Extending the standard model using charge quantization rules

C. D. Froggatt
Department of Physics and Astronomy, University of Glasgow, Glasgow, Scotland

H. B. Nielsen and D. J. Smith
Niels Bohr Institute, Copenhagen, Denmark
(Received 13 June 1996)

We examine extensions of the standard model (SM), basing our assumptions on what has already been observed; we do not consider anything fundamentally different, such as grand unification or supersymmetry, which is not directly suggested by the SM itself. We concentrate on the possibility of additional low mass fermions (relative to the Planck mass) and search for combinations of representations which do not produce any gauge anomalies. Generalizations of the SM weak hypercharge quantization rule are used to specify the weak hypercharge, modulo 2, for any given representation of the non-Abelian part of the gauge group. Strong experimental constraints are put on our models, by using the renormalization group equations to obtain upper limits on fermion masses and to check that there is no U(1) Landau pole below the Planck scale. Our most promising model contains a fourth generation of quarks without leptons and can soon be tested experimentally. [S0556-2821(97)06803-3]

PACS number(s): 12.60.Cn

I. INTRODUCTION

Over the years there have been numerous attempts at extending the standard model (SM). Some of these models have been proposed with the purpose of explaining some particular feature of the SM. For example, grand unified theories (GUT's) ''explain'' the convergence of coupling constants at some energy as a manifestation of a single fundamental unified interaction. Other models such as supersymmetry (SUSY) have been proposed for mainly aesthetic reasons: SUSY introduces a symmetry between bosons and fermions. But so far none of these attempts has been entirely successful, although SUSY GUT's are phenomenologically consistent with the unification of the SM gauge coupling constants and do not suffer from the technical gauge hierarchy problem.

Another approach to extending the SM is to look at the SM itself and look for distinctive features which could be generalized or assumed to hold in an extended theory. The SM has been so successful that, within our experimental and calculational accuracy, it has proved to be a perfect description of nature (except for the gravitational interaction). So a natural method of extending the SM is to look for fundamental features in the SM which could distinguish it from similar and, without experimental evidence, equally plausible models. We propose that one such feature is charge quantization. This can be expressed as

$$\frac{y}{2}+\frac{1}{2}\text{``duality''}+\frac{1}{3}\text{``triality''}\equiv 0 \pmod 1, \tag{1}$$

where y is the conventional weak hypercharge. The duality has value 1 if the representation is an SU(2) doublet (**2**) and 0 if it is an SU(2) singlet (**1**). The triality has value 1 if the representation is an SU(3) triplet (**3**), 0 if it is an SU(3) singlet (**1**), and -1 if it is an SU(3) antitriplet ($\overline{\mathbf{3}}$). In general we can define N-ality to be the number of N-plet representations of SU(N) which must be combined to give the representation of SU(N). In particular N-ality has value 1 if a representation is an SU(N) N-plet (**N**), 0 if it is an SU(N) singlet (**1**), and -1 if it is an SU(N) anti-N-plet ($\overline{\mathbf{N}}$). Note that in SU(2) the $\overline{\mathbf{2}}$ representation is equivalent to the **2** representation. We expect that in an extension of the SM this charge quantization relation or some generalization of it will hold.

An obvious way of extending the SM is to extend the gauge group. The standard model group (SMG) is [1,2]

$$\text{SMG}\equiv S(\text{U}(2)\otimes\text{U}(3))=\text{U}(1)\otimes\text{SU}(2)\otimes\text{SU}(3)/\hat{D}_3, \tag{2}$$

where the discrete group

$$\hat{D}_3\equiv\{(e^{i2\pi/6},-I_2,e^{i2\pi/3}I_3)^n : n\in\mathcal{Z}_6\} \tag{3}$$

ensures the above quantization rule [I_N is the identity of SU(N)]. We argue that the most obvious extension is to add more groups to the sequence $\text{U}(1)\otimes\text{SU}(2)\otimes\text{SU}(3)$ and to use a different discrete group so that the quantization rule above is generalized to involve all the group components. One of the groups we consider is

$$G_5\equiv\text{U}(1)\otimes\text{SU}(2)\otimes\text{SU}(3)\otimes\text{SU}(5)/\hat{D}_5, \tag{4}$$

where the discrete group $\hat{D}_5$ is defined as

$$\hat{D}_5\equiv\{(e^{i2\pi/N_5},-I_2,e^{i2\pi/3}I_3,e^{i2\pi m_5/5}I_5)^n : n\in\mathcal{Z}_{N_5}\}, \tag{5}$$

where $N_5=2*3*5$ and m_5 is an integer which is not a multiple of 5. This group gives a generalized quantization rule

$$\frac{y}{2}+\frac{1}{2}\text{``duality''}+\frac{1}{3}\text{``triality''}+\frac{m_5}{5}\text{``quintality''}\equiv 0 \pmod 1, \quad (6)$$

which is the simplest generalization of the SM charge quantization rule. Further generalizations are obtained by extending the sequence $U(1)\otimes SU(2)\otimes SU(3)$ with a set of SU(N) factors, where the N's are greater than 3 and mutually prime. The latter condition ensures that the generalized quantization rule shares the property with the SM rule, Eq. (1), that a given allowed value of $y/2$ implies a unique combination of N-alities: (duality, triality, ..., N_i-ality, ...).[1]

We will consider the fundamental scale to be the Planck mass (M_{Planck}) and our models will be a full description of physics without gravity below this scale. The assumptions we make about our models essentially lead to the conclusion that all new fermions with a mass significantly below M_{Planck} must have a mass below the TeV scale as explained in Sec. III B. Therefore our models all describe low energy physics (below the TeV scale) and have a desert up to the Planck scale where new physics will occur. We do not specify any details about the Planck scale physics since it is largely irrelevant to low energy physics.

We shall describe the gauge groups considered in this paper and the motivation for choosing such groups in Sec. II A 1. We shall consider general types of gauge groups and also give specific examples, concentrating on the group G_5 defined above. When we also impose the condition that all fermions are in fundamental or singlet representations, as in the SM, we are limited to the models which we shall consider in this paper. After choosing the gauge group we want to examine which low mass fermions (low relative to the Planck scale) can exist in the model. We must check that the model is then consistent, both theoretically and experimentally.

The main theoretical constraint is that there are no anomalies as described in Sec. II C. This greatly limits the choice of fermions and their weak hypercharges in our models. In the Appendix we show how the SM generation of mass-protected fermions can be derived using our assumptions about charge quantization, small representations, and anomaly cancellation.

There is one important fact to keep in mind when proposing any extended model which has extra non-Abelian gauge groups such as SU(N). As we already know from the SM, the SU(3) group acts as a technicolor group [3] and gives a contribution to the $W^{\pm}$ and Z^0 masses. In the SM this contribution is very small but when confining groups with $N>3$ are considered we must carefully consider the effect this will have. Since we are not wanting the complications of extended technicolor in order to generate quark and lepton masses, we assume that there is a Higgs doublet and that the masses of the weak gauge bosons are generated by a combination of the Higgs sector of the theory and the technicolor effects of the gauge groups. This happens in exactly the same way as in the SM where QCD gives a small contribution to the $W^{\pm}$ and Z^0 masses [3].

For our models to be perturbatively valid, all Yukawa couplings at the electroweak scale must be not much greater than 1. However, we will sometimes take a somewhat higher mass threshold for all the new fermions when checking to see if a model could be perturbatively valid up to the Planck scale. For example, we can calculate the running gauge coupling constants, assuming that all the new fermions can be included in the renormalization group equations (RGE's) at the TeV scale. Thus we can check to see if any gauge coupling constant becomes infinite below the Planck scale [i.e., if there are any Landau poles, especially for the U(1) coupling]. If the threshold was lower then the new fermions would effect the coupling constants even more but this would only be a small effect. Obviously we do not want the coupling constants to become infinite or the theory will be inconsistent. When we do this we find that there are few self-consistent models allowed by our assumptions, in the sense that for any particular gauge group only a few combinations of fermions which cancel the anomalies do not cause the U(1) gauge coupling to diverge.

We will show that in the model with gauge group G_5 we can add new fermions with masses accessible to present or planned future accelerators, in particular a fourth generation of quarks without any new leptons. At present this model is consistent and can be tested experimentally in the near future. It can be viewed as the simplest alternative to the SM which has the same characteristic properties as the SM itself.

In Sec. II we shall outline our requirements for a viable model. We will discuss theoretical constraints such as anomaly cancellation as well as aesthetic extrapolations from the SM, including charge quantization as already mentioned. In the Appendix we show how these ideas can be used to derive the SM generation.

In Sec. III we shall discuss the experimental constraints which arise from the consistency of the SM with experiments. This includes the experimental limits on the mass of the top quark and the masses of new, undetected fermions.

In Sec. IV we will discuss the simplification of the anomaly constraints when we assume that all fermions get a mass by the SM Higgs mechanism.

In Sec. V we shall show the difficulty of constructing a model where all the new fermions are in 5-plet or anti-5-plet representations of SU(5). We shall show that such a solution is not possible within the context of our model.

In Sec. VI we will see how the difficulties of Sec. V can be overcome by also adding fermions which are SU(5) singlets; in particular a fourth generation of quarks but no fourth generation of leptons. We will also show how such a solution can be formulated in a more general gauge group.

In Sec. VII we shall discuss the overall merits of such a model and how easily it could be tested experimentally.

II. DISCUSSION OF FORMALISM OF MODELS AND THEORETICAL CONSTRAINTS

First we shall discuss which models we will be considering as viable extensions of the SM and then we shall discuss in detail the requirements for a potentially successful extension of the SM. We shall use some of these constraints when constructing models and the rest to check the consistency of our models.

[1]This corresponds to the global group, associated with the generalized charge quantization rule, having a connected center [1].

A. Extrapolations from the SM

In this section we discuss aesthetic extrapolations from the SM. These are features of the SM which have no obvious explanation but in some way can be used to specify the model almost uniquely. We try to pick out these features and carry them over to or generalize them in our extended model. This is a method of selecting a particular model and our view is that this is the most logical method although the features chosen may of course be subject to personal prejudice.

1. Extending the gauge group and charge quantization

As stated in Sec. I, an obvious way of extending the SM is to extend the gauge group. The SMG is

$$\mathrm{SMG} \equiv U(1) \otimes SU(2) \otimes SU(3)/\hat{D}_3, \tag{7}$$

where the discrete group

$$\hat{D}_3 \equiv \{(e^{i2\pi/6}, -I_2, e^{i2\pi/3}I_3)^n : n \in \mathcal{Z}_6\} \tag{8}$$

ensures the quantization rule, Eq. (1). We believe that the most obvious extension is to add more special unitary groups to the sequence $U(1)\otimes SU(2)\otimes SU(3)$ and to use a different discrete group so that the quantization rule above is generalized. In [1] it is argued that the group should be of the form

$$G_p \equiv U(1)\otimes SU(2)\otimes SU(3)\otimes SU(5)\otimes \cdots \otimes SU(p)/\hat{D}_p, \tag{9}$$

where the product is over all SU(q) where q is a prime number less than or equal to the prime number p. The discrete group $\hat{D}_p$ is defined as

$$\hat{D}_p \equiv \{(e^{i2\pi/N_p}, -I_2, e^{i2\pi/3}I_3, e^{i2\pi m_5/5}I_5, \ldots, e^{i2\pi m_p/p}I_p)^n : n \in \mathcal{Z}_{N_p}\}, \tag{10}$$

where $N_p = 2*3*5*\cdots*p$ and m_N is an integer which is not a multiple of N. In fact we can obviously choose $0 \leq m_N \leq N-1$ since m_N is really only defined modulo N. We also have the freedom to choose that there are, for example, at least as many SU(2) doublets which are **N** representations of SU(N) as $\overline{\mathbf{N}}$ representations since we can conjugate SU(N) and set $m_N \to -m_N$ (mod N). We will use this fact later to eliminate duplicate solutions where all N-plets and anti-N-plets have been interchanged. This also allows us to fix $m_3 = 1$.

This group gives a generalized quantization rule

$$\frac{y}{2} + \frac{1}{2}\text{``duality''} + \frac{1}{3}\text{``triality''} + \frac{m_5}{5}\text{``quintality''} + \cdots + \frac{m_p}{p}\text{``}p\text{-ality''} \equiv 0 \pmod 1. \tag{11}$$

We will also consider the more general groups defined as

$$\mathrm{SMG}_{2N_1N_2\cdots N_k} \equiv U(1)\otimes SU(2)\otimes SU(N_1)\otimes \cdots \otimes SU(N_k)/D_{2N_1\cdots N_k}, \tag{12}$$

where

$$D_{2N_1\cdots N_k} \equiv \{(e^{i2\pi/\hat{N}}, -I_2, e^{i2\pi m_{N_1}/N_1}I_{N_1}, \ldots, e^{i2\pi m_{N_k}/N_k}I_{N_k})^m : m \in \mathcal{Z}_{\hat{N}}\}. \tag{13}$$

Here $\hat{N} = 2*N_1*\cdots*N_k$ and the N_i are odd and mutually prime (we can obviously assume they are arranged in ascending order). So the quantization rule is

$$\frac{y}{2} + \frac{1}{2}d + \frac{m_{N_1}}{N_1}n_1 + \cdots + \frac{m_{N_k}}{N_k}n_k \equiv 0 \pmod 1, \tag{14}$$

where we have defined d to be the duality and n_i to be the N_i-ality of a representation. The groups SMG_{23N} are the minimal extensions of the SMG ($\equiv \mathrm{SMG}_{23}$) which are inspired by the SMG, in the sense that each is also a cross product of U(1) and a set of distinct special unitary groups with a charge quantization rule involving all the direct factors and contains the SMG as a subgroup. The property of the SMG that the value of $y/2$ determines both the duality and triality extrapolates to the principle that $y/2$ should also fix the N-ality, but then it is needed that 2, 3, and N are mutually prime.

It has been suggested that a defining property of the SMG is that it has few outer automorphisms relative to the rank of the group [4,5]. This can be described by saying that it is very skew. The intermingling of the various simple groups SU(2), SU(N_1), ..., SU(N_k) implied by the charge quantization rule, Eq. (14), helps to suppress the number of outer automorphisms and ''generalized automorphisms.'' Thus a group like $\mathrm{SMG}_{2N_1N_2\cdots N_k}$ would indeed be more skew than groups without such intermingling. Alternatively we can derive Eq. (14) directly as a natural generalization of the SM charge quantization rule, Eq. (1).

Of course it is possible that the apparent charge quantization rule in the SM is simply due to chance; i.e., the fermions in the SM just happen to obey that particular rule. However, we believe that the quantization rule is a fundamental feature of the SM; so we argue that it is very difficult to see how there cannot be a generalization of this rule in an extended model, while still retaining the general features of the SM. In fact the form of the generalized quantization rule is suggested from the SM and there seems to be little choice in selecting the rule since the SM rule appears to be the one which involves all the direct factors equivalently. In fact the choice of the most complicated charge quantization rule in some way defines the SMG. This is why we have divided out the discrete groups $\hat{D}_p$ and $D_{2N_1 \cdots N_k}$.

2. Small representations

In the SM, for each SU(N) group, the fermion representations are either N-plet (**N**), anti-N-plet ($\bar{\mathbf{N}}$), or singlet (**1**). This can be described by saying that all the fermions lie in fundamental representations of each SU(N) group to which they couple. We pick this as a feature of the SM which we shall extend to our models. We note here that this is in contrast to some other attempts to extend the SM. For example in SUSY there are fermions in other representations (e.g., gauginos in adjoint representations). Fundamental representations are also suggested in [6] since these make the Weyl equation most stable when considering random dynamics.[2]

Another feature is that the weak hypercharge is in some way minimized in the SM, subject of course to the constraints of anomaly cancellation and charge quantization, as shown in the Appendix. So in our extended model we will choose hypercharge values close to zero when this is possible. More precisely, we choose to minimize the sum of weak hypercharges squared over all fermions. This will also minimize the running of the U(1) gauge coupling constant and so give each model the best chance of being consistent up to the Planck scale, which we require as stated in Sec. II A 3.

3. Higher energies: Desert hypothesis

The SM has been tested at energies up to a few hundred GeV. There have been many theories proposed which would be valid at energy scales ranging from 1 TeV up to the Planck scale around 10^{16} TeV. Many of these theories have a large range of energy where no new physics occurs. One example is GUT's where there is typically no new physics from the SM energy scale up to the grand unification scale around 10^{13} TeV. An alternative is that there is no new physics until the Planck scale where we can be almost certain that quantum gravity will have a significant effect. We shall adopt this view for our extended models. This means that once we have set the mass scale for the fermions in the extended model, we can calculate the running coupling constants and check to see if there is a Landau pole below the Planck scale, i.e., whether the U(1) gauge coupling becomes infinite below the Planck scale. If there is a Landau pole then we will conclude that such a model is not consistent.

B. Fermion representations and alternative groups

In this section we shall describe some alternative extensions of the SM. We will consider groups similar to those we are examining in this paper in the sense that they contain only the SMG and additional special unitary group factors. This obviously does not include models which unify the individual components of the SMG. Models which involve SUSY will not be considered here since we are making the assumption of fundamental or singlet representations for all fermions. models with the non-SUSY. There have been many such models and the additional symmetries are usually used to explain coupling constant unification, the number of families in the SM, or the fermion mass hierarchy in a fairly natural way.

In the models described in Sec. II A 1 the SM fermions cannot couple to any new gauge fields because of the charge quantization rule. This is due to the fact that all values of $y/2$ in the SM are multiples of $\frac{1}{6}$ and so the charge quantization rule, Eq. (14), forces the SM fermions to be singlets of all SU(N) groups where $N>3$ are distinct primes.

However, the situation is more complicated if we allow more than one SU(N) gauge group for any particular N. Where we have $N=2$ or 3 there are two distinct cases. In the first case the SM group SU(N) is an invariant subgroup of the extended group. We then call the extra SU(N) groups a horizontal symmetry. In the other case the SU(N) group in the SMG is not an invariant subgroup and is generally a diagonal subgroup of the extended group.

1. Invariant subgroup case: Horizontal symmetries

If we have one more SU(2) or SU(3) group then we can have a horizontal symmetry (a non-Abelian symmetry which places fermions from different generations in the same multiplet). The idea of a gauged horizontal symmetry is not new and has been used to try and explain the mass hierarchy of the SM fermions [7]. However, an SU(N) group with $N>3$ is not a possible horizontal symmetry without introducing many more fermions because there are only three generations of SM fermions and the smallest nontrivial representation of SU(N) is the N-plet. For example if $N=5$ we would have an SU(5) horizontal symmetry and so we would need at least five generations of SM fermions.

If the horizontal symmetry gauge group is $SU(3)_H$ then we must place fermions from different generations in the same triplet (or antitriplet). It turns out that the only way to do this, avoiding anomalies (see Sec. II C) and not introducing any new fermions, is to put all fermions in the same (or conjugate) representation of $SU(3)_H$ as they are in the color group $SU(3)_C$ of the SM; so that all three generations of left-handed quarks are put in a triplet (or antitriplet) of $SU(3)_H$, etc. However, the SM fermions would not then obey the charge quantization rule which might be expected, similar to Eq. (14):

$$\frac{y}{2}+\frac{1}{2}d+\frac{1}{3}t_C+\frac{1}{3}t_H\equiv 0 \pmod{1}. \tag{15}$$

If the horizontal symmetry group is $SU(2)_H$ then we can make some or all SM fermions triplets of $SU(2)_H$ but this is not the smallest representation and so we do not favor this as explained in Sec. II A 2. We could place some fermions in doublets of $SU(2)_H$. This could be done, without introducing any anomalies, by placing two generations of quarks in the same doublet or taking two generations and placing the

[2] In fact, from this point of view, each representation of the full gauge group should only be nonsinglet with respect to one non-Abelian factor. This is not true for the left-handed quarks but is true for all other fermions in the SM. However, the left-handed quarks are required in order that there are no gauge anomalies. So we can consider that the Weyl equation is as stable as possible if we only have small representations.

fermions in the same representation of $SU(2)_H$ as they are in the $SU(2)_L$ group of the SM. Different doublets could connect fermions from a different pair of generations. For example left-handed quarks from the first and second generations could be in the same doublet, right-handed ''up'' quarks from the first and third generations could be in the same doublet, and right-handed ''down'' quarks from the second and third generations could be in the same doublet. This would not give any anomalies though it is difficult to see how this could be used to explain the fermion masses. The main problem is that fermions in different generations with very different masses are put in the same multiplet. This means that the fermions would naturally get the same mass. It is difficult to break the symmetry in such a way that the masses of all the different fermions are split by realistic amounts [7].

We do not consider these possibilities in this paper because triplets of SU(2) are not fundamental representations and the other possibilities, with gauge group $SU(2)_H$ or $SU(3)_H$, mean that the fermions could not obey the extended charge quantization rule. Of course models involving horizontal symmetries do not enforce such charge quantization rules.

2. Noninvariant subgroup case: SMG as diagonal subgroup

In the case where, for example, the $SU(3)_C$ subgroup of the SMG is not an invariant subgroup of the full gauge group, the only possibility is that it is a diagonal (or antidiagonal) subgroup of $SU(3)^n$. In this type of model different generations can couple to different SU(2) and SU(3) gauge groups in the full gauge group. There would then be symmetry breaking to produce the SMG, in such a way that $SU(3)_C$ could be said to be a diagonal subgroup of all the SU(3) groups in the full group which exists at energies higher than the symmetry breaking scale. In other words, $SU(3)_C$ is then the subgroup in which all the SU(3) groups undergo the same transformations. In this way it is trivial to cancel all the anomalies, since each generation of quarks and leptons cancel all anomalies separately and couple to a $U(1)\otimes SU(2)\otimes SU(3)$ subgroup of the full group in the same way as they couple to the SMG. This is in contrast to the invariant subgroup case, where the SM fermions had to couple to the SMG and also to other subgroups of the full gauge group. Also in the diagonal case, the dimension of each representation is the same as in the SM, whereas, in the invariant subgroup case, the dimensions were larger since different SM representations were combined under the horizontal symmetry.

This type of model has been proposed [8] as an alternative to horizontal symmetries or grand unification. Examples include top-color models [9] and the antigrand unification model [10], where the group $SMG^3 \equiv SMG\otimes SMG\otimes SMG$ has been used to successfully predict the values of the gauge coupling constants. The antigrand unification model has also been analyzed as a model to explain the hierarchy of SM fermion masses [11]. Here the extended model with gauge group $SMG^3\otimes U(1)_f$ has been fairly successful at reproducing the observed fermion masses in an order of magnitude approximation (reproducing all SM fermion masses within a factor of 2 or 3). The extra $U(1)_f$ gauge symmetry is called a flavor symmetry and is required to produce the observed mass differences within the second and third generations, e.g., $m_b \ll m_t$.

We note that the fermions in some of these models obey extended charge quantization rules which we would expect. For example the fermions in the SMG^3 model obey the charge quantization rules,

$$\frac{y_i}{2}+\frac{1}{2}d_i+\frac{1}{3}t_i\equiv 0 \pmod 1, \tag{16}$$

where the three copies of the SMG are labeled by $i=1$, 2, and 3. With three separate charge quantization rules, this is not truly a straightforward extrapolation of the SM charge quantization rule. However it is similar in the sense that these rules are required to produce the group SMG^3 which has as large χ as the group SMG itself.[3] The quantity χ measures how strongly intermingled the U(1) subgroups are with the semi-simple part via the dividing discrete groups [i.e., equivalently via the quantization rule(s)]. It happens that groups of the form SMG^n have the largest possible value of this measure $\chi=\ln(6)/4$. The charge quantization rules are chosen to maximize χ for the group $SMG^3\otimes U(1)_f$ among all those with the same algebra although this group does not have as large a value of χ as the SMG. In fact $\chi=\ln(6^3)/13=\frac{12}{13}\ln(6)/4$ for the group $SMG^3\otimes U(1)_f$.

However, the symmetry breaking scale of the group SMG^3 is taken to be just below the Planck scale in the antigrand unification model and in this paper we wish to study the possibilities of new physics at much lower energies; energies of the same order of magnitude as the electroweak scale rather than the Planck scale. This is still possible in such a model but it then loses its ability to predict the gauge coupling constants. Top-color models do introduce new dynamics at the TeV scale but in this paper we shall not consider such models.

C. Anomalies

1. Gauge anomalies

In any chiral gauge theory, gauge anomalies can arise. These anomalies lead to an inconsistent theory and so they must not be present in a good theory. Each fermion representation makes its contribution to each type of anomaly. We say that there is an anomaly present if the total contribution to an anomaly from all the fermion representations is nonzero.

As explained in Sec. II A 1, the models considered in this paper have gauge groups of the general form

[3]The quantity χ is defined in [5] for any group G as $\chi(G)=\ln[q(G)]/r(G)$ where $r(G)$ is the rank of the group G [really the number of U(1) factors in the maximal Abelian subgroup]. Further, $q(G)$ is defined as the order of the factor group, obtained by dividing the group of all Abelian charge combinations $(y_1,y_2,\ldots,y_r)$ allowed for any representations of the group G, by the group of those Abelian charge combinations allowed for representations trivial under the semisimple part of the group G.

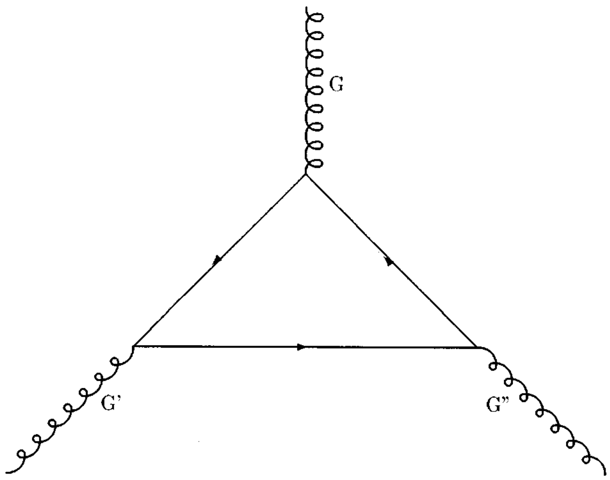

FIG. 1. For the theory to be anomaly free, the amplitude of this Feynman diagram must be zero for all choices of external gauge bosons after summing over all possible fermions in the internal loop (triangle).

$$\mathrm{U}(1)\otimes\prod_i \mathrm{SU}(N_i)/D. \tag{17}$$

The discrete group D leads to charge quantization. We assume all fermions to be in **N**, $\overline{\mathbf{N}}$, or singlet (**1**) representations of each SU(N), as discussed in Sec. II A 2. We define n to be the N-ality of a representation [$n=1$ (-1) for representation **N** ($\overline{\mathbf{N}}$) and $n=0$ for singlet representation]. We can also define the size, S, of each representation as the dimension of the representation [e.g., in the SM, $S=6$ for the (2,3)] representation of SU(2)⊗SU(3) which is equivalent to the fact that there are six left-handed quarks in each generation].

For gauge anomalies we sum the contribution for all left-handed fermions and subtract the sum over all right-handed fermions. This is equivalent to summing over left-handed fermions and left-handed antifermions. We have now introduced all the necessary notation to write down general equations for all types of gauge anomalies.

The requirement that there are no anomalies present in a theory is analogous to the triangle Feynman diagram in Fig. 1 with a fermion loop and three external gauge bosons (labeled by G, G', and G'') having zero amplitude for all possible choices of gauge bosons G, G', and G''. The contribution from each fermion representation is calculated by making particular choices for the fermions in the internal loop. These contributions must then sum to give zero amplitude if there is to be no anomaly.

If each of G, G', and G'' is an SU(N) gauge boson where $N\geqslant 3$ then each representation gives a relative contribution of $Sn^3=Sn$ (since $n=-1$, 0, or 1 in our models). The total contribution is therefore $\Sigma_i S_i n_i$ where i labels each left-handed fermion (and antifermion) representation. We label this type of anomaly $[\mathrm{SU}(N)]^3$ and require

$$\sum_i S_i n_i=0. \tag{18}$$

Another type of anomaly corresponds to the diagram with one U(1) gauge boson and two SU(N) gauge bosons where $N\geqslant 2$, labeled as $[\mathrm{SU}(N)]^2\mathrm{U}(1)$. Each representation gives a relative contribution Sn^2y. Therefore we require

$$\sum_i S_i(n_i)^2 y_i=0. \tag{19}$$

The final type of gauge anomaly corresponds to the diagram with all the gauge bosons G, G', and G'' being U(1) gauge bosons. This is labeled as $[\mathrm{U}(1)]^3$ and each representation gives a relative contribution Sy^3. Therefore we require

$$\sum_i S_i y_i^3=0. \tag{20}$$

2. Other anomalies

There is also a mixed gravitational and gauge anomaly [12] which corresponds to one U(1) gauge boson and two gravitons. We will label this as $[\mathrm{Grav}]^2\mathrm{U}(1)$. Each representation gives a relative contribution Sy and so this leads to the constraint

$$\sum_i S_i y_i=0. \tag{21}$$

Another possible anomaly is the Witten discrete SU(2) anomaly [13]. This states that if the number of left-handed SU(2) doublets is odd then the theory is inconsistent. As we shall see later this anomaly does not give us any problems.

III. EXPERIMENTAL CONSTRAINTS

In this section we shall discuss the constraints on our models which are due to experimental evidence. In particular we are concerned with the possibilities for the existence of more fermions and what restrictions can be imposed both directly and indirectly on their mass. Some difficulty arises since fermions may be confined and so not directly observable. This means that direct experimental restrictions will refer to the mass of particles which are combinations of these fermions, like hadrons in the case for quarks.

A. Direct experimental constraints on fermion masses

First we shall discuss the constraints on fermion masses due to the fact that so far no non-SM fermions have been observed. We shall show that this rules out any extra massless fermions and then give current limits on the masses of different type of new fermions.

1. Massless fermions

Only three massless fermions have been observed and they are the three massless neutrinos described in the SM (even if the neutrinos do have a small mass we know that there are only 3 with a mass less than $\frac{1}{2}M_Z$). Any other massless fermions, which had any significant coupling to the SM fermions or gauge bosons, would have been observed if they were not confined. When we assume that fermions belong only to fundamental and singlet representations (as postulated in Sec. II A 2), the charge quantization rule in our models ensures that the only possible fermions which would not be electrically charged would be neutrinos. A left-handed neutrino without a right-handed neutrino would be massless as in the SM. We already know that there are only three such neutrinos and so we cannot consider this as a possibility for new fermions. A right-handed neutrino would be completely decoupled from the gauge group and so it could get a gauge invariant Majorana mass. So we would expect that it would have a mass $\sim M_{\rm Planck}$ and so it is excluded as a low mass fermion in our models. Therefore any new massless fermions in our models must be electrically charged and so must also be confined by a new interaction well above the QCD scale, on phenomenological grounds. neutrino cannot explanation

If there is a confined gauge group then we assume that fermion condensates will be formed as in QCD. If a fermion does not have a chiral partner with respect to some confined group H, the condensates formed will break the group H. So if we assume that there is no spontaneous gauge symmetry breaking, other than that of the electroweak symmetry group, no fermions can be chiral with respect to G where the full gauge group is $U(1)\otimes SU(2)\otimes G/D$ (where D is some discrete group). In our models the extra SU(N) gauge groups are all confining (with negative beta functions), so that $G\equiv H$.

If the left- and right-handed fermions occur with the same representations of the full gauge group $U(1)\otimes SU(2)\otimes G/D$, then the fermions can form a Dirac mass term in the Lagrangian. So they would be expected to get a mass comparable to the fundamental scale, which we take to be the Planck mass in our models. Such fermions would not contribute to any anomalies and would not be observable because of their high mass. We shall therefore ignore them in our models. If a fermion cannot form such a fundamental Dirac (or Majorana) mass term then we say it is mass protected, since it would be fundamentally massless and could only get a mass indirectly through some interaction such as the Higgs mechanism. All the fermions considered in our models are mass protected by the electroweak interactions.

We conclude that all new fermions in our models must get their mass from the Higgs mechanism. Furthermore, they must couple to the usual SM Higgs particle in the same way as the SM fermions. In other words, the fermion condensates must have the same quantum numbers as the SM Higgs boson; otherwise their contributions to the $W^{\pm}$ and Z^0 masses, via the usual technicolor [3] mechanism, would be analogous to those from the vacuum expectation values of Higgs particles with nonstandard weak isospin and hypercharges. This would lead to a significant deviation of the ρ parameter ($\rho\equiv M_W^2/M_Z^2\cos^2\theta_W$) from unity [14] in contradiction with precision electroweak data.

2. Massive fermions

In the SM there are two different types of fermions, quarks and leptons, which differ by the fact that quarks couple to the SU(3) gauge fields and so are confined, whereas leptons have no direct coupling to the SU(3) gauge fields and are not confined. There are experimental limits on the masses of any quarks and leptons which have not yet been observed. If there are any more leptons then they must have a mass greater than 45 GeV [15]. We shall assume that there are no more leptons, since even the neutrino would have to get a mass larger than this and it is difficult to see how a neutrino could naturally be given a mass greater than 45 GeV but still much lower than the fundamental scale (which is the Planck scale in our models). This is because a right-handed neutrino, as already discussed in Sec. III A 1, would naturally get a Majorana mass and so the see-saw mechanism [16] would leave the left-handed neutrino with a very small mass. For this reason we cannot allow any more generations of SM leptons. However the limits on the quark masses are dependent on the type of quark and its decay modes.

The top quark has recently been observed by the Collider Detector at Fermilab (CDF) and D0 Collaborations [17]. The mass is in the range 150–220 GeV. For the purpose of this paper we take the limit on possible fourth generation quarks, t' and b', to be

$$M_{t'},M_{b'}>130\ \text{GeV}$$

from the dilepton analyses of the CDF and D0 groups [18] (less restrictive limits apply if other decay modes are dominant). Note that experimental limits are taken to apply to the pole masses.

The above experimental limits do not apply to new fermions which are not singlets of the additional SU(N) gauge groups. These fermions would be more difficult to detect experimentally and would anyway be confined inside ''hadrons'' with a confinement scale (generically at the electroweak scale) much higher than the QCD scale. We require our models to remain perturbative in the desert from the TeV scale to the Planck scale. So we can use the RGE's to examine how the Yukawa couplings evolve from the Planck scale down to the electroweak scale. In particular we study the infrared quasi-fixed-point structure of the renormalization group equations (RGE's). In the SM the fixed point values provide upper limits on the mass of the top quark, M_t, and the Higgs scalar, M_H. Similarly in extended models we get upper limits on the masses of the heaviest fermions, though the precise values depend on the relative masses of these fermions and also the unknown gauge coupling strength, g_N, of the SU(N) groups to which the fermions couple. Also we must be careful to point out that the RGE's describe the running of the Yukawa couplings and, as we discuss in Sec. III B, the actual masses will be less than naively expected, due to the technicolorlike contribution from SU(N) to the electroweak vacuum expectation value (VEV), $v=246$ GeV. As we shall see, this will enable us to quite accurately predict the masses of some of the fermions we introduce in our model in Sec. VI, since we have theoretical upper limits and experimental lower limits.

B. Technicolor contributions

Technicolor theories [3] have been proposed as an alternative to the Higgs mechanism to provide a mass for the weak gauge bosons. This is based on the fact that QCD would provide a (very small) mass for these bosons without any Higgs scalars. Similarly any other confining SU(N) gauge groups, with fermions which are nontrivial under $U(1)\otimes SU(2)$, are expected to form fermion condensates which would contribute to the $W^{\pm}$ and Z^0 masses. In our models the charge quantization rule ensures that all fermions are nontrivial under U(1). Thus all SU(N) groups in our models, which are coupled to fermions, will contribute to the weak boson masses.

We stress that we are not proposing a technicolor model as such, but simply taking into account the unavoidable effect that adding an SU(N) group has. We are assuming that the Higgs sector of our models is the same as in the SM, i.e., one Higgs doublet, and that the fermion condensates have the same quantum numbers as the Higgs doublet. Then the VEV due to the Higgs field, $\langle\phi_{\rm WS}\rangle$, is related to the total VEV, v, and the contribution from SU(N) due to fermion condensates, F_{π_N}, by the relation

$$\langle\phi_{\rm WS}\rangle^2 + F_{\pi_N}^2 = v^2 = (246\ \text{GeV})^2, \tag{22}$$

which is exactly the same as in technicolor models with a scalar [19].

The fermion running masses m_f are related to the Higgs field VEV in the usual way:

$$m_f = \frac{y_f}{\sqrt{2}}\langle\phi_{\rm WS}\rangle, \tag{23}$$

where y_f is the Yukawa coupling constant for the fermion f (y is used for both Yukawa coupling and weak hypercharge but it should be obvious from the context which is being referred to). For quarks, the running mass is related to the pole mass, M_f, by

$$M_f = \left(1 + \frac{4\alpha_S(M_f)}{3\pi}\right) m_f(M_f), \tag{24}$$

where $\alpha_S(M_f)$ is the QCD fine structure constant at the pole mass. For quarks with a mass of order M_Z we can approximate $\alpha_S(M_f) \approx \alpha_S(M_Z)$ to give the approximate formula:

$$M_f \approx 1.05 m_f(M_f). \tag{25}$$

This means that the pole mass of a heavy quark will be about 5% higher than the running mass. However, we will use Eq. (24) when calculating the pole masses of the quarks.

Using the Yukawa coupling infrared quasi-fixed-point value as an upper bound, we must avoid any significant suppression of the top quark and possible fourth generation quark masses due to the reduction of $\langle\phi_{\rm WS}\rangle$ below its SM value. We usually imagine taking

$$F_{\pi_N} \leqslant 75\ \text{GeV} \tag{26}$$

and thus

$$\langle\phi_{\rm WS}\rangle > 234\ \text{GeV}. \tag{27}$$

In fact we shall quote limits on fermion pole masses based on taking

$$\langle\phi_{\rm WS}\rangle = 234\ \text{GeV}. \tag{28}$$

This gives the following relation for the pole mass of quark f:

$$M_f = \left(1 + \frac{4\alpha_S(M_f)}{3\pi}\right)\frac{y_f(M_f)}{\sqrt{2}}\langle\phi_{\rm WS}\rangle. \tag{29}$$

In the approximation $M_f \approx M_Z$ we get

$$M_f \approx 174 y_f(M_f)\ \text{GeV}. \tag{30}$$

Upper limits for fermion masses are obtained by using quasi-fixed-point values for the Yukawa coupling constants, y_f, as determined from the RGE's in viable models with a desert above the TeV scale. These infrared fixed point Yukawa couplings are of order unity. However for the purposes of investigating the behavior of the gauge coupling constants, and especially to demonstrate that the U(1) coupling constant develops a Landau pole in our model without new fermions (Sec. V), we take a more generous single threshold of ten times the electroweak scale $\sim$1.7 TeV for all new fermions in that model. For our discussion in Sec. VI of the model with a fourth generation of quarks we take the more stringent lower threshold value of M_Z, in order to demonstrate the absence of Landau poles in this case.

C. Precision electroweak data

Measurements of electroweak interactions are now accurate enough to be sensitive to loop corrections to propagators and vertex corrections. These effects are model dependent and can be sensitive to the values of some parameters such as fermion and Higgs masses. So far the SM seems to be consistent with the precision electroweak measurements and obviously any other viable model should also agree with the data. We note, as discussed in [20], the data impose two important constraints on new fermion SU(2) doublets in our models:

(1) The mass squared differences within any new fermion SU(2) doublets must be small [$\ll(100\ \text{GeV})^2$], in order that the predicted value of the ρ parameter should not deviate too much from its experimental value close to unity; (2) the number of new SU(2) doublets is severely restricted by the measured value of the S parameter or its equivalent [21].

IV. FERMION MASS AND ANOMALY CANCELLATION

In the SM fermions get a mass via the Higgs mechanism. To do this in a general gauge group of the form

$$\text{U}(1)\otimes\text{SU}(2)\otimes G/D,$$

where G is any Lie group and D is a discrete group, using the SM Higgs particle, a left-handed fermion representation $(y,\mathbf{2},\mathbf{R})$ should occur together with the left-handed antifermion representations $(-[y+1],\mathbf{1},\overline{\mathbf{R}})$ and $(-[y-1],\mathbf{1},\overline{\mathbf{R}})$. We shall refer to this as the mass grouping $\{y,\mathbf{R}\}$ where $\mathbf{R}$ is an irreducible representation of G. As explained in Sec. III A 1 we assume that all fermions in our models, other than the leptons which have already been observed, get a mass by this mechanism. We shall now describe what consequences this has for anomaly cancellation in our models, where G is a product of SU(N_i) groups with $N_i \geqslant 3$. We have previously considered [22] the particular case where $G=\text{SU}(N)$ and the special case of the SM [$G=\text{SU}(3)$] is discussed in the Appendix. This case of $G=\text{SU}(N_C)$ (without dividing out the discrete group D) has also been discussed recently [23] in the context of anomaly cancellation in the N_C-extended SM.

We consider the grouping $\{y,\mathbf{R}\}$ for the gauge group

$$\mathrm{U}(1)\otimes\mathrm{SU}(2)\otimes\Pi_i\mathrm{SU}(N_i),$$

where the irreducible representation $\mathbf{R}$ is made up of fundamental ($\mathbf{N}_i$ or $\overline{\mathbf{N}}_i$) or singlet representations of each factor $\mathrm{SU}(N_i)$. The contribution to each type of anomaly from this grouping, $\{y,\mathbf{R}\}$, is easily calculated, using the results of Sec. II C, to be as follows:

$$[\mathrm{SU}(N_i)]^3\rightarrow 2S_Rn_i+S_R(-n_i)+S_R(-n_i)$$
$$=0,$$
$$[\mathrm{SU}(N_i)]^2\mathrm{U}(1)\rightarrow 2S_Rn_i^2y-S_Rn_i^2(y+1)-S_Rn_i^2(y-1)$$
$$=0,$$
$$[\mathrm{Grav}]^2\mathrm{U}(1)\rightarrow 2S_Ry+S_R(-y-1)+S_R(-y+1)$$
$$=0,$$
$$[\mathrm{U}(1)]^3\rightarrow 2S_Ry^3+S_R(-y-1)^3+S_R(-y+1)^3$$
$$=-6S_Ry,$$
$$[\mathrm{SU}(2)]^2\mathrm{U}(1)\rightarrow 2S_Ry.$$

Here n_i is the N_i-ality of the representation $\mathbf{R}$ and S_R is its dimension (size).

So we can see that the above grouping which is necessary to give a mass to the fermions also simplifies the anomaly constraints. In particular, if we take all fermions to be grouped in this way then we are only left with the single constraint for the absence of the mixed gauge-gravitational and gauge anomalies:

$$\sum_j S_jy_j=0, \tag{31}$$

where j labels each grouping $\{y_j,\mathbf{R}_j\}$.

There will also be no Witten anomaly, since we must have an even number of SU(2) doublets to satisfy Eq. (31). This follows from the charge quantization rule, Eq. (14), the fact that N_i are all odd and the assumption of fundamental or singlet representations for each $\mathrm{SU}(N_i)$ subgroup. Using the charge quantization rule and defining

$$\frac{e_j}{d_j}=\sum_i\frac{m_{N_i}}{N_i}(n_i)_j, \tag{32}$$

we can write

$$\frac{y_j}{2}=c_j+\frac{1}{2}+\frac{e_j}{d_j}, \tag{33}$$

where c_j,d_j, and e_j are integers and d_j are odd. Therefore, since Eq. (31) can be written as $\Sigma_jS_j(y_j/2)=0$, we must have $\Sigma_jS_j\frac{1}{2}\equiv 0\pmod 1$. In other words $\Sigma_jS_j\equiv 0\pmod 2$, which means that there are an even number of SU(2) doublets and so no Witten anomaly.

V. THE SMG_{235} MODEL WITHOUT NEW SM FERMIONS

Here we will examine the model based on the gauge group $\mathrm{SMG}_{235}\equiv G_5$ defined in Eqs. (4) and (5), since it is the absolute minimal extension to the SM among all the possible groups we have proposed in Sec. II A 1. In Sec. VI we will consider models based on the groups SMG_{23N} of Eqs. (12) and (13), including new SM fermions to highlight the general features of all such extensions to the SM. However, we will only analyze the consequences in detail for SMG_{235}.

In this section we will discuss the two possibilities: (i) that there are no new fermions beyond those of the SM and (ii) that there are new fermions which all couple to the SU(5) gauge group. This latter possibility may seem to be tantamount to adding a completely separate sector to the SM rather than extending the SM, since the new fermions will be confined under a new gauge group. However, it is really no more a separate sector than the SM is three separate sectors (one for each generation), since these extra fermions will still couple to the electroweak group due to the charge quantization rule. We will discuss the other possibility, that there are new fermions, some coupling to the SU(5) gauge group and others not, in Sec. VI.

A. No new fermions

There is of course the possibility that there are no extra fermions associated with this enlarged group. If this is so then the only possible observations would be the detection of SU(5) ‘‘glueballs.’’ In this case the SU(5) gauge group would be decoupled from the SMG and so the only way to observe the glueballs would be through their gravitational interactions. They could have been produced in the very early universe and the lightest state would be essentially stable since they could only decay via the gravitational interaction. Therefore they would only be observable as dark matter.

So this case is essentially uninteresting and will not be considered further. Instead we turn to the possibility that there exist more types of fermions than have been currently observed and consider whether or not they can be incorporated into a consistent model.

B. New fermions coupling to SU(5)

Of course fermions all contribute to anomalies which must be cancelled. The fermions in the SM cancel all anomalies on their own; so the extra fermions must cancel all anomalies amongst themselves.

As explained in Sec. IV the anomaly equations in our models are greatly simplified when all the fermions are massive due to the SM Higgs mechanism. In fact they are reduced to just one equation, $\Sigma_iS_iy_i=0$. If we label each mass grouping of fermion representations by the label $\{y,\mathbf{R}\}$, where $\mathbf{R}$ is the representation of the group $\mathrm{SU}(3)\otimes\mathrm{SU}(5)$, then Table I shows all six possible groupings, a to f, and their relative contributions, S_iy_i, to the anomaly equation. We use Eq. (6) with the definition $m\equiv m_5$ to simplify the notation, giving us the charge quantization rule,

TABLE I. Allowed mass groupings $\{y,\mathbf{R}\}$ of new fermions in the SMG_{235} model, using the charge quantization rule, Eq. (34), and fundamental representations of SU(5). Their relative contributions to the anomaly equation, Eq. (31), are given in the final column. A particular mass grouping of type t is given by choosing a particular value of weak hypercharge, i.e., by choosing a particular value of the integer N_t.

Type	$\mathbf{R}$	$\frac{y}{2}$	$\frac{1}{10}Sy$
a	$\mathbf{1},\mathbf{5}$	$N_a-\frac{m}{5}-\frac{1}{2}$	$N_a-\frac{m}{5}-\frac{1}{2}$
b	$\mathbf{1},\overline{\mathbf{5}}$	$N_b+\frac{m}{5}+\frac{1}{2}$	$N_b+\frac{m}{5}+\frac{1}{2}$
c	$\mathbf{3},\mathbf{5}$	$N_c-\frac{m}{5}+\frac{1}{6}$	$3N_c-\frac{3m}{5}+\frac{1}{2}$
d	$\mathbf{3},\overline{\mathbf{5}}$	$N_d+\frac{m}{5}+\frac{1}{6}$	$3N_d+\frac{3m}{5}+\frac{1}{2}$
e	$\overline{\mathbf{3}},\mathbf{5}$	$N_e-\frac{m}{5}-\frac{1}{6}$	$3N_e-\frac{3m}{5}-\frac{1}{2}$
f	$\overline{\mathbf{3}},\overline{\mathbf{5}}$	$N_f+\frac{m}{5}-\frac{1}{6}$	$3N_f+\frac{3m}{5}-\frac{1}{2}$

$$\frac{y}{2}+\frac{1}{2}\text{``duality''}+\frac{1}{3}\text{``triality''}+\frac{m}{5}\text{``quintality''} \equiv 0 \pmod 1, \tag{34}$$

where the integer m is fixed in any given model.[4] So we can determine $y/2 \pmod 1$ for any given representation $\mathbf{R}$.

For a solution to the anomaly equation $\Sigma_i S_i y_i=0$, we must obviously combine the fractions $m/5$ so that the 5 is cancelled in the denominator since all N's are integers. We must also have an even number of groupings so that the $\frac{1}{2}$'s combine to give an integer. This automatically ensures that there can be no Witten anomaly as explained in Sec. IV. This can be done by using equal numbers of type a and type b groupings. The two smallest solutions are in fact (i) one type a grouping and one type b grouping and (ii) two groupings of type a and two of type b. The smallest solution, (i), is not possible without giving the fermions a fundamental Dirac mass, since the anomaly constraints require that $N_a+N_b=0$ giving pairs of representations, $(y,\mathbf{2},\mathbf{1},\mathbf{5})$ and $(-y,\mathbf{2},\mathbf{1},\overline{\mathbf{5}})$, etc., which are not mass protected.

The smallest allowed solution with mass protected fermions is therefore solution (ii) with two groupings of type a and two of type b. This solution is shown in detail in Table II. All anomalies cancel provided $\Sigma_{i=1}^{4}N_i=0$. We can now choose values of the N_i.

The fermion contribution to the (first order) beta function for the U(1) running gauge coupling constant is proportional to Σy^2. We therefore want to choose values of N_i so as to minimize Σy^2, in order that any U(1) Landau pole is at as high an energy as possible. This gives us the best chance that the solution of Table II will be perturbatively valid up to the Planck scale and hence that our model will be self-consistent. However, this condition of minimizing Σy^2 is also suggested by the small representation structure of the SM, as explained in Sec. II A 2. Keeping in mind that the N_i are integers, $\Sigma_{i=1}^{4}N_i=0$, and that the particles must be mass protected, we find that the minimum value of Σy^2 is given by

$$N_1=N_2=1, \quad N_3=0, \quad N_4=-2$$

or

$$N_3=N_4=-1, \quad N_1=0, \quad N_2=2,$$

where $m=2$. These values of N_i give $\Sigma y^2=203.2$, for the solution of Table II, which is much larger than the $\frac{40}{3}$ per generation of the SM particles.

TABLE II. Smallest anomaly-free (subject to the constraint $N_1+N_2+N_3+N_4=0$) set of mass protected fermions which all couple to SU(5).

Representation under $SU(2)\otimes SU(3)\otimes SU(5)$	U(1) representation $\frac{y}{2}$
$\mathbf{2},\mathbf{1},\mathbf{5}$	$N_1-\frac{m}{5}-\frac{1}{2}$
$\mathbf{1},\mathbf{1},\overline{\mathbf{5}}$	$-N_1+\frac{m}{5}$
$\mathbf{1},\mathbf{1},\overline{\mathbf{5}}$	$-N_1+\frac{m}{5}+1$
$\mathbf{2},\mathbf{1},\mathbf{5}$	$N_2-\frac{m}{5}-\frac{1}{2}$
$\mathbf{1},\mathbf{1},\overline{\mathbf{5}}$	$-N_2+\frac{m}{5}$
$\mathbf{1},\mathbf{1},\overline{\mathbf{5}}$	$-N_2+\frac{m}{5}+1$
$\mathbf{2},\mathbf{1},\overline{\mathbf{5}}$	$N_3+\frac{m}{5}+\frac{1}{2}$
$\mathbf{1},\mathbf{1},\mathbf{5}$	$-N_3-\frac{m}{5}-1$
$\mathbf{1},\mathbf{1},\mathbf{5}$	$-N_3-\frac{m}{5}$
$\mathbf{2},\mathbf{1},\overline{\mathbf{5}}$	$N_4+\frac{m}{5}+\frac{1}{2}$
$\mathbf{1},\mathbf{1},\mathbf{5}$	$-N_4-\frac{m}{5}-1$

In Sec. III B we explained that it was reasonable to consider that all new fermions could be included at a threshold no higher than 1.7 TeV. This should provide an accurate enough upper limit for the threshold for our purposes. Therefore, since the fermions will have the least effect on the running coupling constants if they are included at the highest possible threshold, we will assume that all these extra fermions can be included with a simple threshold at 1.7 TeV. We can now check whether or not this model has a Landau pole below the Planck scale.

There are four fine structure constants which we shall label by α_1, α_2, α_3, and α_5 corresponding to the four gauge groups U(1), SU(2), SU(3), and SU(5), respectively. The fine structure constants, α_i, are related to the gauge coupling constants, g_i, by the relation $\alpha_i=g_i^2/4\pi$. The equations governing the running coupling constants to first order in pertur-

[4] In fact we can limit m to be 1 or 2 since it is only defined modulo 5 and, by replacing m with $-m \pmod 5$ and all representations of SU(5) with their conjugates, we are left with an equivalent model.

bation theory [24] (a good discussion of RGE's in the SM is given in [25]) can be integrated analytically to give

$$\frac{1}{\alpha_1(\mu)}=\frac{1}{\alpha_1(\mu_0)}-\frac{1}{12\pi}(Y^2+n_H)\ln\left(\frac{\mu}{\mu_0}\right), \quad (35)$$

$$\frac{1}{\alpha_2(\mu)}=\frac{1}{\alpha_2(\mu_0)}+\frac{1}{12\pi}(44-2n_{2f}-n_H)\ln\left(\frac{\mu}{\mu_0}\right), \quad (36)$$

$$\frac{1}{\alpha_3(\mu)}=\frac{1}{\alpha_3(\mu_0)}+\frac{1}{12\pi}(66-2n_{3f})\ln\left(\frac{\mu}{\mu_0}\right), \quad (37)$$

$$\frac{1}{\alpha_5(\mu)}=\frac{1}{\alpha_5(\mu_0)}+\frac{1}{12\pi}(110-2n_{5f})\ln\left(\frac{\mu}{\mu_0}\right), \quad (38)$$

where we calculate $\alpha_i(\mu)$ (the running coupling constants at the energy scale $\mu>\mu_0$) in terms of $\alpha_i(\mu_0)$. $Y^2\equiv\Sigma y^2$ is the sum of the weak hypercharges squared for all fermions with a mass below μ_0 and n_{mf} are the number of fermion **m** and $\overline{\mathbf{m}}$ representations of SU(m) with mass below μ_0. n_H is the number of Higgs doublets with mass below μ_0. These equations assume that there are no fermions or Higgs scalars with a mass between μ_0 and μ. In order to calculate the value of $\alpha_i(\mu)$ when there are fermions or Higgs bosons with masses between μ_0 and μ we must do the calculation in steps, calculating the value of α_i up to the mass of each particle. So we use the experimental values of the fine structure constants at M_Z (including the top quark and Higgs boson in the beta functions at this scale) to calculate the coupling constants at 1.7 TeV, where we include the new fermions, and then run the coupling constants up to the Planck scale. This is a crude method since there would really be complicated threshold effects as each fermion was included. However these effects can reasonably be assumed to be small, relative to the changes in the coupling constants caused by the running from the electroweak scale to the Planck scale, and so we will use this much simpler method. Second order RGE's [26] could be used but the improvement over the first order RGE's would not be significant when compared to the error introduced by the naive assumptions made about threshold effects.

From [15] we find

$$\alpha_1^{-1}(M_Z)=98.08\pm0.16, \quad (39)$$

$$\alpha_2^{-1}(M_Z)=29.794\pm0.048, \quad (40)$$

$$\alpha_3^{-1}(M_Z)=8.55\pm0.37. \quad (41)$$

We can now use the above equations to examine how the coupling constants behave up to the Planck scale. Since there is no experimental value for α_5 at any energy scale we shall assume that $\alpha_5^{-1}(M_Z)=2$, so that the SU(5) interaction is stronger than QCD at M_Z and confines at the electroweak scale. Figure 2 shows what happens for each group. For the graphs, we normalize the U(1) gauge coupling as if the U(1) group was embedded in a simple group. This essentially corresponds to a redefinition of g_1:

$$(g_1^2)_{\rm GUT}\equiv\frac{5}{3}(g_1^2)_{\rm SM}, \quad (42)$$

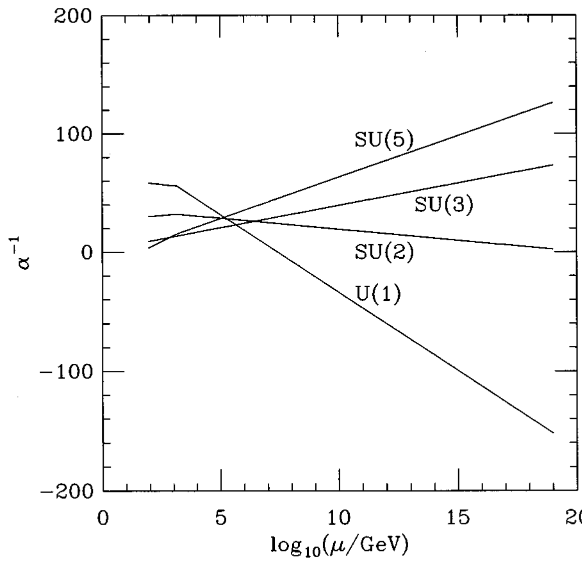

FIG. 2. α^{-1} from M_Z to the Planck scale for each component group in the SMG_{235} model without new SM fermions. There is clearly a U(1) Landau pole at $\mu\sim10^7$ GeV and SU(2) also loses asymptotic freedom. $\alpha_5^{-1}(M_Z)=2$ has been chosen as a specific example.

$$(\alpha_1^{-1})_{\rm GUT}\equiv\frac{3}{5}(\alpha_1^{-1})_{\rm SM}. \quad (43)$$

So henceforth we use the standard GUT normalization. Equations (35) and (39) now become

$$\frac{1}{\alpha_1(\mu)}=\frac{1}{\alpha_1(\mu_0)}-\frac{1}{20\pi}(Y^2+n_H)\ln\left(\frac{\mu}{\mu_0}\right), \quad (44)$$

$$\alpha_1^{-1}(M_Z)=58.85\pm0.10. \quad (45)$$

As we can see from Fig. 2, $1/\alpha_1$ becomes negative at about 10^7 GeV which means that there is a Landau pole. So we can conclude that this theory would be inconsistent, at least as far as perturbation theory is concerned, without new interactions below 10^7 GeV.

In fact we can show that there is no anomaly-free model, having all new fermions coupling to SU(5), with a desert above the TeV scale, which does not have a Landau pole below the Planck scale. The condition for no Landau pole below the Planck scale is $1/\alpha_1(M_{\rm Planck})>0$. Therefore, Eq. (44) can be rearranged to give

$$Y^2+n_H<\frac{20\pi}{\alpha_1(\mu_0)\ln(M_{\rm Planck}/\mu_0)}. \quad (46)$$

Since, for the SM, $Y_{\rm SM}^2=40$ and $n_H=1$,

$$Y^2+n_H\geqslant41 \quad (47)$$

above the electroweak scale and so we can use Eqs. (44) and (45) to calculate an upper limit for $1/\alpha_1(1.7\ \text{TeV})$:

$$\frac{1}{\alpha_1(1.7\ \text{TeV})}\leqslant57. \quad (48)$$

We then use $Y^2 = Y^2_{\mathrm{SM}} + Y^2_{\mathrm{new}}$ in Eq. (46) with $\mu_0 = 1.7$ TeV and conclude that

$$Y^2_{\mathrm{new}} < 57.5, \tag{49}$$

assuming the new fermions can be included naively at a threshold no higher than 1.7 TeV.

For each mass grouping $\{y, \mathbf{R}\}$, with $P_R = 4S_R$ fermions, we can calculate the value of Y^2:

$$Y^2 = S_R[2y^2 + (y+1)^2 + (y-1)^2] = S_R(4y^2 + 2). \tag{50}$$

Therefore, we have

$$Y^2 \geqslant 2S_R = \frac{1}{2} P_R. \tag{51}$$

If there are several mass groupings, $Y^2 \geqslant \frac{1}{2}\Sigma P_R \equiv \frac{1}{2} P$ where P is the total number of fermions. So if we define P_{new} to be the number of non-SM fermions, we can conclude

$$P_{\mathrm{new}} \leqslant 2Y^2_{\mathrm{new}} < 115. \tag{52}$$

So now we have shown that there must be less than 115 extra fermions. However, the smallest solutions, subject to the constraints in this section, larger than two type a and two type b representations are three type a and one type c representations, etc., which contain 120 fermions and so must cause a Landau pole below the Planck scale.[5] Therefore there are no possible anomaly-free models without a Landau pole, where all the new fermions couple to the SU(5) gauge group.

We will now examine the case where we allow some new SU(5) singlet fermions, as well as some fermions which couple to SU(5), in order to cancel the anomalies. We shall show that it is possible to have more SM fermions in such a model.

VI. THE SMG$_{235}$ MODEL WITH NEW SM FERMIONS

In this section we shall first examine sets of fermions (which are generalizations of the SM quarks) in groups, defined by Eqs. (12) and (13), similar to the SMG. We shall then examine the particular case of the group SMG$_{235}$ and discuss the possibility of experimental evidence for and against this self-consistent model.

A. Fermions in the groups SMG$_{2M}$ and SMG$_{2MN}$

In Sec. VI A 1 we shall examine the group SMG$_{2M}$. The SMG is an example of this type of group where $M=3$. We shall show that this general group allows anomaly-free sets of fermions which consist of a generation of SM leptons and a generation of SU(M) ''quarks'' which are a simple generalization of the SU(3) quarks in the SM.

We shall then show in Sec. VI A 2 that we can have anomaly-free sets of fermions in the group SMG$_{2MN}$ without any leptons. We shall then examine the particular case of the group SMG$_{235}$ which we shall discuss in detail.

1. Fermions in the group SMG$_{2M}$

In the SM, each generation is formed by taking the two mass groupings $\{\frac{1}{3}, \mathbf{3}\}$ and $\{-1, \mathbf{1}\}$ [where the representations **3** and **1** are of the group SU(3)] as explained in Sec. IV and the Appendix. We will now consider a more general situation where we have the gauge group SMG$_{2M}$ defined in Sec. II A 1 (where $M>2$ is a prime number) and the fermions are in the groupings $\{y_1, \mathbf{M}\}$ and $\{y_2, \mathbf{1}\}$ [where the representations **M** and **1** are of the group SU(M)].

From Sec. IV all the gauge anomalies will cancel if

$$My_1 + y_2 = 0. \tag{53}$$

Since we also have the charge quantization rule

$$\frac{y}{2} + \frac{1}{2}\text{``duality''} + \frac{m_M}{M}\text{``}M\text{-ality''} \equiv 0 \pmod 1 \tag{54}$$

we can write

$$\frac{y_1}{2} = -\frac{1}{2} - \frac{m_M}{M} + c_1, \tag{55}$$

$$\frac{y_2}{2} = -\frac{1}{2} + c_2, \tag{56}$$

where c_1 and c_2 are integers. We now have the condition that for no anomalies to be present

$$-\frac{M+1}{2} - m_M + Mc_1 + c_2 = 0. \tag{57}$$

In the SM a lepton generation is formed (with the addition of a right-handed neutrino which can be removed without effecting any anomalies) when we have $c_2=0$ as explained in the Appendix. If we insert this value into the above equation then we find

$$c_1 = \frac{1}{M}\left(\frac{M+1}{2} + m_M\right). \tag{58}$$

This can always be solved by setting $m_M = (M-1)/2$. In fact if $M=3$ then this is simply one of the anomaly-free SM quark-lepton generations.

However, this is not a good solution for an extension of the SM (which would be obtained by considering SMG$_{2M} \subset$ SMG$_{23M}$) since it contains an extra massless neutrino which has already been ruled out by experiment. It is difficult to produce a neutrino with a mass so large that it would not already have been detected, as explained in Sec. III A 2. We could choose not to set $c_2=0$ or 1 above, which would force all the extra leptons to be massive [by leptons we mean any fermions which are only coupled to the electroweak subgroup, $\mathrm{SU}(2) \otimes \mathrm{U}(1)$]. This is because there

[5] Using second order RGE's or a more complete analysis of thresholds would obviously change the precise limit in Eq. (49). However, the charge quantization rule in our model means that y cannot be zero and so it is not possible to attain the limit of Eq. (51). In fact, the value of Y^2_{new} will generally be much greater than this limit. For example, three type a and one type c lead to $Y^2_{\mathrm{new}} \geqslant \frac{1708}{15} \approx 114$ which is much greater than the required maximum given by Eq. (49).

would then be two SU(2) singlets which were charged (and at least one would have an electric charge of two or more, which is against our principle of small representations) and so both could get a mass by the usual SM Higgs mechanism since neither could get a Majorana mass. But even if we assumed that these leptons had masses higher than experimental limits this solution is not really favored by our postulate of small values of weak hypercharge discussed in Sec. II A 2. So in order to find a satisfactory solution we shall look at a similar general case.

2. Fermions in the group SMG_{2MN}

Suppose we have the gauge group SMG_{2MN}, where both M and $N>M\geqslant 3$ are mutually prime integers, which has the charge quantization rule

$$\frac{y}{2}+\frac{1}{2}\text{``duality''}+\frac{m_M}{M}\text{``}M\text{-ality''}+\frac{m_N}{N}\text{``}N\text{-ality''}\equiv 0 \pmod 1. \quad (59)$$

Then with fermions in mass groupings $\{y_1,(\mathbf{M},\mathbf{1})\}$ and $\{y_2,(\mathbf{1},\mathbf{N})\}$ [where the representations $(\mathbf{M},\mathbf{1})$ and $(\mathbf{1},\mathbf{N})$ are of the group $SU(M)\otimes SU(N)$] the condition for no anomalies is

$$My_1+Ny_2=0. \quad (60)$$

The charge quantization rule means that we can write

$$\frac{y_1}{2}=-\frac{1}{2}-\frac{m_M}{M}+c_1, \quad (61)$$

$$\frac{y_2}{2}=-\frac{1}{2}-\frac{m_N}{N}+c_2, \quad (62)$$

where c_1 and c_2 are integers. We then find that the condition for no anomalies becomes

$$2Nc_2=N+[2(m_M+m_N)+(1-2c_1)M]. \quad (63)$$

Both N and M are odd and therefore there will always be a solution, since we can choose $(m_M+m_N)=M$ and $3-2c_1$ to be an odd multiple of N. In general there will also be other solutions.

In particular, for the gauge group $G_5\equiv SMG_{235}$ we can have a fourth generation of quarks without any extra leptons, by choosing $M=3$, $N=5$, $m_3=1$, and $c_1=1$ above. Then

$$10c_2=5+[2(1+m_5)-3] \quad (64)$$

or equivalently

$$5c_2=2+m_5. \quad (65)$$

So we have a solution with $c_2=1$ and $m_5=3$.

The representations of the left-handed fermions which couple to the SU(5) subgroup are shown in Table III. This is a generalization of the quarks in the SM, coupling to SU(5) rather than SU(3).

In fact we have a solution with a fourth generation of quarks for the general group SMG_{23N}, where N is any odd integer greater than but not divisible by 3, by choosing $c_2=1$ and $m_N=\frac{1}{2}(N+1)$. This means that, if a fourth generation of quarks without leptons was detected, there would be no immediate way of deducing the value of N. Table IV shows the properties of the left-handed fermions which couple to the SU(N) subgroup. Note that this is a generalization of the SM quarks, coupling to SU(N) with the specific choice of $m_N=\frac{1}{2}(N+1)$. If we set $N=3$ we would in fact get a generation of quarks with the opposite chirality to those in the SM. This is to be expected since we are using these fermions to cancel the anomaly contribution of a four generation of SM quarks (with the usual chirality).

This solution, with a fourth generation of quarks and the fermions of Table III, for the gauge group SMG_{235} is analogous to one SM quark-lepton generation in the gauge group SMG, in the sense that it is the smallest anomaly-free set of mass-protected fermions which couple nontrivially to all the gauge fields. The SM quark-lepton generation is shown to be the smallest such set of fermions for the SMG in the Appendix. Note that although a generation of SM leptons and the fermions conjugate to those in Table III is a smaller anomaly-free set of fermions in the gauge group SMG_{235}, none of these fermions couples to the SU(3) subgroup.

As stated in Sec. III A 2, we take the limits on the masses of a fourth generation of quarks to be $M_{b'}>130$ GeV,

TABLE III. Left-handed fermions coupling to SU(5) in the mass grouping $\{-\frac{1}{5},(\mathbf{1},\mathbf{5})\}$. The electric charges are in units of $\frac{1}{5}$ due to the charge quantization rule.

Representation under $SU(2)\otimes SU(3)\otimes SU(5)$	U(1) representation $\frac{y}{2}$	Electric charge Q
2,**1**,**5**	$-\frac{1}{10}$	$\begin{pmatrix}\frac{2}{5}\\ -\frac{3}{5}\end{pmatrix}$
1,**1**,$\overline{\mathbf{5}}$	$-\frac{4}{10}$	$-\frac{2}{5}$
1,**1**,$\overline{\mathbf{5}}$	$\frac{6}{10}$	$\frac{3}{5}$

TABLE IV. Fermions coupling to SU(N) which would form an anomaly-free set of fermions together with a fourth generation of quarks.

Representation under $SU(2)\otimes SU(3)\otimes SU(N)$	U(1) representation $\frac{y}{2}$	Electric charge Q
2,**1**,**N**	$-\frac{1}{2N}$	$\begin{pmatrix}\frac{N-1}{2N}\\ -\frac{N+1}{2N}\end{pmatrix}$
1,**1**,$\overline{\mathbf{N}}$	$-\frac{N-1}{2N}$	$-\frac{N-1}{2N}$
1,**1**,$\overline{\mathbf{N}}$	$\frac{N+1}{2N}$	$\frac{N+1}{2N}$

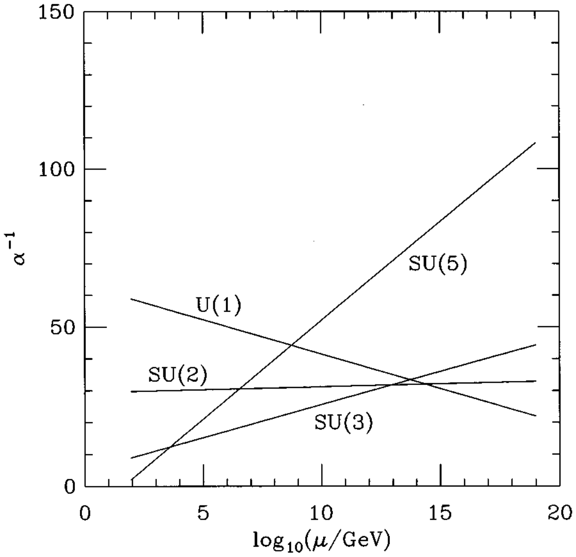

FIG. 3. α^{-1} from M_Z to the Planck scale for each component group in the SMG_{235} model with a fourth generation of quarks and the fermions of Table III which couple to SU(5). The initial value for $\alpha_5^{-1}(M_Z)=2$ was chosen so that it would be confine at the electroweak scale. There are obviously no Landau poles so this model is self-consistent.

$M_{t'}>130$ GeV and the top quark mass to be $M_t \sim 170$ GeV. We can now use the RGE's, first to show that these additional fermions do not cause any inconsistencies such as gauge coupling constants becoming infinite below the Planck scale, and then to estimate upper limits on the values of the Yukawa couplings to the SM Higgs field of these fermions. This will lead to upper limits on the masses, indicating that the t' and b' quarks would be almost within reach of present experiments.

B. No Landau poles

As in Sec. V we can investigate how the gauge coupling constants vary with energy up to the Planck scale. Here we set the thresholds for all the unknown fermions [fourth generation quarks and fermions coupling to SU(5)], as well as for the top quark and Higgs boson, to M_Z. The absence of Landau poles in this case will guarantee their absence if some of the thresholds are set higher than M_Z. From experimental limits we would expect that all these thresholds should be greater than M_Z.

We use Eqs. (36)–(38) and (44) to run the gauge coupling constants up to the Planck scale as shown in Fig. 3. Now we see that with a fourth generation of quarks and the fermions in Table III [i.e., far fewer fermions than the model in Sec. V B where all the new fermions coupled to SU(5)] there are no problems with Landau poles below the Planck scale. So our SMG_{235} model with new SM fermions appears to be consistent.

C. Upper limits for Yukawa couplings

Now we can choose initial values for the Yukawa couplings at the Planck scale and use the RGE's to see how they evolve, as they are run down to the electroweak scale. Assuming no mixing for the quarks and neglecting the masses of all SM fermions except the top quark (a good approximation), the RGE's are, to one loop order in perturbation theory [24],

$$\frac{dy_t}{dt}=y_t\frac{1}{16\pi^2}\left(\frac{3}{2}y_t^2+Y_2(S)-G_{3u}\right), \tag{66}$$

$$\frac{dy_{t'}}{dt}=y_{t'}\frac{1}{16\pi^2}\left(\frac{3}{2}(y_{t'}^2-y_{b'}^2)+Y_2(S)-G_{3u}\right), \tag{67}$$

$$\frac{dy_{b'}}{dt}=y_{b'}\frac{1}{16\pi^2}\left(\frac{3}{2}(y_{b'}^2-y_{t'}^2)+Y_2(S)-G_{3d}\right), \tag{68}$$

$$\frac{dy_{5u}}{dt}=y_{5u}\frac{1}{16\pi^2}\left(\frac{3}{2}(y_{5u}^2-y_{5d}^2)+Y_2(S)-G_{5u}\right), \tag{69}$$

$$\frac{dy_{5d}}{dt}=y_{5d}\frac{1}{16\pi^2}\left(\frac{3}{2}(y_{5d}^2-y_{5u}^2)+Y_2(S)-G_{5d}\right), \tag{70}$$

where the SU(5) fermions have been labeled $5u$ and $5d$ as generalizations of the naming of SU(3) quarks. The other variables are defined as

$$Y_2(S)=5y_{5u}^2+5y_{5d}^2+3y_{t'}^2+3y_{b'}^2+3y_t^2, \tag{71}$$

$$G_{3u}=\frac{17}{20}g_1^2+\frac{9}{4}g_2^2+8g_3^2, \tag{72}$$

$$G_{3d}=\frac{1}{4}g_1^2+\frac{9}{4}g_2^2+8g_3^2, \tag{73}$$

$$G_{5u}=\frac{153}{500}g_1^2+\frac{9}{4}g_2^2+\frac{72}{5}g_5^2, \tag{74}$$

$$G_{5d}=\frac{333}{500}g_1^2+\frac{9}{4}g_2^2+\frac{72}{5}g_5^2. \tag{75}$$

Here $Y_2(S)$ is really $\mathrm{Tr}(Y^\dagger Y)$ where Y is the Yukawa matrix for all the fermions.

We can choose values for the Yukawa couplings at the Planck scale and then use the RGE's to see what values the Yukawa couplings will have at any other scale. We have chosen the low energy scale to be M_Z as shown in Fig. 4. We observe quasifixed points similar to the case for the top quark in the SM [27] and these will provide upper limits on the fermion masses. However, the resulting Yukawa coupling for any fermion at M_Z depends on the Yukawa couplings of the other fermions. Nonetheless there is an approximate infrared fixed point limit on $Y_2(S)$ and so one Yukawa coupling can be increased at the expense of the others. This limit on $Y_2(S)$ is quite precise if there is only one strong interaction at low energies, such as QCD in the SM.[6] We observe numerically that $Y_2(S)\approx 7.5\pm 0.3$, provided the Yukawa couplings of the three heavy quarks are greater than

[6]Detailed results for a general number of heavy SM generations are derived in [28].

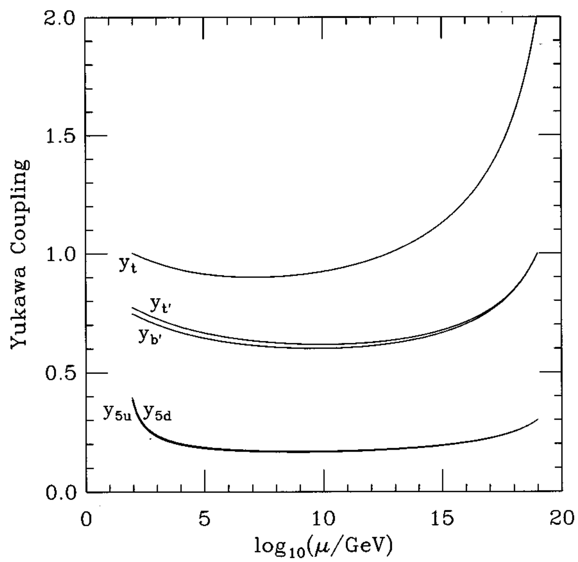

FIG. 4. An example of running Yukawa couplings for all fermions with a mass the same order of magnitude as the electroweak scale. The values were chosen at the Planck scale and run down to M_Z so that all the fermions would have a mass allowed by current experimental limits.

1 at the Planck scale and that the Yukawa couplings of the fermions coupling to the SU(5) gauge group are less than the Yukawa couplings of the heavy quarks at the Planck scale.

The values chosen for Fig. 4 have been chosen so that the top quark pole mass $M_t \approx 170$ GeV and the fourth generation quark pole masses are above the current experimental limit of 130 GeV. Also $M_{b'} \sim M_{t'}$ and $M_{5u} \sim M_{5d}$ have been chosen, so that there is only a small contribution to the ρ parameter described in Sec. III C. We discuss the electroweak radiative corrections in [20] and this SMG_{235} model, with its eight new doublets, is consistent with the experimental data at the 2–3 standard deviation level. However it is clear that any model with significantly more SU(2) doublets must disagree with the current experimental evidence. This rules out the similar models with gauge group SMG_{23N} where N is an odd integer greater than 5 and not divisible by 3.

Table V gives the values of the Yukawa couplings at M_Z and the corresponding pole masses, using Eq. (29), for the quarks. For the fermions coupling to SU(5) we use the equation relating the pole and running masses:

$$M_f = \left(1 + \frac{12\alpha_5(M_f)}{5\pi}\right) m_f(M_f). \tag{76}$$

TABLE V. Infrared fixed point Yukawa couplings and corresponding pole masses (for $F_\pi = 75$ GeV) for a particular choice of Yukawa couplings at the Planck scale.

Fermion	Yukawa coupling at M_Z	Pole mass (GeV)
y_t	1.00	175
$y_{t'}$	0.77	135
$y_{b'}$	0.75	131
y_{5u}	0.38	94
y_{5d}	0.40	97

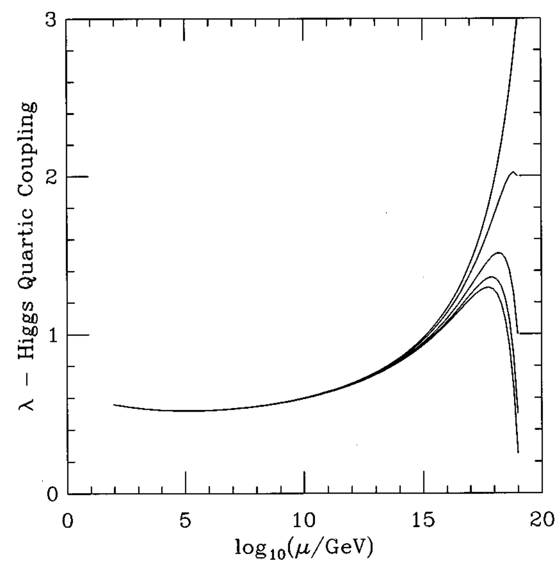

FIG. 5. Fixed point value of λ. This graph, along with the estimated value of $\langle \phi_{WS} \rangle = 234$ GeV, leads to an approximate Higgs mass of 172 GeV.

Therefore these masses should be considered upper limits on the masses of the fermions for this particular choice of Yukawa couplings at the Planck scale. For other choices of Yukawa couplings at the Planck scale we could, for example, increase the mass of the fourth generation of quarks but this would have to be compensated for by a reduction in the mass of some of the other fermions.

These values for the masses are consistent with current experimental limits but are not so high that the new fermions could remain undetected for long. In fact the quark masses may even be within the limits of current accelerators. It is not clear whether the fermions coupling to SU(5) could be observed, since they would obviously be confined by the SU(5) gauge interaction which we take to confine at the electroweak scale. So even if they have masses of about 100 GeV, they would be much more difficult to detect than quarks with greater masses. For this reason we consider the clearest evidence for this model would come from the detection of a fourth generation quark. The masses of some of the new fermions could be increased, but not by much, since this would mean a reduction in the mass of other fermions. This means that this model is consistent and relatively easy to test.

For completeness we show the running of λ, the Higgs quartic coupling. The equation for the running of λ is given by [24]

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\Bigg[12\lambda^2 - \left(\frac{9}{5}g_1^2 + 9g_2^2\right)\lambda + \frac{9}{4}\left(\frac{3}{25}g_1^4 + \frac{2}{5}g_1^2 g_2^2 + g_2^4\right) + 4Y_2(S)\lambda - 4H(S)\Bigg], \tag{77}$$

where we have defined

$$H(S) = 5y_{5u}^4 + 5y_{5d}^4 + 3y_{t'}^4 + 3y_{b'}^4 + 3y_t^4. \tag{78}$$

From Fig. 5 we obtain $\lambda(M_Z) = 0.54$. This graph leads to a

TABLE VI. The lightest SM generation.

Generation	Fermion label	Representation of $SU(2)\otimes SU(3)$	Representation of $U(1)\,\frac{y}{2}$	Electric charge Q
	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	**2**,**3**	$\frac{1}{6}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
Quark	$\overline{u_L}$	**1**,$\overline{\mathbf{3}}$	$-\frac{2}{3}$	$-\frac{2}{3}$
	$\overline{d_L}$	**1**,$\overline{\mathbf{3}}$	$\frac{1}{3}$	$\frac{1}{3}$
Lepton	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	**2**,**1**	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	$\overline{e_L}$	**1**,**1**	1	1

running Higgs boson mass of

$$M_H(M_H)=\sqrt{\lambda}\langle\phi_{WS}\rangle\approx 172\ \text{GeV}. \tag{79}$$

The same low energy value of λ is obtained for any initial choice of λ at the Planck scale since the Yukawa couplings are at the fixed point. This means that the Higgs boson must have this fixed point mass. If the Yukawa couplings were slightly lower than their fixed point values we would obtain a small range of allowable Higgs boson masses. However, this range would always be somewhat below 172 GeV.

VII. CONCLUSIONS

We have discussed extensions of the SM having a similar gauge group structure to the SM itself. In particular we have been guided by the requirement of an anomaly-free theory, with additional mass protected fermions satisfying a generalized charge quantization rule. We were thereby lead to extend the SM cross product group, $U(1)\otimes SU(2)\otimes SU(3)$, by adding extra $SU(N)$ direct factors, with the N's greater than 3 and mutually prime. A generalized charge quantization rule, involving each direct factor, was then obtained by dividing out an appropriate discrete group. Extending the SM in this fairly obvious way produces the groups SMG_{23N}, SMG_{23MN}, etc. Another feature we take over from the SM is the principle of using only small (fundamental or singlet) fermion non-Abelian representations. For the Abelian representations we take the condition that weak hypercharges should be chosen to be close to zero. More precisely, we minimize the sum of weak hypercharges squared over all the fermions. As shown in the Appendix, this idea of small representations (Abelian and non-Abelian) can be used together with the charge quantization rule and anomaly cancellation to uniquely define the SM generation of fermions.

The extra $SU(N)$ groups introduced confine and form fermion condensates having the same quantum numbers as the SM Higgs doublet. It follows that the extra $SU(N)$ groups act as partial technicolor groups and must confine near the electroweak scale. However, the SM Higgs field is still responsible for all the fermion masses, albeit with a somewhat reduced VEV.

We have studied in detail the conditions for anomaly cancellation in our minimal extension of the SM gauge group, SMG_{235}. It is not possible to construct an anomaly-free model using new mass protected fermions which are all nonsinglet under SU(5), without encountering a Landau pole in the U(1) fine structure constant well below the Planck scale. However it is possible to construct a consistent model with a fourth generation of quarks but, instead of an extra generation of leptons, with a generation of the fermions coupling to SU(5) as given in Table III.

A similar solution with a fourth generation of quarks and a generation of $SU(N)$ fermions as given in Table IV is possible for the gauge group SMG_{23N}. However, the number of SU(2) doublets in the model increases with N and hence their contribution to the electroweak radiative corrections becomes more important. The SMG_{235} model is just consistent with the precision electroweak data but SMG_{23N} models with $N>5$ are probably ruled out [20]. Similarly the SMG_{23MN} models would be inconsistent with the precision electroweak data.

The SMG_{235} model with a fourth generation of quarks and a generation of SU(5) fermions seems to be phenomenologically consistent. It requires the existence of t' and b' quarks at or below the top quark mass scale; this is consistent with current experimental limits but they could not remain undetected for long. However it is unlikely that the SU(5) fermions could be observed with current accelerators; they would be confined inside SU(5) ''hadrons,'' with a confinement scale of order 200 GeV and would have a small production cross section at present hadron colliders. Even if this model does not turn out to be correct we hope that the derivation might at least highlight some of the important features of the SM and some of the unique qualities of the SM, which appears (admittedly almost by definition) as the smallest case of our more general models.

ACKNOWLEDGMENTS

We thank John Gunion, Hans Jensen, Victor Novikov, David Sutherland, and Misha Vysotsky for helpful discussions. This work has been supported in part by INTAS Grant No. 93-3316 and PPARC Grant No. GR/J21231, the British Council, Cernføgeforskning, and EF Contract No. SC1 0340 (TSTS).

APPENDIX: DERIVING THE SM GENERATION

1. The SM generation

In the SM there are three generations of fermions which are identical except for their masses. Each generation consists of 15 Weyl fermions and can be divided into a lepton generation and a quark generation. The quarks couple to the SU(3) gauge group, whereas the leptons are SU(3) singlets and so do not ''feel'' the strong force. The properties of these fermions are shown in Table VI. The fermions are labeled as in the first (lightest) generation.

The quark generation is formed by the representations $(\frac{1}{3},\mathbf{2},\mathbf{3})_L$, $(-\frac{4}{3},\mathbf{1},\overline{\mathbf{3}})_L$, and $(\frac{2}{3},\mathbf{1},\overline{\mathbf{3}})_L$ of the gauge group $U(1)\otimes SU(2)\otimes SU(3)$. This is precisely the mass grouping $\{\frac{1}{3},\mathbf{3}\}$ [where the representation **3** is of the gauge group SU(3)] described in Sec. IV. All the quarks get a mass by the Higgs mechanism. The lepton generation is formed by the representations $(-1,\mathbf{2},\mathbf{1})_L$ and $(2,\mathbf{1},\mathbf{1})_L$ of the same gauge group. However, this is not the same as the mass grouping $\{-1,\mathbf{1}\}$ because there is no right-handed neutrino [representation $(0,\mathbf{1},\mathbf{1})_L$] in the SM. This means that the neutrino is massless in the SM but the electron can still get a mass by the Higgs mechanism. However, the lepton generation gives the same contribution to all anomalies as the mass grouping $\{-1,\mathbf{1}\}$ would, since the right-handed neutrino would be totally neutral (i.e., would not interact with any gauge fields).

2. Derivation of the SM generation

In fact, we can derive the SM generation using the following assumptions [1].

(*i*) *The SM gauge group*. $SMG\equiv S(U(2)\otimes U(3))$. This includes the charge quantization rule Eq. (1).

(*ii*) *Mass protection*. This means that no fermions can form a gauge invariant mass term except by the Higgs mechanism. In particular we cannot have left- and right-handed fermions with the same representation of the SMG. Also we cannot have a right-handed neutrino since it can get a Majorana mass.

(*iii*) *Anomaly cancellation*. In addition to the cancellation of gauge anomalies, the Witten global SU(2) anomaly and the mixed gauge and gravitational anomaly must also be absent.

(*iv*) *Small representations*. This means (cf. Sec. II A 2) that all fermions are in either fundamental or singlet representations of the SU(2) and SU(3) subgroups and the sum of weak hypercharge squared for all fermions is as small as possible.

So our aim is to minimize the value of $\Sigma_i S_i(y_i/2)^2$ (where S_i is the dimension of representation i with weak hypercharge y_i) for all possible choices of mass protected fermions in fundamental or singlet representations of SU(2) and SU(3), assuming the charge quantization rule, Eq. (1), and cancelling all relevant anomalies. We note that for one SM generation [which satisfies assumptions (i) to (iii)]

$$\sum_i S_i\left(\frac{y_i}{2}\right)^2=\frac{10}{3}. \tag{A1}$$

and we show that there is no other mass protected solution of the anomaly constraints with

TABLE VII. Contributions of $S(y/2)^2$ for all fundamental and singlet representations of SU(2) and SU(3) for any value of weak hypercharge which satisfies Eq. (1). All N's are integers and S is the dimension of the non-Abelian representation.

Type	Representation of $SU(2)\otimes SU(3)$	$\frac{y}{2}$	$S\left(\frac{y}{2}\right)^2$
a	$\mathbf{2},\mathbf{3}$	$N_a+\frac{1}{6}$	$6N_a^2+2N_a+\frac{1}{6}$
b	$\mathbf{2},\overline{\mathbf{3}}$	$N_b-\frac{1}{6}$	$6N_b^2-2N_b+\frac{1}{6}$
c	$\mathbf{1},\mathbf{3}$	$N_c-\frac{1}{3}$	$3N_c^2-2N_c+\frac{1}{3}$
d	$\mathbf{1},\overline{\mathbf{3}}$	$N_d+\frac{1}{3}$	$3N_d^2+2N_d+\frac{1}{3}$
e	$\mathbf{2},\mathbf{1}$	$N_e-\frac{1}{2}$	$2N_e^2-2N_e+\frac{1}{2}$
f	$\mathbf{1},\mathbf{1}$	N_f	N_f^2

$$\sum_i S_i\left(\frac{y_i}{2}\right)^2\leqslant\frac{10}{3}. \tag{A2}$$

So we shall prove that one SM generation also satisfies assumption (iv) and thus we will show that assumptions (i) to (iv) define the SM generation. Note that in order to satisfy assumption (iv) we must satisfy Eq. (A2). So in the following analysis we will implicitly assume Eq. (A2). Table VII shows all allowed representations and their contribution of $S(y/2)^2$.

In order to satisfy Eq. (A2) we must choose $N_a=N_b=0$, $N_c\in\{0,1\}$, $N_d\in\{-1,0\}$, $N_e\in\{0,1\}$, and $N_f\in\{-1,1\}$. (We do not consider $N_f=0$ because this would be a right-handed neutrino which would not contribute to any anomalies and would be expected to get a Majorana mass of the order of the Planck mass.) This means that we cannot have mass protected fermions of types a and b. So we can choose, without loss of generality, that there are no fermions of type b.[7] So we get Table VIII, which shows all allowed fermions and contributions to some anomalies.

For mass protection we cannot have any of the following combinations; types c_1 and d_1, types c_2 and d_2, types e_1 and e_2, or types f_1 and f_2 (all defined in Table VIII). Also note that all the types of representations in Table VIII contribute to the mixed anomaly, $\Sigma_i S_i y_i$. This means that we cannot use only type f fermions to produce an anomaly-free set of mass protected fermions. Therefore, if no fermions couple to the SU(3) group, there is no way to cancel the $[SU(2)]^2U(1)$ anomaly. So we can conclude that some fermions must couple to $SU(3)$.

Suppose there are no fermions of type a. Then the above arguments mean that, to cancel the $[SU(3)]^3$ anomaly, we must have equal numbers of either types c_1 and d_2 or types c_2 and d_1. But then there is no way to cancel the $[SU(3)]^2U(1)$ anomaly. So we have a contradiction, which means that there must be at least one type a.

[7] Choosing no fermions of type a would lead to an equivalent solution with opposite chirality.

TABLE VIII. All allowed representations of fermions which could be used to satisfy Eq. (A2) and their contributions to some anomalies.

Type	Representation of SU(2)⊗SU(3)	$\frac{y}{2}$	$S\left(\frac{y}{2}\right)^2$	$[SU(3)]^3$	$[SU(3)]^2U(1)$	$[SU(2)]^2U(1)$
a	**2**,**3**	$\frac{1}{6}$	$\frac{1}{6}$	2	$\frac{1}{3}$	$\frac{1}{2}$
c_1	**1**,**3**	$-\frac{1}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$	0
c_2	**1**,**3**	$\frac{2}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	0
d_1	**1**,$\bar{\mathbf{3}}$	$\frac{1}{3}$	$\frac{1}{3}$	-1	$\frac{1}{3}$	0
d_2	**1**,$\bar{\mathbf{3}}$	$-\frac{2}{3}$	$\frac{4}{3}$	-1	$-\frac{2}{3}$	0
e_1	**2**,**1**	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
e_2	**2**,**1**	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
f_1	**1**,**1**	-1	1	0	0	0
f_2	**1**,**1**	1	1	0	0	0

The $[SU(2)]^2U(1)$ anomaly must be cancelled by having as many type e_1 as type a. So there are no type e_2 due to the principle of mass protection. Again using the principle of mass protection, the only way to cancel the $[SU(3)]^3$ and $[SU(3)]^2U(1)$ anomalies is by having the number of types a, d_1, and d_2 the same. We can now cancel the $[U(1)]^3$ and mixed anomalies using Table IX.

So we see that the anomaly-free set of mass-protected fermions which minimizes the sum of the weak hypercharges squared is one of type a, d_1, d_2, e_1, and f_2. This is one SM quark-lepton generation.

3. Alternative derivations of the SM generation

There have been other attempts to derive the SM generation using various assumptions. Most notably Geng and Marshak [29] have tried to derive the SM generation using the constraints due to cancellation of anomalies. They also assume mass protection but not the charge quantization rule, Eq. (1). Instead of minimizing the sum of weak hypercharges squared, they try to find the minimum number of fermions required to satisfy these assumptions.

The smallest number of Weyl fermions found by Geng and Marshak is 14. This solution consists of the following representations of the gauge group U(1)⊗SU(2)⊗SU(3): $(0,\mathbf{2},\mathbf{3})_L$, $(y,\mathbf{1},\bar{\mathbf{3}})_L$, $(-y,\mathbf{1},\bar{\mathbf{3}})_L$, and $(0,\mathbf{2},\mathbf{1})_L$. They rule out this solution because the SU(2) doublet cannot acquire a Dirac or Majorana mass, even with the spontaneous symmetry breaking of the gauge group. However, we know from the SM that the neutrino is massless and so there does not appear to be any reason why massless fermions should be excluded from such an analysis. (We could obviously use phenomenological arguments but that would defeat the purpose of trying to derive the SM generation.) They also object to this solution because they feel it trivializes the cancellation of the mixed gravitational and gauge anomaly. In what sense the anomaly condition is trivial is not entirely clear, since not all fermions have zero weak hypercharge; but also why should it matter if a constraint is trivially satisfied? In our derivation this solution does not occur because of the charge quantization rule. So by enforcing the charge quantization rule, which we have taken as one of the defining properties of the SMG in Sec. II A 1, we can avoid this solution without introducing dubious arguments about fermion masses.

If we then also add the assumption that all subgroups must have some fermion coupling to them, we can almost derive the SM generation. The problem is that we can scale all values of weak hypercharge for the SM fermions by a factor of $(6n+1)$ where n is any integer.[8] The SM generation is obviously the solution with the values of hypercharge closest to zero. We can express this by choosing to minimize the sum of hypercharges squared for this solution. But since we must introduce such an assumption why not use it from the start? This then allows us to drop two of the above assumptions: that all subgroups must have a fermion coupling to them and that we should look for the smallest number of Weyl fermions. We are then left with the four assumptions used in Sec. 2 of this Appendix. This seems more reasonable than introducing more assumptions with no justification.

TABLE IX. Allowed combinations of fermions and their contribution to the remaining anomalies.

Types	$G^2U(1)$	$[U(1)]^3$	$S\left(\frac{y}{2}\right)^2$
$a+d_1+d_2+e_1$	$1+1-2-1=-1$	$\frac{1}{36}+\frac{1}{9}-\frac{8}{9}-\frac{1}{4}=-1$	$\frac{7}{3}$
f_1	-1	-1	1
f_2	1	1	1

[8]Without the charge quantization rule we could scale the weak hypercharges by an arbitrary amount. Then we could not use the procedure of minimizing the sum of hypercharges squared, since this would obviously force all values to zero. There is then no way to fix the scale other than by assuming the fermions get a mass by the Higgs mechanism and fixing the scale to the weak hypercharge of the Higgs boson. So the charge quantization rule effectively introduces a scale for the weak hypercharge independent of any Higgs bosons.

There have also been attempts to explain the charge quantization observed in the SM in terms of anomaly cancellation. A review is given in [30] and references therein. However, in our approach we are trying to understand the origin of the non-Abelian representations of the fermions, not just the values of weak hypercharge. In order to do this it appears necessary to use rather than to derive the charge quantization rule.

[1] C. D. Froggatt and H. B. Nielsen, *Origin of Symmetries* (World Scientific, Singapore, 1991).
[2] L. O'Raifeartaigh, *Group Structure of Gauge Theories* (Cambridge University Press, Cambridge, England, 1986).
[3] E. Fahri and L. Susskind, Phys. Rep. **74**, 277 (1981).
[4] N. Brene and H. B. Nielsen, Phys. Lett. B **223**, 399 (1989).
[5] N. Brene and H. B. Nielsen, Nucl. Phys. **B359**, 406 (1991).
[6] H. B. Nielsen and S. E. Rugh, Proceedings of the XXVI International Symposium Ahrenshoop (Report No. DESY 93-013, 1993) (unpublished), p. 307.
[7] D. S. Shaw and R. R. Volkas, Phys. Rev. D **47**, 241 (1993).
[8] C. D. Froggatt, G. Lowe and H. B. Nielsen, Nucl. Phys. **B420**, 3 (1994).
[9] C. T. Hill, Phys. Lett. B **266**, 419 (1991); **345**, 483 (1995); C. D. Carone and H. Murayama, Phys. Rev. D **52**, 4159 (1995).
[10] D. L. Bennett, H. B. Nielsen, and I. Picek, Phys. Lett. B **208**, 275 (1988).
[11] C. D. Froggatt, G. Lowe, and H. B. Nielsen, Nucl. Phys. **B414**, 579 (1994).
[12] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1984).
[13] E. Witten, Phys. Lett. **117B**, 324 (1982).
[14] M. Veltman, Nucl. Phys. **B123**, 89 (1977).
[15] Particle Data Group, L. Montanet *et al.*, Phys. Rev. D **50**, 1173 (1994).
[16] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, Proceedings of the Workshop, Stony Brook, New York, 1979, edited by P. van Nieuwenhuizen and D. Z. Freedman (North-Holland, New York, 1980), p. 315; T. Yanigada, in *Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe*, edited by A. Sawada and H. Sugawara (KEK, Tsukuba-Gu, Ibaraki-Ken, Japan, 1979), p. 95; C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B164**, 144 (1979).
[17] CDF Collaboration, F. Abe *et al.*, Phys. Rev. Lett. **74**, 2626 (1995); D0 Collaboration, S. Abachi *et al.*, *ibid.* **74**, 2632 (1995).
[18] S. Abachi *et al.*, Phys. Rev. Lett. **72**, 2138 (1994); F. Abe *et al.*, Phys. Rev. D **50**, 2966 (1994).
[19] E. H. Simmons, Nucl. Phys. **B312**, 253 (1989); C. D. Carone and H. Georgi, Phys. Rev. D **49**, 1427 (1994).
[20] C. D. Froggatt, H. B. Nielsen, and D. J. Smith, Z. Phys. C (to be published); Report No. hep-ph 9603436 (unpublished).
[21] M. E. Peskin and T. Tacheuchi, Phys. Rev. D **46**, 381 (1992); G. Altarelli and R. Barbieri, Phys. Lett. B **253**, 161 (1991); V. A. Novikov, L. B. Okun, and M. I. Vysotsky, Nucl. Phys. **B397**, 35 (1993).
[22] C. D. Froggatt, H. B. Nielsen, and D. J. Smith, paper eps0360 contributed to the International Europhysics Conference on High Energy Physics, Brussels, 1995 (unpublished); Report No. GUTPA-95-06-2 (unpublished).
[23] C.-K. Chow and T.-M. Yan, Phys. Rev. D **53**, 5105 (1996); R. Shrock, *ibid.* **53**, 6465 (1996).
[24] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, *ibid.* **30**, 1346 (1973); T. P. Cheng, E. Eichten, and Ling-Fong Li, Phys. Rev. D **49**, 2259 (1974).
[25] H. Arason, D. J. Castaño, B. Kesthelyi, S. Mikaelian, E. J. Piard, P. Ramond, and B. D. Wright, Phys. Rev. D **46**, 3945 (1992).
[26] M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B222**, 83 (1983); **B236**, 221 (1984); **B249**, 70 (1985).
[27] C. T. Hill, Phys. Rev. D **24**, 691 (1981); C. T. Hill, C. N. Leung, and S. Rao, Nucl. Phys. **B262**, 517 (1985).
[28] S. Dimopoulos and S. Theodorakis, Phys. Lett. **154B**, 153 (1985).
[29] C. Q. Geng and R. E. Marshak, Phys. Rev. D **39**, 693 (1989); **41**, 717 (1990); J. Minahau, P. Ramond, and R. C. Warner, *ibid.* **41**, 715 (1990); R. E. Marshak, *Conceptual Foundations of Modern Particle Physics* (World Scientific, Singapore, 1993); P. H. Frampton and R. N. Mohapatra, Phys. Rev. D **50**, 3569 (1994).
[30] R. Foot, H. Lew, and R. R. Volkas, J. Phys. G **19**, 361 (1993).