Applying factorial moments of continuous order to experimental data of 400 GeV/c pp collisions

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(Received 21 February 1996)

Factorial moments of continuous order are studied by using Monte Carlo events and experimental data. A maximum likelihood method is proposed and dynamical fluctuations are studied in normalized pseudorapidity space for 400 GeV/c pp collisions. Multifractal dimensions $D(q)$ and multifractal spectrum $f(\alpha)$ are presented. Monte Carlo simulation indicates that the statistical fluctuations are filtered out. The values of $D(q)$ are consistent with the previous ones obtained by the ordinary method of scaled factorial moment. The observed hierarchy $D(0)>D(1)>D(2)$ provides evidence of multifractal behavior for multiparticle production of 400 GeV/c pp collisions. The multifractal spectrum $f(\alpha)$ can be well reproduced by the random cascade α model. [S0556-2821(97)05403-9]

PACS number(s): 13.85.Hd, 12.40.Ee, 25.75.Dw

I. INTRODUCTION

To investigate intermittent behavior of high-energy multiparticle production, one may calculate the moments of the multiplicity distribution and study their dependence on bin size in pseudorapidity space. The multiplicity distribution originates from two parts, the dynamic fluctuation and the statistical one. In high-energy hadronic collisions, the multiplicity is usually small, especially when the bin size is small. Consequently, the statistical fluctuation might be large and dominate the fluctuation of the multiplicity distribution. In order to extract a genuine dynamic fluctuation and to eliminate the statistical one, Bialas and Peschanski suggested to use the scaled factorial moments [1]

$$F_q=\frac{\langle n(n-1)\cdots(n-q+1)\rangle}{\langle n\rangle^q}, \tag{1}$$

where the angular brackets denote an average weighted by the multiplicity distribution P_n, and the order q is an integer number greater than or equal to 2. They showed that the F_q gives a nonbiased estimation of the moments of dynamic fluctuation, and intermittency can be expressed as a power-law increase of F_q with decreasing bin size. The introduction of F_q greatly stimulated the study of intermittency and intermittency was observed in many experiments [2]. Some mechanics of self-similar multifractal structure were introduced to explain the experimental results. To acquire higher sensitivity, some correlation integral methods have been used in the calculation of F_q [3,4].

It can be seen from Eq. (1) that only events with $n\geqslant q$ can contribute to F_q. So F_q can reveal a spark signal of $n\geqslant q$ and cannot reveal a dip signal of $n=0$, while in some cases, such as in nuclear collisions, abnormal dips are of the same importance to the study of dynamical fluctuations as abnormal sparks. So it is necessary to study F_q for $q<1$. Furthermore, in multifractal analysis, the calculation of the multifractal spectrum requires q to be continuous in order to allow differentiation with respect to q, the calculation of fractal dimension and information dimension requiring q to be 0 and 1. All of the above requires that the range of q be extended.

To obtain moments of arbitrary order, Hwa suggested to use G moments several years ago [5]. It was later modified to achieve better power-law behavior [6]. Many experimental results on G moments have been reported [7–14]. But G moments cannot filter out statistical fluctuation as F moments do. Only a biased estimation is obtained and a statistical component is attached to the moment of genuine dynamical fluctuation. The statistical component cannot be eliminated by increasing the number of events. It can only be subtracted by hand [15], i.e., by comparing the result of experimental data with the result of Monte Carlo (MC) events which have the same multiplicity distribution but no correlation. So the method is not elegant, and the errors of the results are usually large. Some other methods, e.g., the moments proposed by Takagi [16], have also been tried. But they still cannot give out a nonbiased estimation for the moments of dynamical fluctuations [14]. For this reason, Hwa recently proposed a new method [17] to obtain factorial moments of continuous order with statistical fluctuations filtered out. It not only gives a nonbiased estimation for the moments of dynamical fluctuations, but also is defined in continuous order.

In this paper, some efforts are made to apply Hwa's new method of ''factorial moment of continuous order'' to real data. The paper is arranged as follows. In Sec. II, Hwa's new method is briefly reviewed. In Sec. III, Hwa's method is tested and applied to Monte Carlo events. Some problems are presented. In Sec. IV, a maximum likelihood method is proposed to solve the problems, and some results of the maximum likelihood method are presented for Monte Carlo samples. In Sec. V, the variant factorial moment of continuous order is applied to the experimental data to study the multifractal behavior. Some results are compared with that obtained by the ordinary scaled factorial moment. In the last section, Sec. VI, a brief review of this paper is given.

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II. FACTORIAL MOMENT OF CONTINUOUS ORDER

The multiplicity distribution P_n can be represented as [17]

$$P_n=\int_0^\infty dt\frac{t^n}{n!}e^{-t}D(t), \tag{2}$$

where $D(t)$ represents the dynamic fluctuation and the Poisson factor $(t^n/n!)e^{-t}$ describes the statistical component. Denote the numerator of Eq. (1) by f_q: i.e.,

$$f_q=\sum_{n=q}^{\infty}\frac{n!}{(n-q)!}P_n, \tag{3}$$

where q is a positive integer. Substituting Eq. (2) into Eq. (3) and performing the summation over n, one obtains

$$f_q=\int_0^\infty dt\, t^q D(t). \tag{4}$$

It is exactly the qth moment of the dynamic fluctuation $D(t)$. Since $f_1=\langle n\rangle$, one has

$$F_q=f_q/f_1^q. \tag{5}$$

However, for continuous order q, Eq. (3) cannot be applied. So other methods are needed. In [17], it is suggested to expand P_n in a series of negative binomial distributions (NBDs):

$$P_n=\sum_{j=0}^{N}a_jP_n^{\mathrm{NB}}(k_j,x_j), \tag{6}$$

where $n=0,1,\ldots,N$ (N is the maximum observed multiplicity), and

$$P_n^{\mathrm{NB}}(k_j,x_j)=\frac{\Gamma(n+k_j)}{\Gamma(n+1)\Gamma(k_j)}\left(\frac{k_j}{k_j+x_j}\right)^{k_j}\left(\frac{x_j}{k_j+x_j}\right)^n. \tag{7}$$

Using Eqs. (2) and (6) and the relation

$$P_n^{\mathrm{NB}}(k_j,x_j)=\int_0^\infty dt\frac{t^n}{n!}e^{-t}D^{\mathrm{NB}}(t,j), \tag{8}$$

where

$$D^{\mathrm{NB}}(t,j)=\left(\frac{k_j}{x_j}\right)^{k_j}\frac{t^{k_j-1}}{\Gamma(k_j)}e^{-k_jt/x_j}, \tag{9}$$

one gets

$$D(t)=\sum_{j=0}^{N}a_jD^{\mathrm{NB}}(t,j). \tag{10}$$

Substituting Eq. (10) into Eq. (4) and integrating over t, one finds

$$f(q)=\sum_{j=0}^{N}a_jf^{\mathrm{NB}}(q,j), \tag{11}$$

where

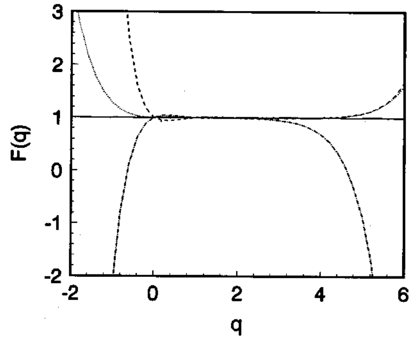

FIG. 1. $F(q)$'s obtained by Hwa's original method for $P_n^{(0)}$ calculated by Eq. (15) with $\langle n\rangle=1$ (solid line) and for P_n's of the three Monte Carlo samples which include 10^5 events (dot-dashed line), 10^6 events (dotted line), and 10^7 events (dashed line), respectively.

$$f^{\mathrm{NB}}(q,j)=\left(\frac{x_j}{k_j}\right)^q\frac{\Gamma(q+k_j)}{\Gamma(k_j)}. \tag{12}$$

To determine the value of a_j, one should assign [17] $N+1$ pairs of x_j and k_j ($j=0,1,\ldots,N$):

$$x_j=x(1+\Delta_j),$$
$$k_j=k(1+\Delta_j), \tag{13}$$

where

$$x=\langle n\rangle=\sum_{n=0}^{N}nP_n,$$

$$k=(F_2-1)^{-1},\quad F_2=\langle n(n-1)\rangle/x^2,$$

$$\Delta_j=\Delta\left(-\frac{1}{2}+\frac{j}{N}\right),$$

$$j=0,1,\ldots,N. \tag{14}$$

Then calculate $P_n^{\mathrm{NB}}(j)$ according to Eq. (7), and solve the $N+1$ linear algebraic equations (6) for a_j. Therefore, $F(q)$ can be obtained by Eqs. (11) and (5).

According to [17], a suitable value for Δ is 0.5, and the value of k could be set to 10^4 for Poissonian distribution, for which $F_2=1$.

III. APPLICATION OF HWA's METHOD

A. Application to the analytical Poissonian distribution

To have a test of Hwa's method, we first apply Hwa's method to the analytical Poissonian distribution

$$P_n^{(0)}=e^{-\langle n\rangle}\langle n\rangle^n/n!. \tag{15}$$

P_n is calculated analytically according to Eq. (15) with $\langle n\rangle=1$, $n=0,1,\ldots,N$, $N=10$. Then we solve a set of $N+1$ algebraic equations (6) to obtain the expansion coefficient a_j. At last, the factorial moments of continuous order $F(q)$ are calculated by Eqs. (5) and (11). In the calculation, MATHEMATICA is adopted to ensure numerical accuracy. The

TABLE I. Results of a Monte Carlo sample (10^5 events) for the Poissonian distribution with $\langle n\rangle=1$. $N_n^{(0)}$ is the calculated value according to Eq. (15). N_n(MC) is counted out from the Monte Carlo sample. $\widetilde{N}_n$ is the value used to fit the N_n(MC) by the maximum likelihood method. a_j is the coefficient to expand N_n(MC) by Eq. (6). $\widetilde{a}_j$ is the coefficient used for the maximum likelihood method.

n	$N_n^{(0)}$	N_n(MC)	$\widetilde{N}_n$	j	a_j	$\widetilde{a}_j$
0	36787.9	36571	36659.0	0	-6.0310967075×10^5	-0.0436
1	36787.9	37130	36887.5	1	5.1107712192×10^6	1.0867
2	18394.0	18274	18456.9	2	-1.8943023020×10^7	-0.0431
3	6131.3	6120	6125.1	3	4.0111196201×10^7	
4	1532.8	1558	1516.5	4	-5.3070399844×10^7	
5	306.6	293	298.7	5	4.4927260786×10^7	
6	51.1	41	48.7	6	-2.3765120027×10^7	
7	7.3	12	6.8	7	7.1816970312×10^6	
8	0.9	1	0.8	8	-9.4927167512×10^5	

results of $F(q)$ are shown in Fig. 1 with a solid line. It can be seen that $F(q)$ has a constant value of 1. For Poissonian multiplicity distribution, which has only pure statistical fluctuations, the obtained ''$F(q)=1$'' means that the statistical fluctuations are filtered out.

B. Application to Monte Carlo samples of the Poissonian distribution

In a practical experiment, the access to an accurate value of P_n would require measurement of an infinite number of events. But in a real experiment, we can only measure a finite number of events. So each P_n is measured with a statistical error, which may dominate at large n, where the event number N_n is small (~1). The statistical error would be a noise to the calculation. In order to see the effect of the statistical noise, we apply Hwa's method to Monte Carlo samples of the Poissonian multiplicity distribution. The multiplicities of events are randomly generated according to the Poissonian distribution. Three samples are generated which include 10^5, 10^6, and 10^7 Monte Carlo events, respectively ($\langle n\rangle=1$). The results of $F(q)$ calculated by Hwa's method for the three samples are shown in Fig. 1 with a dot-dashed line, dotted line, and dashed line, respectively. The $F(q)$'s deviated from 1 obviously and the deviations cannot be reduced only by increasing the event number. It is because the expansion (6) penetrates all the experimental points P_n precisely. So large statistical noises are included in the expansion coefficient a_j. Furthermore, in the calculation, the determinant of the matrix composed by the $P_n^N(k_j,x_j)$'s of Eq. (6) is almost zero, and so the matrix is ill conditioned and the noise is multiplied greatly. A very large a_j with alternating signs may appear and cause great instability for the result of $F(q)$. The a_j's of the 10^5-event sample are listed in Table I.

C. Effect of statistical noise

A simple way to reduce the effect of statistical noise is by cutting P_n of large n which has large noise. The dashed line in Fig. 2 is the result of the previous 10^5-event sample with N set to 6. The deviation of $F(q)$ from 1 is smaller than that of the original one ($N=8$, dot-dashed line in Fig. 2). But in such a way, some important information is lost and there is still obvious deviation.

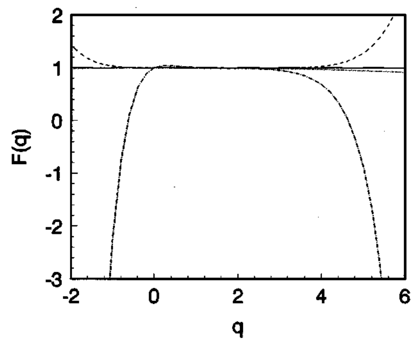

FIG. 2. $F(q)$ obtained by different method (see text) for the previous Monte Carlo sample which includes 10^5 events of the Poissonian distribution with $\langle n\rangle=1$.

On the other hand, for large $\langle n\rangle$, the effect of statistical noise will be much greater. A sample of 10^5 Monte Carlo events is generated according to the Poissonian multiplicity distribution with $\langle n\rangle=6$. The event number N_n, the expansion coefficient a_j, and the calculated result of $F(q)$ are listed in Table II. It can be seen that the $F(q)$ of this sample is totally covered by statistical noise.

So we develop Hwa's method in order to apply it to real experimental data.

IV. VARIANT METHOD

A. Maximum likelihood method

Considering that the negative binomial distribution can fit the experimental multiplicity distribution well, we can choose less a_j to fit the experimental data: i.e.,

$$\widetilde{P}_n=\sum_{j=0}^{J}\widetilde{a}_jP_n^{\rm NB}(k_j,x_j), \tag{16}$$

where

$$x_j=x(1+\Delta_j),$$
$$k_j=k(1+\Delta_j),$$
$$\Delta_j=\Delta\left(-\frac{1}{2}+\frac{j}{J}\right),$$
$$j=0,1,\ldots,J. \tag{17}$$

The fit is performed with $N+1$ points P_n ($P_n=N_n/N_{\rm ev}$). Considering that the event number N_n at large n is very small, the fit should be performed by using the maximum likelihood method. The $\widetilde{a}_j$'s are chosen so that the following likelihood function L reaches its maximum:

$$L=\frac{N_{\rm ev}!}{\prod_{n=0}^{N}N_n!}\prod_{n=0}^{N}(\widetilde{N}_n)^{N_n}, \tag{18}$$

where $\widetilde{N}_n=N_{\rm ev}\widetilde{P}_n$, $N_{\rm ev}$ being the total number of events.

From Eq. (16), one can obtain

TABLE II. Results of a Monte Carlo sample (10^5 events) for the Poissonian distribution with $\langle n\rangle=6$.

n	$N_n^{(0)}$	$N_n(\mathrm{MC})$	j	a_j	q	$F(q)$
0	247.9	245	0	$-9.71817995059087224687 58\times10^{13}$	-5	2.7654×10^{9}
1	1487.3	1513	1	$2.0813635705728971917832\times10^{15}$	-4	-2.7087×10^{7}
2	4461.8	4477	2	$-2.1034475369002476499712\times10^{16}$	-3	1.9012×10^{5}
3	8923.5	8658	3	$1.3328969277313204446820\times10^{17}$	-2	-6.8661×10^{2}
4	13385.3	13439	4	$-5.9349838329157386722709\times10^{17}$	-1	-1.4049
5	16062.3	16106	5	$1.9720438391917844691530\times10^{18}$	0	-2.3309
6	16062.3	16226	6	$-5.0677423407189957937869\times10^{18}$	1	1.0000
7	13767.7	13798	7	$1.0298835875365655044294\times10^{19}$	2	-0.3326
8	10325.8	10382	8	$-1.6779607508456232342107\times10^{19}$	3	0.1025
9	6883.8	6901	9	$2.2082352763024706866285\times10^{19}$	4	-3.0294×10^{-2}
10	4130.3	4098	10	$-2.3530016147247046371315\times10^{19}$	5	8.6394×10^{-3}
11	2252.9	2204	11	$2.0253229219002603969922\times10^{19}$	6	-2.3826×10^{-3}
12	1126.4	1097	12	$-1.3977748551407234062399\times10^{19}$	7	6.3572×10^{-4}
13	519.9	523	13	$7.6301445832105411659127\times10^{18}$	8	-1.6412×10^{-4}
14	222.8	203	14	$-3.2220563852966065065394\times10^{18}$	9	4.1003×10^{-5}
15	89.1	95	15	$1.0156393860442280531282\times10^{18}$	10	-9.9151×10^{-6}
16	33.4	31	16	$-2.2504303410168803389844\times10^{17}$		
17	11.8	2	17	$3.1279928138927774773731\times10^{16}$		
18	3.9	2	18	$-2.0526426342669249645330\times10^{15}$		

$$\sum_{j=0}^{J}\tilde{a}_j=1. \tag{19}$$

It should be fulfilled in the whole process of finding the proper values of $\tilde{a}_j$. At first, we set all $\tilde{a}_j$'s to $1/J$ and assign a step $h=1/(2J)$. Then, we change each $\tilde{a}_j$ in turn by step h to acquire the maximum likelihood. The change could be positive or negative. If a change is made for one $\tilde{a}_j$, an extra factor $1/(1+h)$ or $1/(1-h)$ should be multiplied to each $\tilde{a}_i$ (including $\tilde{a}_j$) in order to satisfy Eq. (19). When the best values of $\tilde{a}_j$ are obtained at step h, we reduce the step and find better values for $\tilde{a}_j$. The last step used in our calculation is 0.0001.

B. Application of the maximum likelihood method to a Monte Carlo sample of the Poissonian distribution

The maximum likelihood method is applied to the previous 10^5-event Monte Carlo sample of the Poissonian distribution with $\langle n\rangle=1$. The results of $\tilde{F}(q)$ are shown in Fig. 2 with a dotted line. It can be seen that $\tilde{F}(q)\approx1$. So the measurement errors are well constrained.

Because the fit of the maximum likelihood method is performed to $N+1$ points P_n, the maximum number of the coefficient $\tilde{a}_j$ is $N+1$, i.e., $J\leqslant N$. When $J=N$, it will come back to the original Hwa's method. When $J=0$, it is equivalent to fitting P_n with only one negative binomial distribution. Generally, the value of J used for the fit is chosen according to the practical multiplicity distribution P_n. The dotted line in Fig. 2 is the result of $J=2$. The fitted value $\tilde{N}_n$ and the fitting coefficient $\tilde{a}_j$ are listed in Table I. It can be seen that $\tilde{N}_n$ is consistent with N_n in their statistical errors.

C. Application to the $P_n^{(2)}$ distribution

We also calculated $F(q)$ for the distribution [17]

$$P_n^{(2)}=(n+1)^{0.5}e^{-n}/z, \tag{20}$$

where z is a normalization factor. Unlike the Poissonian distribution, not only are there statistical fluctuations, but there are also dynamical fluctuations in the $P_n^{(2)}$ distribution.

At first, we generate a sample of 10^5 Monte Carlo events, the multiplicities of which are generated according to the $P_n^{(2)}$ distribution. Then we calculate the multiplicity distribution P_n for this sample and calculate $\tilde{a}_j$ and $\tilde{F}(q)$ by the maximum likelihood method. The event number N_n and the fitted value $\tilde{N}_n$ are listed in Table III. It can be seen from the

TABLE III. Results of a Monte Carlo sample (10^5 events) for the $P_n^{(2)}$ distribution.

n	$N_n^{(2)}$	$N_n(\mathrm{MC})$	$\tilde{N}_n$	q	$F(q)$
0	52016.2	52231	52206.9	-1.0	7.7765×10^{18}
1	27061.9	26955	27047.2	-0.8	2.1717×10^{15}
2	12193.0	12137	12103.6	-0.6	1.5422×10^{13}
3	5179.5	5230	5124.6	-0.4	3.1475×10^{11}
4	2130.3	2043	2107.2	-0.2	9.1117×10^{9}
5	858.5	850	850.9	0.0	0.9978
6	341.1	343	339.3	0.2	-8.5623×10^{7}
7	134.2	132	134.1	1.0	1.0000
8	52.3	51	52.7	1.2	5.1967×10^{4}
9	20.3	14	20.6	2.0	0.7846
10	7.8	10	8.0	3.0	-18.959
11	3.0	2	3.1	4.0	-477.21
12	1.2	1	1.2	5.0	-9.8628×10^{3}
13	0.4	1	0.5	6.0	-1.9438×10^{5}

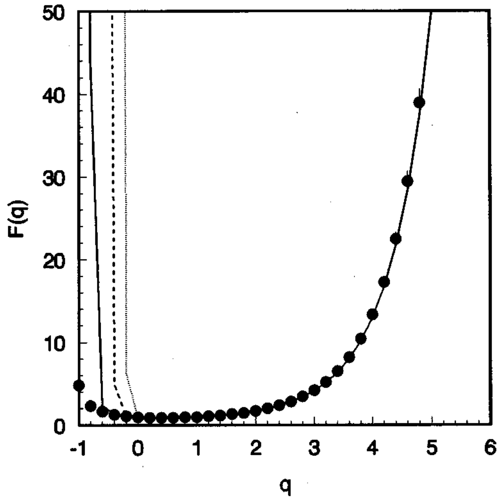

FIG. 3. $F(q)$ obtained by Hwa's original method for $P_n^{(2)}$ calculated by Eq. (20) with $N=10$ (solid line), $N=15$ (dashed line), $N=20$ (dotted line), and $\widetilde{F}(q)$ obtained by the maximum likelihood method for a Monte Carlo sample (10^5 events) generated according to $P_n^{(2)}$ distribution (solid circles).

table that the maximum likelihood method can well fit the distribution which has dynamical fluctuations. The results of $\widetilde{F}(q)$ are shown in Fig. 3 with solid circles. We have also applied the original Hwa's method to this Monte Carlo sample and the results of $F(q)$ are listed in Table III. It can be seen that the $F(q)$ is totally covered by statistical noise. It might suggest that the application of the original Hwa's method might require that the distribution P_n be analytically calculated out, e.g., by Eq. (20).

The application of Hwa's method to P_n analytically calculated by Eq. (20) is done with $N=$ 10, 15, and 20. The results of $F(q)$ are shown in Fig. 3 with a solid line, dashed line, and dotted line, respectively. It can be seen that the results of Hwa's method for analytical $P_n^{(2)}$ are consistent with the results of the maximum likelihood method for a Monte Carlo sample in the range $q>0$, while in the range $q<0$, there are deviations and the deviation is much greater for large N. This might be because the original Hwa's method requires expansion (6) to penetrate all the $N+1$ points $P_n^{(2)}$ analytically calculated out by Eq. (20). But expansion (6) is not the true $P_n^{(2)}$ distribution. And the difference between them may be amplified due to the ill-conditioned character of the matrix composed by the $P_n^{\mathrm{NB}}(k_j,x_j)$ of Eq. (6). The amplitude would be much greater for larger N. So a very large a_j with alternating signs may appear and cause the deviation of $F(q)$. In fact, $P_{10}^{(2)}\sim 10^{-4}$, and is very small. So $F(q)$ of $N=10$ may reveal the main character of $P_n^{(2)}$ better than that of $N=15$, 20 as a result of which the amplitude is decreased for smaller N.

V. EXPERIMENTAL RESULTS

Using the LEBC films offered by the CERN NA27 Collaboration, we measured the pseudorapidity distribution of charged particles produced in 400 GeV/c pp collisions. A sample of 3740 events is obtained. Details of the measurement are described elsewhere [18–20].

TABLE IV. A typical result of P_n and $\widetilde{P}_n$ in one bin at $M=20$.

n	0	1	2	3	4	5
$N_{\mathrm{ev}}P_n$	2609	881	188	52	9	1
$N_{\mathrm{ev}}\widetilde{P}_n$	2611.39	868.62	207.03	42.91	8.23	1.50

A. Experimental results of factorial moments of continuous order

In this paper, the following normalized pseudorapidity [21] is adopted:

$$X(\eta)=\frac{\int_{\eta_{\min}}^{\eta}\rho(\eta')d\eta'}{\int_{\eta_{\min}}^{\eta_{\max}}\rho(\eta')d\eta'}, \tag{21}$$

where $[\eta_{\min},\eta_{\max}]$ is chosen $[-2,2]$, and X is uniformly distributed in $[0,1]$. The dynamical fluctuation of the multiplicity distribution in decreasing the bin size is studied by calculating the factorial moments of continuous order. At one bin size, the X space is divided into M bins of equal size δ and the calculation is done in each bin independently. At first, the multiplicity distribution $P_{n,m}$ of bin m is calculated for the experimental data. Second, $P_{n,m}^{\mathrm{NB}}(k_{j,m},x_{j,m})$ is calculated according to Eq. (7) and $\widetilde{a}_{j,m}$ is determined by the maximum likelihood method according to Eqs. (18) and (16) ($J=4$ for $M\leqslant 4$ and $J=2$ for $M>4$). Then, $\widetilde{f}_m(q)$ is calculated for bin m according to Eq. (11). At last, an average is made over each bin:

$$\widetilde{F}(q,\delta)=\frac{1}{M}\sum_{m=1}^{M}\frac{\widetilde{f}_m(q)}{[\widetilde{f}_m(1)]^q}. \tag{22}$$

Therefore, the factorial moment $\widetilde{F}(q,\delta)$ of continuous order q at bin size δ is obtained. Some information about the quality of the fits can be found in Table IV. The results of $\widetilde{F}(q,\delta)$ are shown in Fig. 4 (solid circle). To see the effect of parameter number, we perform the calculation in the same way but with a smaller J ($J=2$ for $M\leqslant 4$ and $J=0$ for $M>4$). The results are shown in Fig. 4 as open circles. It can be seen that the two results are consistent within their statistical errors for $M>4$. It means that the multiplicity distribution P_n of this experiment can be well described by a negative binomial distribution.

In order to see the statistical contribution to $\widetilde{F}(q,\delta)$, we generate a sample of Monte Carlo events. Compared to the experimental data, the sample of Monte Carlo events has the same multiplicity distribution in X space but no correlations. For each event i of n_i particles, we distribute these particles randomly through X space in a uniform distribution. A total of $100N_{\mathrm{ev}}$ Monte Carlo events is generated. The results of $\widetilde{F}(q,\delta)$ for these Monte Carlo events are also shown in Fig. 4 with dashed lines. It can be seen that if there is only a statistical fluctuation, the moments $\widetilde{F}(q,\delta)$ of continuous orders remain constant when δ is decreased. Hence the inter-

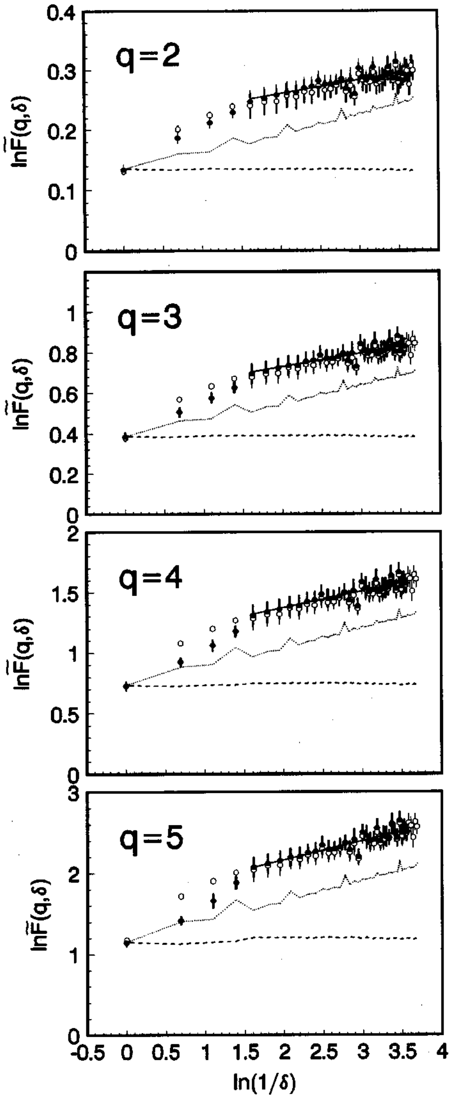

FIG. 4. $\ln\tilde{F}(q,\delta)$ versus $\ln(1/\delta)$. The solid circles represent the experimental results obtained by the maximum likelihood method with $J=4$ for $M\leqslant 4$ and $J=2$ for $M>4$. The solid lines are the results of fitting the $\tilde{F}(q,\delta)$'s (for $M>4$) with Eq. (24). The open circles represent the experimental results obtained by the maximum likelihood method with $J=2$ for $M\leqslant 4$ and $J=0$ for $M>4$. The dashed lines show results for the Monte Carlo events which have no correlation in X space. The dotted lines show results for the Monte Carlo events which are generated according to the α model.

mittency exponents $\phi(q)$ (see below) are equal to zero. This suggests that the statistical fluctuations are filtered out for experimental data.

B. Simulation with the random cascade α model

To describe the intermittent behavior, we made a Monte Carlo simulation with the random cascade α model [1,22]. In the simulation, the multiplicities are generated according to the experimental multiplicity distribution. Having generated the multiplicity n for one Monte Carlo event, n particles are distributed in the X space through a cascading process according to the random cascading α model. In one step of the cascading process, each bin obtained in the previous step is divided into λ sub-bins, and the probability of finding a particle in one previous bin is multiplied by λ factors $W_{(\alpha_j)}/\lambda$ $(1\leqslant\alpha_j\leqslant\lambda)$ so as to obtain the probability of finding a particle in the λ sub-bins, where $W_{(\alpha_j)}$'s are random variables generated for the λ sub-bins. They fulfill the normalized equation $\Sigma_{\alpha_j=1}^{\lambda}W_{\alpha_j}=1$. So, at step ν, X space is divided into λ^{ν} bins. We denote each bin with a set of parameters $\{\alpha_1,\ldots,\alpha_\nu\}$. The probability that one particle is distributed in bin $\{\alpha_1,\ldots,\alpha_\nu\}$ is

$$p\{\alpha_1,\ldots,\alpha_\nu\}=\frac{1}{\lambda^{\nu}}W_{(\alpha_\nu)}W_{(\alpha_{\nu-1})}\cdots W_{(\alpha_2)}W_{(\alpha_1)}. \tag{23}$$

The probability is the same for the n particles of one event. So particle density fluctuates. In the Monte Carlo simulation, we chose $\lambda=2$, $\nu=16$, $W_1=1+ar$, and $W_2=1-ar$, where $a=0.275$, and r is a random number uniformly distributed in range $[-1,1]$. A sample of $100N_{\rm ev}$ Monte Carlo events is generated. The results of $F(q)$ for this sample are shown in Fig. 4 with dotted lines. They give out approximately the same slopes as the experimental results do at small δ.

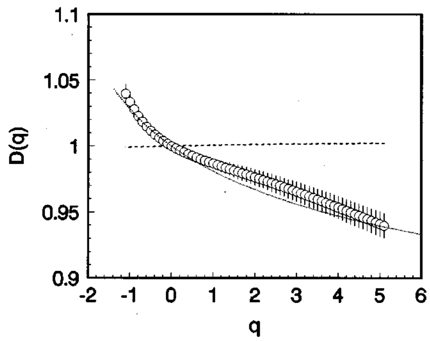

FIG. 5. Multifractal dimension $D(q)$ versus q. The dashed line represents the results for Monte Carlo events which have no correlation in X space. The dotted line represents the results for Monte Carlo events generated according to the α model.

C. Experimental study for multifractal dimensions

When $\delta\to 0$, we obtain the intermittency exponent $\phi(q)$ by fitting $\tilde{F}(q,\delta)$ with the formula (for $M>4$)

$$\tilde{F}(q,\delta)\propto\delta^{-\phi(q)}. \tag{24}$$

From $\phi(q)$ we can calculate the multifractal spectrum $f(\alpha)$ and multifractal dimension $D(q)$ using the relations

$$\tau(q)=q-1-\phi(q),$$

$$\alpha=d\tau(q)/dq,$$

$$f(\alpha)=q\alpha-\tau(q),$$

$$D(q)=\tau(q)/(q-1). \tag{25}$$

$D(q)$ versus q is shown in Fig. 5. It can be seen that $D(0)=1$ for the experimental data. At $q=0$, it can be obtained from Eqs. (4), (5), and (24) that $\phi(0)$ should be zero, even though there is dynamic fluctuation, and consequently,

TABLE V. Multifractal dimension D obtained by different methods.

q	$D(q)(\widetilde{F}(q))$	$D_q(F_q)$	$D_q^{\rm dyn}(G_q)$
0	1		
1	0.9868 ± 0.0004		
2	0.9764 ± 0.0060	0.975 ± 0.002	0.961 ± 0.006
3	0.9654 ± 0.0076	0.964 ± 0.002	0.950 ± 0.006
4	0.9535 ± 0.0087	0.953 ± 0.004	0.966 ± 0.007
5	0.9409 ± 0.0094	0.940 ± 0.010	0.947 ± 0.010

$D(0)$ should be 1. The results of the Monte Carlo samples with particles randomly distributed in X space are shown in Fig. 5 with a dashed line. It can be seen that $D(q)=1$ at each q for the MC data. It can be seen from Eqs. (25) that, if there is no dynamic fluctuation $[\phi(q)=0]$, $D(q)$ should be 1 for all q. The dotted line in Fig. 5 is the Monte Carlo results of the random cascade α model. It approximately reproduces the behavior of $D(q)$ for experimental data. The decrease of $D(q)$ with increasing q for experimental data suggests that there is multifractal behavior in multiparticle productions.

Several values of $D(q)$ at integer q are listed in Table V in comparison with that obtained by the ordinary scaled factorial moment F_q [19] and that obtained by the modified G moment [14]. It can be seen from Table V that the results of $D(q)$ of the present method are well consistent with that of the ordinary method of scaled factorial moments. So the present method is a successful one in filtering out statistical fluctuations as the scaled factorial moments do. The results of $D(q)$ are also consistent with that of the modified G-moment method, the change of $D(q)$ versus q of the present method being smoother than that of the modified G-moment method. It means that the present method has successfully extended the order q to the continuous range.

D. Experimental results of multifractal spectrum

The spectrum $f(\alpha)$ is shown in Fig. 6 (open circles). It is concave downward with a maximum at $q=0$, $f(\alpha(0))=D(0)=1$. The straight line $f(\alpha)=\alpha$ is tangent to the $f(\alpha)$ curve at $q=1$. The solid point represents the results of the Monte Carlo sample with particles randomly distributed in X space, which essentially condense to a single point.

The deviation of the spectrum from parabolic shape might be related to the NBD form multiplicity distribution. The solid curve in Fig. 6 is the Monte Carlo results of the random cascade α model the multiplicities of which are generated according to the experimental multiplicity distribution. It can be seen that the random cascade α model could reproduce the deviation of the spectrum from parabolic shape. Furthermore, we have also made a simulation according to the random cascading α model the multiplicities of which are generated according to the Poissonian distribution, the obtained spectrum having a parabolic shape (not shown).

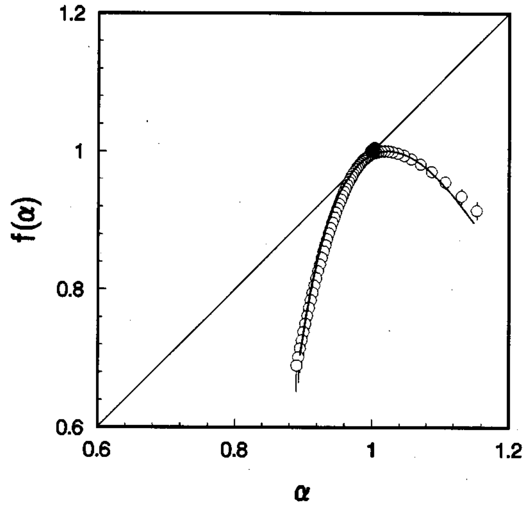

FIG. 6. Multifractal spectrum $f(\alpha)$. The open circles represent the experimental results. The solid circle represents the results for Monte Carlo events which have no correlation in X space. The curved line represents the results for Monte Carlo events generated according to the α model.

VI. CONCLUSION

In summary, we have studied the factorial moments of continuous order by using Monte Carlo events and experimental data. A maximum likelihood method is proposed in order to apply the factorial moments of continuous order to real data. And the maximum likelihood method is used in the study of dynamical fluctuations in decreasing intervals of normalized pseudorapidity space for 400 GeV/c pp collisions. Monte Carlo simulations indicate that the statistical fluctuations are filtered out. The results of multifractal dimensions $D(q)$ are consistent with the ones obtained by the ordinary scaled factorial moment. So the present method is a successful one in extending q to a continuous range and filtering out statistical fluctuations. The observed hierarchy $D(0)>D(1)>D(2)$ provides evidence of multifractal behavior for multiparticle production. The multifractal spectrum $f(\alpha)$ can be well reproduced by the random cascade α model.

ACKNOWLEDGMENTS

The authors are grateful to the CERN NA27 Collaboration for offering the LEBC films and to the National Natural Science Foundation of China for financial support.

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