Instanton calculus and nonperturbative relations in $N=2$ supersymmetric gauge theories

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Using instanton calculus we check, in the weak coupling region, the nonperturbative relation $\langle \mathrm{Tr}\phi^2 \rangle = i\pi[\mathcal{F}-(a/2)\partial\mathcal{F}/\partial a]$ obtained for a $N=2$ globally supersymmetric gauge theory. Our computations are performed for instantons of winding number k, up to $k=2$, and turn out to agree with previous nonperturbative results. [S0556-2821(97)01502-6]

I. INTRODUCTION

In a recent work [1], Seiberg and Witten have managed to compute the quantum moduli space and the Wilsonian effective action for the Yang-Mills theory with global $N=2$ supersymmetry ($N=2$ SUSY from now on). This achievement has been possible by using judiciously a certain number of educated guesses for the behavior of the moduli space of vacua of the theory and by exploiting the unique properties of the $N=2$ SUSY. In fact, the Wilsonian effective action of this theory, after having used the Higgs mechanism, is completely determined once a certain prepotential $\mathcal{F}$ is known [2]. In turn, this prepotential $\mathcal{F}$ is determined if its global structure is known or postulated. This global structure is given by the monodromies around the singular points of the prepotential $\mathcal{F}$: the group generated by these monodromies is a subgroup of SL(2,$\mathbb{Z}$).

In the electric (or Higgs) phase of the theory the form of the prepotential is known since, due to nonrenormalization theorems, only the one-loop term contributes at the perturbative level. Moreover, nonperturbative corrections due to instantons must be considered, leading to the final expression

$$\mathcal{F}(A) = \frac{1}{i\pi}\left(\frac{A^2}{2}\ln\frac{2A^2}{\Lambda^2} - \sum_{k=1}^{\infty} \mathcal{F}_k \frac{\Lambda^{4k}}{A^{4k-2}} \right). \quad (1.1)$$

In Eq. (1.1) Λ is the renormalization-group-invariant scale and A is a chiral superfield whose lowest component squared is $a^2 \equiv -2u$, i.e., it is the gauge-invariant coordinate of the moduli space of vacua, when the gauge group is SU(2) (which will be our choice from now on), at least for large u and a. The coefficients $\mathcal{F}_k$ give the nonperturbative contributions due to instantons. A formidable check of the assumptions made in [1], concerning the symmetries of the moduli space, is thus given by matching the coefficients $\mathcal{F}_k$ against those obtained by instanton calculus. This check must be performed in the weak coupling region in which instanton calculus can be reliably performed. Some work has already been done along these lines. Nonperturbative contributions induced by instantons can in fact be seen, in the

framework of perturbation theory, as effective four-fermion vertices to be added to the tree-level Lagrangian. The computation of these effective vertices has already been performed for the case of instantons of winding number one [3–5] and two [6].

The approach presented in this paper is somewhat different in that we will check the nonperturbative relation

$$\langle \mathrm{Tr}\phi^2 \rangle = i\pi\left(\mathcal{F} - \frac{a}{2}\frac{\partial\mathcal{F}}{\partial a} \right) \quad (1.2)$$

found in [7]. Expanding the left-hand side in Eq. (1.2) as

$$\langle \mathrm{Tr}\phi^2(a) \rangle = -\frac{1}{2}a^2 - \sum_{k=1}^{\infty} \mathcal{G}_k \frac{\Lambda^{4k}}{a^{4k-2}} \quad (1.3)$$

and substituting Eq. (1.1) in Eq. (1.2) we find $\mathcal{G}_k = 2k\mathcal{F}_k$ for a comparison with the results of [8]. The $\mathcal{G}_k$'s can also be straightforwardly checked against the results of the recursion relation found in [7].[1]

Supersymmetric instanton calculus was developed in two distinct ways [9–11] to study supersymmetry (SUSY) breaking. The main difference between these two approaches consists in giving or not giving an expectation value to the scalar (Higgs) field of the $N=2$ multiplet. Given the check we want to perform the right choice is to follow [9,10,12] where such an expectation value for the scalar field is present.

This is the plan of the paper: in Sec. II we shall briefly discuss the basic ingredients of the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instantons which will be useful later on. In Sec. III we introduce the semiclassical expansion of Green functions in SUSY gauge theories, and check the relation (1.2) against a $k=1$ computation. In Sec. IV we extend our considerations to a background of Pontryagin index $k=2$.

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[1]The reader should pay attention to different normalizations. The conventions of this letter for the $\mathcal{F}_k$'s, which have an opposite sign with respect to those of [6], are connected to [7,8] as $\mathcal{F}_k = -2^{6k-2}\mathcal{F}_k^{KLT} = -i\pi 2^{2k}\mathcal{F}_k^M$.

II. A BRIEF REVIEW OF THE ADHM FORMALISM

Before describing the actual computation we need to briefly discuss the ADHM construction and collect some useful formulas.

As it is well known, self-dual SU(2) connections on S^4 can be put into one-to-one correspondence with holomorphic vector bundles of rank 2 over $\mathbb{C}P^3$ admitting a reduction of the structure group to its compact real form. The ADHM construction [13,14] gives all these holomorphic bundles and consequently all SU(2) connections on S^4. The construction is purely algebraic and we find it more convenient to use quaternionic notation. The points x of the one-dimensional quaternionic space $\mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$ can be conveniently represented in the form $x = x^\mu \sigma_\mu$, with $\sigma_\mu = (\mathbb{1}, i\sigma_c)$, $c = 1,2,3$. The σ_c's are the usual Pauli matrices. The conjugate of x is $x^\dagger = x^\mu \sigma_\mu^\dagger$. A quaternion is said to be real if it is proportional to $\mathbb{1}$ and imaginary if it has vanishing real part.

The prescription to find an instanton of winding number k is the following: introduce a $(k+1) \times k$ quaternionic matrix linear in x,

$$\Delta = a + bx. \tag{2.1}$$

The (anti-Hermitian) gauge connection is then written in the form

$$A_\mu^{\text{cl}} = U^\dagger \partial_\mu U, \tag{2.2}$$

where U is a $(k+1) \times 1$ matrix of quaternions providing an orthonormal frame of $\text{Ker}\Delta^\dagger$. In formulas

$$\Delta^\dagger U = 0, \tag{2.3}$$

$$U^\dagger U = \mathbb{1}_2, \tag{2.4}$$

where $\mathbb{1}_2$ is the two-dimensional identity matrix. The constraint (2.4) ensures that A_μ^{cl} is an element of the Lie algebra of the SU(2) gauge group. The condition of self-duality on the field strength of Eq. (2.2) is imposed by restricting the matrix Δ to obey

$$\Delta^\dagger \Delta = f^{-1} \otimes \mathbb{1}_2, \tag{2.5}$$

with f an invertible Hermitian $k \times k$ matrix (of real numbers). In addition to the gauge freedom (right multiplication of U by a unitary quaternion), we have the freedom to perform the transformations

$$\Delta \rightarrow Q\Delta R, \tag{2.6}$$

with $Q \in \text{Sp}(k+1), R \in \text{GL}(k, \mathbb{R})$, which leave Eq. (2.2) invariant.

These symmetries can be used to simplify the expressions of a and b. Exploiting this fact, in the following we will choose the matrix b to be

$$b = - \begin{pmatrix} 0_{1 \times k} \\ \mathbb{1}_{k \times k} \end{pmatrix}. \tag{2.7}$$

From Eq. (2.2), the field strength of the gauge field can be computed and it is

$$F_{\mu\nu} = 2(U^\dagger b f \sigma_{\mu\nu} b^\dagger U), \tag{2.8}$$

where $\sigma_{\mu\nu} = i\eta_{\mu\nu}^a \sigma^a$, with $\eta_{\mu\nu}^a$ the 't Hooft symbols. Using Eq. (2.8) we also compute

$$\text{Tr}(F_{\mu\nu}F_{\mu\nu}) = 2\Box\text{Tr}[b^\dagger(1+P)bf], \tag{2.9}$$

where

$$P = UU^\dagger = 1 - \Delta f \Delta^\dagger \tag{2.10}$$

is the projector on the kernel of $\Delta^\dagger$. Equation (2.9) will turn out to be important in the following.

The bosonic zero modes Z_μ of the gauge-fixed second order differential operator

$$M_{\mu\nu} = -D^2(A^{\text{cl}})\delta_{\mu\nu} - 2F_{\mu\nu}^{\text{cl}}, \tag{2.11}$$

which describes the quantum fluctuations of the gauge fields, can be found by noting that the transverse fluctuations of a self-dual configuration must satisfy the relations

$$*(D_{[\mu}Z_{\nu]}) = D_{[\mu}Z_{\nu]}, \quad D_\mu Z_\mu = 0. \tag{2.12}$$

This allows one to write Z_μ in terms of the quantities appearing in Eq. (2.1), as [15]

$$Z_\mu = U^\dagger C \bar{\sigma}_\mu f b^\dagger U - U^\dagger b f \sigma_\mu C^\dagger U. \tag{2.13}$$

For Eq. (2.13) to satisfy Eq. (2.12), the $(k+1) \times k$ matrix C must obey

$$\Delta^\dagger C = (\Delta^\dagger C)^T, \tag{2.14}$$

where the superscript T stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). In our case, the number of independent Z_μ is $8k$ (the dimension of the moduli space of the instanton). C has $k(k+1)$ quaternionic elements, which are subject to the $4k(k-1)$ constraint (2.14). The number of C's satisfying Eq. (2.14) is thus $8k$ as desired. This is the reason why in the following we will sometimes attach a subscript $r = 1, \ldots, 8k$ to the zero modes.

Fermionic zero modes are easily deduced from Eq. (2.13) by remarking that, due to $N = 2$ SUSY,

$$\lambda_{\beta\dot{A}}^{(r)} = \sigma_{\beta\dot{A}}^\mu Z_\mu^{(r)}, \tag{2.15}$$

where $\dot{A} = 1,2$ labels the two SUSY charges. Furthermore, the superposition of the $Z_\mu^{(r)}$ was computed in [15] to be

$$(Z^{(r)}, Z^{(s)}) = \frac{4\pi^2}{g^2}\text{Tr}[C_r^\dagger(1+P_\infty)C_s] \equiv \frac{4\pi^2}{g^2}(C_r, C_s), \tag{2.16}$$

where $P_\infty = 1 - bb^\dagger$ is the projector P evaluated in the limit $|x| \rightarrow \infty$. The superpositions of the bosonic zero modes are then tied to the Jacobian which yields the integration measure for the bosonic collective coordinates. This integration measure is easily written once the variations of the bosonic fields with respect to the instanton moduli are known. But the zero mode (2.13) is also transverse to allow the factorization of the infinite volume of the gauge group via the introduction of the Faddeev-Popov determinant [16]. It is thus useful to separate, in the expression of the bosonic zero modes, the variation with respect to the instanton moduli

from the gauge part needed to make it transverse [17,18]. In the ADHM formalism this yields to the formula for the variation of the gauge connection [19]

$$\delta_r A_\mu = U^\dagger(\delta_r\Delta)\bar{\sigma}_\mu f b^\dagger U - U^\dagger b f \sigma_\mu (\delta_r\Delta)^\dagger U + [D_\mu, \delta g], \quad (2.17)$$

where δ_r, $r=1,\ldots,8k$ stands for the variations with respect to the instanton moduli, and δg, satisfying $\delta g + \delta g^\dagger = 0$, is an arbitrary infinitesimal gauge transformation. The unconstrained variations $\delta_r\Delta$, which give the integration over the collective coordinates, cannot be easily traded with Eq. (2.16) since the C's appearing in that formula are constrained by Eq. (2.14). The complete relation between the $\delta_r\Delta$ and C's is given by

$$(C_r, C_s) = (\delta_r\Delta, \delta_s\Delta) + 2K_{rji}M_{ij,lm}K_{slm}. \quad (2.18)$$

The explicit expression of the matrices K, M, which parametrize the freedom to transform the ADHM data as in Eq. (2.6), can be found in [15].

III. THE $k=1$ SEMICLASSICAL COMPUTATION

We now briefly review the strategy to perform semiclassical computations in supersymmetric gauge theories, in the context of the constrained instanton method.

The classical potential for the complex scalar field ϕ of an $N=2$ SUSY gauge theory

$$V_D = \frac{g^2}{2}(\epsilon^{abc}\phi^b\phi^{\dagger c})^2 \quad (3.1)$$

has flat directions when ϕ is an SU(2) gauge transform of $\phi = a^c\sigma_c/2i$, where $a^c = a\delta^{c3}$ and a is a complex number. Following 't Hooft [17], we shall then expand the action functional around a properly chosen field configuration, which is the solution of the equations

$$D_\mu(A)F_{\mu\nu} = 0, \quad (3.2)$$

$$D^2(A)\phi_{\text{cl}} = 0, \ \lim_{|x|\to\infty}\phi_{\text{cl}} \equiv \phi_\infty = a\frac{\sigma_3}{2i}. \quad (3.3)$$

The first equation admits instantonic solutions. When $a=0$, Eq. (3.3) admits the trivial solution $\phi=0$ only. When $a\neq 0$, we shall decompose the fields ϕ, $\phi^\dagger$ as

$$\phi = \phi_{\text{cl}} + \phi_Q, \quad \phi^\dagger = (\phi_{\text{cl}})^\dagger + \phi_Q^\dagger, \quad (3.4)$$

and integrate over the quantum fluctuations ϕ_Q, $\phi_Q^\dagger$

The integration over bosonic zero modes can be traded with an integration over collective coordinates, at the cost of introducing the corresponding Jacobian. The existence of fermion zero modes is the way by which Ward identities, related to the group of chiral symmetries of the theory, come into play. When $a=0$, the anomalous $U_R(1)$ symmetry

$$\lambda \to e^{i\alpha}\lambda, \quad \phi \to e^{2i\alpha}\phi, \quad (3.5)$$

and gauge invariance allow a nonzero result for the Green functions with n insertions of the gauge-invariant quantity $(\phi^a\phi^a)(x)$ only when $n=2k$. These correlators possess the

right operator insertions needed to saturate the integration over the Grassmann parameters and, due to supersymmetry, they are also position independent. On the other hand, when $a\neq 0$, the correlator $\langle\phi^a\phi^a\rangle$ has a complete expansion in terms of instanton contributions, as in Eq. (1.3). The $k=1$ term has been computed in [20] in the framework of the constrained superinstanton formalism of [10]. In the following we will use the component formalism, which we find to be simpler, to compute the $k=1,2$ coefficients of the instanton expansion.

Fermion zero modes are found by solving the equation

$$D_\mu\bar{\sigma}_\mu^{\dot{\alpha}\beta}\lambda_{\beta\dot{A}} = 0. \quad (3.6)$$

For instantons of winding number $k=1$ the whole set of solution of this equation is obtained via SUSY and superconformal transformation, which yield

$$\lambda_{\alpha\dot{A}}^a = \frac{1}{2}F_{\mu\nu}^a(\sigma_{\mu\nu})_\alpha{}^\beta\zeta_{\beta\dot{A}}, \quad (3.7)$$

where $\zeta = \xi + (x-x_0)_\mu\sigma_\mu\bar{\eta}/\sqrt{2}\rho$, ξ, $\bar{\eta}$ being two arbitrary quaternions of Grassmann numbers. It is instructive, and useful for the computations to be performed in the next section, to deduce this result using the ADHM expressions (2.13)–(2.15). For $k=1$ the constraint (2.14) is always satisfied since $\Delta^\dagger C$ is a single quaternion. Given a matrix

$$\Delta = \begin{pmatrix} v \\ x_0 - x \end{pmatrix}, \quad (3.8)$$

and choosing C to be

$$C = \begin{pmatrix} \sqrt{2}C_0 \\ 2C_1 \end{pmatrix}, \quad (3.9)$$

with C_0, C_1 two arbitrary quaternions, we can substitute Eq. (3.9) in Eq. (2.13) to find, using Eq. (2.8),

$$Z_\mu = 2F_{\mu\nu}B_\nu, \quad (3.10)$$

where

$$B = B_\nu\sigma^\nu = C_1 + (x-x_0)\frac{\bar{v}}{v^2}\frac{C_0}{\sqrt{2}}. \quad (3.11)$$

Interpreting C_0, C_1 as Grassmann variables and using Eq. (2.15) we obtain Eq. (3.7) with B replacing ζ.

The correct fermionic integration measure is given by the inverse of the determinant of the matrix whose entries are the scalar products of the fermionic zero-mode eigenfunctions. This scalar product is induced by the kinetic terms in the action and for arbitrary SU(2)-valued spinors f,g is

$$(f,g) \equiv \sum_{a=1}^3 \int d^4x (f_\alpha^a)^*\bar{\sigma}_0^{\dot{\alpha}\beta}g_\beta^a. \quad (3.12)$$

The fermionic measure now reads

$$d^4\xi d^4\bar{\eta}\left(\frac{g^2}{32\pi^2}\right)^4, \quad (3.13)$$

where $d^4\xi d^4\bar{\eta} \equiv d^2\xi_{\mathrm{i}} d^2\xi_{\dot{2}} d^2\bar{\eta}_{\dot{i}} d^2\bar{\eta}_{\dot{2}}$. Once auxiliary fields are eliminated, the action is

$$S = S_{\mathrm{G}} + S_{\mathrm{H}} + S_{\mathrm{F}} + S_{\mathrm{Y}} + S_{\mathrm{D}}. \tag{3.14}$$

S_{G} is the usual gauge field action, $S_{\mathrm{F}}[\lambda,\bar{\lambda},A] = \int d^4 x \bar{\lambda}^{\dot{A}a}[\rlap{/}{D}(A)\lambda_{\dot{A}}]^a$ and $S_{\mathrm{H}}[\phi,\phi^\dagger,A] = \int d^4 x(D\phi)^{\dagger a}(D\phi)^a$ are the kinetic terms for the Fermi and Bose fields minimally coupled to the gauge field A_μ. The Yukawa interactions are given by

$$S_{\mathrm{Y}}[\phi,\phi^\dagger,\lambda,\bar{\lambda}] = \sqrt{2} g \epsilon^{abc} \int d^4 x \phi^{a\dagger}(\lambda_{\dot{i}}^b \lambda_{\dot{2}}^c) + \mathrm{H.c.} \tag{3.15}$$

and finally, $S_{\mathrm{D}} = \int d^4 x V_D$ comes from the potential term (3.1) for the complex scalar field ϕ.

The evaluation of the correlator $\langle \phi^a \phi^a \rangle$ in the semiclassical approximation around an instantonic background of winding number $k = 1$ yields

$$\langle \phi^a \phi^a \rangle = \int d^4 x_0 d\rho \left(\frac{2^{10}\pi^6\rho^3}{g^8} \right)$$

$$\times \exp\left(-\frac{8\pi^2}{g^2} - 4\pi^2|a|^2\rho^2 \right)$$

$$\times \int [\delta Q \, \delta\lambda] \delta\bar{\lambda} \delta\phi_Q^\dagger \delta\phi_Q \delta\bar{c} \delta c$$

$$\times \exp\left[-S_{\mathrm{H}}[\phi_Q,\phi_Q^\dagger,A^{\mathrm{cl}}] - S_F[\lambda,\bar{\lambda},A^{\mathrm{cl}}] \right.$$

$$\left. -\frac{1}{2}\int d^4 x Q_\mu M_{\mu\nu} Q_\nu - \int d^4 x \bar{c} D^2(A^{\mathrm{cl}})c \right]$$

$$\times \int d^4\xi d^4\bar{\eta} \left(\frac{g^2}{32\pi^2} \right)^4$$

$$\times \exp\{ -S_{\mathrm{Y}}[\phi_{\mathrm{cl}}+\phi_Q,(\phi_{\mathrm{cl}})^\dagger+\phi_Q^\dagger,\lambda^{(0)},\bar{\lambda}=0] \}$$

$$\times (\phi_{\mathrm{cl}}+\phi_Q)^a (\phi_{\mathrm{cl}}+\phi_Q)^a(x). \tag{3.16}$$

Let us now explain where the different terms in Eq. (3.16) come from.

(1) $d^4 x_0 d\rho (2^{10}\pi^6\rho^3/g^8)$ is the bosonic measure [17,18] after the integration over $SU(2)/\mathbb{Z}_2$ global rotations in color space has been performed. x_0 and ρ are the center and the size of the instanton [see Eq. (3.8), with $\rho \equiv |v|$].

(2) $S_{\mathrm{H}}[\phi_{\mathrm{cl}},(\phi_{\mathrm{cl}})^\dagger,A^{\mathrm{cl}}] = 4\pi^2|a|^2\rho^2$, is the contribution of the classical Higgs action, and has been computed by 't Hooft [17].

(3) The second and third lines include the quadratic approximation of the different kinetic operators for the quantum fluctuation of the fields and the symbol $[\delta\lambda \, \delta Q]$ denotes integration over nonzero modes. $\bar{c}$ and c are the usual ghost fields, $\int d^4 x \bar{c} D^2(A^{\mathrm{cl}})c$ being the corresponding term in the action.

(4) $S_{\mathrm{Y}}[\phi,\phi^\dagger,\lambda^{(0)},\bar{\lambda}=0]$ is the Yukawa action calculated with the complete expansion of the fermionic fields replaced by their projection over the zero-mode subspace. According to the index theorem for the Dirac operator in the back-

ground of a self-dual gauge field configuration, we have only zero modes of one chirality, so that this term reduces to $\sqrt{2} g \epsilon^{abc} \int \phi^{a\dagger}(\lambda_{\dot{i}}^{(0)b} \lambda_{\dot{2}}^{(0)c})$.

After the integration over ϕ, $\phi^\dagger$ and the nonzero modes, the ϕ_Q insertions get replaced by ϕ_{inh}, where

$$\phi_{\mathrm{inh}}^a = \sqrt{2} g \epsilon^{bdc}[(D^2)^{-1}]^{ab}(\lambda_{\dot{i}}^{(0)d} \lambda_{\dot{2}}^{(0)c}), \tag{3.17}$$

and the determinants of the various kinetic operators cancel against each other [21]. The right-hand side (RHS) of Eq. (3.16) now reads

$$\Lambda^4 \int d^4 x_0 d\rho \left(\frac{2^{10}\pi^6\rho^3}{g^8} \right) e^{-4\pi^2|a|^2\rho^2}$$

$$\times \int d^4\xi d^4\bar{\eta} \left(\frac{g^2}{32\pi^2} \right)^4 \exp\left[-\sqrt{2} g \epsilon^{abc} \int \phi_{\mathrm{cl}}^\dagger(\lambda_{\dot{i}}^{(0)b} \lambda_{\dot{2}}^{(0)c}) \right]$$

$$\times (\phi_{\mathrm{cl}}+\phi_{\mathrm{inh}})^a (\phi_{\mathrm{cl}}+\phi_{\mathrm{inh}})^a(x), \tag{3.18}$$

where $\Lambda^4 = \mu^{8-(1/2)(4+4)} e^{-8\pi^2/g^2}$ is the (one-loop) $N=2$ SUSY renormalization-group-invariant scale with gauge group SU(2). μ comes from the Pauli-Villars regularization of the determinants and the exponent is $b_1 k = (n_B - n_F/2)$ where n_B, n_F, b_1 are the number of bosonic, fermionic zero modes, and the first coefficient of the β function of the theory.

A straightforward calculation shows that [22]

$$S_{\mathrm{Y}}[\phi_{\mathrm{cl}},\phi_{\mathrm{cl}}^\dagger,\lambda^{(0)},\bar{\lambda}=0] = \frac{(a^c)^* g}{\sqrt{2}} (\bar{\eta}_{\dot{i}} \sigma^c \bar{\eta}_{\dot{2}}) \left(\frac{g^2}{32\pi^2} \right)^{-1}. \tag{3.19}$$

Moreover, it is also easy to convince oneself that Eq. (3.17) is solved by

$$\phi_{\mathrm{inh}}^a = \sqrt{2}(\zeta_{\dot{i}} \lambda_{\dot{2}}^a), \tag{3.20}$$

as it can be checked by substituting in

$$[D^2]^{ab} \phi_{\mathrm{inh}}^b = \sqrt{2} g \epsilon^{abc}(\lambda_{\dot{i}}^{(0)b} \lambda_{\dot{2}}^{(0)c}). \tag{3.21}$$

The Yukawa action does *not* contain the Grassmann parameters of the zero modes coming from SUSY transformations. As a consequence the only nonzero contributions are obtained by picking out the terms in the ϕ_{inh} insertions which contain the SUSY solutions of the Dirac equation. We thus completely disregard the ϕ_{cl} pieces. Since

$$\phi_{\mathrm{inh}}^a \phi_{\mathrm{inh}}^a = -\zeta_{\dot{i}}^2(\lambda_{\dot{2}}^a \lambda_{\dot{2}}^a) = -\zeta_{\dot{i}}^2 \zeta_{\dot{2}}^2(F_{\mu\nu}^a F_{\mu\nu}^a), \tag{3.22}$$

this amounts to say

$$(\phi_{\mathrm{cl}}+\phi_{\mathrm{inh}})^a (\phi_{\mathrm{cl}}+\phi_{\mathrm{inh}})^a \to -\xi_{\dot{i}}^2 \xi_{\dot{2}}^2(F_{\mu\nu}^a F_{\mu\nu}^a). \tag{3.23}$$

The integration over non-SUSY zero modes is then dealt with by performing the integration

$$\int d^4\bar\eta \left(\frac{g^2}{32\pi^2}\right)^2 \exp\left[\frac{(a^c)^* g}{\sqrt2}(\bar\eta_{\dot1}\sigma^c\bar\eta_{\dot2})\left(\frac{g^2}{32\pi^2}\right)^{-1}\right]$$

$$=\frac{g^2}{2}(a^*)^2. \tag{3.24}$$

Equation (3.16) now becomes

$$\langle\phi^a\phi^a\rangle=\Lambda^4\int d^4x_0 d\rho\left(\frac{2^{10}\pi^6\rho^3}{g^8}\right)$$

$$\times e^{-4\pi^2|a|^2\rho^2}(F^a_{\mu\nu}F^a_{\mu\nu})\frac{g^2}{2}(a^*)^2$$

$$\times\int d^4\xi\left(\frac{g^2}{32\pi^2}\right)^2\xi_1^2\xi_2^2. \tag{3.25}$$

A simple computation using the explicit form of the $k=1$ gauge connection shows that $F^a_{\mu\nu}F^a_{\mu\nu}$ is a function of the difference $x-x_0$. We can then immediately integrate over x_0 remembering that $\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}=32\pi^2/g^2$. The remaining integrations over ξ and ρ in Eq. (3.25) are trivial and yield [3]

$$\langle\phi^a\phi^a\rangle=\frac{2}{g^4}\frac{\Lambda^4}{a^2}. \tag{3.26}$$

This result agrees with the coefficient $\mathcal{G}_1$ found in [7,8].

IV. THE $k=2$ COMPUTATION

We now come to the $k=2$ computation. There are several modifications to take into account with respect to the $k=1$ case but the general strategy is unchanged. We start by giving the form of the matrix Δ of Eq. (2.1),

$$\Delta=\begin{pmatrix} v_1 & v_2 \\ x_1-x & d \\ d & x_2-x \end{pmatrix}=\begin{pmatrix} v_1 & v_2 \\ e & d \\ d & -e \end{pmatrix}+b(x-x_0). \tag{4.1}$$

The constraint (2.5) is obeyed if

$$d=\frac{1}{2}\frac{z}{z^2}(\bar v_2 v_1-\bar v_1 v_2), \tag{4.2}$$

with $z=x_1-x_2$ [23]. We find it more convenient to expose the role of the center of the instanton, the part proportional to the matrix b of Eq. (2.7), because this will be central in the integration that we will perform later. This is achieved with the substitutions $x_0=(x_1+x_2)/2, e=(x_1-x_2)/2$ which give the other form of the matrix Δ in Eq. (4.1).

We also need the form of the matrix C appearing in Eq. (2.15),[2] which is constrained by Eq. (2.14). Since this constraint is very similar to Eq. (2.5) [to get convinced of this

fact just think that two solutions of Eq. (2.14) are given by $C=a,b$], it is convenient to choose a form of C which parallels Eq. (4.1),

$$C=\begin{pmatrix} \nu_1 & \nu_2 \\ \xi_1 & \delta \\ \delta & \xi_2 \end{pmatrix}=\begin{pmatrix} \nu_1 & \nu_2 \\ \eta & \delta \\ \delta & -\eta \end{pmatrix}-b\xi_0. \tag{4.3}$$

The constraint (2.14) is satisfied imposing

$$\delta=\frac{z}{z^2}[2\bar d\,\eta+\bar v_2\nu_1-\bar v_1\nu_2]. \tag{4.4}$$

In analogy with the points 2, 4 of the previous calculation for the $k=1$ case, we have to compute[3]

$$S_H[\phi_{cl}+\phi_{inh},(\phi_{cl})^\dagger,A^{cl}]+S_Y[\phi=0,(\phi_{cl})^\dagger,\lambda^{(0)},\bar\lambda=0]. \tag{4.5}$$

This computation involves only the contributions of the ϕ_{cl} function and of the ϕ_{inh} field at the boundary of the physical space. It has been performed in [6] and it yields

$$S_H+S_Y=4\pi^2|a|^2(|v_1|^2+|v_2|^2)$$

$$-4\pi^2\frac{[\mathrm{Tr}(v_1\bar v_2-v_2\bar v_1)\phi_\infty]^2}{|v_1|^2+|v_2|^2+4(|d|^2+|c|^2)}$$

$$+2\sqrt2\pi^2\epsilon^{\dot A\dot B}\epsilon^{\alpha\gamma}\Big[(v_i)_{\gamma\dot A}(\phi_\infty)_\alpha{}^\beta(v_i)_{\beta\dot B}$$

$$+\left(\frac{\mathrm{Tr}(v_1\bar v_2-v_2\bar v_1)\phi_\infty}{|v_1|^2+|v_2|^2+4(|d|^2+|c|^2)}\right)$$

$$\times|(v_1)_{\gamma\dot A}[(v_2)_{\alpha\dot B}+2\eta_{\gamma\dot A}\delta_{\alpha\dot B}]], \tag{4.6}$$

where ϕ_∞ was defined in Eq. (3.3).

Let us comment on Eq. (4.6): in the Yukawa action the variable ξ_0 is missing. In fact, the expectation value of the scalar field has broken the conformal but not the translational invariance of the action. The effect is the appearance, in the action, of the collective coordinates related to these symmetries. As SUSY is still a symmetry, the collective coordinates associated to it must be missing in Eq. (4.6), which is what we observe. In complete analogy with the $k=1$ case, the Grassmann parameters can now be divided into two sets: those which do not appear in the action (connected to SUSY) must be isolated in the ϕ_{inh} piece to cancel against the measure. Those which appear in the action will not appear in the insertion of $\phi^a\phi^a$: all fermionic zero modes are lifted but the SUSY ones. There is another consequence of this observation. On the RHS of Eq. (3.17) there are fermionic fields expanded in the basis of the zero modes given by Eq. (2.15). See also the $k=1$ case [Eqs. (3.10), (3.11)] for a comparison. The previous observation thus suggests us to solve Eq. (3.21) only for those fermionic fields containing the Grassmann parameters related to SUSY transformations. As a conse-

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[2]The elements of this matrix must be interpreted as Grassmann numbers from the point of view of the functional integration.

[3]The following expression contains the prescription for computing S_H+S_Y on the saddle point for which the only zero modes are the left-handed ones.

quence, Eq. (3.23) still holds. The term $F_{\mu\nu}^a F_{\mu\nu}^a$ is given by Eq. (2.9), and in Eq. (4.1) Δ was parametrized in order to be a function of $x-x_0$. Since Eq. (2.9) depends only on Δ, $F_{\mu\nu}^a F_{\mu\nu}^a$ is a function of $x-x_0$ too.

Given all these observations we can write the correlator for the case $k=2$ as

$$\langle \phi^a \phi^a \rangle = \frac{\Lambda^8}{\mathcal{S}} \int d^4 e\, d^4 v_1 d^4 v_2 d^4 \eta\, d^4 \nu_1 d^4 \nu_2$$

$$\times \left(\frac{J_{\text{Bose}}}{J_{\text{Fermi}}} \right)^{1/2} e^{-S_{\text{H}} - S_{\text{Y}}} \int d^4 \xi_0 (\xi_0)_{\hat{1}}^2 (\xi_0)_{\hat{2}}^2$$

$$\times \int d^4 x_0 F_{\mu\nu}^a F_{\mu\nu}^a, \tag{4.7}$$

where $\mathcal{S}=16$ is a statistical weight computed in [15]. The integrations in the last line of Eq. (4.7) can be performed immediately after trading the x_0 with the x integration by shifting variable, and give

$$\int d^4 \xi_0 [(\xi_0)_{\hat{1}}]^2 [(\xi_0)_{\hat{2}}]^2 \int d^4 x_0 F_{\mu\nu}^a F_{\mu\nu}^a = 4 \frac{64\pi^2}{g^2}. \tag{4.8}$$

The fermionic Jacobian J_{Fermi} is obtained from Eqs. (2.15) and (2.16) while J_{Bose} was computed in [15,6]. Putting these results together, we get

$$\left(\frac{J_{\text{Bose}}}{J_{\text{Fermi}}} \right)^{\frac{1}{2}} = \frac{2^{10}}{\pi^8} \frac{\|e|^2 - |d|^2|}{|v_1|^2 + |v_2|^2 + 4(|d|^2 + |c|^2)}. \tag{4.9}$$

After substituting Eq. (4.9) in Eq. (4.7), the remaining integrations can be performed and give $5/(32\pi^2 a^6 g^6)$ [6].

Collecting all these results, we finally find

$$\langle \text{Tr}\phi^2 \rangle = -\frac{5}{4} \frac{\Lambda^8}{g^8 a^6}, \tag{4.10}$$

which is in agreement with the results of [7,8]

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