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Depolarization of the cosmic microwave background by a primordial magnetic field and its effect upon temperature anisotropy

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We estimate the depolarizing effect of a primordial magnetic field upon the cosmic microwave background radiation due to differential Faraday rotation across the last scattering surface. The degree of linear polarization of the CMB is significantly reduced at frequencies around and below 30 GHz $(B_*/10^{-2}\ \mathrm{G})^{1/2}$, where B_* is the value of the primordial field at recombination. The depolarizing mechanism reduces the damping of anisotropies due to photon diffusion on small angular scales. The $l\approx 1000$ multipoles of the CMB temperature anisotropy correlation function in a standard cold dark matter cosmology increase by up to 7.5% at frequencies where depolarization is significant. [S0556-2821(97)03204-9]

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I. INTRODUCTION

The cosmic microwave background (CMB) radiation is expected to have acquired a small degree of linear polarization through Thomson scattering [1], which polarizes the radiation if there is a quadrupole anisotropy in its distribution function [2]. Typically, the CMB degree of linear polarization is expected to be more than ten times smaller than the relative temperature anisotropy on comparable angular scales, at least within a standard ionization history of the Universe. The CMB has not yet been observed to be polarized, the upper limit on its degree of linear polarization on large angular scales being $P<6\times 10^{-5}$ [3]. When measured, the CMB polarization will provide a wealth of information about the early Universe, additional to that revealed by the CMB anisotropy.

The polarization properties of the CMB may prove particularly valuable to either constrain or detect a hypothetical primordial magnetic field [4,5]. A cosmological magnetic field could leave significant imprints upon the CMB polarization through the effect of Faraday rotation. After traversing a distance L in a direction $\hat{q}$ within a homogeneous magnetic field $\vec{B}$, linearly polarized radiation has its plane of polarization rotated an angle

$$\varphi=\frac{e^3 n_e x_e \vec{B}\cdot\hat{q}}{8\pi^2 m^2 c^2}\lambda^2 L \quad . \tag{1.1}$$

n_e is the total number density of electrons and x_e its ionized fraction. λ is the wavelength of the radiation, m is the electron mass, and c is the speed of light. We work in Heaviside-Lorentz electromagnetic units ($\alpha=e^2/4\pi\approx 1/137$ is the fine structure constant if we take $\hbar=c=1$).

Faraday rotation of synchrotron emission by distant galaxies serves, for instance, to estimate the value of galactic and extragalactic magnetic fields [6]. Faraday rotation acts also as a depolarizing mechanism. If an extended source emits polarized radiation, the total outcome may become significantly depolarized by a magnetic field, after the radiation emanating from points at different depths within the source experiences different amounts of Faraday rotation. This process affects significantly the radio emission of galaxies and quasars [7].

In this paper we analyze the depolarizing effect exerted by a primordial magnetic field upon the CMB across the last scattering surface. We consider a Robertson-Walker universe with scalar, energy-density fluctuations, and assume a standard thermal history. We make use of an analytic approach [8], based on a recent refinement and extension [9] of the tight-coupling approximation [10], that highlights the physical process responsible for the CMB polarization and its dependence upon various cosmological parameters, while still yielding reasonably accurate results. The polarization of the CMB is proportional to the width of the last scattering surface (LSS), the interval of time during which most of the CMB photons that we observe today last-scattered off free electrons. A primordial magnetic field could prevent the polarization from growing across the full width of the LSS. We shall see that the effect is controlled by the dimensionless and time-independent parameter

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$$F \equiv \frac{3}{2\pi e}\frac{B}{\nu^2} \approx 0.7\left(\frac{B_*}{10^{-3}\ \mathrm{G}}\right)\left(\frac{10\ \mathrm{GHz}}{\nu_0}\right)^2. \quad (1.2)$$

The coefficient F represents the average Faraday rotation (in radians) between Thomson scatterings [4]. ν_0 is the CMB frequency observed today. $B_* = B(t_*)$ is the strength of the primordial magnetic field at a redshift $z_* = 1000$, around the time of decoupling of matter and radiation. Current bounds suggest that a magnetic field pervading cosmological distances, if it exists, should have a present strength below $B_0 \approx 10^{-9}$ G [6]. It is conceivable that the large scale magnetic fields observed in galaxies and clusters have their origin in a primordial field, and several theoretical speculations exist about its possible origin [11]. A primordial magnetic field is expected to scale as $B(t) = B(t_0)a^2(t_0)/a^2(t)$, where $a(t)$ is the Robertson-Walker scale factor. Thus, a primordial field with strength $B_* = 10^{-3}$ G at recombination would have a present strength

$$B_0 = \frac{B_*}{(1+z_*)^2} \approx 10^{-9}\ \mathrm{G}\left(\frac{B_*}{10^{-3}\ \mathrm{G}}\right). \quad (1.3)$$

A primordial magnetic field may significantly depolarize the CMB right before its decoupling from matter. The effect is sensitive to the strength of the magnetic field at recombination, not to its present strength. A value of B_* somewhat larger than 10^{-3} G is not ruled out. Compatibility with big-bang nucleosynthesis, for instance, places an upper bound that, extrapolated to the time of recombination, is at most $B_* = 0.1$ G [12]. Recent proposals for either detecting or constraining a primordial field at recombination were suggested in [5,13]. In [5], Faraday rotation of the CMB polarization was analyzed in the limit of small rotation angles, concluding that a measurement of the effect could provide evidence for magnetic fields of order $B_* \approx 10^{-3}$ G at recombination. In [13] the change in the photon-baryon sound speed in the presence of a magnetic field of order $B_* = 0.2$ G was claimed to distort the structure of the acoustic peaks in the CMB anisotropy power spectrum at a level detectable by currently planned CMB experiments.

We shall entertain in our discussions the possibility that the strength of the primordial magnetic field at recombination be somewhat larger than $B_* = 10^{-3}$ G. We will show that currently planned CMB experiments might be sensitive to the effect of depolarization upon the temperature anisotropy power spectrum on small angular scales if B_* is around or larger than 0.01 G, while experiments at somewhat lower frequencies would be sensitive to primordial fields of strength around $B_* \approx 10^{-3}$ G.

The impact of depolarization upon anisotropy comes about as a consequence of the polarization dependence of Thomson scattering, which feeds back polarization into anisotropy [8,14,15]. The dominant effect is a reduction in the exponential damping due to photon diffusion, which results in an increase of the anisotropy at those frequencies for which depolarization is significant. We shall perform an analytic estimate of the effect, based on the tight-coupling approximation. In order to make more quantitative and specific predictions about the impact and potential measurability of the effect of depolarization upon temperature anisotropy, we shall also use a recently developed numerical code [16] to integrate the Boltzmann equations in a standard cold dark matter model. We shall see that the temperature anisotropy correlation function multipoles at $l \approx 1000$ increase by up to 7.5% at frequencies where depolarization is significant. We conclude that a primordial magnetic field of strength around 10^{-2} G at recombination is worth of membership in the list of multiple cosmological parameters that one may attempt to determine through CMB anisotropy measurements on small angular scales [17].

The paper is structured as follows. In Sec. II, we write down and describe the radiative transfer equations for the total and polarized photon-distribution function in the presence of a single Fourier mode of the scalar metric fluctuations. We include the term describing Faraday rotation by a primordial magnetic field. We solve these equations in the tight-coupling approximation, and find the dependence of the degree of polarization upon the frequency of the CMB photons in the presence of a primordial magnetic field. In Sec. III we discuss the effects of the depolarizing mechanism upon the anisotropy of the CMB on small angular scales, both analytically as well as numerically. We discuss the possibility that the effect be detected by currently planned CMB experiments. Section IV is the discussion and conclusion.

II. DEPOLARIZATION BY A MAGNETIC FIELD

A. Boltzmann equations

We begin by considering the radiative transfer equations for a single Fourier mode of the temperature and polarization fluctuations in a Robertson-Walker spatially flat universe with scalar (energy-density) metric fluctuations, described in terms of the gauge-invariant gravitational potentials Ψ and Φ. We follow the notation and formalism of Ref. [8]. The total temperature fluctuation is denoted by Δ_T, while the fluctuations in the Stokes parameters Q and U are denoted by Δ_Q and Δ_U, respectively. The degree of linear polarization is given by $\Delta_P = (\Delta_Q^2 + \Delta_U^2)^{1/2}$. All three quantities are expanded in Legendre polynomials as $\Delta_X = \Sigma_l (2l+1)\Delta_{X_l} P_l(\mu)$, where $\mu = \cos\theta = \vec{k}\cdot\hat{q}/|\vec{k}|$ is the cosine of the angle between the wave vector of a given Fourier mode $\vec{k}$, and the direction of photon propagation $\hat{q}$. The evolution equations for the Fourier mode of wave vector $\vec{k}$ of the gauge-invariant temperature and polarization fluctuations [8,9,18,19], including the Faraday rotation effect of a primordial magnetic field [5], read

$$\dot{\Delta}_T + ik\mu(\Delta_T + \Psi) = -\dot{\Phi} - \dot{\kappa}[\Delta_T - \Delta_{T_0} - \mu V_b + \tfrac{1}{2}P_2(\mu)S_P], \quad (2.1)$$

$$\dot{\Delta}_Q + ik\mu\Delta_Q = -\dot{\kappa}[\Delta_Q - \tfrac{1}{2}(1 - P_2(\mu))S_P] + 2\omega_B\Delta_U, \quad (2.2)$$

$$\dot{\Delta}_U + ik\mu\Delta_U = -\dot{\kappa}\Delta_U - 2\omega_B\Delta_Q. \quad (2.3)$$

We have defined

$$S_P \equiv -\Delta_{T_2} - \Delta_{Q_2} + \Delta_{Q_0}, \quad (2.4)$$

which acts as the effective source term for the polarization. V_b is the bulk velocity of the baryons, which verifies the continuity equation

$$\dot{V}_b = -\frac{\dot{a}}{a}V_b - ik\Psi + \frac{\dot{\kappa}}{R}(3\Delta_{T_1} - V_b). \tag{2.5}$$

An overdot means derivative with respect to the conformal time $\tau = \int dt a_0/a$, with $a(t)$ the scale factor of the spatially flat Robertson-Walker metric, and $a_0 = a(t_0)$ its value at the present time. $R \equiv 3\rho_b/4\rho_\gamma$ coincides with the scale factor $a(t)$ normalized to 3/4 at the time of equal baryon and radiation densities. $\dot{\kappa} = x_e n_e \sigma_T a/a_0$ is the Thomson-scattering rate, or differential optical depth, with n_e the electron number density, x_e its ionized fraction, and σ_T the Thomson-scattering cross section. Finally, ω_B is the Faraday rotation rate [5]

$$\omega_B \equiv \frac{d\varphi}{d\tau} = \frac{e^3 n_e x_e \vec{B}\cdot\hat{q}}{8\pi^2 m^2 \nu^2}\frac{a}{a_0}. \tag{2.6}$$

If there were axial symmetry around $\vec{k}$ and no Faraday rotation, one could always choose a basis for the Stokes parameters such that $U=0$. A magnetic field with arbitrary orientation breaks the axial symmetry, and Faraday rotation mixes Q and U.

B. Tight-coupling approximation

We now solve the Eqs. (2.1)–(2.3) in the tight-coupling approximation, which amounts to an expansion in powers of $k\tau_C$, where $\tau_C \equiv \dot{\kappa}^{-1}$ is the average conformal time between collisions.

At times earlier than decoupling, Thomson scattering is very efficient, and the mean free path of the photons is very short. The lowest order tight-coupling expression constitutes in that case an excellent approximation. It implies that the photon-distribution function is isotropic in the baryon's rest frame, and hence the polarization vanishes [8]. To first order in $k\tau_C$ there is a small quadrupole anisotropy, and thus a small polarization. As decoupling of matter and radiation proceeds, the tight-coupling approximation breaks down. Still, for wavelengths longer than the width of the last scattering surface, it provides a very accurate approximation to the exact result.

In the absence of a magnetic field ($\omega_B = 0$), the tight-coupling solutions, to first order in $k\tau_C$, are such that [8]

$$\Delta_U = 0 \quad , \quad \Delta_Q = \frac{3}{4}S_P \sin^2\theta, \tag{2.7}$$

$$S_P = -\frac{5}{2}\Delta_{T_2} = \frac{4}{3}ik\tau_C\Delta_{T_1} = -\frac{4}{3}\tau_C\dot{\Delta}_0, \tag{2.8}$$

where we defined $\Delta_0 \equiv \Delta_{T_0} + \Phi$. Notice that $\Delta_{Q_0} = -5\Delta_{Q_2} = -\frac{5}{4}\Delta_{T_2} = \frac{1}{2}S_P$, while all multipoles with $l \geqslant 3$ vanish to first order in $k\tau_C$. All quantities of interest can be expressed, in the tight-coupling approximation, in terms of Δ_0, which in turn verifies the equation of a forced and damped harmonic oscillator [9]

$$\ddot{\Delta}_0 + \left[\frac{\dot{R}}{1+R} + \frac{16}{45}\frac{k^2\tau_C}{(1+R)}\right]\dot{\Delta}_0 + \frac{k^2}{3(1+R)}\Delta_0 = \frac{k^2}{3(1+R)}[\Phi - (1+R)\Psi], \tag{2.9}$$

where we have neglected $O(R^2)$ corrections.

Now, consider the effect of the magnetic field ($\omega_B \neq 0$), assumed spatially homogeneous over the scale of a perturbation with wave vector $\vec{k}$. Faraday rotation breaks the axial symmetry around the direction of the wave vector. The depolarizing effect of Faraday rotation depends not only upon the angle between the magnetic field and the direction in which the radiation propagates, but also upon the angle between the magnetic field and the wave vector $\vec{k}$. Nevertheless, we shall only be interested in the stochastic superposition of all Fourier modes of the density fluctuations, with a Gaussian spectrum that has no privileged direction. Average quantities thus depend only upon the angle between the line of sight and the direction of the magnetic field, but not upon the angle between the magnetic field and the wave vector $\vec{k}$, which is integrated away. For simplicity of the calculation, when computing the evolution of perturbations with wave vector $\vec{k}$ we shall consider a magnetic field with no component perpendicular to $\vec{k}$. This choice also satisfies the condition of axial symmetry around $\vec{k}$, under which Eqs. (2.1), (2.2), and (2.3) for Δ_T, Δ_Q, and Δ_U, respectively, were derived. We shall later use the result of this calculation for the stochastic superposition of all Fourier modes with arbitrary orientation relative to the magnetic field. This simplification will result at most in an underestimate of the net depolarizing effect, since the case $\vec{B} \| \vec{k}$ is that for which depolarization is less effective, the magnetic field being perpendicular to the direction in which polarization is maximum.

To first order in $k\tau_C$, the tight-coupling solutions in the presence of a homogeneous magnetic field $\vec{B} \| \vec{k}$ are such that

$$\Delta_U = -F\cos\theta\Delta_Q, \quad \Delta_Q = \frac{3}{4}\frac{S_P \sin^2\theta}{(1+F^2\cos^2\theta)}, \tag{2.10}$$

where we have defined the coefficient F as

$$F\cos\theta \equiv 2\omega_B\tau_C \tag{2.11}$$

and so

$$F = \frac{e^3}{4\pi^2 m^2 \sigma_T}\frac{B}{\nu^2} \approx 0.7\left(\frac{B_*}{10^{-3}\ \mathrm{G}}\right)\left(\frac{10\ \mathrm{GHz}}{\nu_0}\right)^2. \tag{2.12}$$

The coefficient F represents the average Faraday rotation between collisions, since $2\omega_B$ is the Faraday rotation rate and $\tau_C = \dot{\kappa}^{-1}$ is the photons mean free path (in conformal time units). When calculating the evolution of each mode $\vec{k}$ we have assumed that the strength of the primordial magnetic field scales as $B(t) = B(t_*)a^2(t_*)/a^2(t)$, which is justified by flux conservation and because the Universe behaves as a good conductor [6]. Since the frequency also redshifts as $\nu = \nu_0 a(t_0)/a(t)$, the parameter F is time independent. ν_0 is the frequency of the CMB photons at present time, while B_* is the strength of the magnetic field at a redshift $z_*=1000$, around recombination. Within a standard thermal history, with no early reionization, depolarization is only significant across the LSS, and it thus depends only upon the value of the primordial magnetic field around the time of recombination. Notice that Faraday rotation between collisions becomes considerably large, paving the way to an efficient depolarizing mechanism, at frequencies around and below $\nu_{\rm d}$ defined such that

$$F\equiv\left(\frac{\nu_d}{\nu_0}\right)^2 \qquad (2.13)$$

so that

$$\nu_d\approx 8.4\ \ \mathrm{GHz}\left(\frac{B_*}{10^{-3}\ \ \mathrm{G}}\right)^{1/2}\approx 27\ \ \mathrm{GHz}\left(\frac{B_*}{0.01\ \ \mathrm{G}}\right)^{1/2}. \qquad (2.14)$$

From Eqs. (2.10) we can read the values of Δ_{Q_0} and Δ_{Q_2}. They reduce to Eqs. (2.7) with $O(F^2)$ corrections for small F, while they vanish as F^{-1} for large F. We write them as

$$\Delta_{Q_0}=\frac{1}{2}d_0(F)S_P,\quad \Delta_{Q_2}=-\frac{1}{10}d_2(F)S_P. \qquad (2.15)$$

The coefficients d_0,d_2 are defined so that $d_i\approx 1+O(F^2)$ for small F, while $d_i\to O(1/F)$ as $F\to\infty$, and represent the effect of depolarization. They read

$$d_0(F)=\frac{3}{2}\left[\frac{\arctan(F)}{F}\left(1+\frac{1}{F^2}\right)-\frac{1}{F^2}\right], \qquad (2.16)$$

$$d_2(F)=\frac{15}{4}\left[\frac{\arctan(F)}{F}\left(1+\frac{4}{F^2}+\frac{3}{F^4}\right)-\frac{3}{F^2}-\frac{3}{F^4}\right]. \qquad (2.17)$$

In terms of the combination

$$d\equiv\frac{5}{6}\left(d_0+\frac{d_2}{5}\right)=\frac{15}{8}\left[\frac{\arctan(F)}{F}\left(1+\frac{2}{F^2}+\frac{1}{F^4}\right)-\frac{5}{3F^2}-\frac{1}{F^4}\right] \qquad (2.18)$$

and using the definition of S_P, we find the relation

$$\Delta_{T_2}=-S_P(1-\tfrac{3}{5}d) \qquad (2.19)$$

and from the equation for Δ_T in the tight-coupling limit we get

$$S_P=\frac{4}{3(3-2d)}ik\tau_C\Delta_{T_1}=-\frac{4}{3(3-2d)}\tau_C\dot{\Delta}_0. \qquad (2.20)$$

Notice that $d\approx 1-F^2/7$ if $F\ll 1$ while $d\to\frac{15}{16}\pi F^{-1}$ for large F. We stress here again that in a general case the depolarizing coefficient d depends upon the angle between $\vec{k}$ and $\vec{B}$. The net anisotropy and polarization being the outcome of the stochastic superposition of all Fourier modes of the density fluctuations, with a spectrum that has no privileged direction, the average depolarizing factor, after superposition of all wave vectors in arbitrary orientations with respect to the magnetic field, depends only upon F. The average depolarizing factor might slightly differ from that calculated with $\vec{k}\|\vec{B}$, which at most underestimates the average effect.

Equations (2.10), (2.19), and (2.20) condense the main effects of a magnetic field upon polarization. When there is no magnetic field $(F=0,d=1)$, $\Delta_U=0$ and $\Delta_Q=-\frac{15}{8}\Delta_{T_2}\sin^2\theta$. A magnetic field generates Δ_U, through Faraday rotation, and reduces Δ_Q. In the limit of very large F (large Faraday rotation between collisions), the polarization vanishes. The quadrupole anisotropy Δ_{T_2} is also reduced by the depolarizing effect of the magnetic field, by a factor 5/6 in the large F limit, because of the feedback of Δ_Q upon the anisotropy or, in other words, because of the polarization dependence of Thomson scattering. The dipole Δ_{T_1} and monopole Δ_{T_0} are affected by the magnetic field only through its incidence upon the damping mechanism due to photon diffusion for small wavelengths, that we shall discuss in detail in Sec. III. Indeed, the equation for $\Delta_0=\Delta_{T_0}+\Phi$, neglecting $O(R^2)$ contributions, now reads

$$\ddot{\Delta}_0+\left[\frac{\dot{R}}{1+R}+\frac{16}{90}\frac{(5-3d)}{(3-2d)}\frac{k^2\tau_C}{(1+R)}\right]\dot{\Delta}_0+\frac{k^2}{3(1+R)}\Delta_0 =\frac{k^2}{3(1+R)}[\Phi-(1+R)\Psi]. \qquad (2.21)$$

The damping term is reduced by a factor 5/6 at frequencies such that $d\ll 1$, for which depolarization is significant.

We have assumed that the magnetic field is spatially homogeneous. We can expect corrections to our result if the field is inhomogeneous over scales smaller than τ_C at any given time around decoupling. Indeed, if the field reverses its direction N times along a photon path during a time τ_C, Faraday rotation will not accumulate as assumed above. In that case depolarization would start to be significant only at those frequencies such that Faraday rotation is large over the scale on which the magnetic field reverses its direction. The frequencies at which depolarization starts to be significant would thus be reduced by a factor $1/\sqrt{N}$.

C. Frequency dependence of the degree of polarization

The anisotropy and polarization observed at present time can be evaluated using the formal solutions of Eqs. (2.1)–(2.3)

$$\Delta_T(\tau_0)=\int_0^{\tau_0}d\tau e^{ik\mu(\tau-\tau_0)}g(\tau)[\Delta_{T_0}(\tau)+\mu V_b(\tau) -\tfrac{1}{2}P_2(\mu)S_P(\tau)]+\int_0^{\tau_0}d\tau e^{ik\mu(\tau-\tau_0)} \times e^{-\kappa(\tau_0,\tau)}(\dot{\Psi}-\dot{\Phi}), \qquad (2.22)$$

$$\Delta_Q(\tau_0)=\int_0^{\tau_0}d\tau e^{ik\mu(\tau-\tau_0)}g(\tau)\{\tfrac{1}{2}[1-P_2(\mu)]S_P(\tau)+F\Delta_U(\tau)\}, \tag{2.23}$$

$$\Delta_U(\tau_0)=-\int_0^{\tau_0}d\tau e^{ik\mu(\tau-\tau_0)}g(\tau)F\Delta_Q(\tau), \tag{2.24}$$

where

$$g(\tau)\equiv\dot{\kappa}e^{-\kappa(\tau_0,\tau)} \tag{2.25}$$

is the visibility function. It represents the probability that a photon observed at τ_0 last scattered within $d\tau$ of a given τ. For a standard thermal history, with no significant early reionization after recombination, $g(z)$ is well approximated by a Gaussian centered at a redshift of about $z\approx 1000$ and width $\Delta z\approx 80$ [20]. In conformal time, we shall denote the center and width of the Gaussian which approximately describes the process of decoupling by τ_D and $\Delta\tau_D$, respectively.

The visibility function being strongly peaked around the time of decoupling, the first integral in Eq. (2.22) for the anisotropy is well approximated, at least for wavelengths longer than the width of the last scattering surface, by its instantaneous recombination limit. In that case it reduces to the tight-coupling expression of its integrand evaluated at time $\tau=\tau_D$ [9]. This first integral is dominated by its first two terms, proportional to the monopole Δ_{T_0} and the baryon velocity V_b (in turn proportional to Δ_{T_1}), respectively. The quadrupole term S_P gives a negligible contribution for long wavelengths, but becomes relatively significant on small scales. The second integral in Eq. (2.22) corresponds to the anisotropies induced by time-dependent potentials after the time of last scattering.

Equations (2.23) and (2.24) for the polarization can be approximated replacing the integrand by its tight-coupling expression. Then,

$$\Delta_Q(\tau_0)=\frac{3}{4}\frac{\sin^2\theta}{(1+F^2\cos^2\theta)}\int_0^{\tau_0}d\tau e^{ik\cos\theta(\tau-\tau_0)}g(\tau)S_P(\tau), \tag{2.26}$$

$$\Delta_U(\tau_0)=-F\cos\theta\Delta_Q(\tau_0), \tag{2.27}$$

while the total polarization, $\Delta_P=(\Delta_Q^2+\Delta_U^2)^{1/2}$, reads

$$\Delta_P(\tau_0)=\sqrt{1+F^2\cos^2\theta}\Delta_Q(\tau_0). \tag{2.28}$$

Evaluation of the time integral in Eq. (2.26) requires a more detailed knowledge of the time dependence of the integrand than in the case of the anisotropy. Indeed, the tight-coupling expression (2.20) for the quadrupole term S_P being proportional to the mean free path τ_C, which varies rapidly during decoupling, the instantaneous recombination approximation becomes inappropriate. The induced polarization is, indeed, proportional to the width of the last scattering surface. Adapting the method of [8] to include also the effect of the primordial magnetic field, we write down the equation satisfied by S_P when all other quantities are already approximated by their first-order tight-coupling expressions

$$\dot{S}_P+\frac{3}{10}(3-2d)\dot{\kappa}S_P=\frac{2}{5}ik\Delta_{T_1}. \tag{2.29}$$

Neglect of $\dot{S}_P$ returns the tight-coupling result of Eq. (2.20). Instead, the formal solution to Eq. (2.29),

$$S_P(\tau)=\frac{2}{5}ik\int_0^{\tau'}d\tau'\Delta_{T_1}\exp\left[-\frac{3}{10}\kappa(\tau,\tau')(3-2d)\right], \tag{2.30}$$

tracks down the time dependence of S_P through the decoupling process with better accuracy.

For wavelengths longer than the width of the LSS we can neglect the time variation of Δ_{T_1} and that of $e^{ik\cos\theta(\tau-\tau_0)}$ around decoupling. We also approximate the visibility function by a Gaussian, which justifies the approximation $\dot{\kappa}(\tau_0,\tau)\approx-\kappa(\tau_0,\tau)/\Delta\tau_D$ [21]. Then,

$$S_P(\tau)\approx\frac{2}{5}ik\Delta_{T_1}(\tau_D)\Delta\tau_D\exp\left[\frac{3}{10}\kappa(\tau_0,\tau)(3-2d)\right]\times\int_1^\infty\frac{dx}{x}\exp\left[-\frac{3}{10}x\kappa(3-2d)\right], \tag{2.31}$$

where the integration variable has been changed to $x=\kappa(\tau_0,\tau)/\kappa(\tau_0,\tau')$. Thus, within these approximations,

$$\int_0^{\tau_0}d\tau g(\tau)S_P(\tau)=-\frac{2}{5}ik\Delta_{T_1}(\tau_D)\Delta\tau_D\int_0^\infty d\kappa\times\exp\left[-\frac{1+6d}{10}\kappa\right]\mathrm{Ei}\left[-\frac{3}{10}(3-2d)\kappa\right]=\frac{4}{1+6d}ik\Delta_{T_1}(\tau_D)\Delta\tau_D\left[\ln\left(\frac{10}{3}\right)-\ln(3-2d)\right]. \tag{2.32}$$

Finally, the total polarization induced at an angle θ with respect to the wave vector $\vec{k}$, reads

$$\Delta_P(\tau_0)=\frac{3}{(1+6d)}[\ln(\tfrac{10}{3})-\ln(3-2d)]\times\frac{\sin^2\theta e^{ik\cos\theta(\tau_D-\tau_0)}}{\sqrt{1+F^2\cos^2\theta}}ik\Delta_{T_1}(\tau_D)\Delta\tau_D. \tag{2.33}$$

It can also be written as follows, in terms of the polarization that would be induced if there were no magnetic field (or equivalently, in terms of the polarization at frequencies large enough such that the depolarizing effect is negligible):

$$\Delta_P(\theta,F)=D(\theta,F)\Delta_P(B=0), \tag{2.34}$$

where we have defined the depolarizing factor as

$$D(\theta,F)=\frac{1}{\sqrt{1+F^2\cos^2\theta}}f(F), \tag{2.35}$$

with

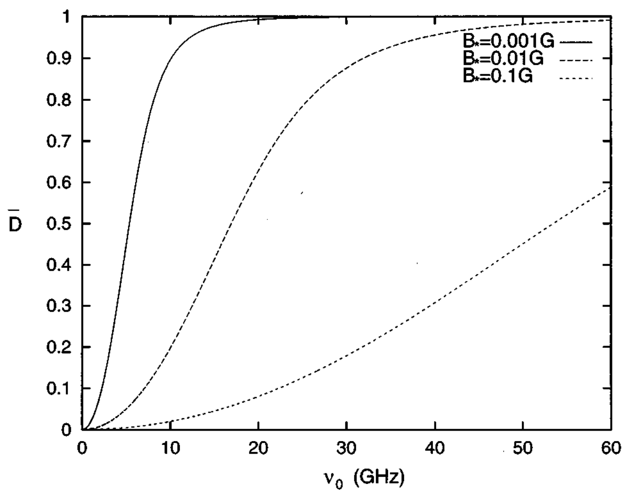

FIG. 1. The average depolarizing factor $\overline{D}$ as a function of the CMB frequency ν_0. The corresponding figure for an arbitrary value of B_* is identical to any of these after a scaling of the frequency units, proportional to $B_*^{1/2}$.

$$f(F)=\frac{7}{1+6d}\left[1-\frac{\ln(3-2d)}{\ln(10/3)}\right]. \tag{2.36}$$

Equation (2.35), together with the defining Eqs. (2.12), (2.13) and (2.18) for F and d, respectively, summarize the main result of this section. Notice that

$$f\to 1 \quad \text{as} \quad F\to 0\ , \quad f\to 0.61 \quad \text{as} \quad F\to\infty. \tag{2.37}$$

The polarization observed at present times depends upon the angle between the line of sight and the orientation of the magnetic field at the time of decoupling. There will be no depolarization if the magnetic field is perpendicular to the line of sight. The magnetic field is likely to change orientation randomly over scales longer than the Hubble radius at the time of decoupling, which subtends an angle of order 1° in the sky, so that after averaging over many regions separated by more than a few degrees, we can always expect a net average depolarizing effect. To roughly estimate its order of magnitude we could assume an average component of $\vec{B}$ parallel to the observation direction of order $B/\sqrt{2}$ and define an average $\overline{D}$ as

$$\overline{D}=\frac{1}{\sqrt{1+F^2/2}}f(F). \tag{2.38}$$

Figure 1 displays the depolarizing factor $\overline{D}$ as a function of the CMB frequency ν_0. We have plotted it for three different values of the magnetic field B_* to help visualize the relevant frequency range, but notice that since depolarization depends only upon $F=(\nu_d/\nu_0)^2$, the plot for an arbitrary value of B_* is identical to that corresponding to another value of the magnetic field after an appropriate scaling of the frequency units, proportional to the square root of the magnetic field.

At low frequencies, those for which the effect is large, the average depolarizing factor scales as

$$\overline{D}\approx 0.6\frac{\sqrt{2}}{F}\approx 0.85\left(\frac{\nu_0}{\nu_d}\right)^2 \quad \text{if} \quad \nu_0\ll\nu_d. \tag{2.39}$$

At comparatively large frequencies, instead,

$$\overline{D}\approx 1-0.36F^2=1-0.36\left(\frac{\nu_d}{\nu_0}\right)^4 \quad \text{if} \quad \nu_0\gg\nu_d. \tag{2.40}$$

III. EFFECTS UPON THE ANISOTROPY

Depolarization by a primordial magnetic field has significant and potentially measurable effects upon the anisotropy of the CMB on small angular scales. Indeed, the polarization properties of the CMB feed back into its anisotropy, as evidenced in Eq. (2.1), due to the polarization dependence of Thomson scattering. The dominant effect of polarization upon anisotropy derives from its impact upon the photon diffusion length [8,14,15], which damps anisotropies on small angular scales [9,22,23]. It was shown in [14], through numerical integration of the Boltzmann equations, that neglect of the polarization properties of the CMB leads to an overestimate of its anisotropy on small angular scales as large as 10%. We thus expect depolarization by a primordial magnetic field to introduce a significant frequency-dependent distortion of the CMB anisotropy power spectrum. Notice that a different (frequency-independent) distortion of the CMB anisotropy power spectrum by a primordial magnetic field, due to its impact upon the photon-baryon fluid sound speed, was recently discussed in [13].

A. Reduced diffusion damping

Photon diffusion damps anisotropies on small angular scales [9,22,23]. The effect is described by the term proportional to $k^2\tau_C\dot{\Delta}_0$ in Eq. (2.21). The photon-diffusion length depends upon the degree of polarization of the CMB [8,14,15]. Thus, the photon-diffusion length is different at frequencies where the depolarizing effect is significant.

The damping of anisotropies on small angular scales due to photon diffusion can be found, now including the full R dependence, by solving the tight-coupling equations to second order, assuming solutions of the form

$$\Delta_X(\tau)=\Delta_X e^{i\omega\tau}, \tag{3.1}$$

for $X=T$, Q, and U, and similarly for the baryon velocity V_b. One then finds that

$$\omega=\frac{k}{\sqrt{3(1+R)}}+i\gamma, \tag{3.2}$$

with the photon-diffusion damping length scale determined by

$$\gamma(d)\equiv\frac{k^2}{k_D^2}=\frac{k^2\tau_C}{6(1+R)}\left(\frac{8}{15}\frac{(5-3d)}{(3-2d)}+\frac{R^2}{1+R}\right). \tag{3.3}$$

The depolarizing effect of the magnetic field reduces the viscous damping of anisotropies. In the case of small R, such that R^2 terms can be neglected, the damping factor γ is smaller by a factor 5/6 at those frequencies for which the depolarizing effect is large.

We make now an analytic estimate of the effect of the frequency dependence of the photon-diffusion length, in the presence of a primordial magnetic field, upon the CMB anisotropy power spectrum. The temperature anisotropy correlation function is typically expanded in Legendre polynomials as

$$C(\theta)=\langle\Delta_T(\hat{n}_1)\Delta_T(\hat{n}_2)\rangle_{\hat{n}_1\cdot\hat{n}_2=\cos\theta}$$
$$=\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_lP_l(\cos\theta). \quad (3.4)$$

The multipole coefficients of the anisotropy power spectrum are given by

$$C_l=(4\pi)^2\int k^2dkP(k)|\Delta_{T_l}(k,\tau_0)|^2, \quad (3.5)$$

with $P(k)$ the power-spectrum of the scalar fluctuations assumed scale invariant in the standard cold dark matter (SCDM) model. The largest contribution to a given multipole C_l comes from those wavelengths such that $l=k(\tau_0-\tau_D)$, where τ_0 is the conformal time at present and τ_D the conformal time at decoupling. The average damping factor due to photon diffusion upon the C_l's is given by an integral of $e^{-2\gamma}$ times the visibility function across the last scattering surface [9]. It depends upon cosmological parameters, notably R, and upon the recombination history. Approximately, and for a standard cold dark matter model, we can take $2\gamma(d=1)\approx(l/1500)^2$. The relative change in the C_l's due to the change in the photon-diffusion length, as we move down from frequencies where depolarization is insignificant $(d=1)$ to lower frequencies $(d\ll 1)$, is then given by

$$\Delta C_l=\frac{C_l(d)}{C_l(d=1)}-1\approx\exp\left(\frac{(l/1500)^2(1-d)}{(6-4d)}\right)-1. \quad (3.6)$$

In Fig. 2 we plot ΔC_l (expressed as a percentage) at $l=1000$ as a function of frequency, for three different values of the magnetic field $B_*=$ 0.001, 0.01, and 0.1 G. Once again, the graph for an arbitrary value of B_* can be read from any of these with an appropriate scaling of the frequency units. We have chosen to display the effect at $l=1000$, that will be accessible by the recently funded CMB satellite experiments, MAP [24] and COBRAS/SAMBA [25].

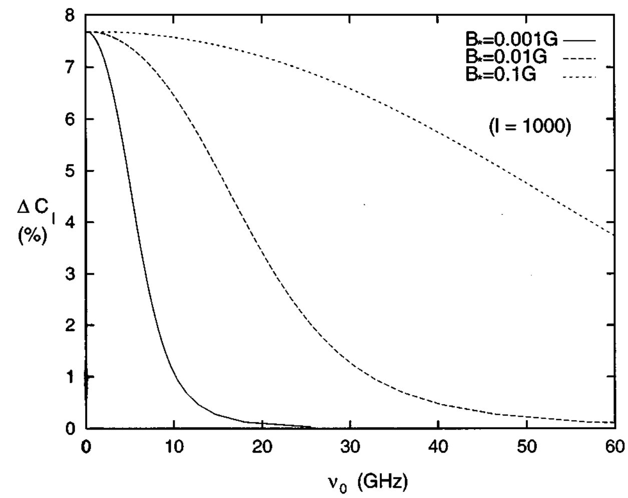

FIG. 2. Analytic estimate of the percentual change due to reduction in diffusion damping of the $l=1000$ anisotropy correlation function multipoles as a function of the CMB frequency for different strengths of the primordial magnetic field at recombination. The corresponding figure for arbitrary B_* can be obtained from any of these after a scaling of the frequency units, proportional to $B_*^{1/2}$.

B. Reduced quadrupole contribution

The depolarizing effect of a primordial magnetic field also changes the strength of the quadrupole term S_P around decoupling, and thus its incidence upon the anisotropy of the CMB on small angular scales. Indeed, the quadrupole anisotropy and the polarization of the CMB at the time of recombination contribute to the presently observed anisotropy through the following term of Eq. (2.22):

$$\Delta_{S_P}(\tau_0)\equiv-\frac{1}{2}P_2(\cos\theta)\int_0^{\tau_0}d\tau e^{ik\cos\theta(\tau-\tau_0)}g(\tau)S_P(\tau). \quad (3.7)$$

This term is negligible for long wavelengths, those that dominate the lowest multipoles of the present anisotropy, but becomes non-negligible on small angular scales (large multipoles). Indeed, in the tight-coupling approximation $S_P\propto\tau_C\dot{\Delta}_0$ and thus, barring a very strong time dependence of the scalar potential, the contribution of S_P is well below that of the monopole term, except for small wavelengths.

The depolarizing effect of a magnetic field modifies the value of S_P around decoupling, compared to what it would have been if there were no magnetic field, and then,

$$\Delta_{S_P}(F)=f(F)\Delta_{S_P}(B=0), \quad (3.8)$$

with f as defined in Eq. (2.36). Recall that $f\approx 1$ if $\nu\gg\nu_d$ while $f\approx 0.6$ if $\nu\ll\nu_d$. Thus, at frequencies such that the depolarizing effect of the magnetic field is significant, the partial contribution of the quadrupole term S_P to the total anisotropy is reduced by a factor 0.6 compared to that at frequencies where depolarization is unimportant. On small angular scales this could represent a decrease of the anisotropy by a few percent. The effect is opposite to that of the change in diffusion damping, but is likely to be less significant on small angular scales.

C. Numerical estimate of the effect upon the anisotropy

In order to accurately ascertain the net effect of the depolarizing mechanism upon the CMB anisotropy and to make definite quantitative predictions within a standard cosmological model, we turn now to the numerical integration of the Boltzmann equations (2.1)–(2.3). We use the recently developed code CMBFAST [16], that integrates the sources over the photon past light cone. Its starting points are the formal so-

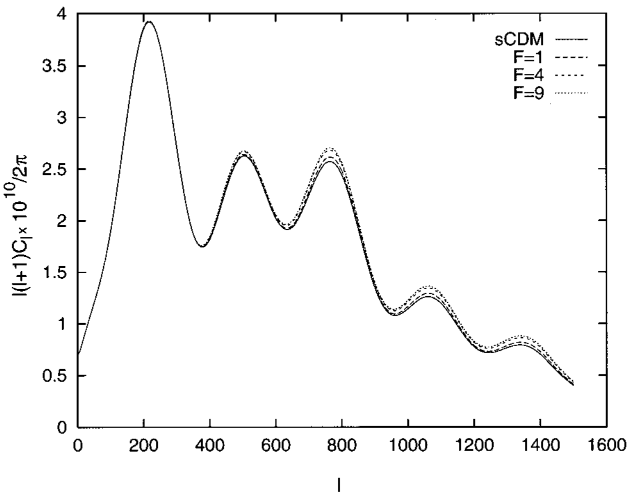

FIG. 3. Numerical integration for the multipoles of the anisotropy correlation function in a standard CDM model without a primordial magnetic field ($F=0$), and with $F=1,\ 4,\ 9$, which correspond to $\nu_0=\nu_d,\ \nu_d/2,\ \nu_d/3$, respectively, with $\nu_d\approx$ 27 GHz $(B_*/0.01\ \mathrm{G})^{1/2}$.

lutions (2.22)–(2.24), where the geometrical and dynamical contributions are separately handled to improve efficiency. As in our analytic estimates, when computing the evolution of each Fourier mode, we introduce in the code the Faraday rotation term with the angular dependence corresponding to the case where $\vec{B}$ has no component perpendicular to $\vec{k}$.

Figures 3 and 4 summarize the numerical calculation of the effect of depolarization upon temperature anisotropy, in a standard cold dark matter model (SCDM).

The quantity plotted in Fig. 3 is $l(l+1)C_l$, for the SCDM model without a magnetic field and with a magnetic field and at frequencies such that $F=1,4,9$, corresponding to $\nu_0=\nu_d,\nu_d/2$, and $\nu_d/3$, respectively. Figure 3 clearly shows that the CMB anisotropy on small angular scales increases at

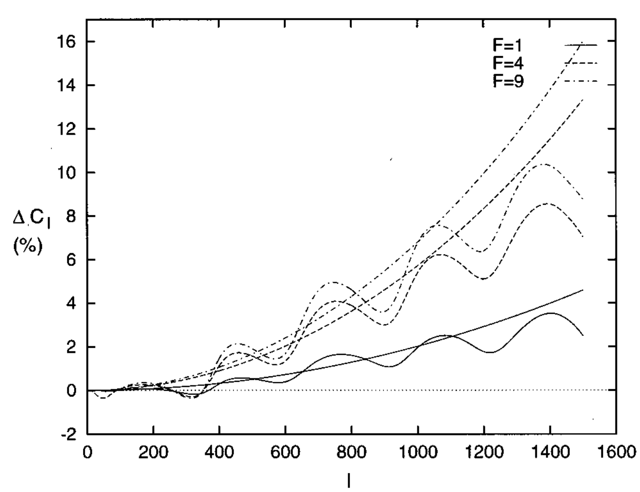

FIG. 4. Numerical result for the percentual change of C_l as a function of l relative to its value without magnetic field in a standard CDM model. The monotonic curves also shown for comparison purposes correspond to the analytic estimate of the effect of reduced diffusion damping.

frequencies where depolarization is significant. This result indicates that the reduction in diffusion damping due to depolarization is the dominant effect among the two opposite effects discussed in the previous sections.

Figure 4 displays the same results but expressed in terms of ΔC_l, the percentual increase in C_l relative to the case without magnetic field. The monotonic curves in the same figure, included for comparison purposes, correspond to the analytic estimate of the effect of reduced diffusion damping, Eq. (3.6). As expected, the effect is larger on smaller angular scales (larger l). The numerical result approximately follows the analytic estimate of the effect of the reduction in diffusion damping. The total effect, however, does not increase monotonically with l. This can be understood as a consequence of the nature of the subdominant quadrupole contribution S_P, which oscillates out of phase with the C_l's [8] (remember that $S_P\propto\dot{\Delta}_0$), and is reduced by depolarization through a factor $f(F)$.

It is also clear from Fig. 4 that the analytic result for the change in diffusion damping due to depolarization overestimates the total effect at high l. This is because the actual damping in the C_l spectra has two contributions, one from Silk damping and the other due to cancellations in the integral across the last scattering surface produced by the oscillations in the exponential and sources in Eq. (2.22). Only Silk damping is reduced by the magnetic field, and that is why Eq. (3.6) slightly overestimates the net effect.

The analytic and numeric calculations are in very good agreement around $l\approx1000$. The frequency dependence of ΔC_l at $l=1000$ plotted in Fig. 2 fits very well the analogous result after the full numerical integration of the Boltzmann equations.

We conclude that the depolarizing effect of the magnetic field results in an increase of the anisotropy correlation function multipoles of up to 7.5% (for sufficiently low frequencies) on small angular scales ($l\approx1000$), those that will be accessible by future CMB satellite experiments such as MAP and COBRAS/SAMBA. The frequencies at which the effect is significant, however, depend on the strength and coherence length of the primordial magnetic field at the time of recombination.

Depolarization depending upon frequency, the effect must be carefully separated from foreground contamination. We have expressed our results, for the sake of simplicity, in terms of a frequency-dependent temperature anisotropy. We could have developed our formalism in terms of the complete photon-distribution function, that deviates from a perfect blackbody distribution. We could then switch from a description in terms of the photon brightness function to another in terms of a frequency-dependent temperature, and define frequency-dependent multipoles of the temperature anisotropy [26]. The spectral distortion away from an absolute blackbody spectrum is, however, very difficult to measure with the required accuracy. Anisotropy detection, based on differential rather than absolute measurements, reaches instead higher sensitivities, at least if foreground contamination is well under control through multifrequency determinations.

The relative change of the C_l's at $l=1000$ is larger than 2% on frequencies below 30 GHz (accessible to the first two channels in MAP), if $B_*=0.02$ G or larger. The first two channels in COBRAS/SAMBA being at 31.5 and 53 GHz, the signal would reach a 2% level within this range only if B_* is around or larger than 0.1 G. However, COBRAS/SAMBA might reach out to larger values of l, where ΔC_l is larger, and might thus have a sensitivity to the depolarizing effect of B_* comparable to that of MAP. In any case, both experiments will be sensitive to a magnetic field around $B_*=0.1$ G, and would thus at least be able to place a direct constraint on B_* comparable or better than the one obtained from extrapolation of the nucleosynthesis bound.

Experiments searching CMB anisotropy and polarization at smaller frequencies, which currently operate down to 5 GHz [27,28], may play a significant role to detect the depolarizing effect of a primordial magnetic field.

IV. CONCLUSION

The CMB is expected to have a small degree of linear polarization. Several estimates were made for the predicted polarization, both in the context of anisotropic cosmological models [1,29], as well as in isotropic and homogeneous cosmologies perturbed with either energy-density fluctuations or gravitational waves [18,21,30]. The polarization of the CMB remains undetected, its upper limit being $P<6\times10^{-5}$ [3].

A primordial magnetic field depolarizes the CMB radiation on those frequencies that experience a significant amount of Faraday rotation around the time of decoupling. In this paper, we have applied the analytic method developed in Ref. [8] to estimate the depolarizing effect of a primordial magnetic field across the last scattering surface, assuming a standard ionization history. The result is expressed by Eqs. (2.35), (2.39), and (2.40) and is represented in Fig. 1. The CMB becomes significantly depolarized at frequencies around and below 30 GHz $(B_*/0.01\ \mathrm{G})^{1/2}$, below which the degree of polarization decreases quadratically with frequency. B_* is the value of the primordial field at a redshift $z_*=1000$, around recombination, likely to be 10^6 times larger than a hypothetical cosmological magnetic field at present times.

The average depolarizing factor depends only upon the parameter F, as defined by Eq. (2.12), which represents the average Faraday rotation between collisions. We have calculated the depolarizing factor $d(F)$, as given by Eq. (2.18), in the particular case of a wave vector $\vec{k}\|\vec{B}$. In a general case, the factor d depends upon the angle between $\vec{k}$ and $\vec{B}$. This dependence integrates away in average quantities, after the stochastic superposition of all Fourier modes of the density fluctuations. The value derived here for d is at most an underestimate of the average depolarizing effect, which would eventually start to be significant at slightly larger frequencies. Our derivation also assumed that Faraday rotation accumulates over the width of the last scattering surface. If the primordial magnetic field is very entangled over that scale, the depolarizing effect starts to be significant at smaller frequencies.

The depolarizing mechanism has a significant effect upon the anisotropy of the CMB on small angular scales. On those angular scales and at frequencies such that the depolarizing effect is large, the damping of anisotropies by photon diffusion is reduced, which results in a significant increase of the anisotropy at a fixed angular scale. In addition, depolarization reduces the contribution of the intrinsic quadrupole anisotropy. Figure 2 displays the estimate for the percentual change of the anisotropy power spectrum at $l=1000$ due to the reduction in diffusion damping, as a function of frequency and for different values of the primordial magnetic field at recombination.

We conclude that a primordial magnetic field increases the anisotropy of the CMB by up to 7.5% at $l\approx1000$ in a standard CDM cosmology. The asymptotic strength of the effect is independent of the intensity of the magnetic field, but the frequencies at which it starts to be significant are those around and below 30 GHz $(B_*/0.01\ \mathrm{G})^{1/2}$. Measurements of anisotropy and polarization at sufficiently low frequencies could probe primordial magnetic fields in an interesting range.

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