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Propagation of cool pions

Robert D. Pisarski and Michel Tytgat
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973-5000
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For an exact chiral symmetry that is spontaneously broken at zero temperature, we show that, at nonzero temperature, generally pions travel at *less* than the speed of light. This effect first appears at next-to-leading order in an expansion around low temperature. When the chiral symmetry is approximate we obtain two formulas, like that of Gell-Mann, Oakes, and Renner, for the static and dynamic pion masses. [S0556-2821(96)50315-2]

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Pions are light because they are (almost) Goldstone bosons: in QCD, quarks have an (approximate) chiral symmetry of $SU(2)_\ell \times SU(2)_r$ that is spontaneously broken to the usual isospin symmetry of $SU(2)_V$ by the dynamical generation of a quark condensate [1–3]. Notably, the pion mass squared is proportional to the up and down quark masses through the formula of Gell-Mann, Oakes, and Renner [4,5].

In this paper we consider how pions propagate in a thermal bath [6]; similar results should also hold for pions propagating in a Fermi sea of nucleons. In the limit of exact chiral symmetry, pions are true Goldstone modes and so massless. At zero temperature relativistic invariance then requires pions to travel at the speed of light. Our basic point is elementary: since the presence of a medium provides a privileged rest frame, relativistic invariance no longer applies, and so typically pions travel at *less* than the speed of light. We also derive how the formula of Gell-Mann, Oakes, and Renner generalizes to nonzero temperature [7]. Because the pion's velocity is less than c, in a medium the pion dispersion relation, as a function of momentum, is ''flattened'' from that at zero temperature. Such a flattening has been found in a wide variety of models [8–10], due apparently to the detailed dynamics. Our results show that at least some of the flattening arises on *very* general grounds, as a consequence of chiral symmetry breaking in a medium.

While we speak of pions throughout, our conclusions apply to Goldstone bosons in any system that is relativistically invariant at zero temperature. Indeed, our results for the changes in the pion dispersion relation have exact analogies with spin waves in antiferromagnets [11,12]. We think that our manner of derivation — in terms of the pion decay constants — is novel and illuminating. The detailed calculations that we perform to demonstrate this effect in the linear σ model extend and complement previous results by Itoyama and Mueller [13].

We begin with a heuristic derivation, in the limit of exact chiral symmetry. At zero temperature, the matrix element of the axial vector current, A^μ_a, sandwiched between the vacuum and a pion of momentum $P^\mu=(p^0,\vec{p})$ is

$$\langle 0|A^\mu_a|\pi^b(P)\rangle = i f_\pi \delta^{ab} P^\mu, \tag{1}$$

with a and b isospin indices. The pion decay constant $f_\pi \sim 93$ MeV; whenever we write f_π, we mean its value at zero temperature.

At nonzero temperature, because of the presence of the medium, we expect that there are *two* distinct pion decay constants: one for the timelike component of the current f^t_π,

$$\langle 0|A^0_a|\pi^b(P)\rangle_T = i f^t_\pi \delta^{ab} p^0,$$

and one for the spatial f^s_π,

$$\langle 0|A_a^i|\pi^b(P)\rangle_T = i f_\pi^s \delta^{ab} p^i. \quad (2)$$

Both matrix elements are computed at a temperature T in the imaginary time formalism. Implicitly the timelike component of the momentum, p^0, is analytically continued from Euclidean values (pions, as bosonic fields, have $p^0 = 2\pi n T$ for integral n) to Minkowski values, $p^0 = -i\omega + 0^+$. In Eq. (2), f_π^t and f_π^s are defined about zero momentum, ω and $p \to 0$.

The possibility of two distinct pion decay constants is familiar from nonrelativistic systems, such as discussed by Leutwyler [12]; in this context it was recognized previously by Kirchbach and Riska [14] and by Thorsson and Wirzba [7].

By assumption the chiral symmetry is exact, and only broken spontaneously by the vacuum. Consequently, while the axial vector current acts nontrivially on the vacuum, it nevertheless is conserved on the pion mass shell. At zero temperature this is trivial: the divergence of the matrix element in Eq. (1) is $\langle 0|\partial_\mu A^\mu|\pi\rangle \sim f_\pi P^2$, which vanishes when $P^2 = -\omega^2 + p^2 = 0$, as expected for a massless, relativistically invariant field.

At nonzero temperature, however, the condition that the axial vector current is conserved on the pion mass shell leads to interesting restrictions on the two pion decay constants. The divergence of the axial current in Eq. (2) vanishes when

$$f_\pi^t p_0^2 + f_\pi^s p^2 = 0|_{\pi \text{ mass shell}}. \quad (3)$$

At nonzero temperature each pion decay constant, f_π^t and f_π^s, has a real and an imaginary part. The pion mass shell then lies in the complex plane, at $p^0 = -i\omega - \gamma$. Equating the real parts of Eq. (3) gives

$$\omega^2 = v^2 p^2 \approx \frac{\mathrm{Re} f_\pi^s}{\mathrm{Re} f_\pi^t} p^2. \quad (4)$$

The requirement that pions travel at less than (or equal to) the speed of light, $v \leq 1$, implies $\mathrm{Re} f_\pi^s \leq \mathrm{Re} f_\pi^t$. To obtain this, we assume that the imaginary parts can be neglected relative to the real parts:

$$\mathrm{Im} f_\pi^{t,s} \ll \mathrm{Re} f_\pi^{t,s}. \quad (5)$$

Physically, $v \leq 1$ is most familiar: pions move through a medium as if it has an index of refraction greater than or equal to 1.

The imaginary part of the mass shell is given by

$$\gamma \approx \frac{1}{2\omega \ \mathrm{Re} f_\pi^t}(+\ \mathrm{Im} f_\pi^t \omega^2 - \mathrm{Im} f_\pi^s p^2) \geq 0. \quad (6)$$

The requirement that pions are damped, and not antidamped, fixes γ to be semipositive definite; using Eq. (4), this then constrains the real and imaginary parts of f_π^t and f_π^s

Our analysis only applies to "cool" pions, where the components of the pion momenta, ω and p, are small relative to the real parts of f_π^t and f_π^s. If the chiral phase transition is of second order at $T = T_\chi$, then as $T \to T_\chi^-$, $f_\pi^t(T)$ and $f_\pi^s(T) \to 0$ [15], and the region in which cool pions dominate shrinks to zero. About T_χ, over large distances the behavior of pions (and the σ meson) is controlled by an O(4) critical point, as appropriate for two massless flavors.

Assuming that the imaginary parts of f_π^t and f_π^s are nonzero at zero momentum, from Eq. (6) the damping rate vanishes linearly about zero momentum, $\gamma \sim p$ as $p \to 0$. This is consistent with Goldstone's theorem [1]: at zero momentum, the complete inverse pion propagator must vanish, including both the real and the imaginary parts. If $\gamma \sim p$, then the imaginary part of the pion self energy, $\mathrm{Im}\Pi(P)$ in Eq. (10), and so the complete inverse pion propagator $\Delta^{-1}(P)$, vanishes $\sim p^2$ as $p \to 0$. This implies that even when pions are damped, about zero momentum they still dominate the correlation functions of axial vector currents.

In a nonlinear σ model, the first contribution to the damping rate appears at two-loop order [16–18]. Using a virial expansion, such as Eq. (2.4) of Ref. [17], we estimate that about zero momentum in the chiral limit, $\gamma \sim p(T^4/f_\pi^4)$. In the linear σ model considered below, the damping rate vanishes exponentially, Eq. (27), but this is special to the kinematics at one-loop order in this model [19].

To make our conclusions rigorous, and to extend them to an approximate chiral symmetry, we follow Shore and Veneziano [3] by using a chiral Ward identity of QCD. Take two flavors of quarks, each with a (current) quark mass $= m$. A chiral Ward identity between the form factors and the propagators of the quark composite operator $\phi_5^a \hat{=} i\overline{q t^a}\gamma_5 q$ is [3]

$$\partial_\mu \langle 0|A_a^\mu|\phi_5^b\rangle_T + \langle\overline{q}q\rangle_T \langle 0|T^* \phi_5^a \phi_5^b|0\rangle_T^{-1} = 2m\,\delta^{ab}, \quad (7)$$

where $\langle\overline{q}q\rangle_T$ is the quark condensate and $\langle 0|T^*\phi_5^a\phi_5^b|0\rangle_T^{-1}$ the inverse propagator for the ϕ_5^a field. As usual, this chiral Ward identity has the same structure as at zero temperature, except that now thermal expectation values enter. Assume that ϕ_5 is directly proportional to the pion field,

$$\pi^a = b\,\phi_5^a. \quad (8)$$

The normalization constant "b" is a function of both temperature and momentum. The temperature dependence follows from our analysis, while we neglect any momentum dependence. As discussed by Shore and Veneziano [3], dropping this momentum dependence is equivalent to the usual assumptions which give the partial conservation of the axial vector current.

Using Eqs. (2) and (8) in Eq. (7), the chiral Ward identity becomes

$$-b(f_\pi^t p_0^2 + f_\pi^s p^2) + b^2 \langle\overline{q}q\rangle_T \Delta_\pi^{-1}(P) = 2m. \quad (9)$$

For the inverse pion propagator $\Delta_\pi^{-1}(P)$ we take

$$\Delta_\pi^{-1}(P) = p_0^2 + v^2 p^2 + m_\pi^2 - i\ \mathrm{Im}\Pi(P). \quad (10)$$

Similar forms of the pion propagator have appeared previously [8–10]; for us this form is motivated by the need to satisfy the chiral Ward identity.

Reference [3] requires that the pion field is canonically normalized, so the coefficient of p_0^2 in the pion propagator must be unity. We allow for a pion velocity which is less than 1 by introducing the velocity "v." Since the quark mass $m \neq 0$, we introduce a pion mass, m_π. Lastly, we introduce an imaginary part of the pion self-energy,

$\mathrm{Im}\Pi(P)$, which is a function of momentum. This form of the propagator should be valid in an expansion about zero momentum.

The chiral Ward identity shows that the assumption used to derive Eq. (3) is correct: in the chiral limit, $m=0$, the divergence of the axial vector current vanishes on the pion mass shell, as defined by the condition $\Delta_\pi^{-1}(P)=0$. Since the chiral Ward identity holds for arbitrary (small) momentum, however, we can derive several identities by matching the coefficients of p_0^2, p^2, and 1, for both the real and imaginary parts.

Equating the terms $\sim p_0^2$ fixes the constant of proportionality between the quark operator and the pion field to be

$$b=\frac{\mathrm{Re}f_\pi^t}{\langle\overline{q}q\rangle_T}. \tag{11}$$

Since both terms on the right-hand side of Eq. (11) change with temperature, so does the factor ''b.'' Matching the terms $\sim p^2$ fixes the velocity as in Eq. (4). Lastly, matching the imaginary parts in (9) gives $\mathrm{Im}\Pi(P)=2\omega\gamma$, with γ as in Eq. (6).

Away from the chiral limit, we match the real parts at zero momentum, $p_0=p=0$, to obtain the generalization of the relation of Gell-Mann, Oakes, and Renner to nonzero temperature:

$$m_\pi^2=\frac{2m\langle\overline{q}q\rangle_T}{(\,\mathrm{Re}f_\pi^t)^2}. \tag{12}$$

This is the same expression as Dashen [5] found at zero temperature, except that instead of f_π, at nonzero temperature the real part of f_π^t enters. A relation such as Eq. (12) was obtained by Thorsson and Wirzba [7]; they did not recognize, however, that in general f_π^t has an imaginary part, and so wrote just f_π^t instead of $\mathrm{Re}f_\pi^t$.

The pion mass in Eq. (12) is the dynamic pion mass, defined as the position of the singularity in the pion propagator in the complex p_0 plane at $p=0$. Alternately, we can introduce the static pion mass, as the position of the singularity in the pion propagator for $p_0=0$ in the complex p plane. From the form of the pion propagator, $m_\pi^{\mathrm{static}}=m_\pi/v$, and so by Eqs. (4) and (12) this is just

$$(m_\pi^{\mathrm{static}})^2=\frac{2m\langle\overline{q}q\rangle_T}{\mathrm{Re}f_\pi^s\,\mathrm{Re}f_\pi^t}. \tag{13}$$

Obviously, $v\leqslant 1$ implies that

$$m_\pi^{\mathrm{static}}\geqslant m_\pi. \tag{14}$$

We now consider where these effects first appear in an expansion about zero temperature. Using either a nonlinear [20] or a linear [21] σ model, to leading order in T^2/f_π^2,

$$f_\pi^t(T)=f_\pi^s(T)=\left(1-\frac{T^2}{12f_\pi^2}\right)f_\pi. \tag{15}$$

Hence to leading order in low temperature, pions move at the speed of light and are undamped. This was established by Dey, Eletsky, and Ioffe [22], who showed that to $\sim T^2/f_\pi^2$, the thermal average of the two point function of either vector or axial vector currents is directly proportional to a linear combination of those at zero temperature. Since these two point functions are Lorentz covariant at zero temperature, they remain so to $\sim T^2/f_\pi^2$.

Thus the first place where the effects which we are discussing can enter is at next-to-leading order, $\sim T^4$ [23]. The pion damping rate [16] and self-energy [17,18] have been computed to $\sim T^4/f_\pi^4$ in a nonlinear σ model. In particular, Schenk [17,18] computed the pion self-energy not in the chiral limit, but using physically reasonable approximations. His results imply that for $T=150$ MeV, $v\sim 0.87$ [24].

Instead of computing to two-loop order in a nonlinear σ model, to illustrate the effect we calculate, in weak coupling, to one-loop order in a linear σ model. In Euclidean space time the Lagrangian is

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{\mu^2}{2}\phi^2+\frac{\lambda}{4}(\phi^2)^2-h\sigma, \tag{16}$$

where $\phi=(\sigma,\vec{\pi})$ is an O(4) isovector field. We introduce a background magnetic field h which is proportional to the current quark mass m. For $h=0$, the vacuum expectation value of the σ is $\sigma_0=\sqrt{\mu^2/\lambda}$, where we then shift $\sigma\to\sigma_0+\sigma$; for two flavors, $f_\pi=\sigma_0$.

We compute terms of order $\sim T^4/(f_\pi^2 m_\sigma^2)$. There are many terms of order $\sim T^4/f_\pi^4$, at both one- and two-loop order. In weak coupling, however, the σ meson is light relative to f_π, $m_\sigma^2=2\lambda f_\pi^2$ Thus in weak coupling, which we assume, the terms of order $\sim T^4/f_\pi^4$ are smaller by $\sim\lambda$ than those computed. Conversely, in the limit of strong coupling, $m_\sigma\to\infty$, the only terms are those $\sim T^4/f_\pi^4$.

The diagrams for the pion self-energy have been computed to $\sim T^4/(f_\pi^2 m_\sigma^2)$ by Itoyama and Mueller [13]; $f_\pi^t=f_\pi^s$ has been computed to $\sim T^2/f_\pi^2$ by Bochkarev and Kapusta [21], so all we have to do is extend the calculation of f_π^t and f_π^s to this order. Consequently, we merely sketch the simplest way of performing the calculations. For the real parts, the terms of interest arise from diagrams involving a virtual σ and a π in a loop, such as

$$\mathcal{I}(P)=\mathrm{tr}_K\frac{1}{K^2((P-K)^2+m_\sigma^2)}, \tag{17}$$

where $\mathrm{tr}_K=T\Sigma_{n=-\infty}^{+\infty}\int d^3k/(2\pi)^3$. To $\sim T^2$, it suffices to approximate this integral by its value at zero momentum, neglecting the K dependence in the σ propagator, so Eq. (17) becomes

$$\mathcal{I}(P)\sim\frac{1}{m_\sigma^2}\mathrm{tr}_K\frac{1}{K^2}\sim\frac{1}{m_\sigma^2}\frac{T^2}{12}. \tag{18}$$

In the integral we have ignored apparent ultraviolet divergences to concentrate on the term $\sim T^2$. Of course renormalization is taken care of as usual at zero temperature.

To compute terms of $\sim T^4$, it is necessary to expand the integral in Eq. (17) to $\sim P^2$, including both terms $\sim P^2$ and terms $\sim P^\mu P^\nu$. In addition to the integral in Eq. (18), we also need

$$\mathrm{tr}_K \frac{K^\mu K^\nu}{K^2} \sim (\delta^{\mu\nu} - 4n^\mu n^\nu)\frac{\pi^2 T^4}{90}, \tag{19}$$

where $n^\mu = (1,\vec{0})$.

The imaginary part of expressions cannot be extracted so easily. We evaluate the imaginary part only near the pion mass shell, which is for $\omega \sim p$. In this region, the only contribution to the imaginary part of Eq. (18) is from

$$\mathrm{Im}\mathcal{I}(P) = \int \frac{d^3k}{(2\pi)^3} \frac{\pi(n_1 - n_2)}{4E_1E_2} \delta(\omega + E_1 - E_2). \tag{20}$$

In this expression $E_1 = k$ is the energy of the pion, $E_2 = \sqrt{(p-k)^2 + m_\sigma^2}$ is the energy of the σ, and $n_1 = n(E_1)$, $n_2 = n(E_2)$ are the corresponding Bose-Einstein distribution functions. This result can be obtained in various ways, such as following [25]. In all there are four possible δ functions in energy which contribute to $\mathrm{Im}\mathcal{I}(P)$. For $\omega \sim p \ll T$ only that in Eq. (20) contributes, and corresponds to Landau damping. In this region, the δ function requires

$$k = \frac{m_\sigma^2}{2(\omega + p\cos\theta)}. \tag{21}$$

We assume that $k \gg m_\sigma$, and then expand the energies accordingly; this is justified, since from Eq. (21), when $m_\sigma \gg \omega, p$, then $k \gg m_\sigma$. The result for the imaginary part is

$$\mathrm{Im}\mathcal{I}(P)|_{\omega \sim p \ll m_\sigma} \sim \frac{1}{16\pi} \exp\left(-\frac{m_\sigma^2}{4pT}\right). \tag{22}$$

Because the fields being scattered have large momentum, the Bose-Einstein distribution functions are essentially Boltzmann, which generates the exponential suppression seen in Eq. (22).

These integrals are sufficient to reproduce the results of Ref. [13] for the pion self-energy. To evaluate the corresponding terms for the pion structure constants, we need the axial vector current in the linear σ model:

$$A_\mu^a = (\sigma_0 + \sigma)\partial_\mu \pi^a - \pi^a \partial_\mu \sigma. \tag{23}$$

The diagrams which contribute at one-loop order to f_π^t and f_π^s are given in Fig. (5) of [21]. In addition to the pion self-energy, there is a contribution from a σ-π loop at the vertex for A_a^μ These contributions can be evaluated expanding integrals like Eq. (17) and using Eq. (19). For the imaginary parts, we need the integrals

$$\mathrm{tr}_K \frac{k^0}{K^2((P-K)^2 + m_\sigma^2)}\Big|_{\omega \sim p \ll m_\sigma} \sim \frac{i}{16\pi}\left(\frac{m_\sigma^2}{4p} + T\right)\exp\left(-\frac{m_\sigma^2}{4pT}\right) \tag{24}$$

and

$$\mathrm{tr}_K \frac{k^i}{K^2((P-K)^2 + m_\sigma^2)}\Big|_{\omega \sim p \ll m_\sigma} \sim \frac{p^i}{16p\pi}\left(\frac{m_\sigma^2}{4p} - T\right)\exp\left(-\frac{m_\sigma^2}{4pT}\right). \tag{25}$$

The results of the computations are as follows. At one-loop order the quantity

$$t_1 = \frac{T^2}{12 f_\pi^2} \tag{26}$$

typically enters. To the order we work, we also need

$$t_2 = \frac{\pi^2}{45}\frac{T^4}{f_\pi^2 m_\sigma^2},$$

$$t_3 = \frac{1}{32\pi}\frac{m_\sigma^4}{f_\pi^2 p^2}\exp\left(-\frac{m_\sigma^2}{4pT}\right). \tag{27}$$

Then at weak coupling in the linear σ model, to $\sim T^4/(f_\pi^2 m_\sigma^2)$,

$$f_\pi^t \sim (1 - t_1 + 3t_2 + it_3) f_\pi,$$

$$f_\pi^s \sim (1 - t_1 - 5t_2 - it_3) f_\pi. \tag{28}$$

By Eq. (4) the pion velocity is

$$v^2 \sim 1 - 8t_2, \tag{29}$$

while from Eq. (6) the pion mass shell is

$$ip^0 \sim vp - ipt_3. \tag{30}$$

Including the one-loop self-energy computed to this order, the pion propagator is

$$Z_\pi \Delta^{-1}(P)|_{\omega \sim p \ll m_\sigma} \sim (1 + t_1 + 6t_2)p_0^2 + (1 + t_1 - 2t_2)p^2 + m_\pi^2(1 + 3t_1/2) - 2ip^2 t_3. \tag{31}$$

To ensure that $\Delta^{-1}(P)$ has canonical normalization we introduce a factor for wave-function renormalization of the pion:

$$Z_\pi \sim 1 + t_1 + 6t_2. \tag{32}$$

It is elementary to check that the zero of Eq. (31) agrees with Eq. (30). We have included the results to leading order in the external field h, when the pion mass is nonzero. Assuming that the quark condensate is proportional to the vacuum expectation value of the σ field,

$$\langle \overline{q}q \rangle_T \sim \sigma_0(T) \sim \sigma_0(0)(1 - 3t_1/2), \tag{33}$$

we also verify our generalization of the formula of Gell-Mann, Oakes, and Renner in Eq. (12) for the dynamic pion mass:

$$m_\pi^2(T) \sim m_\pi^2(1 + t_1/2 - 6t_2). \tag{34}$$

We conclude with some general comments. First, while the effects computed at low temperature (26)–(34) are small, that does not mean that they remain so for temperatures of physical interest, as seen in the results of [8–10]. Secondly, the form of the inverse propagator in Eq. (10) applies not just to Goldstone bosons, but to any scalar field at nonzero temperature. For example, in numerical simulations on the lattice in Euclidean spacetime, typically what is measured is only the static mass, not the dynamic.

Finally, we note that the coefficient of v^2-1 in Eq. (29) is proportional to the free energy density for pions, $=\pi^2T^4/30$. (It would be interesting to know what the analogous coefficient is for the nonlinear σ model in the chiral limit.) This and other examples [26] hint of a general relation, valid for all temperatures, where the deviation of the velocity squared from unity is proportional to the free energy density [6].

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