], \qquad (126)$$

as $t\to\infty$. Again, we observe power-law approach to the GR Milne solution at large t since $\alpha<2$.

2. Early-time behavior

In the early-time limit we find $u_1\to 1$ and Eq. (120) becomes

$$\ln(1-u_1)\to\lambda_1\ln\left|\frac{(A^2-\Gamma^2)^{1/2}e^{2(\eta+\eta_0)}-\Gamma-A}{(A^2-\Gamma^2)^{1/2}e^{2(\eta+\eta_0)}-\Gamma+A}\right|, \qquad (127)$$

and as $u_1\to 1$, $\ln(1-u_1)\to-\infty$. To simplify the analysis we pick η_0 such that $u_1\to 1$ as $\eta\to 0$, which requires

$$e^{2\eta_0}=\left(\frac{A+\Gamma}{A-\Gamma}\right)^{1/2}. \qquad (128)$$

Combined with the $k=-1$ version of $y(\eta)$ from Eq. (21), this leads to the limiting forms

$$\phi(\eta)\to\phi_0\left(\frac{A+\Gamma}{A}\right)\eta^{\lambda_1}, \qquad (129)$$

$$a^2(\eta)\to\frac{A}{\phi_0}\left(\frac{A+\Gamma}{A}\right)^{-\lambda_1}\eta^{1-\lambda_1}, \qquad (130)$$

and the t-parametrized evolution will be that of the flat vacuum model described by Eqs. (68)–(73) after we apply the relabeling

$$K_1\to\frac{A+\Gamma}{A}. \qquad (131)$$

3. Minima

To complete the study of this class of solutions, we search for points where the gradient of a^2 vanishes. Equation (27) gives the condition

$$\left(1-\frac{\Gamma^2}{A^2}\right)^{1/2}=\frac{\lambda_1 u_1{}^{\alpha/2}}{\cosh[2(\eta+\eta_0)]}, \qquad (132)$$

for a stationary point to exist. Since $\alpha>0$, $u_1<1$, and $\cosh[2(\eta+\eta_0)]>1$ this expression is equivalent to the inequalities

$$\left(1-\frac{\Gamma^2}{A^2}\right)^{1/2}<\lambda_1,\quad \lambda_1>0, \qquad (133)$$

$$\left(1-\frac{\Gamma^2}{A^2}\right)^{1/2}>\lambda_1,\quad \lambda_1<0. \qquad (134)$$

Since $A^2>\Gamma^2$, we know that $0<(1-\Gamma^2/A^2)^{1/2}<1$. Hence, models with $\lambda_1>1$, in which a begins collapsing from infinity, will bounce to mimic GR expansion at late times.

F. Radiation solutions ($k=+1$)

If we select the necessary right-hand side from Eq. (22) for the finite, closed-universe models with $k=+1$, then Eq. (56) is

$$-\frac{1}{\lambda_1}\int\frac{du_1}{u_1{}^{\alpha/2}(1-u_1)}=\ln\left[K_1\left(\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right)\right]. \qquad (135)$$

As $u_1\to 0$, we obtain the following approximate expressions for the behavior of ϕ and a:

$$\phi(\eta)\to\phi_0\left\{1-\left[\lambda_1\left(\frac{\alpha-2}{2}\right)\right]^{2/(2-\alpha)}\times\ln^{2/(2-\alpha)}\left[K_1\left(\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right)\right]\right\}, \qquad (136)$$

$$a^2(\eta)\to\frac{1}{2\phi_0}\{\Gamma+(\Gamma^2+A^2)^{1/2}\sin[2(\eta+\eta_0)]\}$$
$$\times\left\{1-\left[\lambda_1\left(\frac{\alpha-2}{2}\right)\right]^{2/(2-\alpha)}\times\ln^{2/(2-\alpha)}\left[K_1\left(\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right)\right]\right\}^{-1}, \qquad (137)$$

which are valid as $\phi \to \phi_0$.

G. Perfect-fluid solutions $(k=0)$

We shall analyze the late-time behavior of Theory 1, specified in Sec. IV, by the device of using the solution defined by the choice of field evolution

$$\phi(\xi) = \phi_0 \exp(H \ln^B \xi) , \tag{138}$$

where H and B are constants. This gives rise to the generating function

$$g(\xi) = \frac{\xi^2}{HB} \ln^{1-B} \xi , \tag{139}$$

and the scale-factor

$$a^{3(2-\gamma)} = \frac{a_0^{3(2-\gamma)}}{HB\phi_0} \xi^2 \ln^{1-B} \xi \exp(H \ln^B \xi) . \tag{140}$$

From Eq. (140) it is clear that to keep the left-hand side positive we require the combination $(\xi-1)^{B-1} HB > 0$. This leads to

$$f(\xi) = \frac{\xi^2}{HB} \ln^{1-B} \xi + \frac{4-3\gamma}{4} \xi^2 - D , \tag{141}$$

and hence

$$2\omega(\xi) + 3 = \frac{4-3\gamma}{3(2-\gamma)^2} \frac{[(2/HB)\xi \ln^{1-B}\xi + ([1-B]/HB)\xi \ln^{-B}\xi + ([4-3\gamma]/2)\xi]^2}{(\xi^2/HB)\ln^{1-B}\xi + ([4-3\gamma]/4)\xi^2} . \tag{142}$$

We require $\phi \to \phi_0$ at late times and this occurs as $\xi \to 1$ when $B>0$ and as $\xi \to \infty$ when $B<0$. When $B=0$, $\phi = \phi_0 e^H$ and when $H=0$, $\phi = \phi_0$; in both cases the theory is GR at all times.

1. Late-time behavior

When $0<B<1$, we find from Eq. (32)

$$t \propto \ln^{(1-B)/(2-\gamma)} \xi , \tag{143}$$

as $\xi \to 1$. The power of the log is always positive for $0 \leqslant \gamma < 2$ and $t \to 0$ in this limit. Since we are only concerned with limits corresponding to large t we exclude the range $0<B<1$. When $B>1$ the coupling function tends to

$$2\omega(\xi) + 3 \to \frac{4-3\gamma}{3(2-\gamma)^2} \frac{(1-B)^2}{HB} \ln^{-(B+1)} \xi , \tag{144}$$

and Eq. (32) becomes

$$\begin{aligned} dt = {} & \frac{1-B}{3(2-\gamma)} \frac{a_0^{3(\gamma-1)}}{(HB\phi_0)^{(\gamma-1)/(2-\gamma)}} \left(\frac{4-3\gamma}{HB} \right)^{1/2} \\ & \times \xi^{2[(\gamma-1)/(2-\gamma)]} \ln^{(3\gamma-4-B\gamma)/[2(2-\gamma)]} \xi \\ & \times \exp\left[H \left(\frac{\gamma-1}{2-\gamma} \right) \ln^B \xi \right] d\xi . \end{aligned} \tag{145}$$

As $\xi \to 1$ this tends to

$$\begin{aligned} dt \simeq {} & \frac{1-B}{3(2-\gamma)} \frac{a_0^{3(\gamma-1)}}{(HB\phi_0)^{(\gamma-1)/(2-\gamma)}} \left(\frac{4-3\gamma}{HB} \right)^{1/2} \\ & \times (\xi-1)^{(3\gamma-4-B\gamma)/[2(2-\gamma)]} d\xi , \end{aligned} \tag{146}$$

and

$$t(\xi) \simeq F^{-1} (\xi-1)^{[\gamma(1-B)]/[2(2-\gamma)]} , \tag{147}$$

where

$$F = \frac{3\gamma}{2} \frac{(HB\phi_0)^{(\gamma-1)/(2-\gamma)}}{a_0^{3(\gamma-1)}} \left(\frac{HB}{4-3\gamma} \right)^{1/2} . \tag{148}$$

Eventually, this leads to

$$\xi(t) \to 1 + F^{[2(2-\gamma)]/[\gamma(1-B)]} t^{[2(2-\gamma)]/[\gamma(1-B)]} , \tag{149}$$

$$\phi(t) \to \phi_0 [1 + H F^{[2B(2-\gamma)]/[\gamma(1-B)]} t^{[2B(2-\gamma)]/[\gamma(1-B)]}] , \tag{150}$$

$$\begin{aligned} a(t) \to {} & \frac{a_0 F^{2/3\gamma}}{(HB\phi_0)^{1/[3(2-\gamma)]}} t^{2/3\gamma} \\ & \times \left[1 + \frac{2F^{[2(2-\gamma)]/[\gamma(1-B)]}}{3(2-\gamma)} t^{[2(2-\gamma)]/[\gamma(1-B)]} \right] , \end{aligned} \tag{151}$$

as $t \to \infty$, and there is power-law approach to the GR solutions in this limit. The coupling, in terms of the field, approaches

$$2\omega(\phi) + 3 \to \frac{4-3\gamma}{3(2-\gamma)^2} \frac{(1-B)^2}{B} H^{1/B} \ln^{-[(B+1)/B]} \left(\frac{\phi}{\phi_0} \right) . \tag{152}$$

The negativity of $d\xi/dt$ implied by Eq. (145) for $B>1$ implies that ξ approaches unity from above. As we noted earlier [after Eq. (140)], when $\xi>1$ positivity of B implies positivity of H and thus Eq. (138) confirms that $\phi \to \phi_0$ from above in these theories, i.e., $\phi \in (\phi_0, \infty)$. When $B<0$ the coupling function, as $\xi \to \infty$, is

$$2\omega(\xi) + 3 \to \frac{4-3\gamma}{3(2-\gamma)^2} \frac{4}{HB} \ln^{1-B} \xi , \tag{153}$$

and the temporal line element becomes

$$dt=\frac{2}{3(2-\gamma)}\frac{a_0^{3(\gamma-1)}}{(HB\phi_0)^{\gamma-1/2-\gamma}}\left(\frac{4-3\gamma}{HB}\right)^{1/2}\times\xi^{[2(\gamma-1)]/(2-\gamma)}\ln^{[\gamma(1-B)]/[2(2-\gamma)]}\xi d\xi\ . \tag{154}$$

Integrating, this gives

$$t(\xi)\simeq C^{-1}\xi^{\gamma/(2-\gamma)}\ln^{[\gamma(1-B)]/[2(2-\gamma)]}\xi\ , \tag{155}$$

where

$$C=\frac{3\gamma}{2}\frac{(HB\phi_0)^{(\gamma-1)/(2-\gamma)}}{a_0^{3(\gamma-1)}}\left(\frac{HB}{4-3\gamma}\right)^{1/2}. \tag{156}$$

Inverting asymptotically in t, this gives

$$\xi(t)\simeq C^{(2-\gamma)/\gamma}t^{(2-\gamma)/\gamma}\ln^{(B-1)/2}t^{(2-\gamma)/\gamma}\ , \tag{157}$$

and hence there is logarithmic approach to the GR perfect-fluid solutions

$$\phi(t)\to\phi_0[1+H\ln^B t^{(2-\gamma)/\gamma}]\ , \tag{158}$$

$$a(t)\to\frac{a_0C^{2/3\gamma}}{(HB\phi_0)^{1/[3(2-\gamma)]}}t^{2/3\gamma}\left[1+\frac{H}{3(2-\gamma)}\ln^B t^{(2-\gamma)/\gamma}\right], \tag{159}$$

as $t\to\infty$. The late-time behavior of the coupling as a function of ϕ is given by

$$2\omega(\phi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\frac{4H^{-1/B}}{B}\ln^{(1-B)/B}\left(\frac{\phi}{\phi_0}\right). \tag{160}$$

At large t, $\xi\to\infty$ and must do so from below to maintain positivity. Consequentially, $B<0$ implies $H<0$ and from Eq. (138) $\phi\to\phi_0$ from below, i.e., $\phi\in(0,\phi_0)$ in these models.

2. Early-time behavior

When $B>1$, $a(\xi)$ has a zero as $\xi\to0$ if $(-1)^BH\leqslant0$. At late times, $\xi\to1$ from above; however, this is a manifestation of the fact that $\xi(t)$, for the exact theory defined by Eq. (138), is not monotonic. That this is correct may be demonstrated using Eq. (32) and solving $\sqrt{2\omega+3}=0$ for ξ when $B=2$ then further showing that $(\sqrt{2\omega+3})_\xi\neq0$ at that point. As $\xi\to0$, $\ln\xi<0$ and $(-1)^{B-1}HB\geqslant0$ so that $a(\xi)\geqslant0$. When $B>0$, $(-1)^{B-1}H\geqslant0$ and hence $(-1)^BH\leqslant0$, i.e., the universe is always singular at $\xi=0$ in $B>1$ theories. In this limit

$$2\omega(\xi)+3\to\frac{(4-3\gamma)^2}{3(2-\gamma)^2}\ , \tag{161}$$

i.e., a BD theory.

When $B<0$, $a(\xi)$ has a zero as $\xi\to1$ if $H<0$. This arises since $\sqrt{2\omega+3}>0$ as $\xi\to1$ and ξ approaches unity from above ($\ln\xi>0$). As in the preceding paragraph, $a(\xi)>0$ implies $HB>0$, $B<0$ enforces $H<0$ and the evolution begins from a singularity at $\xi=0$. Again, the form of the coupling function at early times is given by Eq. (161), BD theory.

We will not present explicit solutions showing the approach to BD theory since these models are not early-time limits of the theories, defined in Sec. IV, that we are interested in; they are merely early-time limits of other theories which happen to asymptote (at late times) to the theories with which we are concerned.

We now highlight some special cases of these models for particular values of γ.

3. Dust models

These arise by substituting the choice $\gamma=1$ into the asymptotic relations already derived. When $B>1$ we find

$$\phi(t)\to\phi_0\left[1+\left(\frac{3}{2}\right)^{2B/(1-B)}H^{1/(1-B)}B^{B/(1-B)}t^{2B/(1-B)}\right], \tag{162}$$

$$a(t)\to\left(\frac{3}{2}\right)^{2/3}\frac{a_0}{\phi_0^{1/3}}t^{2/3}\left[1+\left(\frac{3}{2}\right)^{(1+B)/(1-B)}\times(HB)^{1/(1-B)}t^{2/(1-B)}\right]. \tag{163}$$

When $B<0$ the solutions tend to the GR solution only logarithmically

$$\phi(t)\to\phi_0[1+H\ln^Bt]\ , \tag{164}$$

$$a(t)\to\left(\frac{3}{2}\right)^{2/3}\frac{a_0}{\phi_0^{1/3}}t^{2/3}\left[1+\frac{H}{3}\ln^Bt\right]. \tag{165}$$

All of these expressions are valid as $t\to\infty$.

4. Inflationary models

Inflationary models driven by a false vacuum equation of state may be derived from the choice $\gamma=0$. Although there are varieties of inflationary universe with $-1/3>\gamma>0$, and these can easily be found from the formula for the general γ solutions given above, we shall confine our attention to the $\gamma=0$ case which is not described by the previous formulae. It offers an excellent approximation to many slowly changing scalar-field potentials. In this case we can view the scalar-tensor coupling as providing a second scalar field, thereby offering the chance for double inflation to occur. However, it is not sufficient simply to substitute $\gamma=0$ into the existing expressions since the qualitative structure of the solutions is different. When $B>1$, the form of $2\omega+3$ is as presented in Eq. (144) and the temporal line element of Eq. (145) can be approximated by

$$dt\simeq\frac{(1-B)\phi_0^{1/2}}{3a_0^3}\ln^{-1}\xi d\ln\xi\ , \tag{166}$$

as $\xi\to1$. Integrating this expression we find

$$t(\xi)\simeq\frac{(1-B)\phi_0^{1/2}}{3a_0^3}\ln(\ln\xi)\ , \tag{167}$$

which $\to\infty$ as $\xi\to 1$. This can be inverted, yielding

$$\xi(t)\simeq\exp\left\{\exp\left[\frac{3a_0^3 t}{(1-B)\phi_0^{1/2}}\right]\right\}, \tag{168}$$

and hence,

$$\phi(t)\to\phi_0\left[1+H\exp\left\{\frac{3a_0^3 Bt}{(1-B)\phi_0^{1/2}}\right\}\right], \tag{169}$$

$$a(t)\to\frac{a_0}{(HB\phi_0)^{1/6}}\exp\left(\frac{a_0^3 t}{2(\phi_0)^{1/2}}\right)\times\left[1+\frac{1}{3}\exp\left\{\frac{3a_0^3 t}{(1-B)\phi_0^{1/2}}\right\}\right], \tag{170}$$

as $t\to\infty$. Here we see explicitly the possibility of double inflation arising from the sequential effects of the ϕ field and the $p=-\rho$ stress. If $B<0$, $t\to\infty$ as $\xi\to\infty$ and $2\omega+3$ is given by Eq. (153). The differential $\xi-t$ relation, Eq. (154), is then well approximated by

$$dt\simeq\frac{2\phi_0^{1/2}}{3a_0^3}\exp\left(-\frac{H}{2}\ln^B\xi\right)d\ln\xi . \tag{171}$$

Making the substitution $\zeta=\ln^B\xi$ we can integrate the above equation to obtain

$$t(\xi)\simeq\frac{2\phi_0^{1/2}}{3a_0^3}\ln\xi\left[1-\frac{H}{2(B+1)}\ln^B\xi\right], \tag{172}$$

as ξ, and hence t, tend to infinity. Asymptotically, we obtain

$$\xi(t)\to\exp\left(\frac{3a_0^3 t}{2\phi_0^{1/2}}\right), \tag{173}$$

$$\phi(t)\to\phi_0\left[1+H\left(\frac{3a_0^3}{2\phi_0^{1/2}}\right)^B t^B\right], \tag{174}$$

$$a(t)\to\frac{a_0}{(HB\phi_0)^{1/6}}\left(\frac{3a_0^3}{2\phi_0^{1/2}}\right)^{(1-B)/6} t^{(1-B)/6}\exp\left(\frac{a_0^3 t}{2\phi_0^{1/2}}\right), \tag{175}$$

as $t\to\infty$.

5. The connection to the parameters of Theory 1

We now use the results derived earlier in this section to model the late-time behavior of Theory 1. Consider first universes in which $\phi\to\phi_0$ from above, i.e.,

$$\phi\in(\phi_0,\infty). \tag{176}$$

The approach of Theory 1 to the relativistic limit in this direction can be accurately modeled using the theory defined by Eq. (138) with $H>0$, $B>1$. From Eq. (152),

$$2\omega(\phi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\frac{(1-B)^2}{B}H^{1/B}\times\left[\left(\frac{\phi}{\phi_0}\right)-1\right]^{-[(B+1)/B]}, \tag{177}$$

as $\phi\to\phi_0$. This expression is essentially the same as definition of the Theory 1 coupling for $\phi\in(\phi_0,\infty)$ introduced in Sec. IV. Explicitly, we may obtain the asymptotic behavior of Theory 1 by making the following identifications between its parameters and the parameters of Eq. (138):

$$H=\left[\frac{6B_1(2-\gamma)^2(\alpha-1)}{(4-3\gamma)(\alpha-2)^2}\right]^{1/(\alpha-1)}, \tag{178}$$

$$B=\frac{1}{\alpha-1} . \tag{179}$$

The constraint on the $B>0$ models such that they approach GR at late times, namely $B>1$, is equivalent to $-(B+1)/B>-2$. This is a very restrictive condition because the function $-(B+1)/B$ is naturally bounded above by the value -1. Thus, for perfect-fluid universes with $0<\gamma<4/3$ Theory 1 can only be expected to converge to the general relativistic value of ϕ *from above* if $1<\alpha<2$. When ϕ converges to ϕ_0 from below, i.e., $\phi\in(0,\phi_0)$ we can approximate the late-time behavior of Theory 1 using the solutions for $H<0$, $B<0$. The asymptotic form of the coupling in this case is given by Eq. (160)

$$2\omega(\phi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\frac{4(-H)^{-1/B}}{(-B)}\left[1-\left(\frac{\phi}{\phi_0}\right)\right]^{(1-B)/B}, \tag{180}$$

as $\phi\to\phi_0$. The behavior of Theory 1 at late times may be found by substituting the expressions

$$H=\left[\frac{3B_1(2-\gamma)^2(-1)^\alpha}{2(4-3\gamma)(1-\alpha)}\right]^{1/(\alpha-1)}, \tag{181}$$

$$B=\frac{1}{1-\alpha}, \tag{182}$$

into the formulas describing the asymptotic evolution of the theory defined by Eq. (138). The choice $B<0$ leads to the constraint $(1-B)/B<-1$, which becomes $\alpha>1$ for Theory 1 as $\phi\to\phi_0$ from below.

VI. THEORY 2: $2\omega(\phi)+3=B_2|\ln(\phi/\phi_0)|^{-2\delta}$; $\delta>1/4$, $B_2>0$ const

The left-hand side of Eq. (20) for this choice of the coupling function, when $\phi\in(0,\phi_0)$, is

$$\int\frac{[2\omega(\phi)+3]^{1/2}}{\phi}d\phi=\begin{cases}-(\sqrt{B_2}/1-\delta)|\ln(\phi/\phi_0)|^{1-\delta}-\sqrt{3}\ln K_2, & \delta\neq 1,\\ -\sqrt{B_2}\ln|\ln(\phi/\phi_0)|-\sqrt{3}\ln K_2, & \delta=1,\end{cases} \tag{183}$$

where K_2 is an integration constant. We now investigate vacuum and radiation solutions for this case, under the assumption that $\phi \in (0,\phi_0)$.

A. Vacuum solutions ($k=0$)

Selecting the zero-curvature right-hand side from Eq. (20) and the necessary form for $y(\eta)$ from Eq. (19), Eq. (183) for $\delta \neq 1$ reads

$$\left[-\ln\left(\frac{\phi}{\phi_0}\right)\right]^{1-\delta} = -\lambda_2(1-\delta)\ln[K_2(\eta+\eta_0)] \ , \quad (184)$$

where $\lambda_2 = \sqrt{3/B_2}$. The left-hand side is positive, for the right-hand side to follow suit as $\eta \to \infty$ requires $\lambda_2(1-\delta)<0$. As $\eta \to \infty$, the right-hand side$\to\infty$ and we must have $\delta>1$ for this to occur on the left as $\phi \to \phi_0$. This implies $\lambda_2>0$ and we obtain for the field and the scale-factor, when $\delta \neq 1$, the exact expressions

$$\phi(\eta) = \phi_0 \exp[-\{\lambda_2(\delta-1)\ln[K_2(\eta+\eta_0)]\}^{1/(1-\delta)}] \ , \quad (185)$$

$$a^2(\eta) = \frac{A}{\phi_0}(\eta+\eta_0)\exp\{[\lambda_2(\delta-1)\ln \times [K_2(\eta+\eta_0)]]^{1/(1-\delta)}\} \ . \quad (186)$$

When $\delta=1$, the conformal time-evolution of the field and the scale-factor are given by

$$\phi(\eta) = \phi_0 \exp[-K_2(\eta+\eta_0)]^{-\lambda_2} \ , \quad (187)$$

$$a^2(\eta) = \frac{A}{\phi_0}(\eta+\eta_0)\exp\{[K_2(\eta+\eta_0)]^{-\lambda_2}\} \ . \quad (188)$$

1. Late-time behavior

As $\eta \to \infty$ the late-time behavior can be modeled by Eqs. (62)–(64) under the simultaneous transformations, $\lambda \to \lambda_2$, $\alpha \to 2\delta$. Setting $\eta_0=0$ for simplicity, we see that the scale-factor tends to zero as $\eta \to 0$ when $\delta>1$. When $\delta \neq 1$ we require $\lambda_2>0$ if ϕ is to tend to ϕ_0 as $\eta \to \infty$. We find

$$\phi(t) \to \phi_0\left[1 - K_2^{-\lambda_2}\left(\frac{9\phi_0}{4A}\right)^{-\lambda_2/3} t^{-2\lambda_2/3}\right] \ , \quad (189)$$

which approaches ϕ_0 as $t \to \infty$. The asymptotic form of the scale-factor is

$$a(t) \to \left(\frac{3A}{2\phi_0}\right)^{1/3} t^{1/3}\left[1 + K_2^{-\lambda_2}\left(\frac{1-\lambda_2}{3-2\lambda_2}\right) \times \left(\frac{9\phi_0}{4A}\right)^{-\lambda_2/3} t^{-2\lambda_2/3}\right] \ , \quad (190)$$

with limiting behavior $a \propto t^{1/3}$ as $t \to \infty$.

2. Early-time behavior

Examining the form of $a(\eta)$ when $\delta \neq 1$ in Eq. (186), we see that the choice $\eta_0=0$ ensures $a(0)=0$. As $\eta \to 0$, $\ln\eta \to -\infty$ and the exponential factor in a either tends to zero or a constant, depending upon the sign of $1-\delta$. However, the bound $1-\delta<0$ on the power of the logarithm in the exponential guarantees that we always obtain $a(\eta) \propto \sqrt{A/\phi_0}\,\eta^{1/2}$ as $\eta \to 0$ and hence $\eta \to (9\phi_0/4A)^{1/3}t^{2/3}$. This final expression implies $t \to 0$ as $\eta \to 0$ and the early-time formulae for $\phi(t)$ and $a(t)$ are identical to Eqs. (60) and (63), under the transformations $\lambda \to \lambda_2$, $\alpha \to 2\delta$.

When $\delta=1$, the dominant behavior in $a(\eta)$ as $\eta \to 0$ is contained in the exponential, from which we may conclude (after setting $\eta_0=0$) that

$$t \propto \eta^{(3/2)+\lambda_2}\exp\left[\frac{1}{2}(K_2\eta)^{-\lambda_2}\right] \ . \quad (191)$$

This may be inverted asymptotically in η^{-1}. To first order we obtain $\eta \sim K_2^{-1}(2\ln t)^{-1/\lambda_2}$ as η and t tend to zero [noting the sign of λ_2 and the necessary monotonicity of $\eta(t)$]. The next-order corrections at early-times follow by substituting this lowest-order result into the weakest dependence in Eq. (191), i.e., the power-law factor, and solving for η. This yields

$$\eta \sim K_2^{-1}\left\{2\left[\ln t + \left(1+\frac{3}{2\lambda_2}\right)\ln(\ln t)\right]\right\}^{-1/\lambda_2} \ , \quad (192)$$

and hence

$$\phi(t) \to \phi_0 t^{-2}\ln^{-(3/\lambda_2)-2} t \ , \quad (193)$$

$$a(t) \propto t(\ln t)^{1+1/\lambda_2} \ , \quad (194)$$

as $t \to 0$.

3. Minima

Examining the $\delta \neq 1$ vacuum models for the presence of minima we find

$$(a^2)_\eta = \frac{A}{\phi_0}\exp\{-[\lambda_2(\delta-1)\ln(K_2\eta)]^{1/(1-\delta)}\} \times\left[1+\left\{-\frac{[\lambda_2(\delta-1)]^{1/(1-\delta)}}{1-\delta}\ln^{\delta/(1-\delta)}K_2\eta\right\}\right] \ . \quad (195)$$

For the exponential to tend to zero we require its argument to tend to $-\infty$. The bound on δ, namely $\delta>1$, implies that $\eta \to K_2^{-1}$ for this to happen. At this point $\ln^{\delta/(1-\delta)}(K_2\eta) \to \infty$; nevertheless, this logarithmic divergence will be insufficient to counter the exponential convergence of the prefactor. We can also expect to see turning points in the evolution of a when the factor in square brackets vanishes. This happens when $\eta = \eta_*$, where

$$\eta_* = K_2^{-1}\exp\left[\frac{(-\lambda_2)^{-1/\delta}}{1-\delta}\right] \ . \quad (196)$$

One can show that the conditions for the factor in square brackets in Eq. (195) to vanish also guarantee that the exponential will be well behaved. When $\delta=1$ we have

$$(a^2)_\eta = \frac{A}{\phi_0}[1+\lambda_2 K_2^{-\lambda_2}\eta^{-\lambda_2}]\exp\{-K_2^{-\lambda_2}\eta^{-\lambda_2}\} \ . \quad (197)$$

$K_2\eta>0$ and so the exponential will vanish when $\eta\to 0$. When this happens the factor in square brackets diverges, however, its divergence is quashed by the rapid convergence of the exponential, confirming the presence of a stationary point at $\eta=0$. There will also exist a stationary point in the evolution of a when the square-bracketed factor itself vanishes in Eq. (197). This occurs at η_*, where

$$\eta_*=(-\lambda_2)^{1/\lambda_2}K_2{}^{-1}\ . \tag{198}$$

At this point $-(K_2\eta)^{-\lambda_2}=\lambda_2^{-1}>0$ and the exponential is well behaved.

B. Vacuum solutions ($k=-1$)

Similarly, we analyze the behavior of the negatively curved models. Selecting the $k=-1$ versions of Eqs. (20) and (19), we obtain for the field and the metric, when $\delta\neq 1$, the expressions

$$\phi(\eta)=\phi_0\exp\{-[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\times[K_2\tanh(\eta+\eta_0)]\}\ , \tag{199}$$

$$a^2(\eta)=\frac{A}{2\phi_0}\sinh[2(\eta+\eta_0)t]\times\exp\{[[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\times[K_2\tanh(\eta+\eta_0)]]\}\ . \tag{200}$$

This solution will approach the Milne model at late times under the conditions $0<\delta<1$, $K_2=1$. The approach of ϕ to ϕ_0 from below requires $\lambda_2<0$. When $\delta=1$, setting $\eta_0=0$, we have

$$\phi(\eta)=\phi_0\exp\{-K_2{}^{-\lambda_2}\tanh^{-\lambda_2}\eta\}\ , \tag{201}$$

$$a^2(\eta)=\frac{A}{2\phi_0}\sinh(2\eta)\exp\{K_2{}^{-\lambda_2}\tanh^{-\lambda_2}\eta\}\ . \tag{202}$$

Because there is no choice of K_2 which allows ϕ to tend to ϕ_0 at late times we do not pursue this model any further.

1. Late-time behavior

The late-time behavior is given by Eqs. (84)–(86) under the substitutions $\lambda_1=\lambda_2$, $\alpha=2\delta$.

2. Early-time behavior

Picking $\eta_0=0$, $K_2=1$ in Eqs. (199) and (200), we have, as $\eta\to 0$,

$$\phi(\eta)\to\phi_0\exp\{-[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\eta\}\ , \tag{203}$$

$$a^2(\eta)\to\frac{A}{\phi_0}\,\eta\exp\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\eta\}\ . \tag{204}$$

We know $0<\delta<1$ and hence $1/(1-\delta)>1$ and $a^2(\eta)$ will be dominated by the exponential factor as $\eta\to 0$, and hence $a\to 0$. Integrating and inverting $a(\eta)$, we obtain

$$\eta(t)\simeq\exp\left\{\frac{2^{1-\delta}\ln^{1-\delta}(-t)}{\lambda_2(\delta-1)}\left[1+3\left(\frac{2^{-\delta}}{\lambda_2}\right)\ln^{-\delta}(-t)\right]\right\}\ , \tag{205}$$

where $t\to-\infty$ as $\eta\to 0$. Using this relation the early-time behavior is

$$\phi(t)\to\phi_0t^{-2}\exp\left\{-3\left(\frac{2^{1-\delta}}{\lambda_2(1-\delta)}\right)\ln^{1-\delta}(-t)\right\}\ , \tag{206}$$

$$a^2(t)\to\sqrt{\frac{A}{\phi_0}}t\exp\left\{\frac{2^{1-\delta}\ln^{1-\delta}(-t)}{\lambda_2(1-\delta)}\right\}\ , \tag{207}$$

as $t\to-\infty$.

3. Minima

Differentiating Eq. (200) yields

$$(a^2)_\eta=\frac{A}{\phi_0}\{\cosh(2\eta)+(-\lambda_2)^{1/(1-\delta)}(1-\delta)^{\delta/(1-\delta)}\ln^{\delta/(1-\delta)}\times(\tanh\eta)\}\exp\{[\lambda_2(1-\delta)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\times[K_2\tanh(\eta+\eta_0)]\}\ . \tag{208}$$

The exponential cannot tend to zero, since its argument is always positive. If we examine the prefactor we find stationary points exist at $\eta=\ \eta_*$, where

$$\cosh(2\eta_*)=-(-\lambda_2)^{1/(1-\delta)}(1-\delta)^{\delta/(1-\delta)}\times[\ln(\tanh\eta_*)]^{\delta/(1-\delta)}\ . \tag{209}$$

This is not soluble analytically, although we may gain a bound on its value by demanding that the cosh function always be greater than unity. We obtain

$$\eta_*<\operatorname{arctanh}\left\{\exp\left(-\frac{\lambda_2^{-1/\delta}}{1-\delta}\right)\right\}\ . \tag{210}$$

C. Vacuum solutions ($k=+1$)

When ϕ lies in the range $0<\phi<\phi_0$, and $\delta\neq 1$ we have the exact solution

$$\phi(\eta)=\phi_0\exp\{-[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\times[K_2\tan(\eta+\eta_0)]\}\ , \tag{211}$$

$$a^2(\eta)=\frac{A}{2\phi_0}\sin[2(\eta+\eta_0)]\exp\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\times[K_2\tan(\eta+\eta_0)]\}\ . \tag{212}$$

In the particular case $\delta=1$, we have instead

$$\phi(\eta)=\phi_0\exp\{-K_2{}^{-\lambda_2}\tan^{-\lambda_2}(\eta+\eta_0)\}\ , \tag{213}$$

$$a^2(\eta)=\frac{A}{2\phi_0}\sin[2(\eta+\eta_0)]\exp\{-K_2{}^{-\lambda_2}\tan^{-\lambda_2}(\eta+\eta_0)\}\ . \tag{214}$$

D. Radiation solutions $(k=0)$

Fixing the origin of conformal time in Eqs. (22) and (21) such that $2\Gamma\eta_0=A$, we obtain the exact results for $\delta\neq 1$:

$$\phi(\eta)=\phi_0\exp\left\{-[\lambda_2(\delta-1)]^{1/(1-\delta)}\times\ln^{1/(1-\delta)}\left[K_2\left|\frac{\eta}{\eta+2\eta_0}\right|\right]\right\}, \tag{215}$$

$$a^2(\eta)=\frac{\Gamma}{\phi_0}\eta(\eta+2\eta_0)\exp\left\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\times\left[K_2\left|\frac{\eta}{\eta+2\eta_0}\right|\right]\right\}, \tag{216}$$

In the special case of $\delta=1$, we have the exact relations

$$\phi(\eta)=\phi_0\exp\left[-\left|\frac{K_2\eta}{\eta+2\eta_0}\right|^{-\lambda_2}\right], \tag{217}$$

$$a^2(\eta)=\frac{\Gamma}{\phi_0}\eta(\eta+2\eta_0)\exp\left[\left|\frac{K_2\eta}{\eta+2\eta_0}\right|^{-\lambda_2}\right]. \tag{218}$$

To ensure $a^2>0$, we require $\eta(\eta+2\eta_0)>0$ and the modulus signs in the above expressions can be dropped.

1. Late-time behavior

The asymptotic behavior may be obtained from Eqs. (98)–(100), by applying the transformations $\alpha\to 2\delta$, $\lambda_1\to\lambda_2$. When $\delta\neq 1$ Eq. (215) serves to bound the allowed parameter values in order that $\phi\to\phi_0$ as $\eta\to\infty$. We can see that we require $\delta\leqslant 1$ and $K_2=1$, delimiting the allowed range of δ to $1/4<\delta<1$. We take $\lambda_2\eta_0>0$ to ensure both sides of Eq. (22) exist on the correct domains. Examining Eq. (217), we see that when $\delta=1$ there is no nontrivial choice of K_2 which permits $\phi\to\phi_0$ at late times. Therefore, we exclude it from further analysis.

2. Early-time behavior

At early times when $\delta\neq 1$ the behavior is harder to ascertain and we need to make use of logarithmic approximations. From Eq. (217), by demanding that $\phi\in(0,\phi_0)$, we see that the lower limit of η occurs as

$$\text{(i)}\quad \eta\to 0,\quad \lambda_2>0,\ \eta_0>0 \tag{219}$$

or

$$\text{(ii)}\quad \eta\to -2\eta_0,\quad \lambda_2<0,\ \eta_0<0. \tag{220}$$

Case (i) gives

$$a(\eta)\simeq\eta^{1/2}\exp\left\{\frac{1}{2}[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\left(\frac{\eta}{2\eta_0}\right)\right\}, \tag{221}$$

which, for the range of values of δ to which we are confined, is dominated by the exponential at small η. Integrating this expression and inverting approximately then leads to

$$\eta(t)=2\eta_0\exp\left\{-\frac{2^{-\delta}}{\lambda_2(1-\delta)}\left[\ln(-t)+3\frac{2^{-\delta}}{\lambda_2(1-\delta)}\times\ln^{1-\delta}(-t)\right]^{1-\delta}\right\}, \tag{222}$$

and so $t\to-\infty$ as $\eta\to 0$. The field and the metric are then given by

$$\phi(t)\simeq t^{-2}\exp\left\{3\frac{2^{1-\delta}}{\lambda_2(\delta-1)}\ln^{1-\delta}(-t)\right\}, \tag{223}$$

$$a(t)\simeq t\exp\left\{\frac{2^{1-\delta}}{\lambda_2(1-\delta)}\ln^{1-\delta}(-t)\right\}, \tag{224}$$

as $t\to-\infty$. Case (ii) can be modeled by applying the transformations: $\eta_0\to-\eta_0$, $\eta\to\eta+2\eta_0$, $\lambda_2\to-\lambda_2$, in this order, to Eqs. (222)–(224).

3. Minima

Since the scale-factor is infinite at early-times, we know there must exist at least one minimum in its evolution, in order that we obtain general-relativistic expansion at late times. Differentiating Eq. (216), we find

$$(a^2)_\eta=\frac{2\Gamma}{\phi_0}\left[\eta+\eta_0+\eta_0(-\lambda_2)^{1/(1-\delta)}(1-\delta)^{\delta/(1-\delta)}\times\ln^{\delta/(1-\delta)}\left(\frac{\eta}{\eta+2\eta_0}\right)\right]\times\exp\left\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\left(\frac{\eta}{\eta+2\eta_0}\right)\right\}. \tag{225}$$

For the parameter choices we are confined to, the exponential factor in the above expression is a monotonic function, existing in the range $(1,\infty)$. Thus we search for zeros of the prefactor, finding them to exist at η_*, where

$$\eta_*+\eta_0+\eta_0(-\lambda_2)^{1/(1-\delta)}(1-\delta)^{\delta/(1-\delta)}\times\ln^{\delta/(1-\delta)}\left(\frac{\eta_*}{\eta_*+2\eta_0}\right)=0, \tag{226}$$

which is nonanalytic and must be solved numerically for particular δ,λ_2, and η_0.

E. Radiation solutions $(k=-1)$

For the negatively-curved models with $\delta\neq 1$ we have exactly

$$\phi(\eta)=\phi_0\exp\left\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\times\ln^{1/(1-\delta)}\left[K_2\left|\frac{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]\right\}, \tag{227}$$

$$a^2(\eta)=\frac{1}{2\phi_0}[-\Gamma+(\Gamma^2+A^2)^{1/2}\sinh[2(\eta+\eta_0)]]\times\exp\left\{-[\lambda_2(\delta-1)]^{1/(1-\delta)}\times\ln^{1/(1-\delta)}\left[K_2\left|\frac{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]\right\}. \tag{228}$$

When $\delta=1$ these become

$$\phi(\eta)=\phi_0\exp\left\{-K_2{}^{-\lambda_2}\left|\frac{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|^{-\lambda_2}\right\}, \tag{229}$$

$$a^2(\eta)=\frac{1}{2\phi_0}\{-\Gamma+(A^2-\Gamma^2)^{1/2}e^{2(\eta+\eta_0)}\}\exp\left\{K_2{}^{-\lambda_2}\left|\frac{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tanh(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|^{-\lambda_2}\right\}. \tag{230}$$

1. Late-time behavior

The late-time behavior may be derived from the Theory 1 solutions, by the simultaneous transformations $\lambda_1\to\lambda_2$, $\alpha\to2\delta$. When $\delta\neq1$, Eqs. (125) and (126) require $A\lambda_2>0$ and $\delta<1$ to ensure late-time approach to GR. When $\delta=1$, there exists no value of K_2 that will admit a tendency of ϕ to ϕ_0 at large η, and for this reason we pursue these models no further.

2. Early-time behavior

We make the simplifying choice for the origin of η time:

$$e^{2\eta_0}=\left(\frac{A+\Gamma}{A-\Gamma}\right)^{1/2}. \tag{231}$$

Equation (227) for the $\delta\neq1$ solution then becomes

$$\phi(\eta)\simeq\phi_0\exp\left\{-\left[\lambda_2(\delta-1)\ln\left[\left(\frac{\Gamma+A}{A}\right)\eta\right]\right]^{1/(1-\delta)}\right\}, \tag{232}$$

as $\eta\to0$. When $\lambda_2>0$, $\phi\to0$ as $\eta\to0$. When $\lambda_2<0$, however, there only exist solutions when $(-1)^{1/(1-\delta)}$ is real. For the limiting form of the solution, we find

$$\eta(t)\propto t^{2/3}, \tag{233}$$

$$\phi(t)\propto\exp\left\{-\left[\frac{2\lambda_2}{3}(\delta-1)\ln t\right]^{1/(1-\delta)}\right\}, \tag{234}$$

$$a(t)\propto t^{1/3}, \tag{235}$$

as η, and hence t, tend to zero.

3. Minima

We define

$$\Psi\equiv\frac{e^{2\eta}-1}{e^{2\eta}-e^{-4\eta_0}}, \tag{236}$$

such that

$$(a^2)_\eta=\frac{1}{\phi_0}\left\{(A^2-\Gamma^2)^{1/2}\cosh[2(\eta+\eta_0)]+[-\Gamma+(A^2-\Gamma^2)^{1/2}\sinh[2(\eta+\eta_0)]](-\lambda_2)^{1/(1-\delta)}\times(1-\delta)^{\delta/(1-\delta)}\left[\frac{e^{2\eta}(1-e^{-4\eta_0})}{(e^{2\eta}-e^{-4\eta_0})^2}\right]\ln^{\delta/(1-\delta)}\Psi\right\}\times\exp\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\Psi\}. \tag{237}$$

At a stationary point in the evolution of the scale-factor we require the expression on the right-hand side of Eq. (237) to vanish. Since a zero of the exponential would require $\phi\to\infty$, which is outside our range of consideration, we search for zeros of the prefactor. Minima exist at η_*, given by the implicit formula

$$(A^2-\Gamma^2)^{1/2}\cosh[2(\eta_*+\eta_0)]+[-\Gamma+(A^2-\Gamma^2)^{1/2}\times\sinh[2(\eta_*+\eta_0)]](-\lambda_2)^{1/(1-\delta)}(1-\delta)^{\delta/(1-\delta)}\times\left[\frac{e^{2\eta_*}(1-e^{-4\eta_0})}{(e^{2\eta_*}-e^{-4\eta_0})^2}\right]\ln^{\delta/(1-\delta)}\left(\frac{e^{2\eta_*}-1}{e^{2\eta_*}-e^{-4\eta_0}}\right)=0. \tag{238}$$

F. Radiation models $(k=+1)$

For the closed models with $\delta\neq 1$ we have

$$\phi(\eta)=\phi_0\exp\left\{-[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\right.\\ \left.\times\left[K_2\left|\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]\right\}, \tag{239}$$

$$a^2(\eta)=\frac{1}{2\phi_0}(\Gamma+(\Gamma^2+A^2)^{1/2}\sin[2(\eta+\eta_0)])\\ \times\exp\left\{[\lambda_2(\delta-1)]^{1/(1-\delta)}\ln^{1/(1-\delta)}\right.\\ \left.\times\left[K_2\left|\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]\right\}, \tag{240}$$

and when $\delta=1$ these become

$$\phi(\eta)=\phi_0\exp\left\{-\exp\left(-\lambda_2\ln\left[K_2\left|\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]\right)\right\}, \tag{241}$$

$$a^2(\eta)=\frac{1}{2\phi_0}(\Gamma+(\Gamma^2+A^2)^{1/2}\sin[2(\eta+\eta_0)])\exp\left\{\exp\left(-\lambda_2\ln\left[K_2\left|\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]\right)\right\}. \tag{242}$$

G. Perfect-fluid solutions $(k=0)$

When $\phi\to\phi_0$ from above, i.e., $\phi\in(\phi_0,\infty)$ the approach of Theory 2 to the general relativistic limit can be accurately modeled using the theory defined by Eq. (138) with $H>0$, $B>1$. Recall Eq. (152):

$$2\omega(\phi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\frac{(1-B)^2}{B}H^{1/B}\ln^{-(B+1)/B}\left(\frac{\phi}{\phi_0}\right), \tag{243}$$

as $\phi\to\phi_0$. Thus we can deduce the late-time evolution of Theory 2 by enforcing the relations on the perfect-fluid solutions presented in Sec. V

$$H=\left[\frac{3B_2(2-\gamma)^2(2\delta-1)}{4(4-3\gamma)(\delta-1)^2}\right]^{1/(2\delta-1)}, \tag{244}$$

$$B=\frac{1}{2\delta-1}. \tag{245}$$

when $\phi\to\phi_0$ from above. Theory 2 can only tend to GR at late times with ϕ going to ϕ_0 from above if $1/2<\delta<1$, as follows from the constraint $B>0$. When $\phi\to\phi_0$ from below the behavior of the coupling is given by Eq. (160):

$$2\omega(\phi)+3\to\frac{(4-3\gamma)}{3(2-\gamma)^2}\frac{4H^{-1/B}}{B}\ln^{(1-B)/B}\left(\frac{\phi}{\phi_0}\right), \tag{246}$$

The late-time behavior of Theory 2 when ϕ is in the range $\phi\in(0,\phi_0)$ is then given by Eqs. (138)–(175) under the substitutions

$$H=\left[\frac{3(2-\gamma)^2B_2}{4(4-3\gamma)(1-2\delta)}\right]^{1/(2\delta-1)}, \tag{247}$$

$$B=\frac{1}{1-2\delta}. \tag{248}$$

The bound $B<0$ leads to the constraint $\delta>1/2$, hence there exists a wide spectrum of models with $0<\gamma<4/3$ in which $\phi\to\phi_0$ from below.

VII. THEORY 3: $2\omega(\phi)+3=B_3|1-(\phi/\phi_0)^\beta|^{-1}$, $\beta>0$, $B_3>0$ const

For this choice of the coupling function, we find

$$\int\frac{(2\omega+3)^{1/2}}{\phi}d\phi=-\frac{\sqrt{B_3}}{\beta}\ln\left|\frac{1+\sqrt{u_3}}{1-\sqrt{u_3}}\right|-\sqrt{3}\ln K_3, \tag{249}$$

where $0<\phi<\phi_0$ and

$$u_3=1-\left(\frac{\phi}{\phi_0}\right)^\beta. \tag{250}$$

A. Vacuum solutions $(k=0)$

Starting with the flat $k=0$ models with $\phi\in(0,\phi_0)$, we obtain

$$\frac{1-\sqrt{u_3}}{1+\sqrt{u_3}}=(K_3\eta)^{\lambda_3\beta}, \tag{251}$$

with $\lambda_3=\sqrt{3/B_3}$ and fixing $\eta_0=0$. Using Eq. (19) we can then deduce the evolution of the field and the metric to be

$$\phi(\eta)=\frac{4^{1/\beta}\phi_0(K_3\eta)^{\lambda_3}}{[1+(K_3\eta)^{\lambda_3\beta}]^{2/\beta}}, \tag{252}$$

$$a^2(\eta)=\frac{AK_3{}^{-\lambda_3}}{4^{1/\beta}\phi_0}\eta^{1-\lambda_3}[1+(K_3\eta)^{\lambda_3\beta}]^{2/\beta}. \tag{253}$$

1. Late-time behavior

Examining the form of Eq. (252) we see that there is no choice of K_3 for which $\phi\to$const at large η, precluding any possible approach to GR at late times. In spite of this, we find for the asymptotic behavior

$$\phi(\eta)\to 4^{1/\beta}\phi_0 K_3^{-|\lambda_3|}\eta^{-|\lambda_3|}, \tag{254}$$

$$a^2(\eta)\to \frac{AK_3^{|\lambda_3|}}{4^{1/\beta}\phi_0}\eta^{1+|\lambda_3|}. \tag{255}$$

At late times we find

$$\eta(t)\to\left[\left(\frac{3+|\lambda_3|}{2}\right)\left(\frac{4^{1/\beta}\phi_0}{AK_3^{|\lambda_3|}}\right)^{1/2}\right]^{2/(3+|\lambda_3|)} t^{2/(3+|\lambda_3|)}, \tag{256}$$

$$\phi(t)\to\left(\frac{4^{1/\beta}\phi_0}{K_3^{|\lambda_3|}}\right)^{3/(3+|\lambda_3|)} \times\left(\frac{3+|\lambda_3|}{2A^{1/2}}\right)^{-[(2|\lambda_3|)]/(3+|\lambda_3|)]} t^{-[(2|\lambda_3|)]/(3+|\lambda_3|)]}, \tag{257}$$

$$a(t)\to\left(\frac{AK_3^{|\lambda_3|}}{4^{1/\beta}\phi_0}\right)^{1/(3+|\lambda_3|)} \times\left(\frac{3+|\lambda_3|}{2}\right)^{(1+|\lambda_3|)/(3+|\lambda_3|)} t^{(1+|\lambda_3|)/(3+|\lambda_3|)}. \tag{258}$$

Since there is no late-time approach to GR, we do not pursue the early-time behavior or probe for the existence of minima in these models.

B. Vacuum solutions $(k=-1)$

Similarly, for the $k=-1$ cases we have

$$\frac{1-\sqrt{u_3}}{1+\sqrt{u_3}}=K_3^{\lambda_3\beta}\tanh^{\lambda_3\beta}\eta, \tag{259}$$

and convergence to GR as $\eta\to\infty$ demands $K_3=1$. By direct comparison with the solution for flat models we obtain the exact results for the evolution of ϕ and a

$$\phi(\eta)=\frac{4^{1/\beta}\phi_0\tanh^{\lambda_3}\eta}{(1+\tanh^{\lambda_3\beta}\eta)^{2/\beta}}, \tag{260}$$

$$a^2(\eta)=\frac{A}{2^{\beta+2/\beta}\phi_0}\sinh(2\eta)\frac{(1+\tanh^{\lambda_3\beta}\eta)^{2/\beta}}{\tanh^{\lambda_3}\eta}. \tag{261}$$

1. Late-time behavior

Asymptotically, these relationships tend to the pair

$$\phi(\eta)\to\phi_0[1-\lambda_3^2\beta e^{-4\eta}], \tag{262}$$

$$a(\eta)\to\frac{1}{2}\sqrt{\frac{A}{\phi_0}}e^{\eta}\left[1+\frac{\lambda_3^2\beta}{2}e^{-4\eta}\right], \tag{263}$$

as $\eta\to\infty$, and we find

$$\eta(t)\simeq\ln\left[2\sqrt{\frac{\phi_0}{A}}t\left(1+\frac{A^2\lambda_3^2\beta}{96\phi_0^2}t^{-4}\right)\right], \tag{264}$$

leading to

$$\phi(t)\to\phi_0\left[1-\frac{A^2\lambda_3^2\beta}{16\phi_0^2}t^{-4}\right], \tag{265}$$

$$a(t)\to t\left[1+\frac{A^2\lambda_3^2\beta}{24\phi_0^2}t^{-4}\right], \tag{266}$$

as $t\to\infty$, and so there is power-law approach to the Milne universe of GR at late times.

2. Early-time behavior

Setting $\eta_0=0$, we can expand Eqs. (260) and (261) about $\eta=0$. We find

$$\phi(\eta)\simeq 4^{1/\beta}\phi_0\eta^{|\lambda_3|}, \tag{267}$$

$$a^2(\eta)\simeq 4^{-1/\beta}\frac{A}{\phi_0}\eta^{1-|\lambda_3|}. \tag{268}$$

The latter of these leads to the early-time $\eta(t)$ relation,

$$\eta(t)=2^{[2(1-\beta)]/[\beta(3-|\lambda_3|)]}\left(\frac{\phi_0}{A}\right)^{1/(3-|\lambda_3|)} \times(3-|\lambda_3|)^{2/(3-|\lambda_3|)}t^{2/(3-|\lambda_3|)}. \tag{269}$$

Studying the form of the exponent in Eq. (269) reveals that when $|\lambda_3|<3$, $t\to 0$ as $\eta\to 0$ and when $|\lambda_3|\geqslant 3$, $t\to-\infty$ as $\eta\to 0$. The early-time evolution of the universe as a function of t is then given by

$$\phi(t)\to 2^{[2(3-|\lambda_3|\beta)]/[\beta(3-|\lambda_3|)]}\phi_0^{3/(3-|\lambda_3|)}A^{|\lambda_3|/(|\lambda_3|-3)} \times(3-|\lambda_3|)^{2|\lambda_3|/(3-|\lambda_3|)}t^{2|\lambda_3|/(3-|\lambda_3|)}, \tag{270}$$

$$a(t)\to 2^{[\beta(|\lambda_3|-1)-2]/[\beta(3-|\lambda_3|)]}\left(\frac{A}{\phi_0}\right)^{1/(3-|\lambda_3|)} \times(3-|\lambda_3|)^{(1-|\lambda_3|)/(3-|\lambda_3|)}t^{(1-|\lambda_3|)/(3-|\lambda_3|)}, \tag{271}$$

as $t\to 0$, $-\infty$ accordingly. When $|\lambda_3|=3$, the t-dependent evolution is given by

$$\eta(t)\to\exp\left[2^{1/\beta}\sqrt{\frac{\phi_0}{A}}t\right], \tag{272}$$

$$\phi(t)\to 4^{1/\beta}\phi_0\exp\left[3.2^{1/\beta}\sqrt{\frac{\phi_0}{A}}t\right], \tag{273}$$

$$a(t)\to 2^{-1/\beta}\sqrt{\frac{A}{\phi_0}}\exp\left[-2^{1/\beta}\sqrt{\frac{\phi_0}{A}}t\right], \tag{274}$$

as $t\to-\infty$.

3. Minima

Differentiating Eq. (261), we find

$$(a^2)_\eta=\frac{a^2}{\sinh(2\eta)}\left[\cosh(2\eta)+\frac{2\lambda_3}{\tanh^{-\lambda_3\beta}\eta+1}-\lambda_3\right]. \tag{275}$$

At early times, i.e., as $\eta\to 0$, $a^2\to 0$ if $|\lambda_3|<1$. Since $\sinh(2\eta)\simeq 2\eta$ as $\eta\to 0$, Eq. (268) implies

$$(a^2)_\eta \propto \eta^{-|\lambda_3|}[\cdots]\ , \tag{276}$$

as $\eta\to 0$, and thus the power-law prefactor in $(a^2)_\eta$ can never vanish at early times. The field ϕ evolves monotonically, by Eq. (17), and tends to a constant at late times, which guarantees that ϕ_η cannot diverge during the subsequent evolution. Equation (17), in conjunction with the bound $0<\phi<\phi_0$, then ensures that there can be no further zeros of a as the universe evolves. Equation (275) ensures $(a^2)_\eta$ is nonzero as $\eta\to\infty$. In general, stationary points in the evolution of a arise at η_*, given by

$$\cosh(2\eta_*)+\frac{2\lambda_3}{\tanh^{-\lambda_3\beta}\eta_*+1}-\lambda_3=0\ , \tag{277}$$

obtained from the square-bracketed factor in Eq. (275). Remembering the positivity of both $\cosh(2\eta_*)-1$ and η_*, we can derive the interesting result that when $|\lambda_3|>1$ the location of the minimum is constrained by

$$\eta_*<\mathrm{arctanh}\left(\frac{\lambda_3-1}{\lambda_3+1}\right)^{1/\lambda_3\beta}\ , \tag{278}$$

and when $|\lambda_3|\leqslant 1$ stationary points do not exist.

C. Vacuum solutions $(k=+1)$

Finally, we note the existence of $k=+1$ closed-universe solutions, governed by

$$\frac{1-\sqrt{u_3}}{1+\sqrt{u_3}}=K_3{}^{\lambda_3\beta}\tan^{\lambda_3\beta}\eta\ . \tag{279}$$

This leads to the exact solution, in conformal time,

$$\phi(\eta)=\frac{4^{1/\beta}\phi_0 K_3{}^{\lambda_3}\tan^{\lambda_3}\eta}{(1+K_3{}^{\lambda_3\beta}\tan^{\lambda_3\beta}\eta)^{2/\beta}}\ , \tag{280}$$

$$a^2(\eta)=\frac{A}{2^{\beta+2/\beta}\phi_0}\sin(2\eta)\frac{(1+K_3{}^{\lambda_3\beta}\tan^{\lambda_3\beta}\eta)^{2/\beta}}{K_3{}^{\lambda_3}\tan^{\lambda_3}\eta}\ . \tag{281}$$

D. Radiation solutions $(k=0)$

The behavior of the $k=0$ radiation models for this choice of the coupling function is determined by

$$\ln\left(\frac{1-\sqrt{u_3}}{1+\sqrt{u_3}}\right)=\lambda_3\beta\ln\left[\frac{K_3\eta}{\eta+2\eta_0}\right]\ , \tag{282}$$

where we have exploited our freedom in η_0 to set $A=2\Gamma\eta_0$ and u_3 is as defined by Eq. (250). We also require $K_3=1$, so that $\phi\to\phi_0$ as $\eta\to\infty$. Using Eq. (21) we can calculate the exact evolution of these models

$$\phi(\eta)=4^{1/\beta}\phi_0\left(\frac{\eta}{\eta+2\eta_0}\right)^{\lambda_3}\left[1+\left(\frac{\eta}{\eta+2\eta_0}\right)^{\lambda_3\beta}\right]^{-2/\beta}\ , \tag{283}$$

$$a^2(\eta)=\frac{\Gamma}{4^{1/\beta}\phi_0}\eta^{1-\lambda_3}(\eta+2\eta_0)^{1+\lambda_3}\times\left[1+\left(\frac{\eta}{\eta+2\eta_0}\right)^{\lambda_3\beta}\right]^{2/\beta}\ . \tag{284}$$

1. Late-time behavior

At late times these equations may be approximated by

$$\phi(\eta)\to\phi_0\left[1-\lambda_3^2\beta\frac{\eta_0^2}{\eta^2}\right]\ , \tag{285}$$

$$a(\eta)\to\sqrt{\frac{\Gamma}{\phi_0}}\eta\left[1+\frac{\eta_0}{\eta}+\frac{1}{2}(\lambda_3^2\beta-1)\frac{\eta_0^2}{\eta^2}\right]\ . \tag{286}$$

It is necessary to extend the computation of $a(\eta)$ to second-order since the first-order contributions will later vanish. Equation (286) allows asymptotic calculation of conformal time as a function of cosmic time,

$$\eta(t)\to\sqrt{2}\left(\frac{\phi_0}{\Gamma}\right)^{1/4}t^{1/2}\left[1-\frac{\eta_0}{\sqrt{2}}\left(\frac{\Gamma}{\phi_0}\right)^{1/4}t^{-1/2}-\frac{\eta_0^2}{8}\sqrt{\frac{\Gamma}{\phi_0}}(\lambda_3^2\beta-1)\frac{\ln t}{t}\right]\ , \tag{287}$$

and hence there is power-law approach to the GR solution

$$\phi(t)\to\phi_0\left[1-\frac{\lambda_3^2\beta}{2}\eta_0^2\sqrt{\frac{\Gamma}{\phi_0}}t^{-1}\right]\ , \tag{288}$$

$$a(t)\to\sqrt{2}\left(\frac{\Gamma}{\phi_0}\right)^{1/4}t^{1/2}\left[1-\frac{\eta_0^2}{8}\sqrt{\frac{\Gamma}{\phi_0}}(\lambda_3^2\beta-1)\frac{\ln t}{t}\right]\ , \tag{289}$$

as $t\to\infty$.

2. Early-time behavior

We now probe the early-time behavior of these models, finding two distinct cases. There exist zeros of ϕ as $\eta\to 0$ and $\eta\to -2\eta_0$. Since we require $\phi>0$ during the portion of the evolution in which we are interested, and ultimately at late times, we shall take our early-time limit to be the most recent of these zeros. We know from Eq. (17) that

$$\phi_\eta a^2=\lambda_3 A\left|1-\left(\frac{\phi}{\phi_0}\right)^\beta\right|^{1/2}\ , \tag{290}$$

and since $\phi\in(0,\phi_0)$ and $\phi\to\phi_0$ at late times we deduce that $\phi_\eta>0$. From Eq. (290), we derive $\lambda_3 A>0$. We have fixed $A=2\Gamma\eta_0$, with $\Gamma>0$, so we may also say $\lambda_3\eta_0>0$, i.e., when $\lambda_3>0$, $\eta\to 0$ is the more recent zero of ϕ and when $\lambda_3<0$, $\eta\to -2\eta_0$ is most recent. When $\lambda_3>0$ we find

$$\phi(\eta)\to 4^{1/\beta}\phi_0\left(\frac{\eta}{2\eta_0}\right)^{\lambda_3}\ , \tag{291}$$

$$a^2(\eta)\to\frac{A}{4^{1/\beta}\phi_0}(2\eta_0)^{\lambda_3}\eta^{1-\lambda_3}\ , \tag{292}$$

as $\eta\to 0$. We can model the t-parametrized early-time behavior by Eqs. (269)–(271), for the $k=-1$ vacuum models, via the transformation

$$\phi_0 \to \frac{\phi_0}{(2\eta_0)^{\lambda_3}} . \tag{293}$$

When $\lambda_3<0$

$$\phi(\eta)\to 4^{1/\beta}\phi_0\left(\frac{\eta+2\eta_0}{-2\eta_0}\right)^{-\lambda_3} , \tag{294}$$

$$a^2(\eta)\to -\frac{A}{4^{1/\beta}\phi_0}(-2\eta_0)^{-\lambda_3}(\eta+2\eta_0)^{1+\lambda_3} , \tag{295}$$

as $\eta\to -2\eta_0$. Note that both A and η_0 are negative here. The behavior as a function of t can be gleaned from Eqs. (269)–(271), by the redefinitions

$$\phi_0\to\phi_0(-2\eta_0)^{\lambda_3-2} , \tag{296}$$

$$\eta\to\eta+2\eta_0, \tag{297}$$

$$A\to -A . \tag{298}$$

3. Minima

Differentiating Eq. (284) yields

$$(a^2)_\eta = \frac{a^2}{\eta(\eta+2\eta_0)}\left[2\eta_0(1+\lambda_3)+2\eta - \frac{4\lambda_3\eta_0}{1+[\eta/(\eta+2\eta_0)]^{\lambda_3\beta}}\right] . \tag{299}$$

If the early-time limit is given by $X\to 0$ where X is either η or $\eta+2\eta_0$, then

$$(a^2)_\eta \propto X^{-|\lambda_3|}[\cdots] , \tag{300}$$

and thus never vanishes at early times. From Eq. (290) we deduce that when $0<\phi<\phi_0$, ϕ is bound to evolve monotonically. The necessity that it tend to a constant at late times then occludes the possibility of it diverging after the early-time regime. The non-zero right-hand side of Eq. (290) on the interval $0<\phi<\phi_0$ then guarantees that a is non-zero. Consequently, the stationary points in the evolution are obtained by setting the square-bracketed factor in Eq. (299) to zero. They lie at η_* where

$$\frac{2\lambda_3\eta_0}{[\eta_*/\eta_*+2\eta_0)]^{\lambda_3\beta}+1} - \eta_* = \eta_0(1+\lambda_3) , \tag{301}$$

which must be solved numerically for the particular values of β, λ_3, and η_0.

E. Radiation solutions $(k=-1)$

When the curvature is negative, the evolution is determined by the equation

$$\frac{1-\sqrt{u_3}}{1+\sqrt{u_3}} = K_3{}^{\lambda_3\beta}\left|\frac{\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}-A}{\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}+A}\right|^{\lambda_3\beta} ,$$

$$A^2\geqslant\Gamma^2 , \tag{302}$$

with u_3 and λ_3 as defined earlier. We make the simplifying choice of η_0:

$$\tanh(2\eta_0)=\frac{\Gamma}{A} , \tag{303}$$

which ensures that the early-time behavior (i.e., $\phi\to 0$) occurs at $\eta=0$. To recover GR at late times, we fix

$$K_3=\frac{\Gamma+(A^2-\Gamma^2)^{1/2}+A}{\Gamma+(A^2-\Gamma^2)^{1/2}-A} . \tag{304}$$

Substituting Eq. (303) into the $k=-1$ right-hand side of Eq. (21) we find

$$y(\eta)=\frac{\Gamma}{2}[\cosh(2\eta)-1]+\frac{A}{2}\sinh(2\eta) . \tag{305}$$

Last, we note that with η_0 given by Eq. (303), the expression within the moduli on the right-hand side of Eq. (302) is monotonically increasing from zero and thus positive. Dropping the moduli, we find

$$\phi(\eta)=\frac{4^{1/\beta}\phi_0 K_3{}^{\lambda_3}\{[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]\}^{\lambda_3}}{[1+K_3{}^{\lambda_3\beta}\{[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]\}^{\lambda_3\beta}]^{2/\beta}} , \tag{306}$$

$$a^2(\eta)=\frac{4^{-1/\beta}K_3{}^{-\lambda_3}}{2\phi_0}[\Gamma[\cosh(2\eta)-1]+A\sinh(2\eta)] \times\frac{[1+K_3{}^{\lambda_3\beta}\{[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]\}^{\lambda_3\beta}]^{2/\beta}}{\{[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]\}^{\lambda_3}} . \tag{307}$$

1. Late-time behavior

As $\eta\to\infty$ we find

$$\phi(\eta)\to\phi_0\left[1-\frac{\lambda_3^2A^2\beta}{A^2-\Gamma^2}\left(\frac{A-\Gamma-(A^2-\Gamma^2)^{1/2}}{A+\Gamma-(A^2-\Gamma^2)^{1/2}}\right)^2e^{-4\eta}\right], \tag{308}$$

$$a^2(\eta)\to\frac{\Gamma+A}{4\phi_0}e^{2\eta}\left[1-\frac{2\Gamma}{\Gamma+A}e^{-2\eta}\right], \tag{309}$$

as $\eta\to\infty$. These lead to the t-dependent behaviors:

$$\eta(t)\to\ln\left[2\sqrt{\frac{\phi_0}{\Gamma+A}}t\left(1-\frac{\Gamma}{4\phi_0}t^{-2}\right)\right], \tag{310}$$

$$\phi(t)\to\phi_0\left[1-\frac{\lambda_3^2A^2\beta}{16\phi_0^2}\left(\frac{A+\Gamma}{A-\Gamma}\right)\times\left(\frac{A-\Gamma-(A^2-\Gamma^2)^{1/2}}{A+\Gamma-(A^2-\Gamma^2)^{1/2}}\right)^2t^{-4}\right], \tag{311}$$

$$a(t)\to t\left[1-\frac{\Gamma}{4\phi_0}t^{-2}\right], \tag{312}$$

as $t\to\infty$. Again, we see the approach of this model to the Milne universe as t increases.

2. Early-time behavior

Analyzing the behavior in the neighborhood of the last zero of ϕ (i.e., around $\eta=0$) we obtain a description of the early-time behavior. We find

$$\phi(\eta)\to4^{1/\beta}\phi_0K_3{}^{\lambda_3}\left[\frac{\Gamma}{A}\sqrt{\frac{A^2}{\Gamma^2}-1}\left(\frac{A}{\Gamma}-\sqrt{\frac{A^2}{\Gamma^2}-1}\right)\right]^{\lambda_3}\eta^{\lambda_3}, \tag{313}$$

$$a^2(\eta)\to\frac{4^{-1/\beta}AK_3{}^{-\lambda_3}}{\phi_0}\left[\frac{\Gamma}{A}\sqrt{\frac{A^2}{\Gamma^2}-1}\times\left(\frac{A}{\Gamma}-\sqrt{\frac{A^2}{\Gamma^2}-1}\right)\right]^{-\lambda_3}\eta^{1-\lambda_3}, \tag{314}$$

with K_3 as given by Eq. (304). The reality condition $A^2>\Gamma^2$ and the inequality $\lambda_3A>0$ [from Eq. (290), using $\phi_\eta>0$] ensure all of the prefactors in the above relations are real and positive. We obtain the t-parametrized behavior at early times by applying the substitutions

$$|\lambda_3|\to\lambda_3, \tag{315}$$

$$\phi_0\to\phi_0K_3{}^{\lambda_3}\left[\frac{\Gamma}{A}\sqrt{\frac{A^2}{\Gamma^2}-1}\left(\frac{A}{\Gamma}-\sqrt{\frac{A^2}{\Gamma^2}-1}\right)\right]^{\lambda_3}, \tag{316}$$

to Eqs. (269)–(273).

3. Minima

If $\lambda_3>1$ then the scale factor diverges at early times. For such models to look like GR as $t\to\infty$ requires the presence of a minimum. Using Eq. (27) we find

$$\cosh[2(\eta_*+\eta_0)]=\lambda_3\cosh(2\eta_0)\times\left\{\frac{1-K_3{}^{\lambda_3\beta}\{[\Gamma\tanh(\eta_*+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta_*+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]\}^{\lambda_3\beta}}{1+K_3{}^{\lambda_3\beta}\{[\Gamma\tanh(\eta_*+\eta_0)+(A^2-\Gamma^2)^{1/2}-A]/[\Gamma\tanh(\eta_*+\eta_0)+(A^2-\Gamma^2)^{1/2}+A]\}^{\lambda_3\beta}}\right\}, \tag{317}$$

defines η_*, where the minimum is situated. Although this expression is not soluble analytically, we may obtain bounds on the value of η_*. Equation (27), for Theory 3, may be written

$$\Gamma\sinh(2\eta_*)+A\cosh(2\eta_*)=\lambda_3Au_3{}^{1/2}. \tag{318}$$

The bounds on u_3, namely $u_3\in(0,1)$ then imply

$$0<\cosh[2(\eta_*+\eta_0)]<\lambda_3\cosh(2\eta_0), \tag{319}$$

after using Eq. (303). The cosh function is greater than unity and monotonically increasing when its argument is positive, allowing us to strengthen the above inequality to

$$0<\eta_*<\frac{1}{2}\text{arccosh}[\lambda_3\cosh(2\eta_0)]-\eta_0, \tag{320}$$

where the lower limit follows from the positivity of η.

F. Radiation solutions ($k=+1$)

For the positively-curved models ($k=+1$) we obtain

$$\frac{1-\sqrt{u_3}}{1+\sqrt{u_3}}=\left[K_3\left|\frac{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}-A}{\Gamma\tan(\eta+\eta_0)+(\Gamma^2+A^2)^{1/2}+A}\right|\right]^{\lambda_3\beta}\equiv\sigma(\eta). \tag{321}$$

This leads to the conformal time-parametrized set of equations

$$\phi(\eta)=\phi_0\left[\frac{4\sigma(\eta)}{[1+\sigma(\eta)]^2}\right]^{1/\beta}, \tag{322}$$

$$a^2(\eta)=\frac{1}{2\phi_0}\left[\frac{[1+\sigma(\eta)]^2}{4\sigma(\eta)}\right]^{1/\beta}\{\Gamma+(\Gamma^2+A^2)^{1/2}\times\sin[2(\eta+\eta_0)]\}\ . \tag{323}$$

G. Perfect-fluid solutions ($k=0$)

We examine the late-time form of the solutions resulting from a scalar-tensor cosmology driven by the third type of coupling function using the theory first studied by Barrow and Mimoso [33] and defined by

$$\phi(\xi)=\phi_0\exp(P\xi^Q)\ , \tag{324}$$

where P and Q are constants. This choice of $\phi(\xi)$ arises from the generating function

$$g(\xi)=\frac{1}{PQ}\xi^{2-Q}\ , \tag{325}$$

and results in the scale-factor

$$a^{3(2-\gamma)}(\xi)=\frac{a_0^{3(2-\gamma)}}{PQ\phi_0}\xi^{2-Q}\exp(-P\xi^Q)\ . \tag{326}$$

Positivity of the left-hand side of Eq. (326) requires that $\xi^{2-Q}PQ>0$. Hence, we may derive

$$f(\xi)=\frac{1}{PQ}\xi^{2-Q}+\frac{4-3\gamma}{4}\xi^2-D\ , \tag{327}$$

and

$$2\omega(\xi)+3=\frac{4-3\gamma}{3(2-\gamma)^2}\frac{[(2-Q)/PQ]\xi^{1-Q}+[(4-3\gamma)/2]\xi]^2}{[(1/PQ)\xi^{2-Q}+[(4-3\gamma)/4]\xi^2]}\ . \tag{328}$$

Again, at late times we demand $\phi\to\phi_0$. This can be realized as $\xi\to 0$ when $Q>0$ or as $\xi\to\infty$ when $Q<0$. When $P=0$ and/or $Q=0$ we have GR.

1. Late-time behavior

The limiting forms of the coupling function as $\xi\to 0$ when $Q>0$, and as $\xi\to\infty$ when $Q<0$, are identical:

$$2\omega(\xi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\frac{(2-Q)^2}{PQ}\xi^{-Q}\ . \tag{329}$$

In this case, Eq. (32) is

$$dt\simeq\left(\frac{2-Q}{2-\gamma}\right)\frac{a_0^{3(\gamma-1)}}{3(PQ\phi_0)^{(\gamma-1)/(2-\gamma)}}\times\left(\frac{4-3\gamma}{PQ}\right)^{1/2}\xi^{(4\gamma-4-Q\gamma)/[2(2-\gamma)]}\times\exp\left[-P\left(\frac{\gamma-1}{2-\gamma}\right)\xi^Q\right]d\xi\ , \tag{330}$$

as $\xi^Q\to 0$. This expression may be integrated approximately for small ξ^Q, giving

$$t(\xi)\simeq S^{-1}\xi^{[\gamma(2-Q)]/[2(2-\gamma)]}\left[1-P\left(\frac{\gamma-1}{2-\gamma}\right)\xi^Q\right]\ , \tag{331}$$

where S is defined by

$$S\equiv\frac{3\gamma}{2}\frac{(PQ\phi_0)^{(\gamma-1)/(2-\gamma)}}{a_0^{3(\gamma-1)}}\left(\frac{PQ}{4-3\gamma}\right)^{1/2}\ . \tag{332}$$

Ignoring the square brackets in Eq. (331), we obtain the lowest-order inversion of this expression

$$\xi(t)\simeq S^{[2(2-\gamma)]/[\gamma(2-Q)]}t^{[2(2-\gamma)]/[\gamma(2-Q)]}\ . \tag{333}$$

Introducing the corrections associated with the terms in the square bracket leads, at the next-order, to

$$\xi(t)\simeq S^{[2(2-\gamma)]/[\gamma(2-Q)]}t^{[2(2-\gamma)]/[\gamma(2-Q)]}\left[1+\frac{2P}{\gamma}\left(\frac{\gamma-1}{2-Q}\right)\times S^{[2Q(2-\gamma)]/[\gamma(2-Q)]}t^{[2Q(2-\gamma)]/[\gamma(2-Q)]}\right]\ . \tag{334}$$

From this we can see that $\xi\to 0$ as $t\to\infty$ if $Q>2$. As $\xi\to\infty$, $t\to\infty$ iff $Q<2$; the condition $\phi\to\phi_0$ as $\xi\to\infty$ requires $Q<0$, we thus deduce that the range of theories demarcated by $0<Q<2$ will not approach GR as $t\to\infty$. Substituting the above $\xi(t)$ into Eqs. (324) and (326) we obtain the t-dependent evolution

$$\phi(t)\to\phi_0[1-PS^{[2Q(2-\gamma)]/[\gamma(2-Q)]}t^{[2Q(2-\gamma)]/[\gamma(2-Q)]}]\ , \tag{335}$$

$$a(t)\to\frac{a_0S^{2/3\gamma}}{(PQ\phi_0)^{1/[3(2-\gamma)]}}t^{2/3\gamma}\times\left[1-\frac{1}{3}PS^{[2Q(2-\gamma)]/[\gamma(2-Q)]}t^{[2Q(2-\gamma)]/[\gamma(2-Q)]}\right]\ , \tag{336}$$

as $t\to\infty$. The corresponding late-time evolution of the coupling as a function of ϕ is given by

$$2\omega(\phi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\frac{(2-Q)^2}{Q}\frac{1}{\ln(\phi/\phi_0)}\ . \tag{337}$$

When $Q>0$ the requirement that a be positive ensures $P\xi^Q>0$. From Eq. (32) we see that for $Q>2$, $d\xi/dt<0$, i.e., $\xi\to 0$ from above and hence $P>0$. Equation (324) confirms that $\phi\to\phi_0$ from above when P and Q lie in these domains. Conversely, when $Q<0$ the requirement $a>0$ implies $P\xi^Q<0$. At late times $\xi\to\infty$ in these models and so $P<0$ and ϕ approaches the GR value, ϕ_0, from below.

We remarked earlier that the $Q=0$ case gives pure GR at all times. When $Q=2$ Eq. (328) for $2\omega+3$ does not approach the limit given in Eq. (329) as $\xi\to 0$, instead we have

$$2\omega(\xi)+3\to\frac{2(4-3\gamma)^3P}{12(2-\gamma)^2}\xi^2\ , \tag{338}$$

and $2\omega+3$ decays to zero as $\xi\to 0$. The range of Q incompatible with current observations, that GR is a good approximation to the time description of gravitation today, is then $0<Q\leqslant 2$.

2. Early-time behavior

When $Q>2$, $P>0$ the scale-factor a approaches zero as $\xi\to\pm\infty$ and

$$2\omega+3\to\frac{(4-3\gamma)^2}{3(2-\gamma)^2}\,, \tag{339}$$

i.e., BD theory. The square root of this limit is positive and so it follows from Eq. (32) that $d\xi/dt$ must also be positive, demanding that ξ approach $-\infty$. The positivity of a, Q and P imply $(-1)^Q>0$ and hence $\phi\to\infty$ as $\xi\to-\infty$, $a\to 0$.

When $Q<0$, a necessary condition for the scale-factor to approach zero is $\xi\to 0$ and in this limit we recover BD theory again, the coupling given by Eq. (339). As before $\sqrt{2\omega+3}>0$, $d\xi/dt>0$ and $\xi\to 0$ from above. Positivity of a thus requires $PQ>0$, which is guaranteed from the late-time behavior. However, for a to converge to zero as $\xi\to 0$ requires $P>0$ and hence $Q>0$. This direct contradiction ensures that $a(\xi)$ will be nonsingular when $Q<0$. Explicitly, $a(\xi)$ possesses a minimum at

$$\xi_*=\left(\frac{2-Q}{PQ}\right)^{1/Q}. \tag{340}$$

For reasons given in Sec. V G 2 we refrain from presenting the explicit form of the solution here.

3. Dust models

The late-time evolution of the $\gamma=1$ models is, from Eqs. (335) and (336), that

$$\phi(t)\to\phi_0\left[1-P\left(\frac{3}{2(PQ)^{1/2}}\right)^{2Q/(2-Q)}t^{2Q/(2-Q)}\right], \tag{341}$$

$$a(t)\to a_0\left(\frac{9}{4\phi_0(PQ)^2}\right)^{1/3}t^{2/3}\times\left[1-\frac{P}{3}\left(\frac{3}{2(PQ)^{1/2}}\right)^{2Q/(2-Q)}t^{2Q/(2-Q)}\right], \tag{342}$$

as $t\to\infty$.

4. Inflationary models

As was the case in Sec. V G 4, the solutions when $\gamma=0$ are qualitatively different to their more general counterparts, Eqs. (335) and (336). In this case Eq. (330) is

$$dt\simeq\frac{(2-Q)\phi_0^{1/2}}{3a_0^3}\,\xi^{-1}\exp\left[\frac{P}{2}\xi^Q\right]d\xi\,, \tag{343}$$

as $\xi^Q\to 0$, which integrates approximately in this limit to give

$$t(\xi)\simeq\frac{(2-Q)\phi_0^{1/2}}{3a_0^3}\ln\xi\left[1+\frac{P\xi^Q}{2Q\ln\xi}\right]. \tag{344}$$

The first-order inversion of this at large t gives

$$\xi(t)\simeq\exp\left(\frac{3a_0^3t}{(2-Q)\phi_0^{1/2}}\right), \tag{345}$$

which arises by neglecting the square brackets on the right of Eq. (344). The next-order correction to this result is

$$\xi(t)\simeq\left(\frac{3a_0^3t}{(2-Q)\phi_0^{1/2}}\right)\left[1-\frac{P}{2Q}\exp\left(\frac{3Qa_0^3t}{(2-Q)\phi_0^{1/2}}\right)\right], \tag{346}$$

as $t\to\infty$. Substituting this into Eqs. (324) and (326) yields

$$\phi(t)\to\phi_0\left[1+P\exp\left(\frac{3Qa_0^3t}{(2-Q)\phi_0^{1/2}}\right)\right], \tag{347}$$

$$a(t)\to\frac{a_0}{(PQ\phi_0)^{1/6}}\exp\left(\frac{a_0^3t}{2\phi_0^{1/2}}\right)\times\left[1-\frac{P(2+Q)}{12Q}\exp\left(\frac{3Qa_0^3t}{(2-Q)\phi_0^{1/2}}\right)\right]. \tag{348}$$

5. Connection to the parameters of Theory 3

At late times we have for all of the models considered in this section

$$2\omega(\phi)+3\to\frac{4-3\gamma}{3(2-\gamma)^2}\,\frac{(2-Q)^2E}{Q}\left[\left(\frac{\phi}{\phi_0}\right)^E-1\right]^{-1},\quad\text{as}\quad\phi\to\phi_0, \tag{349}$$

$$\to\frac{4-3\gamma}{3(2-\gamma)^2}\,\frac{(2-Q)^2E}{(-Q)}\left[1-\left(\frac{\phi}{\phi_0}\right)^E\right]^{-1}, \tag{350}$$

where E is a constant. When $\phi>\phi_0$ and $Q>0$ and we can model the form of the coupling as a function ϕ for Theory 3 by Eq. (349). When $\phi<\phi_0$ and $Q<0$ we can use Eq. (350) for the coupling as a function of ϕ. This leads to the consistency relations

$$\frac{(2-Q)^2}{\mathrm{sgn}(Q)}=\frac{3(2-\gamma)^2}{4-3\gamma}\,\frac{B_3}{\beta}\,, \tag{351}$$

$$E=\beta\,, \tag{352}$$

connecting Theory 3 as defined in Sec. IV with the solutions presented here.

6. Exact solution

We note the existence of an exact solution, defined by the choice

$$\phi(\xi)=\phi_0(1-\xi^{2-\nu})\,, \tag{353}$$

where ν is a constant. This leads, by Eq. (43), to

$$g(\xi)=\frac{1}{\nu-2}\,\xi^\nu(1-\xi^{2-\nu})\,, \tag{354}$$

and by Eq. (45) to

$$a^{3(2-\gamma)}=\frac{a_0^{3(2-\gamma)}}{(\nu-2)\phi_0}\xi^{\nu}. \tag{355}$$

The exact form of coupling driving this behavior is given by Eq. (50) as

$$2\omega(\xi)+3=\frac{4-3\gamma}{3(2-\gamma)^2(\nu-2)}\frac{[\nu\xi^{\nu-2}-2J]^2}{\xi^{\nu-2}-J}, \tag{356}$$

or, using Eq. (353),

$$2\omega(\phi)+3=\frac{4-3\gamma}{3(2-\gamma)^2(\nu-2)}\frac{[\nu(1-\phi/\phi_0)^{-1}-2J]}{(1-\phi/\phi_0)^{-1}-J}, \tag{357}$$

where

$$J=1+\frac{(3\gamma-4)(\nu-2)}{4}. \tag{358}$$

VIII. DISCUSSION

In this paper we have supplied a comprehensive study of isotropic cosmological models in scalar-tensor theories, extending the earlier work of Refs. [12] and [33]. We have explored the behavior of isotropic cosmological models in these theories using a combination of two basic mathematical techniques introduced in Sec. III. In the case where the trace of the energy-momentum tensor of matter vanishes (i.e., vacuum or radiation) exact solutions can be found directly for all curvatures if the requisite integrals can be performed. Asymptotic forms are easily derived in all cases. This technique exploits the conformal relationship between scalar-tensor theories and general relativity that exists when the trace of the energy-momentum tensor vanishes. However, when the energy-momentum tensor is not trace-free the conformal equivalence disappears and the indirect method of Barrow and Mimoso must be used to find exact solutions. This only works for zero curvature cosmological models but includes the important cases of all $p=0$ universes and inflationary universes with $-\rho\leqslant p\leqslant -\frac{1}{3}\rho$. It also permits a simple means of comparing the behavior of cosmological solutions in any scalar-tensor theory with those of Brans-Dicke theory at early and late times. Since this procedure does not commence with the specification of $\omega(\phi)$, but with a choice of generating function that produces the entire solution by nonlinear transformations, it is necessary to build up intuition by a thorough exploration of the results of employing particular classes of generating function. In particular, we were able to find generating functions which gave rise to dust universes in scalar-tensor theories for which the exact radiation and vacuum solutions can be found exactly by the direct method. This provides us with descriptions of scalar-tensor cosmologies throughout the entire radiation and dust and vacuum dominated eras. We were also able to find a wide range of new inflationary universe solutions with $p=-\rho$ in these theories.

In Sec. IV we introduced three classes of scalar-tensor theory which permit asymptotic approach to general relativity at late times when $\phi\to\phi_0$. The parameters defining the functional form of $\omega(\phi)$ which specify these gravity theories can be restricted further if we require the theory to approach general relativity in the weak-field limit ($\omega\to\infty$ and $\omega'\omega^{-3}\to 0$), and describe expanding universes. For each of these general classes of theory we have determined the behavior of flat, open, and closed universes by a combination of exact solutions and asymptotic studies of the early and late-time behaviors.

The catalogue of solutions and asymptotes that we have found will enable scalar-tensor theories to be constrained in new ways because they enable complete cosmological histories to be constructed through initial vacuum, radiation, dust, and final vacuum-dominated eras. The standard sequence of physical processes responsible for events like monopole production, inflation, baryosynthesis, primordial black hole formation, electroweak unification, the quark-hadron phase transition, and nucleosynthesis can be explored in the cosmological environment provided by scalar-tensor gravity theories. The constraints derived from these considerations can be compared directly with those imposed by weak-field tests in the solar system and observations of astrophysical objects like white dwarfs and the binary pulsar. The ubiquity of scalar fields in current string theories of high-energy physics has led to continued interest in the detailed behavior of scalar-tensor gravity theories and their associated cosmologies. In this paper we have displayed some of the diversity that these cosmologies possess together with a collection of methods for solving other specific theories that may be motivated by future developments in high-energy physics.

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APPENDIX: CONFORMAL EQUIVALENCE

The action for trace-free matter is conformally invariant and we may exploit this fact to study the behavior of vacuum and radiation models with $\phi>\phi_0$. Under a conformal transformation to a new metric $\tilde{g}_{\mathrm{ab}}$, with components

$$g_{\mathrm{ab}}=\phi^{-2}\tilde{g}_{\mathrm{ab}}, \tag{A1}$$

and a redefinition of the field

$$\tilde{\phi}=\tilde{\phi}_0\left(\frac{\phi_0}{\phi}\right), \tag{A2}$$

the action becomes

$$S_G=\int d^4x\sqrt{-\tilde{g}}\left[-\tilde{\phi}\tilde{\mathcal{R}}+\frac{\omega(\phi_0\tilde{\phi}_0/\tilde{\phi})}{\tilde{\phi}}\tilde{g}^{\mathrm{ab}}\partial_{\mathrm{a}}\tilde{\phi}\partial_{\mathrm{b}}\tilde{\phi}\right], \tag{A3}$$

neglecting overall constant factors. When $0<\phi<\phi_0$ the coupling function for Theory 2, as defined in Sec. IV, is

$$2\omega(\phi)+3=B_2\left[-\ln\left(\frac{\phi}{\phi_0}\right)\right]^{-2\delta}, \tag{A4}$$

and so

$$2\omega(\phi_0\tilde{\phi}_0/\tilde{\phi})+3=B_2\left[-\ln\left(\frac{\tilde{\phi}_0}{\tilde{\phi}}\right)\right]^{-2\delta}, \qquad \text{(A5)}$$

with $\tilde{\phi}>\tilde{\phi}_0$ when $0<\phi<\phi_0$. Equation (A5) is exactly the form of the coupling for Theory 2 with a gravitational scalar field $\tilde{\phi}>\tilde{\phi}_0$. Thus we may obtain solutions when $\phi>\phi_0$ in Theory 2 by applying the transformations

$$a\rightarrow\phi a, \qquad \text{(A6)}$$

$$\phi\rightarrow\frac{1}{\phi}, \qquad \text{(A7)}$$

in this order, to the solutions presented in Sec. VI. These are transformations which render the form of y in Eq. (15) invariant.

Last, we remark that the asymptotic behaviors of Theories 1 and 3 when $\phi>\phi_0$ may also be examined in this way, since the coupling functions for both these theories may be approximated by logarithms as $\phi\rightarrow\phi_0$.

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