Order $\alpha_s^2 \beta_0$ correction to the charged lepton spectrum in $b \rightarrow c \ell \bar{\nu}_{\ell}$ decays

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We compute the $\alpha_s^2 \beta_0$ part of the two-loop QCD corrections to the charged lepton spectrum in $b \rightarrow c \ell \overline{\nu}_{\ell}$ decays and find them to be about 50% of the first order corrections at all lepton energies, except those close to the end point. Including these corrections we extract the central values $\overline{\Lambda} = 0.33$ GeV and $\lambda_1 = -0.17$ GeV² for the HQET matrix elements and use them to determine the $\overline{\text{MS}} b$ and c quark masses, and $|V_{cb}|$. [S0556-2821(97)00603-6]

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I. INTRODUCTION

In the last few years numerous theoretical and experimental studies have focused on the electron spectrum in semileptonic inclusive $B \rightarrow X_{c} \ell \overline{\nu}_{\ell}$ decays. The electron spectrum from free quark decays receives both perturbative and nonperturbative corrections. Knowledge of the shape of the spectrum can provide insights into nonperturbative effects in B meson decays, and thereby also give some information on the weak mixing angle $|V_{cb}|$. In the framework of heavy quark effective theory (HQET) it is possible to show that the quark level decay rate is the first term in a power series expansion in the small parameter Λ_{OCD}/m_b [1]. For infinitely heavy quarks the free quark model is an exact description of heavy meson physics. At finite quark masses the first few terms in the heavy quark expansion have to be taken into account. Expressions for these nonperturbative corrections to the lepton spectrum are known to order $(\Lambda_{\text{OCD}}/m_b)^3$ [2–5] and the $O(\alpha_s)$ perturbative corrections to the free quark decay are given in [6].

The dominant remaining uncertainties are the two-loop corrections to the quark level decay rate and the perturbative corrections to the coefficients of the HQET matrix elements in the operator product expansion. Here, we examine the former. While a full two-loop calculation of the electron spectrum is a rather daunting task, it is possible to calculate the piece of the two-loop correction that is proportional to $\beta_0 = 11 - 2/3n_f$ with relative ease by performing the oneloop QCD calculation with a massive gluon. The $\alpha_s^2 \beta_0$ parts of the two-loop correction may then be obtained from a dispersion integral over the gluon mass [7]. If there are no gluons in the tree-level graph, the $\alpha_s^2 \beta_0$ part of the two-loop contribution is believed to dominate the full α_s^2 result because β_0 is rather large. Several examples supporting this belief are listed in [8], while one counterexample can be found in [9].

A recent calculation [8] of the $\alpha_s^2 \beta_0$ correction to the total inclusive rate for $B \rightarrow X_c \ell \overline{\nu}_{\ell}$ decays showed that the $\alpha_s^2 \beta_0$ parts of the two-loop correction are approximately half as big as the one-loop contribution, resulting in a rather low Brodsky-Lepage-Mackenzie (BLM) scale [10] of $\mu_{\text{BLM}} = 0.13m_b$. For the electron spectrum we find that this part of the second order correction also amounts to about 50% of the order α_s contribution, at all electron energies except those close to the end point. Close to the end point the corrections are roughly equal in magnitude.

The HQET matrix elements λ_1 and Λ can be extracted from the electron spectrum in $B \rightarrow X_c / \bar{\nu}_\ell$ decays [11]. Even though this method of obtaining HQET matrix elements was found to be rather insensitive to the first order perturbative corrections, it is useful to extract $\lambda_1, \bar{\Lambda}$ including the $\alpha_s^2 \beta_0$ corrections. Then, these matrix elements can be used to relate the pole quark mass to the modified minimal subtraction scheme ($\overline{\text{MS}}$) masses at order $\alpha_s^2 \beta_0$. Similarly, one can include the $\alpha_s^2 \beta_0$ parts of the two-loop contribution in the theoretical prediction for the total rate, which is needed for the determination of $|V_{cb}|$. Since the $\alpha_s^2 \beta_0$ corrections are rather large, the resulting changes in the quark masses and $|V_{cb}|$ are not negligible.

In Secs. II and III we give analytic expressions for the contributions from virtual and real gluon radiation. The last phase-space integral in the virtual correction and the last two integrals in the bremsstrahlung are done numerically. Readers not interested in calculational details are advised to skip these sections. In Sec. IV we combine the results from the previous two sections to obtain the $\alpha_s^2 \beta_0$ corrections to the electron spectrum, and discuss the implications for the extraction of $\overline{\Lambda}, \lambda_1$, the $\overline{\text{MS}}$ quark masses, and $|V_{cb}|$. In Appendix B we give an interpolating polynomial which reproduces the two-loop correction calculated here.

II. VIRTUAL CORRECTIONS

The corrections from massive virtual gluons can be calculated in complete analogy to the usual one-loop QCD corrections. The ultraviolet divergence in the vertex correction cancels when combined with the quark wave function renormalizations. There is no infrared divergence since we do the calculation with a massive gluon. The virtual one-loop correction to the differential rate can be written as

$$\frac{d\Gamma_{\text{virt}}^{(1)}(\hat{\mu})}{dy} = \alpha_s^{(V)} \frac{|V_{cb}|^2 G_F^2 m_b^5}{48\pi^4} \int d\hat{q}^2 \\ \times \{2(y-\hat{q}^2)(\hat{q}^2+1-r^2-y)(a_1+a_{wr}) \\ -2r\hat{q}^2 a_2 + [\hat{q}^2(y-1)+y(1-r^2)-y^2]a_3\},$$
(1)

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where $\hat{\mu} = \mu/m_b$ is the rescaled gluon mass, and $y = 2E_e/m_b$, $r = m_c/m_b$, and $\hat{q}^2 = q^2/m_b^2$ are the rescaled electron energy, charm mass, and momentum transfer, respectively. The limits for the integration over \hat{q}^2 are

$$0 \leq \hat{q}^2 \leq \frac{y(1-y-r^2)}{1-y}, \quad 0 \leq y \leq 1-r^2.$$
 (2)

The functions $a_{wr}(\hat{q}^2)$ and $a_i(\hat{q}^2)$, i=1,2,3 are the contributions from the wave function renormalization and the vertex correction, respectively. They can be expressed in terms of the scalar two- and three-point functions B_0 and C_0 [12], and the derivative $B'_0 = \partial B_0(a,b,c)/\partial a$. Explicit expressions for these functions are given in Appendix A. Using the standard decomposition for the vector and tensor loop integrals [12], we obtain

$$a_{1} = -2 + 4C_{00} + 2(C_{11} + C_{1} + r^{2}C_{22} + r^{2}C_{2})$$

+ 2(1 - $\hat{q}^{2} + r^{2}$)(C₁₂ + C₀ + C₁ + C₂),
 $a_{2} = 2r(C_{1} + C_{2}),$
 $a_{3} = -4(C_{11} + C_{12} + C_{1}) - 4r^{2}(C_{12} + C_{22} + C_{2}),$ (3)

$$a_{wr} = \frac{1}{2} \bigg[2 - B_0(1, \hat{\mu}^2, 1) - B_0(r^2, \hat{\mu}^2, r^2) + (1 - \hat{\mu}^2) \\ \times [B_0(1, \hat{\mu}^2, 1) - B_0(0, \hat{\mu}^2, 1)] + \frac{(r^2 - \hat{\mu}^2)}{r^2} \\ \times [B_0(r^2, \hat{\mu}^2, r^2) - B_0(0, \hat{\mu}^2, r^2)] \\ + 2(2 + \hat{\mu}^2) B_0'(1, \hat{\mu}^2, 1) + 2(2r^2 + \hat{\mu}^2) B_0'(r^2, \hat{\mu}^2, r^2) \bigg].$$
(4)

Defining $f_1 = 1 + r^2 - \hat{q}^2$ and $f_2 = (f_1^2 - 4r^2)$, the coefficient functions take the form

$$C_{00} = \frac{1}{4f_2} [f_2 + \hat{\mu}^2 (f_1 - 2)B_0(1, 1, \hat{\mu}^2) + \hat{\mu}^2 (f_1 - 2r^2) \\ \times B_0(r^2, r^2, \hat{\mu}^2) + (f_2 + 2\hat{q}^2\hat{\mu}^2)B_0(\hat{q}^2, 1, r^2) \\ + 2\hat{\mu}^2 (f_2 + \hat{q}^2\hat{\mu}^2)C_0(1, \hat{q}^2, r^2, \hat{\mu}^2, 1, r^2)],$$
(5)

$$\begin{split} C_{11} &= \frac{r^2}{f_2} + \frac{(f_1 - 2r^2)(1 - r^2)}{2\hat{q}^2 f_2} B_0(0, 1, r^2) + \frac{f_1(\hat{\mu}^2 - 1)}{2f_2} \\ &\times B_0(0, 1, \hat{\mu}^2) + \frac{3r^2\hat{\mu}^2(f_1 - 2r^2)}{f_2^2} B_0(r^2, r^2, \hat{\mu}^2) \\ &+ \frac{2\hat{q}^2\hat{\mu}^2(f_2 + 6\hat{q}^2r^2) - f_2(f_2 + 2\hat{q}^2r^2)}{2\hat{q}^2 f_2^2} B_0(\hat{q}^2, 1, r^2) \\ &+ \frac{\hat{\mu}^2[6r^2(f_1 - 2) - f_2(f_1 + 2)]}{2f_2^2} B_0(1, 1, \hat{\mu}^2) \\ &+ \frac{2\hat{\mu}^2r^2f_2 + \hat{\mu}^4(f_2 + 6\hat{q}^2r^2)}{f_2^2} C_0(1, \hat{q}^2, r^2, \hat{\mu}^2, 1, r^2), \end{split}$$
(6)

$$C_{22} = \frac{1}{f_2} + \frac{(1-r^2)(2-f_1)}{2\hat{q}^2 f_2} B_0(0,1,r^2) + \frac{2\hat{q}^2 \hat{\mu}^2 (f_2 + 6\hat{q}^2) - f_2 (f_2 + 2\hat{q}^2)}{2\hat{q}^2 f_2^2} \times B_0(\hat{q}^2,1,r^2) + \frac{(\hat{\mu}^2 - r^2)f_1}{2r^2 f_2} B_0(0,r^2,\hat{\mu}^2) + \frac{3\hat{\mu}^2 (f_1 - 2)}{f_2^2} B_0(1,1,\hat{\mu}^2) + \frac{\hat{\mu}^2 [6r^2 (f_1 - 2r^2) - f_2 (f_1 + 2r^2)]}{2r^2 f_2^2} B_0(r^2,r^2,\hat{\mu}^2) + \frac{\hat{\mu}^2 [2f_2 + \hat{\mu}^2 (f_2 + 6\hat{q}^2)]}{f_2^2} C_0(1,\hat{q}^2,r^2,\hat{\mu}^2,1,r^2),$$
(7)

$$C_{12} = \frac{-f_1}{2f_2} + \frac{(1-r^2)(2r^2-f_1)}{2\hat{q}^2f_2} B_0(0,1,r^2) + \frac{r^2-\hat{\mu}^2}{f_2} \\ \times B_0(0,r^2,\hat{\mu}^2) + \frac{\hat{\mu}^2[6(f_1-2r^2)-f_2]}{2f_2^2} B_0(1,1,\hat{\mu}^2) \\ + \frac{\hat{\mu}^2[6r^2(f_1-2)-f_2]}{2f_2^2} B_0(r^2,r^2,\hat{\mu}^2) \\ + \frac{f_2(f_2+\hat{q}^2f_1)-2\hat{\mu}^2\hat{q}^2(3\hat{q}^2f_1+f_2)}{2\hat{q}^2f_2^2} B_0(\hat{q}^2,1,r^2) \\ + \frac{\hat{\mu}^2[-f_1f_2-\hat{\mu}^2(3\hat{q}^2f_1+f_2]}{f_2^2} C_0(1,\hat{q}^2,r^2,\hat{\mu}^2,1,r^2),$$
(8)

$$C_{1} = \frac{1}{f_{2}} [f_{1}B_{0}(1,1,\hat{\mu}^{2}) + (2r^{2} - f_{1})B_{0}(\hat{q}^{2},1,r^{2}) - 2r^{2} \\ \times B_{0}(r^{2},r^{2},\hat{\mu}^{2}) + \hat{\mu}^{2}(2r^{2} - f_{1})C_{0}(1,\hat{q}^{2},r^{2},\hat{\mu}^{2},1,r^{2})],$$
(9)

$$C_{2} = \frac{1}{f_{2}} \left[-2B_{0}(1,1,\hat{\mu}^{2}) + (2-f_{1})B_{0}(\hat{q}^{2},1,r^{2}) + f_{1}B_{0}(r^{2},r^{2},\hat{\mu}^{2}) + \hat{\mu}^{2}(2-f_{1})C_{0}(1,\hat{q}^{2},r^{2},\hat{\mu}^{2},1,r^{2}) \right].$$
(10)

The infinite parts of the regularized two-point functions can be shown to cancel in Eq. (1). In the limit $\hat{\mu} \rightarrow 0$, the vertex correction diverges logarithmically. This divergence will be canceled by corresponding divergences in the bremsstrahlung contributions discussed in the next section.

III. BREMSSTRAHLUNG

The bremsstrahlung correction is found in the usual manner, by inserting a real gluon on the *c* and *b* quark lines. The calculation here is complicated by the four-body phase space with two massive final states. We follow the standard procedure of decomposing the four-body phase space into a twoand a three-body phase space by introducing the four-momentum $P = p_c + p_g$. In the rest frame of the *b* quark this decomposition reads

$$dR_4 = dP^2 dR_3(m_b; p_e, p_{\overline{\nu}}, P) dR_2(P; p_c, p_g).$$
(11)

The $O(\alpha_s)$ bremsstrahlung correction to the differential rate is given in terms of dimensionless variables $(\hat{P}^0 = P^0/m_b, \hat{P}^2 = P^2/m_b^2)$ by

$$\frac{d\Gamma_{\rm brems}^{(1)}(\hat{\mu})}{dy} = \alpha_s^{(V)} \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^4} \int d\hat{P}^2 d\hat{P}^0 (\hat{P}^{02} - \hat{P}^2)^{-5/2} \\ \times [2b_1(1 - 2\hat{P}^0 + \hat{P}^2) + b_2(2 - 2\hat{P}^0 - y)y \\ + b_3(1 - y - \hat{P}^2)(2\hat{P}^0 + y - \hat{P}^2 - 1) \\ + b_4(1 - y - \hat{P}^2)y + b_5(2\hat{P}^0 + y - 2) \\ \times (1 - 2\hat{P}^0 - y + \hat{P}^2)].$$
(12)

For convenience, the above rate has been written in terms of the coefficients b_i :

$$b_1 = (\hat{P}^{02} - \hat{P}^2) [\hat{P}^2(c_2 - c_1) + \hat{P}^{02}c_1 + c_3 - \hat{P}^0(c_4 + c_5)],$$
(13)

$$b_{2} = (\hat{P}^{02} - \hat{P}^{2})\hat{P}^{2}c_{1} + 3\hat{P}^{4}c_{2} + (\hat{P}^{2} + 2\hat{P}^{02})$$
$$\times c_{3} - 3\hat{P}^{0}\hat{P}^{2}(c_{4} + c_{5}), \qquad (14)$$

$$b_{3} = (\hat{P}^{02} - \hat{P}^{2})c_{1} + (\hat{P}^{2} + 2\hat{P}^{0})c_{2} + 3c_{3} - 3\hat{P}^{0}(c_{4} + c_{5}),$$
(15)

$$b_{4} = -\hat{P}^{0}(\hat{P}^{02} - \hat{P}^{2})c_{1} - 3\hat{P}^{0}\hat{P}^{2}c_{2} - 3\hat{P}^{0}c_{3} + (\hat{P}^{02} + 2\hat{P}^{2})c_{4} + 3\hat{P}^{02}c_{5}, \qquad (16)$$

$$b_{5} = -\hat{P}^{0}(\hat{P}^{02} - \hat{P}^{2})c_{1} - 3\hat{P}^{0}\hat{P}^{2}c_{2} - 3\hat{P}^{0}c_{3} + 3\hat{P}^{02}c_{4} + (\hat{P}^{02} + 2\hat{P}^{2})c_{5}, \qquad (17)$$

which are linear combinations of

$$c_{1} = \frac{4(v_{+}^{2} - v_{-}^{2})}{h} + \frac{2[(h + \hat{\mu}^{2} - 2\hat{P}^{0})^{2} + (\hat{\mu}^{2} - 2\hat{P}^{0})^{2} + 2\hat{\mu}^{2}(1 + r^{2})]}{h} \ln\left(\frac{2v_{+} - \hat{\mu}^{2}}{2v_{-} - \hat{\mu}^{2}}\right) + \frac{4(2 + \hat{\mu}^{2})(h - 2\hat{P}^{0})(v_{+} - v_{-})}{(2v_{+} - \hat{\mu}^{2})(2v_{-} - \hat{\mu}^{2})} - \frac{8[(z + \hat{\mu}^{2})(\hat{P}^{0} - h) + z\hat{P}^{0}](v_{+} - v_{-})}{h^{2}},$$
(18)

$$c_{2} = \frac{2(v_{+}^{2} - v_{-}^{2})(2-h)}{h} - \frac{[h\hat{\mu}^{2}(3\hat{\mu}^{2} - 4\hat{P}^{0}) + 4\hat{\mu}^{2}(2\hat{P}^{0} - 1) - 16\hat{P}^{02}]}{h} \ln\left(\frac{2v_{+} - \hat{\mu}^{2}}{2v_{-} - \hat{\mu}^{2}}\right) + \frac{2(\hat{\mu}^{4} - 4)(2\hat{P}^{0} - \hat{\mu}^{2})(v_{+} - v_{-})}{(2v_{+} - \hat{\mu}^{2})(2v_{-} - \hat{\mu}^{2})} - \frac{4[h^{2}(\hat{\mu}^{2} - \hat{P}^{0}) + 2\hat{P}^{0}(\hat{\mu}^{2} + 2\hat{P}^{2})](v_{+} - v_{-})}{h^{2}},$$
(19)

$$c_{3} = \frac{\{(\hat{P}^{2} + r^{2})[h^{2} + 2(\hat{\mu}^{2} - 2\hat{P}^{0})(h - 2\hat{P}^{0})] - h\hat{\mu}^{4} + 4r^{2}\hat{P}^{2}\hat{\mu}^{2}\}}{h} \ln\left(\frac{2v_{+} - \hat{\mu}^{2}}{2v_{-} - \hat{\mu}^{2}}\right) \\ + \frac{2(2 + \hat{\mu}^{2})(\hat{\mu}^{2} - r^{2} - \hat{P}^{2})(2\hat{P}^{0} - \hat{\mu}^{2} - h)(v_{+} - v_{-})}{(2v_{+} - \hat{\mu}^{2})(2v_{-} - \hat{\mu}^{2})} - \frac{4[(\hat{P}^{2} + r^{2})(h\hat{P}^{0} - hz + \hat{\mu}^{2}\hat{P}^{0}) + 4r^{2}\hat{P}^{2}\hat{P}^{0}](v_{+} - v_{-})}{h^{2}}, \quad (20)$$

$$c_{4} = \frac{4(P^{0} - P^{2})(v_{+}^{2} - v_{-}^{2})}{h} + \frac{2(2 + \hat{\mu}^{2})(2\hat{P}^{0} - \hat{\mu}^{2})(\hat{\mu}^{2} - 2\hat{P}^{0} + h)(v_{+} - v_{-})}{(2v_{+} - \hat{\mu}^{2})(2v_{-} - \hat{\mu}^{2})} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(2\hat{P}^{2} + \hat{\mu}^{2}) + (\hat{\mu}^{2} - r^{2})h(2\hat{P}^{2} + h - 4\hat{P}^{0}) + h\hat{P}^{2}(\hat{P}^{2} + r^{2} - 8\hat{P}^{0})](v_{+} - v_{-})}{h^{2}} - \frac{2[4\hat{P}^{02}(\hat{P}^{2} - \hat{P}^{0})(h\hat{\mu}^{2} - 2\hat{P}^{0}h + 4\hat{P}^{02}) + h^{2}(1 - \hat{P}^{0} + \hat{\mu}^{2}) - 2\hat{\mu}^{2}(r^{2} + r^{2}\hat{P}^{0} + \hat{P}^{0}) + \hat{\mu}^{4}(1 + r^{2} - 2\hat{P}^{0})]\ln\left(\frac{2v_{+} - \hat{\mu}^{2}}{2v_{-} - \hat{\mu}^{2}}\right)}{h^{2}},$$
(21)

$$c_{5} = \frac{2(\hat{\mu}^{4} - 4)(\hat{P}^{2} + r^{2} - \hat{\mu}^{2})(v_{+} - v_{-})}{(2v_{+} - \hat{\mu}^{2})(2v_{-} - \hat{\mu}^{2})} - \frac{2[\hat{\mu}^{2}h^{2} + (\hat{P}^{2} + r^{2})(2\hat{\mu}^{2} + 2h - h^{2}) + 8r^{2}\hat{P}^{2}](v_{+} - v_{-})}{h^{2}} - \frac{2[\hat{\mu}^{4}h - (\hat{P}^{2} + r^{2})(\hat{\mu}^{2}h + 4\hat{P}^{0}) + 2\hat{P}^{2}\hat{\mu}^{2}]}{h} \ln\left(\frac{2v_{+} - \hat{\mu}^{2}}{2v_{-} - \hat{\mu}^{2}}\right).$$
(22)

In the expressions for the c_i we have put $h = \hat{P}^2 - r^2$ and

$$v_{\pm} = \frac{(\hat{P}^2 + \hat{\mu}^2 - r^2)\hat{P}^0 \pm \sqrt{\hat{P}^{02} - \hat{P}^2}\sqrt{(\hat{P}^2 + \hat{\mu}^2 - r^2)^2 - 4\hat{\mu}^2\hat{P}^2}}{2\hat{P}^2}.$$
(23)

The integrals in Eq. (12) are done numerically between the kinematic limits

$$\frac{(1-y)^2 + \hat{P}^2}{2(1-y)} \leqslant \hat{P}^0 \leqslant \frac{1+\hat{P}^2}{2}, \tag{24}$$

$$(\hat{\mu}+r)^2 \le \hat{P}^2 \le 1-y.$$
 (25)

To improve the numerical stability for small $\hat{\mu}^2$ we found it useful to do the \hat{P}^0 integral with the variable $\ln(\hat{P}^0 - r^2)$. The remaining limits for this four-body decay are

$$0 \le \hat{\mu} \le \sqrt{1-y} - r, \quad 0 \le y \le 1 - r^2.$$
 (26)

IV. THE $\alpha_s^2 \beta_0$ CORRECTION

Combining the corrections from virtual and real gluon radiation, Eqs. (1) and (12), we obtain

$$\frac{d\Gamma^{(1)}(\hat{\mu})}{dy} = \frac{d\Gamma^{(1)}_{\text{virt}}(\hat{\mu})}{dy} + \frac{d\Gamma^{(1)}_{\text{brems}}(\hat{\mu})}{dy}\Theta(\sqrt{1-y} - r - \hat{\mu}).$$
(27)

In the $\hat{\mu} \rightarrow 0$ limit, Eq. (27) yields the one-loop correction to the electron spectrum. We have checked that our expression reproduces the result in [6] in this limit. The $\alpha_s^2 \beta_0$ part of the two-loop correction is related to the one-loop expression calculated with a massive gluon by [7]

FIG. 1. Perturbative QCD corrections to the lepton spectrum. The differential rates are given in units of Γ_0 , with r=0.29, $\overline{\alpha_s}=0.2$, and $n_f=3$.

$$\frac{d\Gamma^{(2)}}{dy} = -\frac{\alpha_s^{(V)}\beta_0}{4\pi} \int_0^\infty \frac{d\hat{\mu}^2}{\hat{\mu}^2} \left(\frac{d\Gamma^{(1)}(\hat{\mu})}{dy} - \frac{1}{1+\hat{\mu}^2}\frac{d\Gamma^{(1)}(0)}{dy}\right).$$
(28)

Note that $\alpha_s^{(V)}$ defined in the V scheme of BLM [10], is related to the more familiar $\overline{\alpha}_s$, defined in the MS scheme by

$$\alpha_s^{(V)} = \overline{\alpha}_s + \frac{5}{3} \frac{\overline{\alpha}_s^2}{4\pi} \beta_0 + \cdots .$$
 (29)

 α_s is evaluated at m_b unless stated otherwise. In the $\overline{\text{MS}}$ scheme the $\overline{\alpha}_s^2 \beta_0$ part of the two-loop correction reads

$$\frac{d\Gamma^{(2)}}{dy} = \frac{5}{3} \frac{\overline{\alpha}_s \beta_0}{4\pi} \frac{d\Gamma^{(1)}(0)}{dy} - \frac{\overline{\alpha}_s \beta_0}{4\pi} \int_0^\infty \frac{d\hat{\mu}^2}{\hat{\mu}^2} \\ \times \left(\frac{d\Gamma^{(1)}(\hat{\mu})}{dy} - \frac{1}{1+\hat{\mu}^2} \frac{d\Gamma^{(1)}(0)}{dy}\right).$$
(30)

The dispersion integral has to be done with some care. We found that using $\ln(\hat{\mu}^2)$ instead of $\hat{\mu}^2$ as the integration variable simplifies the numerical evaluation considerably.

In Fig. 1 we plot the $\overline{\alpha}_s^2 \beta_0$ part of the two-loop correction¹ and for comparison the one-loop correction to the electron spectrum, using r = 0.29, $\overline{\alpha}_s = 0.2$, $n_f = 3$, and dividing by $\Gamma_0 = G_F^2 |V_{cb}|^2 m_b^5 / 192 \pi^3$. Except for electron energies close to the end point, the $\alpha_s^2 \beta_0$ corrections are about half as big as the first order corrections. The perturbation series appears to be controlled but the higher order corrections clearly are not negligible. Integrating over the electron energy we reproduce the result for the correction to the total rate given in Ref. [8].

In [11] the HQET matrix elements Λ, λ_1 were extracted from the lepton spectrum using the experimentally accessible observables

$$R_{1} = \frac{\int_{1.5 \text{ GeV}} dE_{\ell} E_{\ell} d\Gamma/dE_{\ell}}{\int_{1.5 \text{ GeV}} dE_{\ell} d\Gamma/dE_{\ell}}, \quad R_{2} = \frac{\int_{1.7 \text{ GeV}} dE_{\ell} d\Gamma/dE_{\ell}}{\int_{1.5 \text{ GeV}} dE_{\ell} d\Gamma/dE_{\ell}}.$$
(31)

It is straightforward to calculate the $\bar{\alpha}_s^2 \beta_0$ corrections to these quantities. In the spirit of HQET, we use the spinaveraged meson masses $\bar{m}_B = 5.314$ GeV, and $\bar{m}_D =$ 1.975 GeV instead of quark masses. Keeping only two-loop corrections that are proportional to β_0 , and neglecting terms of order $\bar{\alpha}_s^2 \beta_0 \Lambda_{\text{OCD}} / \bar{m}_B$, we find

¹In Appendix B we give a polynomial fit to our results that reproduces the numerical results to better than 1% for mass ratios in the range $0.29 \le r \le 0.37$.



$$R_1 = 1.8059 - 0.035 \frac{\overline{\alpha}_s}{\pi} - 0.082 \frac{\overline{\alpha}_s^2 \beta_0}{\pi^2} + \cdots, \qquad (32)$$

$$R_2 = 0.6581 - 0.039 \frac{\overline{\alpha}_s}{\pi} - 0.098 \frac{\overline{\alpha}_s^2 \beta_0}{\pi^2} + \cdots$$

where the ellipses denote the other contributions including nonperturbative corrections discussed in [5,11]. The BLM scales for these quantities are $\mu_{\text{BLM}}(R_1) = 0.01\overline{m}_B$, and $\mu_{\text{BLM}}(R_2) = 0.007\overline{m}_B$, reflecting the fact that the second order corrections are larger than the first order. This is a result of the almost complete cancellation of the first order perturbative corrections from the denominators and numerators in $R_{1,2}$. In Eq. (32) the BLM scales for the numerators and denominators are separately comparable to the BLM scale for the total rate $\mu_{\text{BLM}} \approx 0.1\overline{m}_B$. Therefore, the very low BLM scales of $R_{1,2}$ do not necessarily indicate badly behaved perturbative series.

In order to demonstrate the impact of the $\overline{\alpha}_s^2 \beta_0$ corrections on the extraction of $\overline{\Lambda}, \lambda_1$, we repeat the analysis of [11], neglecting nonperturbative corrections of order $(\Lambda_{\rm OCD}/m_b)^3$ which may be substantial [5]. Because of the higher order nonperturbative corrections, large theoretical uncertainties have to be assigned to the extracted values of Λ, λ_1 . We find that the central values are moved from $\overline{\Lambda}$ =0.39±0.11 GeV, λ_1 =-0.19±0.10 GeV² to $\overline{\Lambda}$ =0.33 GeV, $\lambda_1 = -0.17$ GeV². The shift in the values of the HQET matrix elements lies well within the 1σ statistical error of the previously extracted values. However, using the values of the HQET matrix elements extracted at a given order in α_s to predict physical observables at the same order in α_s , guarantees that the renormalon ambiguity in Λ and λ_1 will cancel [13,14] if the expansion is continued to sufficiently high orders in α_s . Thus, including the $\overline{\alpha}_s^2 \beta_0$ parts in the determination of $\overline{\Lambda}, \lambda_1$ allows one to calculate the \overline{MS} quark masses consistently at order $\bar{\alpha}_s^2 \beta_0$. To second order in Λ_{QCD}/m_q and to order $\overline{\alpha}_s^2 \beta_0$, we have

$$\overline{m}_{q}(m_{q}) = \left(\overline{m}_{\text{meson}} - \overline{\Lambda} + \frac{\lambda_{1}}{2m_{q}} + \dots\right) \\ \times \left(1 - \frac{4\overline{\alpha}_{s}(m_{q})}{3\pi} - 1.56\frac{\overline{\alpha}_{s}^{2}(m_{q})\beta_{0}}{\pi^{2}} + \dots\right),$$
(33)

where m_q is the *b* or *c* quark pole mass and \overline{m}_{meson} is the corresponding spin-averaged meson mass. With $\overline{\alpha}_s(m_b) = 0.22$, $\overline{\alpha}_s(m_c) = 0.39$ this yields $\overline{m}_b(m_b) = 4.16$ GeV, $\overline{m}_c(m_c) = 0.99$ GeV for the $\overline{\text{MS}}$ quark masses, albeit with large theoretical uncertainties due to the effect of the higher order nonperturbative corrections on the extraction of $\overline{\Lambda}, \lambda_1$ [5]. The value of $\overline{m}_b(m_b) = 4.17 \pm 0.06$ GeV and $\overline{m}_b(m_b) = 4.0 \pm 0.01$ GeV [15]. The weak mixing angle $|V_{cb}|$ can be determined by comparing the theoretical prediction for the total rate with experimental measurements. Including all corrections discussed in [11] we find, at order $\alpha_s^2 \beta_0$,

$$|V_{cb}| = 0.043 \left(\frac{\mathcal{B}(B \to X_c \ell \bar{\nu}_{\ell})}{0.105} \frac{1.55 \text{ ps}}{\tau_B} \right)^{1/2}$$
. (34)

V. CONCLUSIONS

We have calculated the $O(\alpha_s^2 \beta_0)$ corrections to the electron spectrum in $b \rightarrow c \ell \overline{\nu}_{\ell}$ decays which turn out to be rather large, about 50% of the one-loop corrections. These corrections can be included in the extraction of the HQET matrix elements $\overline{\Lambda}, \lambda_1$. We obtain $\overline{\Lambda} = 0.33$ GeV and $\lambda_1 = -0.17$ GeV², both somewhat lower than the values extracted at $O(\alpha_s)$. Using these values and including $O(\alpha_s^2 \beta_0)$ corrections we obtain $\overline{m}_b(m_b) = 4.16$ GeV, $\overline{m}_c(m_c) = 0.99$ GeV for the $\overline{\text{MS}}$ quark masses and $|V_{cb}| = 0.043 [\mathcal{B}(B \rightarrow X_c \ell \overline{\nu}_c)/0.105 \times 1.55 \text{ ps}/\tau_B]^{1/2}$. These results have large theoretical uncertainties due to the effect of nonperturbative corrections of order $(\Lambda_{\text{QCD}}/m_b)^3$ on the extraction of $\overline{\Lambda}, \lambda_1$.

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APPENDIX A: SCALAR TWO- AND THREE-POINT FUNCTIONS

Here, we list expressions for the scalar functions B_0 and C_0 [12] needed for the calculation in Sec. II. With the cut for logarithms along the negative real axis, we have

$$B_0(a,b,c) = \frac{2}{\epsilon} - \ln\left(\frac{\mu^2}{4\pi\Lambda^2 e^{-\gamma}}\right) - \int_0^1 dx \ln\left(\frac{ax^2 - x(a+b-c) + b}{\mu^2}\right)$$
(A1)

$$= \frac{2}{\epsilon} - \ln\left(\frac{\mu^2}{4\pi\Lambda^2 e^{-\gamma}}\right) + 2 - \ln\left(\frac{c}{\mu^2}\right) + x_+ \ln\left(\frac{x_+ - 1}{x_+}\right) + x_- \ln\left(\frac{x_- - 1}{x_-}\right),$$
(A2)

where

$$x_{\pm} = \frac{a+b-c \pm \sqrt{(a+b-c)^2 - 4ab}}{2a};$$
(A3)

$$C_{0}(1,\hat{q}^{2},r^{2},\hat{\mu}^{2},1,r^{2}) = \frac{1}{1+r^{2}-\hat{q}^{2}-2\alpha r^{2}} \Big\{ \operatorname{Li}_{2} \Big(\frac{z_{1}}{z_{1}-z_{4}}\Big) - \operatorname{Li}_{2} \Big(\frac{z_{1}-1}{z_{1}-z_{4}}\Big) + \operatorname{Li}_{2} \Big(\frac{z_{1}}{z_{1}-z_{5}}\Big) - \operatorname{Li}_{2} \Big(\frac{z_{1}-1}{z_{1}-z_{5}}\Big) - \operatorname{Li}_{2} \Big(\frac{z_{2}}{z_{2}-z_{6}}\Big) \\ + \operatorname{Li}_{2} \Big(\frac{z_{2}-1}{z_{2}-z_{6}}\Big) - \operatorname{Li}_{2} \Big(\frac{z_{2}}{z_{2}-z_{7}}\Big) + \operatorname{Li}_{2} \Big(\frac{z_{2}-1}{z_{2}-z_{7}}\Big) + \operatorname{Li}_{2} \Big(\frac{z_{3}}{z_{3}-z_{8}}\Big) - \operatorname{Li}_{2} \Big(\frac{z_{3}-1}{z_{3}-z_{8}}\Big) \\ + \operatorname{Li}_{2} \Big(\frac{z_{3}}{z_{3}-z_{9}}\Big) - \operatorname{Li}_{2} \Big(\frac{z_{3}-1}{z_{3}-z_{9}}\Big) \Big\},$$
(A4)

where

$$\alpha = \frac{(\hat{q}^2 - 1 - r^2) + \sqrt{(\hat{q}^2 - 1 - r^2)^2 - 4r^2}}{-2r^2},$$

$$z_0 = \frac{\hat{\mu}^2 - 2 - \alpha(\hat{q}^2 - 1 - r^2 + \hat{\mu}^2)}{1 + r^2 - \hat{q}^2 - 2r^2\alpha},$$
(A5)

$$z_1 = z_0 + \alpha, \quad z_2 = \frac{z_0}{(1 - \alpha)}, \quad z_3 = -\frac{z_0}{\alpha},$$

$$z_{4,5} = \frac{\hat{\mu}^2 \pm \sqrt{\hat{\mu}^4 - 4\hat{\mu}^2 r^2}}{2r^2},$$
 (A6)

$$z_{6,7} = \frac{\hat{q}^2 + 1 - r^2 \pm \sqrt{(\hat{q}^2 + 1 - r^2)^2 - 4\hat{q}^2 + i\epsilon}}{2\hat{q}^2},$$

$$z_{8,9} = \frac{2 - \hat{\mu}^2 \pm \sqrt{(2 - \hat{\mu}^2)^2 - 4}}{2}.$$
(A7)

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The dilogarithms here are defined as

 $z_{8,9} = -$

$$\mathrm{Li}_{2}(z) = -\int_{0}^{1} dt \, \frac{\ln(1-zt)}{t}.$$
 (A8)

TABLE I. Coefficients $A_{n,m}$ for the interpolation function $f^{(2)}(y,r)$ given in Appendix B.

$\overline{A_{n,m}}$	m = 0	1	2
$\overline{n=1}$	-4.9	19.3	-20.4
2	-2.72	12.4	-14.7
3	3.55	-12.4	11.5
4	2.21	-10.1	11.9
5	1.97	-9.59	11.9
6	1.11	-5.08	6.06
7	-0.159	0.448	-0.274
8	-0.336	1.44	-1.62
9	-0.319	1.47	-1.76
10	-0.174	0.818	-0.994
11	-0.0881	0.417	-0.509
12	-0.0507	0.247	-0.309

APPENDIX B: INTERPOLATING FUNCTION

In this appendix we give an interpolation scheme that makes it easy to reproduce our $\overline{\alpha}_s^2 \beta_0$ corrections to the lepton spectrum for $b \rightarrow c \ell \bar{\nu}_{\ell}$ decays. The interpolation is done as a function of y and r with all other parameters left explicit:

$$\frac{d\Gamma}{dy} = \Gamma_0 \left(f^{(0)}(y,r) + \frac{\overline{\alpha}_s}{\pi} f^{(1)}(y,r) + \frac{\overline{\alpha}_s^2 \beta_0}{\pi^2} f^{(2)}(y,r) \right).$$
(B1)

The well-known tree-level result is given by the exact equation

$$f^{(0)}(y,r) = \frac{2y^2(y+r^2-1)^2(-3-3r^2+5y+r^2y-2y^2)}{(y-1)^3},$$
(B2)

while the first order function $f^{(1)}$ can be obtained from Ref. [6]. The interpolating function $f^{(2)}$ is found by fitting Chebyshev polynomials T_n to the energy dependence, and quadratic polynomials to the mass ratio r. To improve the accuracy of the fit near the end point of the lepton spectrum the expansion is given in terms of the variable

$$y' = \frac{\ln[(1-y)^2/r^2]}{\ln(r^2)}.$$
 (B3)

Our fit is accurate to better than 1% for the region

$$0.09 \le y \le 0.99(1 - r^2), \quad 0.29 \le r \le 0.37.$$
 (B4)

The second order function is given by

$$f^{(2)}(y,r) = \sum_{n=1}^{12} \left(\sum_{m=0}^{2} A_{n,m} r^{m} \right) T_{n-1}(y'), \qquad (B5)$$

where $T_n(y') = \cos(n \arccos y')$. The 36 coefficients $A_{i,j}$ are given in Table I. Both the coefficients as well as FORTRAN code evaluating $d\Gamma^{(2)}/dy$ and $d\Gamma^{(1)}/dy$ from the expressions in Secs. II-IV are available from the authors.²

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