BRST-invariant boundary conditions for gauge theories

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A systematic way of generating sets of local boundary conditions on the gauge fields in a path integral is presented. These boundary conditions are suitable for one-loop effective action calculations on manifolds with a boundary and for quantum cosmology. For linearized gravity, the general procedure described here leads to new sets of boundary conditions. [S0556-2821(97)05002-9]

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I. INTRODUCTION

The aim of the work reported here is to characterize sets of local boundary conditions on the fields in a path integral. This is a nontrivial problem for gauge theories, where the boundary conditions have to be consistent with the gauge symmetry. In the Becchi-Rouet-Stora-Tyutin (BRST) approach [1,2], which we examine, this consistency with the gauge symmetry translates into BRST invariance. The gauge fields are augmented by extra families of ghosts, antighosts, and auxiliary fields that also require boundary conditions.

Boundary conditions are needed for effective action calculations on manifolds with boundary and for the evaluation of wave functions in quantum cosmology [3–5]. In many of these applications the geometry is curved, and this is where local boundary conditions are especially useful. Boundary conditions that required separating transverse from longitudinal photons, for example, would be nonlocal. This would not be a problem in flat spacetime, because the separation is local in momentum space. In curved spacetimes, however, these nonlocal operations are best avoided.

There is another important reason for considering local boundary conditions. To first order in Planck's constant, the result of a path integral is closely related to the asymptotic behavior of the eigenvalues of an operator. With local boundary conditions, the asymptotic behavior of the eigenvalues is determined by local tensors through a heat kernel expansion [6–11].

Local boundary conditions have been described before for Maxwell gauge theory, where the fields of interest include perturbations in the vector potential, ghosts, and antighost fields [12]. There are two sets of boundary conditions corresponding to fixing the magnetic or electric field on the boundary. Each set has mixtures of Dirichlet and Robin boundary conditons. If we split the fields into two subsets by using projection operators $P_\pm$,

$$(\mathcal{L}+\psi)P_+\phi=0, \tag{1}$$

$$P_-\phi=0, \tag{2}$$

where $\mathcal{L}$ is the Lie derivative along the normal to the boundary Σ and ψ is a matrix. These boundary conditions are now widely used [13–22].

A similar set of boundary conditions was found for gravitational fields [13,23], but it was soon discovered that this set of boundary conditions was not invariant under BRST transformations [12,25]. A set found by Barvinski [24] is invariant under BRST, but not quite of the same form. In this set, ψ in Eq. (1) includes a first order differential operator restricted to the boundary [25–28].

The gauge fixing in both of the cases mentioned above is a covariant function of the gravitational background. By contrast, allowing noncovariant gauge fixing allows a set of boundary conditions that is both BRST invariant and of the mixed type [28]. These noncovariant approaches are not applicable, so far, to all topological situations. Other possibilities have also been considered [29,30]

It appears that gravity with covariant gauge-fixing terms in the Lagrangian requires us to generalize the original class of mixed boundary conditions to new classes $\mathcal{M}_n$, where ψ is a differential operator of order n. The asymptotic behavior of the heat kernel is known for mixed boundary conditions $\mathcal{M}_0$ [31]. It should be possible to extend these results to classes $\mathcal{M}_1$ and $\mathcal{M}_2$ without too much difficulty.

In the next section we shall see that a set of boundary conditions of type $\mathcal{M}_n$ can always be generated, based upon a standard idea of having the ghost and antighost fields vanish on the boundary [32]. We shall also see how this gives rise to a means of generating new sets of boundary conditions through the application of canonical transformations between the ghosts and antighosts.

For linearized gravity with 't Hooft–Veltman gauge fixing (sometimes called harmonic gauge) [33], the general procedure described above leads to two new sets of boundary conditions in class $\mathcal{M}_2$. With certain restrictions on the extrinsic curvature of the boundary, one new set of boundary conditions arises that is $\mathcal{M}_0$ and is, therefore, the first BRST-invariant set of boundary conditions of the original mixed type.

In this paper we set the signature of the background four-metric to be $(++++)$.

II. VANISHING GHOSTS

In the BRST approach to the path integral the original fields q are augmented by ghosts c, antighosts $\bar{c}$, and auxiliary fields b (see [32] for a review). The path integral over the fields on a manifold with a boundary Σ will result in an amplitude in which the fields are specified on Σ:

$$\Psi=\Psi(q,c,\bar{c},b;\Sigma). \tag{3}$$

If Σ has only one connected component, then the amplitude would be a wave function in the sense adopted in the study of quantum cosmology [3].

When evaluating the path integral, a classical term is usually subtracted from the fields so that the residual fields satisfy simplified boundary conditions. The result of the path integral can then be written in terms of operators acting on the fields.

Our aim is to find what additional restrictions have to be placed on the fields in order to recover the correct number of physical degrees of freedom. In most applications, for example, an immediate restriction follows from the elimination of the auxiliary field, leading to boundary conditions $b^i = \overline{E^i}(q,\mathcal{L}q)$, where $\mathcal{L}$ is the Lie derivative along the normal to the boundary.

We will regard events on the boundary as simultaneous and $\mathcal{L}q$ as the time derivative of q. The importance of these time derivatives indicates that Hamiltonian methods should be useful.

In the classical Hamiltonian approach we introduce the Poisson brackets

$$[q_n, p^m]_{\rm PB} = \delta_n^m, \quad [c_i, p^j]_{\rm PB} = -\delta_i^j, \quad [\overline{c^i}, \overline{p_j}]_{\rm PB} = -\delta_i^j. \tag{4}$$

The momenta are distinguished by their indices, m and n for the fields and i and j for the ghosts. For field theories, the index also includes the coordinates on Σ and summation over a repeated index includes integration over Σ.

Two important operators that we shall use are constructed from classical generating functions [32]. The ghost-number generator keeps track of the number of ghosts:

$$G = c_i p^i - \overline{c^i p_i}. \tag{5}$$

The BRST generator Ω generates BRST symmetries s:

$$s^R z = [z, \Omega]_{\rm PB}, \tag{6}$$

where s^R is used to denote BRST acting from the right. The BRST generator depends on constraints $E^i(q,p)$ and their structure constants $C^{ij}{}_k$ In the type of theory known as rank 1 the gauge-fixed action leads to a BRST generator which has the form

$$\Omega = \overline{p_i} E^i + c_i \overline{E^i} + \tfrac{1}{2} c_i c_j C^{ij}{}_k p^k. \tag{7}$$

We shall assume that the theory has rank 1 for notational convenience.

Vanishing ghost number and BRST invariance are imposed as fundamental requirements on the quantum theory. In terms of operators and amplitudes we set

$$G\Psi = 0, \tag{8}$$

$$\Omega\Psi = 0. \tag{9}$$

These conditions, which reduce the space of states to those that may be regarded as physical, serve as boundary conditions on the path integral.

The simplest way to satisfy the constraints (8) and (9) is to set the ghost fields to zero on the boundary of the path integral. The Poisson brackets,

$$[\overline{c^i}, Q]_{\rm PB} = -\overline{E^i}, \tag{10}$$

when expressed as a commutator acting on Eq. (9), imply that $\overline{E^i}$ also has to vanish on Σ. The set of boundary conditions so far is, therefore,

$$c_i = \overline{c^i} = b^i = \overline{E^i} = 0. \tag{11}$$

The BRST variation of the fields q when $c = 0$ is given by

$$s^R q_n = \overline{p_j}[q_n, \overline{E^j}]. \tag{12}$$

Those fields which commute with $\overline{E}$ belong to a set we call $\mathcal{Q}$ and can be fixed on the boundary. The boundary conditions on the fields which do not commute with $\overline{E}$ are determined by the vanishing of $\overline{E}(q,p)$. The vanishing-ghost boundary conditions on the gauge-fixed path integral are, therefore,

$$c_i = \overline{c^i} = 0, \tag{13}$$

$$b^i = \overline{E^i}(q,p) = 0, \tag{14}$$

$$q \quad \text{fixed for } q \in \mathcal{Q}, \tag{15}$$

where $\mathcal{Q}$ is the set of fields whose momenta do not appear in the gauge-fixing functions $\overline{E}$. These boundary conditions are invariant under BRST transformations by construction.

Our principal concern is to list all of the possible sets of boundary conditions, subject to specific restrictions. An obvious place to begin is the division of phase space into ghosts and their conjugate momenta, which is not preserved by canonical transformations (defined below). One way of creating further sets of boundary conditions would, therefore, be to perform an arbitrary canonical transformation before applying the vanishing-ghost conditions.

In actual fact, not all canonical transformations turn out to be suitable. Some lead to vanishing-ghost boundary conditions that are not BRST invariant. This is due to structural changes in the BRST generator Ω. Because of this fact, we consider a restricted class of transformations that satisfy the following conditions: (1) The transformation is canonical; (2) it preserves the number of ghosts; (3) the variation of a vanishing ghost vanishes.

Condition (3) means that $[c,\Omega] = 0$ when $c = 0$, where c is the new ghost field. This condition arises from requiring that setting the BRST variation of c to zero should not imply any further restrictions on the fields.

We will consider how these restrictions apply to transformations between the ghosts, antighosts, and auxiliary fields. For this purpose it is convenient to blur the distinction between ghosts and antighosts and write

$$\eta_i = \begin{pmatrix} c_i \\ \overline{p_i} \end{pmatrix}, \quad \mathcal{P}^i = \begin{pmatrix} p^i \\ \overline{c^i} \end{pmatrix}, \quad \mathcal{E}^i = \begin{pmatrix} E^i \\ \overline{E^i} \end{pmatrix}. \tag{16}$$

We also define $\lambda_i = \lambda_i(q,p)$ to be the set of fields canonically conjugate to $b^i = \overline{E^i}(q,p)$.

Canonical transformations from $\{\eta, \mathcal{P}, \lambda, b\}$ to $\{\eta', \mathcal{P}', \lambda', b'\}$ are generated by $F(\eta', \mathcal{P}, \lambda', b)$:

$$\eta_i = \frac{\partial F}{\partial \mathcal{P}^i}, \quad \lambda_i = -\frac{\partial F}{\partial b^i},$$

$$\mathcal{P}'^i = -\frac{\partial F}{\partial \eta'_i}, \quad b'^i = -\frac{\partial F}{\partial \lambda'_i}. \tag{17}$$

The ghost-number operator can now be written

$$G = \eta_i \mathcal{P}^i = \frac{\partial F}{\partial \mathcal{P}^i}\mathcal{P}^i. \tag{18}$$

In the new coordinate system,

$$G' = \eta'_i \mathcal{P}'^i = -\eta'_i \frac{\partial F}{\partial \eta'_i}. \tag{19}$$

Setting $G' = G$, therefore, leads to transformations of the form

$$F \equiv F(\mathcal{P}^i \eta'_j, \lambda', b). \tag{20}$$

The allowed linear transformations on the ghost and antighost fields are covered by the following theorem.

Transformations generated by

$$F = \mathcal{A}_i^j(\lambda')\mathcal{P}^i \eta'_j + b^i \lambda'_i, \tag{21}$$

where the matrix $\mathcal{A}$ has the properties

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad dA = dC = dD = 0, \quad dB_i{}^j = \tfrac{1}{2} C^{kl}{}_i B_l{}^j \, d\lambda'_k, \tag{22}$$

satisfy conditions (1)–(3).

These transformations are manifestly of the form given in Eq. (20). The rest of the proof is by direct application of Eqs. (17). These allow the generator Ω to be written in the form

$$\Omega = \mathcal{A}_i{}^l \eta'_l \mathcal{E}^i + \tfrac{1}{2}\mathcal{A}_i{}^l \mathcal{A}_j{}^m \eta'_l \eta'_m C^{ij}{}_k \mathcal{P}^k. \tag{23}$$

We are also able to replace b^i by b'^i:

$$\mathcal{E}^i = \mathcal{E}'^i + \frac{\partial B_j{}^k}{\partial \lambda'_i} p^j \overline{p'_k}. \tag{24}$$

The linear term in Ω commutes with c'_i and gives no further boundary conditions. The condition on B in Eq. (22) removes terms beginning $\overline{p_i p_j} C^{ij}{}_k$ which are the only ones that violate condition (3). This completes the proof of the theorem.

We can now write the vanishing-ghost boundary condition in terms of the original variables and obtain new sets of boundary conditions:

$$B_i{}^j p^i + D_i{}^j \overline{c^i} = 0, \tag{25}$$

$$B_i{}^j \overline{p^i} - D_i{}^j c^i = 0, \tag{26}$$

$$B_i{}^j E^i + D_i{}^j \overline{E^i} = \tfrac{1}{2} D_i{}^j C^{il}{}_k p^k c_l, \tag{27}$$

$$q \quad \text{fixed for } q \in \mathcal{Q}, \tag{28}$$

where $\mathcal{Q}$ is now the set of fields whose momenta do not appear in the other boundary conditions.

For an Abelian theory, these boundary conditions allow any linear combination of the ghosts and their momenta to be set to zero as long as it is consistent with ghost number. The remaining boundary conditions are then determined uniquely. These boundary conditions, therefore, include all the possible sets of linear boundary conditions. Since quantum field theories are effectively Abelian up to order $\hbar$, these also exhaust sets of linear boundary conditions for one-loop quantum field theory.

There is still one further restriction to impose, namely, that the boundary conditions are in a class $\mathcal{M}_n$ of the mixed boundary conditions mentioned in the introduction. This means that the linear transformations must now be local, and the momenta should be replaced by normal derivatives. We can then proceed by the following rules.

(1) Each set of linear combinations of the constraints $E(q,\mathcal{L}q)$ and gauge-fixing conditions $\overline{E}(q,\mathcal{L}q)$ that can be written in the form of Eqs. (1) and (2), possibly after removing an overall surface derivative, defines a set of mixed boundary conditions.

(2) The boundary conditions on the ghosts are fixed by Eq. (25) once the linear combinations are given.

(3) The set $\mathcal{Q}$ of fields that can be fixed on the boundary is finally identified by examining the combinations $E(q,p)$ and $\overline{E}(q,p)$ for any missing momenta.

The resulting sets of boundary conditions depend on the choice of gauge-fixing term in the action. This is to be expected, because the path integral is usually expressed in terms of operators which themselves depend on the choice of gauge-fixing term. On the other hand, there is still some freedom in the choice of Lagrangian density even when the gauge-fixing term is fixed. For example, it is possible to eliminate the auxiliary field b from the action at an early stage or to leave it in. This affects the form of the constraints, gauge-fixing condition, and even the momenta, but it does not affect the final form of the boundary conditions.

III. ELECTRODYNAMICS

A simple example of the preceding ideas is provided by electrodynamics in curved spacetime. For Lorentz gauges, the Maxwell field A_a is accompanied by one ghost field c and one antighost field $\overline{c}$.

In order to set up a phase space associated with the hypersurface Σ we need to decompose the Maxwell field into normal and tangential components,

$$A_a = \phi_a + \phi n_a. \tag{29}$$

(The index structure alone distinguishes different vector and scalar quantities. We find this preferable to a profusion of notation.) Momenta conjugate to ϕ_a, ϕ, c, and $\overline{c}$ are denoted by π^a, π, p, and $\overline{p}$, respectively. The extrinsic curvature will be denoted by K_{ab} and a vertical bar denotes covariant differentiation in Σ.

The Lagrangian density L can be taken to be the sum of three terms, the Maxwell, ghost, and gauge-fixing terms

$$L_A = -\tfrac{1}{4}F_{ab}F^{ab}, \quad L_{\rm gh} = -\overline{c^{;a}}c_{;a},$$
$$L_{\rm GF} = -bA_a{}^{;a} + \tfrac{1}{2}b^2. \qquad (30)$$

The field b can be eliminated by

$$b = A_a{}^{;a}, \qquad (31)$$

restricting the nilpotency of the BRST transformations to solutions of the field equations.

Starting from the Lagrangian density, one way to find the BRST generator is to compute the Noether current J^a. Using left BRST transformations s^L $[s^L z = (-)s^R z$ for even (odd) fields $z]$,

$$J^a = s^L z \frac{\partial L}{\partial z_{;a}} - j^a, \qquad (32)$$

where $s^L L = j_a{}^{;a}$. For the present example,

$$s^L A_a = c_{;a}, \quad s^L \bar{c} = b, \quad s^L c = s^L b = 0, \quad j_a = -bc_{;a}. \qquad (33)$$

The Noether current is, therefore,

$$J^a = -bc^{;b} + F^{ab}c_{;b}. \qquad (34)$$

The Noether charge Ω is the volume integral of the local charge density $\omega = n_a J^a$.

Decomposition of the Lagrangian density, following the outline given in the Appendix, results in the momenta

$$\pi^a = g^{ab}(\phi_{|b} - \mathcal{L}\phi_a), \quad \pi = -b,$$
$$p = \mathcal{L}\bar{c}, \quad \bar{p} = -\mathcal{L}c. \qquad (35)$$

Using these expressions, Eq. (34) leads trivially to the BRST charge density

$$\omega = \bar{p}b - c\pi^a_{|a}. \qquad (36)$$

In the notation used in the previous section, $\bar{E} = -\pi$ (Lorentz gauge condition) and $E = \pi^a_{|a}$ (Gauss' law constraint).

The vanishing-ghost boundary condition is given by

$$c = \bar{c} = b = \pi = 0, \quad \phi_a = 0. \qquad (37)$$

Using Eq. (31) and eliminating momenta puts this into mixed form,

$$c = \bar{c} = 0, \quad \mathcal{L}\phi + K\phi = 0, \quad \phi_a = 0. \qquad (38)$$

This set of boundary conditions fixes the magnetic field on the boundary.

No other linear combination of $\bar{E}$ and E can be put into mixed form, except for E itself, which is a total divergence. The momentum π does not appear in E and ϕ can be fixed on the boundary by rule (3) of Sec. II. The only other set of mixed boundary conditions is, therefore,

$$\mathcal{L}c = \mathcal{L}\bar{c} = 0, \; \mathcal{L}\phi_a = 0, \; \phi = 0. \qquad (39)$$

This set of boundary conditions fixes the electric field on the boundary.

IV. LINEARIZED GRAVITY

Linearized gravity forms the starting point for order $\hbar$ quantum gravity calculations based on Einstein gravity, as well as having wider applications to supergravity and superstring theories by taking various spacetime dimensions. We are seeking sets of local boundary conditions for the path integral, using 't Hooft–Veltman gauges because they are widely used and are covariant.

For 't Hooft–Veltman gauges, the metric fluctuation γ_{ab} is accompanied by ghost fields C_a and antighost fields $\bar{C}^a$. The metric fluctuation is defined in terms of the perturbed metric

$$g_{ab} + 2\kappa\gamma_{ab}, \qquad (40)$$

where $\kappa^2 = 8\pi G$. We will also make use of the dual quantity

$$\overline{\gamma^{ab}} = g^{(ab)(ef)}\gamma_{ef}, \qquad (41)$$

defined by the metric

$$g^{(ab)(cd)} = \tfrac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd}). \qquad (42)$$

In order to set up a phase space associated with the hypersurface Σ we need to decompose all of these fields into normal and tangential components:

$$\gamma_{ab} = \phi_{ab} + 2\phi_{(a}n_{b)} + \phi n_a n_b,$$
$$\overline{\gamma_{ab}} = \overline{\phi_{ab}} + 2\overline{\phi_{(a}}n_{b)} + \overline{\phi}n_a n_b,$$
$$C_a = c_a + cn_a, \quad \overline{C^a} = \overline{c^a} + \overline{cn^a}. \qquad (43)$$

(The index structure distinguishes different vector and scalar quantities.) Momenta conjugate to ϕ_X, c_X, and $\overline{c^X}$ are denoted by π^X, p^X, and $\overline{p_X}$, respectively.

The background metric on Σ will be denoted by h_{ab}. Variations in the surface metric correspond to variations in both ϕ_{ab} and ϕ_a, but variations in the surface geometry depend only on ϕ_{ab}.

The Lagrangian density L can be taken to be the sum of two terms, the gauge-fixed Einstein-Hilbert term and the ghost terms. For a 't Hooft–Veltman gauge-fixing term,

$$L_\gamma = -\tfrac{1}{2}{}^{ab;c}\gamma_{ab;c} + R^{acbd}\overline{\gamma_{ab}}\gamma_{cd} + G^{ac}g^{bd}\overline{\gamma_{ab}}\gamma_{cd},$$
$$L_{\rm gh} = -\overline{C^{a;b}}C_{a;b} + R_a{}^b\overline{C^a}C_b, \qquad (44)$$

where G_{ab} is the Einstein tensor. The auxiliary field b has already been eliminated.

The nonvanishing BRST transformations are

$$s^L\gamma_{ab} = 2C_{(a;b)}, \quad s^L\bar{C} = 2{}^{ab}{}_{;b}. \qquad (45)$$

The BRST charge density can be calculated as in the last section, using Eq. (7) and the decompositions in Appendix A. The result can be written in the form

$$\omega = \overline{p^a}\overline{E_a} + \bar{p} + c_a E^a + cF. \qquad (46)$$

Explicit expressions for $\overline{E_a}$, $\bar{F}$, E^a, and F appear in Appendix C, Eqs. (C7)–(C10). Whilst $\overline{E_a}$ and $\bar{F}$ are already in the correct form [given in Eq. (1)], E^a and F are not. (Even in

TABLE I. Four sets of boundary conditions for linearized gravity with extrinsic curvature $K_{ab}=Kh_{ab}/3$. Each entry is equated to zero, quantities listed under $\mathcal{Q}$ denoting Dirichlet boundary conditions which are combined with the entries under $\mathcal{R}$ to form the mixed class $\mathcal{M}_n$. Special combinations of fields are denoted by $\phi^T=h^{ab}\phi_{ab}$, $\phi^K_{ab}=\phi_{ab}-(1/K)\phi^T K_{ab}$, $\phi^L_{ab}=\phi_{ab}-\phi^T h_{ab}$, and $\phi^L=-(2/3)\phi^T$. The operator $\Delta=(\mathcal{L}K)+K^2-\nabla^2$ and F is defined in the second line.

Set	n	$\mathcal{Q}$	$\mathcal{R}$
I	1	ϕ_{ab}	$\mathcal{L}\phi_a+K\phi_a-\frac{1}{2}\nabla_a\phi$
		c_a, c	$\mathcal{L}\phi+2K+\nabla^a\phi_a=\overline{F}$
II	2	$\phi+\alpha\phi^T, \phi^K_{ab}$	$\mathcal{L}\phi_a+K\phi_a+\nabla_a(\phi^L-\phi)$
			$K\mathcal{L}\phi^L-\Delta\phi^L-(K_L^{\vert a}+2K_L\nabla^a)\phi_a-(\alpha+1)K\overline{F}$
		c_a	$\mathcal{L}c+\alpha Kc$
III	0	$\phi+\beta\phi^T, \phi_a$	$\mathcal{L}\phi^L_{ab}-K_L\phi^L_{ab}$
			$\mathcal{L}\phi+2K\phi-K\phi^L$
		c	$\mathcal{L}c_a+\beta Kc_a$
IV	2	ϕ_a	$K\mathcal{L}\phi^L-\Delta\phi^L-(K_L^{\vert a}+2K_L\nabla^a)\phi_a-(\alpha+1)K\overline{F}$
			$\mathcal{L}\phi^L_{ab}-K_L\phi^L_{ab}-(\beta+1)F$
			$\mathcal{L}c_a+\alpha Kc_a+\beta K\nabla_a c$

this form they can be used to obtain nonlocal boundary conditions which are potentially useful for particular backgrounds.)

We still have the freedom to perform the linear transformations described in rule (1) at the end of Sec. II. We first of all perform linear transformations on E^a and F to separate divergences of momenta from gradients of momenta:

$$F'=F-\overline{E_c}^{|c}+\alpha K\overline{F}, \tag{47}$$

$$E_a{}'=E_a-K_a{}^c\overline{E_c}-\beta\overline{F}_{|a}. \tag{48}$$

The new F' commutes with ϕ_a and the fields

$$\phi^K_{ab}=\phi_{ab}-K^{-1}K_{ab}\phi_c{}^c, \quad \phi^{(\alpha)}=\phi+\alpha\phi_a{}^a. \tag{49}$$

A final linear transformation allows a choice of $\overline{F'}$ and $\overline{E_a}'$ from the set $\{F',E_a{}',\overline{F},\overline{E_a}\}$.

What happens depends very much on whether the extrinsic curvature K_{ab} is proportional to the surface metric. If $K_{ab}=Kh_{ab}/3$, then we have the following boundary conditions:

(I) $\{\overline{E_a},\overline{F},\phi_{ab},\overline{c^a},\overline{c},c_a,c\}=0$.

(II) $\{E_a,F',\phi^K_{ab},\phi^{(\alpha)},c,\overline{c_a},p+\alpha K\overline{c},\overline{p}-\alpha Kc\}=0$.

If, in addition, K is constant then

(III) $\{E'_a,\overline{F},\phi_a,\overline{c},c,p_a-K_a{}^b\overline{c_b},\overline{p_a}+K_a{}^b c_b\}=0$, and

(IV) $\{E'_a,F',\phi_a,p+\alpha K\overline{c},\overline{p}-\alpha Kc,p_a-K_a{}^b\overline{c_b}-\beta\overline{c}_{|a},\overline{p_a}+K_a{}^b c_b+\beta c_{|a}\}=0$.

In cases (III) and (IV), the expression for E'_a is a total divergence which can be integrated to obtain boundary conditions of the correct type.

The boundary conditions are written explicitly in Table I. Boundary condition (I) has been applied previously to applications in quantum cosmology. The other boundary conditions are new, to the best of our knowledge. Boundary condition (III) is especially interesting because it contains no spatial derivative terms.

Difficulties arise when the extrinsic curvature is not proportional to the surface metric. The function E'_a can be written as a total divergence, but not of a symmetric tensor [see Eq. (C13)]. Boundary conditions (III) and (IV) belong to a wider class of boundary conditions where the projection operators in Eqs. (1) and (2) include surface derivatives. This leaves boundary conditions (I) and (II). The resulting expressions are listed in Table II.

V. CONCLUSIONS

We have assumed that the boundary conditions on the path integral are local, linear, and BRST invariant. Locality means that the boundary conditions at a point depend only on the fields and their derivatives and has been imposed because it is useful for quantum field theory with nontrivial

TABLE II. Two sets of boundary conditions for linearized gravity with extrinsic curvature $K_{ab}\neq Kh_{ab}/3$. Each entry is equated to zero as before. Special combinations of fields are as in Table I, except for $\phi^L=K^{-1}K^{ab}\phi^L_{ab}$ and $\Delta^{ab}=[2\mathcal{L}K^{ab}-K^{-1}(\mathcal{L}K)K^{ab}-4K^{ac}K_c{}^b+KK^{ab}-\nabla^a\nabla^b]$.

Set	n	$\mathcal{Q}$	$\mathcal{R}$
I	1	ϕ_{ab}	$\mathcal{L}\phi_a+K\phi_a-\frac{1}{2}\nabla_a\phi$
		c_a, c	$\mathcal{L}\phi+2K\phi+\nabla^a\phi_a=\overline{F}$
II	2	$\phi+\alpha\phi^T, \phi^K_{ab}$	$\mathcal{L}\phi_a+K\phi_a+\nabla^b\phi_{ab}$
			$K\mathcal{L}\phi^L-\Delta^{ab}\phi^L_{ab}-2K^L_{ab}K^{ab}\phi-[(K_L)^{ab}{}_{\vert b}+2(K_L)^{ab}\nabla_b]\phi_a-(\alpha+1)K\overline{F}$
		c_a	$\mathcal{L}c+\alpha Kc$

background fields. Linearity is imposed for the same reason, since linear theory is the starting point of the $\hbar$ expansion in quantum field theory.

BRST invariance is meant in the sense that the BRST operator annihilates the result of the path integral. The boundary conditions themselves are BRST invariant in the sense that, when written in terms of momenta, they commute with the BRST generator.

With these assumptions, the boundary conditions can all be generated by following the rules given at the end of Sec. II. Using these rules it has been possible to find all of the boundary conditions for linearized gravity with 't Hooft–Veltman gauge fixing that are of the mixed Robin-Dirichlet-type, generalized to include surface derivative terms. These are given in Tables I and II. Set I of boundary conditions which fixes the surface geometry is known already [24–27] and the other sets are new. Set III has no surface derivatives.

Boundary conditions for linearized gravity are useful in quantum cosmology. The first set of boundary conditions fixes the scale factor of the universe. The second set of boundary conditions would correspond to fixing the expansion rate of the universe instead of the scale factor. The expansion rate has the advantage over the scale factor in being a single-valued function of time in classical cosmological models.

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APPENDIX A: HYPERSURFACES

Introducing a hypersurface Σ into the manifold $\mathcal{M}$ leads to a natural decomposition of the tangent space of $\mathcal{M}$ into the tangent space of Σ and its compliment along the normal vector n^a. We denote the intrinsic metric by

$$h_{ab}=g_{ab}-n_a n_b . \tag{A1}$$

The Lie derivative of the intrinsic metric along the normal direction defines the extrinsic curvature K_{ab}:

$$\mathcal{L}h_{ab}=2K_{ab} . \tag{A2}$$

The covariant derivative on $\mathcal{M}$, expressed by $\phi_{a;b}$, induces a covariant derivative on Σ. The definition

$$\phi_{a|b}=\phi_{a;b}-n_b\mathcal{L}\phi_a+\Gamma^c{}_{ab}\phi_c , \tag{A3}$$

where

$$\Gamma^c{}_{ab}=K^c{}_a n_b+K^c{}_b n_a+(\mathcal{L}n)^c n_a n_b \tag{A4}$$

is particularly useful. This expression extends to tensors on Σ. A particular example is the surface metric itself, which is easily seen to satisfy $h_{ab|c}=0$.

Decomposition of the Riemann tensor is straightforward if we take $\mathcal{L}n=0$. Two applications of Eq. (A3) give

$$R_{abc0}=K_{cb|a}-K_{ca|b} , \quad R_{a0b0}=K_a{}^c K_{cb}-\mathcal{L}K_{ab} ,$$
$$R_{abcd}=r_{abcd}-K_{ac}K_{bd}+K_{ad}K_{bc} , \tag{A5}$$

where $r^a{}_{bcd}$ is the Riemann tensor for h_{ab}.

APPENDIX B: MOMENTA

Equation (A3) can be used to express any Lagrangian, that is second order in derivatives, as a function $L(\mathcal{L}\phi_X,\phi_X)$. Momenta π^X are defined by differentiation of Lagrangian densities L with respect to $\mathcal{L}\phi_X$,

$$\pi^X=\frac{1}{\det g}\frac{\partial(L\det g)}{\partial(\mathcal{L}\phi_X)} . \tag{B1}$$

Because of the linear form of Eq. (A3), it is also possible to write this as

$$\pi^X=n_a\frac{\partial L}{\partial\phi_{X;a}} . \tag{B2}$$

For gauge-fixed electrodynamics in curved spaces with the Lagrangian given by Eqs. (30),

$$L_A=-\tfrac{1}{2}g^{ab}(\phi_{|a}-\mathcal{L}\phi_a)(\phi_{|b}-\mathcal{L}\phi_b)+\cdots ,$$
$$L_{\rm gh}=-(\mathcal{L}\overline{c})(\mathcal{L}c)+\cdots ,$$
$$L_{\rm GF}=-b(\phi_a{}^{|a}+\mathcal{L}\phi+K\phi)+\tfrac{1}{2}b^2 . \tag{B3}$$

These allow the momenta to be read off using Eq. (B1).

For gravity, with the Lagrangian density given by Eqs. (44), it is best to use Eq. (B2):

$$\pi^X=-n^c\overline{\gamma^{ab}}_{;c} , \quad p^X=n^b\overline{C^a}_{;b} , \quad \overline{p_X}=n^b C_{a;b} . \tag{B4}$$

After application of Eq. (A3), the momenta become

$$\pi_{ab}=-(\mathcal{L}\overline{\phi_{ab}}-2K_{(a}{}^c\overline{\phi_{b)c}}) , \tag{B5}$$

$$\pi^a=-2g^{ab}(\mathcal{L}\phi_b-K_b{}^c\phi_b) , \tag{B6}$$

$$\pi=-\mathcal{L}\overline{\phi,} \tag{B7}$$

$$p^a=+(\mathcal{L}\overline{c^a}+K^a{}_b\overline{c^b}) , \quad p=\mathcal{L}\overline{c,} \tag{B8}$$

$$\overline{p_a}=-(\mathcal{L}c_a-K_a{}^b c_b) , \quad \overline{p}=-\mathcal{L}c . \tag{B9}$$

APPENDIX C: BRST CHARGE FOR GRAVITY

Under the BRST variations, the Lagrange densities (44) transform by a divergence plus extra terms:

$$s^L L=j^a{}_{;a}+2E^{ab}(2C^d{}_{;b}\gamma_{ad}+C^d\gamma_{ab;d}) , \tag{C1}$$

where

$$j_a=-2\overline{\gamma_{ab;c}}C^{b;c}+2(R_a{}^b{}_d{}^c+R_d{}^b\delta^c{}_a-E^{bc}g_{ad})_{bc}C^d . \tag{C2}$$

The tensor E^{ab} depends on the Einstein tensor of the background fields and also the stress-energy tensor if a matter Lagrangian is included:

$$E^{ab}=G^{ab}-\kappa^2 T^{ab} . \tag{C3}$$

This tensor vanishes for background fields that satisfy the Einstein equations, which will be assumed throughout.

The BRST generator ω can be obtained from the Noether current, $2\omega=n^c J_c$, where

$$J^c = \frac{\partial L}{\partial \gamma_{ab;c}} s^L \gamma_{ab} + s^L \overline{C^a} \frac{\partial L}{\partial \overline{C^a}_{;c}} - j^c. \tag{C4}$$

For the Lagrange densities (44), this becomes

$$J_c = -2\overline{\gamma_{ab;c}} C^{a;b} - 2\overline{\gamma^{ab}}_{|b} C_{a;c} - j_c\,. \tag{C5}$$

Using the decomposition rule (A3) and the momenta (B9), the BRST generator can be written in the form

$$\omega = \overline{p^a}\overline{E_a} + \overline{p} + c_a E^a + cF. \tag{C6}$$

The functions appearing here are evaluated on phase space (π^X, ϕ_X). The dependence of the functions on the momenta is given explicitly by

$$\overline{E_a}(\pi^X, 0) = -\tfrac{1}{2}\pi^a, \tag{C7}$$

$$\overline{F}(\pi^X, 0) = -\pi, \tag{C8}$$

$$E_a(\pi^X, 0) = -\pi_{ab}{}^{|b} - \tfrac{1}{2} K_{ab}\pi^b, \tag{C9}$$

$$F(\pi^X, 0) = K_{ab}\pi^{ab} - \tfrac{1}{2}\pi^a{}_{|a}\,. \tag{C10}$$

For the boundary conditions we need to eliminate the momenta. This leads to the expressions

$$\overline{E_a} = \mathcal{L}\phi_a + K\phi_a + \nabla^b \overline{\phi_{ab}}\,, \tag{C11}$$

$$\overline{F} = \mathcal{L}\overline{\phi} + K\overline{\phi} - K^{ab}\overline{\phi_{ab}} + \nabla^a \phi_a\,. \tag{C12}$$

The linear combinations of E_a and F that come closest to the form that we require are

$$E_a - K_a{}^b \overline{E_b} = \nabla^b \mathcal{L}\overline{\phi_{ab}} + 2\nabla^b (K_a{}^c \overline{\phi_{bc}} - \overline{\phi} h_{ab}) + K^{bc}{}_{|a}(\overline{\phi_{bc}} + \overline{\phi} h_{bc}) + (K_a{}^c K_c{}^b - r_a{}^b - \nabla^2)\phi_b\,, \tag{C13}$$

$$F - \overline{E_a}^{|a} = -K^{ab}\mathcal{L}\overline{\phi_{ab}} + (\mathcal{L}K_{ab} - \nabla_a \nabla_b)(\overline{\phi^{ab}} + \overline{\phi} h^{ab}) + (K_b{}^{a|b} - K^{|a} + 2K^{ab} - 2Kh^{ab} + K\nabla^a)\phi_a\,. \tag{C14}$$

Surface derivatives on ϕ_X are denoted now by ∇^a.

[1] C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. **42**, 127 (1975).
[2] I. V. Tyutin (unpublished).
[3] J. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[4] J. J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).
[5] J. Louko, Phys. Rev. D **38**, 478 (1988).
[6] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem* (Chemical Rubber, Boca Raton, 1995).
[7] H. C. Luckock and I. G. Moss, Class. Quantum Grav. **6**, 1993 (1989).
[8] I. G. Moss, Class. Quantum Grav. **6**, 759 (1989).
[9] I. G. Moss and J. S. Dowker, Phys. Lett. B **229**, 261 (1989).
[10] T. P. Branson and P. B. Gilkey, Commun. Partial Differ. Equ. **15**, 245 (1990).
[11] D. V. Vassilevich, J. Math. Phys. (N.Y.) **36**, 3174 (1995).
[12] I. G. Moss and S. J. Poletti, Phys. Lett. B **245**, 335 (1990).
[13] I. G. Moss and S. J. Poletti, Nucl. Phys. **B341**, 155 (1990).
[14] S. J. Poletti, Phys. Lett. B **249**, 249 (1990).
[15] P. D. D'Eath and G. Esposito, Phys. Rev. D **43**, 3234 (1991).
[16] P. D. D'Eath and G. Esposito, Phys. Rev. D **44**, 1713 (1991).
[17] A. O. Barvinski, A. Yu. Kamenshchik, and I. P. Karmazin, Ann. Phys. (N.Y.) **219**, 201 (1992).
[18] A. O. Barvinski, A. Yu. Kamenshchik, I. P. Karmazin, and I. V. Mishakov, Class. Quantum Grav. **9**, L27 (1992).
[19] G. Esposito, Class. Quantum Grav. **11**, 905 (1994).
[20] A. Yu. Kamenshchik and I. V. Mishakov, Int. J. Mod. Phys. A **7**, 3713 (1992).
[21] A. Yu. Kamenshchik and I. V. Mishakov, Phys. Rev. D **47**, 1380 (1993).
[22] A. Yu. Kamenshchik and I. V. Mishakov, Phys. Rev. D **49**, 816 (1994).
[23] H. C. Luckock, J. Math. Phys. (N.Y.) **32**, 1755 (1991).
[24] A. O. Barvinski, Phys. Lett. B **195**, 344 (1987).
[25] G. Esposito, A. Yu. Kamenshchik, I. V. Mishakov, and G. Pollifrone, Class. Quantum Grav. **11**, 2939 (1994).
[26] G. Esposito, A. Yu. Kamenshchik, I. V. Mishakov, and G. Pollifrone, Phys. Rev. D **52**, 3457 (1995).
[27] G. Esposito, G. Gionti, A. Yu. Kamenshchik, I. V. Mishakov, and G. Pollifrone, Int. J. Mod. Phys. D **4**, 735 (1995).
[28] Avramidi, G. Esposito, and A. Yu. Kamenshchik, Class. Quantum Grav. **13**, 2361 (1996).
[29] G. Esposito and A. Yu. Kamenshchik, Class. Quantum Grav. **12**, 2715 (1995).
[30] V. N. Marachevsky and D. V. Vassilevich, Class. Quantum Grav. **13**, 645 (1996).
[31] I. G. Moss, *Quantum Theory, Black Holes and Inflation* (Wiley, New York, 1996).
[32] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
[33] G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré, A **20**, 69 (1974).