PHYSICAL REVIEW D VOLUME 54, NUMBER 12 15 DECEMBER 1996

Summation over histories for the Friedmann universe

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(Received 15 April 1996)

An explicit path-integral treatment of the Friedmann minisuperspace model with minimally coupled massless scalar fields is presented. Unlike the usual approach where a semiclassical approximation is used, an exact path integration is performed. [S0556-2821(96)04022-2]

PACS number(s): 04.60.Gw, 98.80.Hw

I. INTRODUCTION

Feynman's summation over histories, or path integration, has commonly been employed [1–5] to quantize gravity most especially in minisuperspace models where all degrees of freedom are ''frozen'' except the one selected to be quantized [2–6]. To evaluate the path integral, however, various authors either reduce the path integral to its corresponding differential equation [2,3], or apply a semiclassical approximation [3–5] and, consequently, no explicit path integration is performed. In this paper, we evaluate the Friedmann minisuperspace model with minimally coupled massless scalar fields by directly path integrating the system from which a closed form for the propagator is obtained.

In the path-integral approach to quantize gravity, one evaluates the propagator of the form

$$K=\int \exp\{(i/\hbar)S\}D[g_{\mu\nu}]D[\phi], \tag{1.1}$$

where $g_{\mu\nu}$ is the spacetime metric, and ϕ a scalar field. The action, $S=S_g+S_m$, for the system we will consider here consists of the gravitational part given by

$$S_g=(1/16\pi G)\int R\sqrt{-g}d^4x \tag{1.2}$$

and the action for the matter fields of the form

$$S_m=\int \tfrac{1}{2}[g_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi)-m^2\phi^2]\sqrt{-g}d^4x. \tag{1.3}$$

Our task, therefore, is to evaluate Eq. (1.1) given Eqs. (1.2) and (1.3). In Sec. II, we briefly discuss the specific form of the action when we consider the Friedmann universe, and, in Sec. III, an explicit path-integral calculation will be carried out for the case where the scalar fields are massless. The propagator for a Friedmann universe coupled to a scalar field is then obtained. Section IV contains the conclusions.

II. THE ACTION FOR THE FRIEDMANN UNIVERSE WITH SCALAR FIELDS

In order to get a suitable form for the gravitational action (1.2), we note that a spacetime metric may be written locally in the form

$$ds^2=-(N^2-N_iN^i)dt^2+2N_idx^idt+h_{ij}dx^idx^j, \tag{2.1}$$

where the three-metric is denoted by h_{ij} $(i,j=1,2,3)$. With Eq. (2.1), the gravitational action (1.2) can be written as

$$S_g=(1/16\pi G)\int (K_{ij}K^{ij}-K^2+{}^{(3)}R)N\sqrt{h}d^4x, \tag{2.2}$$

where terms corresponding to total derivatives have been dropped [6]. Here, $h=\det(h_{ij})$, ${}^{(3)}R$ is the scalar curvature constructed from h_{ij}, and

$$K_{ij}=(1/2N)[-(\partial h_{ij}/\partial t)+N_{i;j}+N_{j;i}], \tag{2.3a}$$

$$K=h^{ij}K_{ij}, \tag{2.3b}$$

where the semicolon denotes covariant differentiation based on the three-metric h_{ij}.

Let us now consider a homogeneous, isotropic, closed universe with a metric of the form

$$ds^2=-dt^2+a^2(t)d\Omega_3^2, \tag{2.4}$$

where $a(t)$ is the time-varying radius and, $d\Omega_3^2=\Omega_{ij}^0dx^idx^j$ $(i,j=1,2,3)$, for Ω_{ij}^0 a metric of a three-sphere of unit radius. With Eq. (2.4) the gravitational action (2.2) becomes

$$S_g=(3\pi/4G)\int[-a(da/dt)^2+a]dt. \tag{2.5}$$

In this minisuperspace model [6] described by Eq. (2.5), we have frozen all degrees of freedom except the time-varying parts like the spherical radius $a(t)$, and the integral of the volume of the Friedmann universe given by $2\pi^2a^3$, as well as ${}^{(3)}R=6/a^2$, have been used.

The action for the scalar field (1.3), on the other hand, for this Friedmann minisuperspace model becomes

$$S_m=\pi^2\int[-(d\phi/dt)^2-m^2\phi^2]a(t)^3dt. \tag{2.6}$$

Note that, having frozen all degrees of freedom except the time-dependent parts, we have $(\partial^i\phi)=0$, in Eq. (1.3) and with the factor $a(t)^3$ in Eq. (2.6), which comes from the volume integral of the Friedmann universe, the matter field does not completely decouple from the gravitational part.

With Eqs. (2.5) and (2.6), the total action becomes

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$$S=\int\{(3\pi/4G)[-a(da/dt)^2+a]+\pi^2[-(d\phi/dt)^2-m^2\phi^2]a(t)^3\}dt. \quad (2.7)$$

To facilitate the path integration of Eq. (1.1) for the action (2.7), we introduce a rescaled time $d\tau$ defined by

$$d\tau=dt/a(t), \quad (2.8)$$

which allows us to write Eq. (2.7) as

$$S=\int\{-\tfrac{1}{2}\mu(da/d\tau)^2+\tfrac{1}{2}\mu a^2-\pi^2a^2(d\phi/d\tau)^2-\pi^2m^2a^4\phi^2\}d\tau, \quad (2.9)$$

where $\mu=6\pi/4G$. We note that the rescaling of the form (2.8) was also used by the authors of Ref. [3] [e.g. Eq. (5.16)]. The case discussed in Ref. [3], however, has an action where the matter fields completely decouple, or separate, from the gravitational part and, hence, differs from our case. With Eq. (2.9), we shall now evaluate the path integral (1.1) in the next section.

III. SUMMATION OVER HISTORIES

Using Eq. (2.9), the path integral (1.1) now becomes

$$K(a'',\phi'';a',\phi')=\int \exp\left[(i/\hbar)\int\{-\tfrac{1}{2}\mu(da/d\tau)^2+\tfrac{1}{2}\mu a^2-\pi^2a^2(d\phi/d\tau)^2-\pi^2m^2a^4\phi^2\}d\tau\right]D[a]D[\phi]. \quad (3.1)$$

In Eq. (3.1), we have $D[a]$ instead of $D[g_{\mu\nu}]$ since quantization has been restricted to the time-varying $a(t)$ having frozen the other degrees of freedom. The $K(a'',\phi'';a',\phi')$ gives the probability amplitude for the variables a and ϕ to have values of a', ϕ' at time τ', and a'', ϕ'' at time τ''. This propagator may be used to study the quantum evolution of the system where the wave function at time τ'' is obtained from the relation

$$\Psi(a'',\phi'')=\int K(a'',\phi'';a',\phi')\Psi(a',\phi')d[a',\phi']. \quad (3.2)$$

Note that the integral in Eq. (3.2) may be carried out given a choice of the initial wave function $\Psi(a',\phi')$. For example, an initial wave function may be chosen to be a Gaussian wave packet [7] or, alternatively, the no-boundary proposal of Ref. [3] may be adopted.

Let us now proceed to evaluate the path integral (3.1) by writing it as

$$K(a'',\phi'';a',\phi')=\int \exp\left[(i/\hbar)\int\{-\tfrac{1}{2}\mu(da/d\tau)^2+\tfrac{1}{2}\mu a^2\}d\tau\right]K(\phi'',\phi')D[a], \quad (3.3)$$

where the integral over ϕ involves the expression

$$K(\phi'',\phi')=\int \exp\left[(i/\hbar)\int\{-\pi^2a^2(d\phi/d\tau)^2-\pi^2m^2a^4\phi^2\}d\tau\right]D[\phi]. \quad (3.4)$$

At this stage, an exact path-integral treatment will be done for the case where $m=0$. Using Feynman's prescription, we divide the time into N subintervals: i.e., $(\tau''-\tau')/N=\varepsilon_j=\tau_j-\tau_{j-1}$, and defining $a_j=a(\tau_j)$, $\phi_j=\phi(\tau_j)$, and $\Delta\phi_j=\phi_j-\phi_{j-1}$ for $j=1,\ldots,N$, Eq. (3.4) becomes

$$K(\phi'',\phi')=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\{-i\pi^2\hat{a}_j^2(\Delta\phi_j)^2/\hbar\varepsilon_j\}\times\prod_{j=1}^{N}[-\pi\hat{a}_j^2/i\hbar\varepsilon_j]^{1/2}\prod_{j=1}^{N-1}d(\phi_j), \quad (3.5)$$

where $\hat{a}_j^2=(a_ja_{j-1})$, and we defined $D[\phi]$ with normalization factors as

$$D[\phi]=\lim_{N\to\infty}\prod_{j=1}^{N}[-\pi\hat{a}_j^2/i\hbar\varepsilon_j]^{1/2}\prod_{j=1}^{N-1}d(\phi_j). \quad (3.6)$$

If we next introduce a new time variable given by

$$\sigma_j=-\varepsilon_j/\hat{a}_j^2, \quad (3.7)$$

Eq. (3.5) can be written as

$$K(\phi'',\phi')=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\{i\pi^2(\Delta\phi_j)^2/\hbar\sigma_j\}\times\prod_{j=1}^{N}[\pi/i\hbar\sigma_j]^{1/2}\prod_{j=1}^{N-1}d(\phi_j). \quad (3.8)$$

Equation (3.8) for the path integration over ϕ, where $-\infty<\phi<+\infty$, is analogous to the time-sliced propagator for a free particle [8] evolving in σ time and can then be easily evaluated with the result

$$K(\phi'',\phi')=(1/2\pi)\int_{-\infty}^{+\infty}dp\,\exp\{ip(\phi''-\phi')-i(p^2/4\pi^2)\hbar\sigma\}, \quad (3.9)$$

where $\sigma=\Sigma_j\sigma_j$. We note from Eq. (3.7) that σ contains the variable a and this term in the exponential must be included when path integration over the a variable is carried out. With Eq. (3.9), Eq. (3.3) becomes

$$K(a'',\phi'';a',\phi')=(1/2\pi)\int_{-\infty}^{+\infty}dp\,\exp[ip(\phi''-\phi')]\times K(a'',a'), \quad (3.10)$$

where the a-dependent part, $K(a'',a')$, in time-sliced form appears as

$$K(a'',a') = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\{[-i\mu(\Delta a_j)^2/2\hbar\varepsilon_j] + [i\mu a_j^2 \varepsilon_j/2\hbar] + [ip^2\hbar\varepsilon_j/4\pi^2\hat{a}_j^2]\} \times \prod_{j=1}^{N} [-\mu/2\pi i\hbar\varepsilon_j]^{1/2} \prod_{j=1}^{N-1} d(a_j). \quad (3.11)$$

Equation (3.11) is a one-dimensional path integral over the a variable where we defined the $D[a]$, with normalization factors as

$$D[a] = \lim_{N\to\infty} \prod_{j=1}^{N} [-\mu/2\pi i\hbar\varepsilon_j]^{1/2} \prod_{j=1}^{N-1} d(a_j). \quad (3.12)$$

If we allow the time-varying radius a to have the range $0 \leqslant a < \infty$, we can evaluate Eq. (3.11) by using results from path integration in spherical polar coordinates where the radial path integral has the form [9]

$$\int \exp\left\{(i/\hbar)\int [\tfrac{1}{2}m\dot{r}^2 - (\nu^2-\tfrac{1}{4})/2mr^2 - \tfrac{1}{2}m\omega^2 r^2]d\tau\right\} D[r] = -i(r'r'')^{1/2} m\omega \csc(\omega\tau)\exp[(1/2\hbar)im\omega(r'^2+r''^2)\cot(\omega\tau)] \times I_\nu[-im\omega r'r'' \csc(\omega\tau)/\hbar] \quad (3.13)$$

for $\mathrm{Re}(\nu) > -1$. Equation (3.11) when written in the unsliced-time form (and taking $\tau \to -\tau$) appears as

$$K(a'',a') = \int \exp\left\{(i/\hbar)\int [\tfrac{1}{2}\mu\dot{a}^2 - (\nu^2-\tfrac{1}{4})/2\mu a^2 - \tfrac{1}{2}\mu\omega^2 a^2]d\tau\right\} D[a], \quad (3.14)$$

which is identical to the left-hand side of Eq. (3.13) where $\nu = \frac{1}{2}[1+(3p^2\hbar^2/\pi G)]^{1/2}$ and $\omega = 1$. Hence the path integral over the variable a with the help of Eq. (3.13) yields the result

$$K(a'',a') = -i(a'a'')^{1/2}\mu \csc(\tau)\exp[(1/2\hbar)i\mu(a'^2+a''^2)\cot(\tau)]I_\nu[-i\mu a'a'' \csc(\tau)/\hbar]. \quad (3.15)$$

An alternative form for Eq. (3.15) can be obtained using the relation

$$[1/i\hbar \sin(\tau)]\exp\{(i/2\hbar)(\rho'^2+\rho''^2)\cot(\tau)\}I_\nu[\rho'\rho'' \csc(\tau)/i\hbar] = 2\hbar^{-(\nu+1)}(\rho'\rho'')^\nu \exp\{-(1/2\hbar)(\rho'^2+\rho''^2)\}\sum_{n=0}^{\infty}\{[n!/\Gamma(n+\nu+1)]L_n^\nu(\rho'^2/\hbar)L_n^\nu(\rho''^2/\hbar)\exp[-i\tau(2n+\nu+1)]\}, \quad (3.16)$$

where $L_n^\nu(x)$ are the generalized Laguerre polynomials, which allows us to write Eq. (3.15) as

$$K(a'',a') = \sum_{n=0}^{\infty} R_{nl}(\rho')R_{nl}(\rho'')\exp[-i(2n+\nu+1)\tau], \quad (3.17)$$

where $\rho = \sqrt{\mu}a = (6\pi/4G)^{1/2}a$, and

$$R_{nl}(\rho) = (\mu\hbar^{-2\nu})^{1/4}\sqrt{2n!/\Gamma(n+\nu+1)}\rho^{\nu+1/2} \times \exp(-\rho^2/2\hbar)L_n^\nu(\rho^2/\hbar). \quad (3.18)$$

With Eq. (3.17), we arrive at a closed form for the full propagator, Eq. (3.1) or (3.10), given by

$$K(a'',\phi'';a',\phi') = \sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} dp\, B(\phi'',a'')B(\phi',a')^* \times \exp[-i(2n+\nu+1)\tau], \quad (3.19)$$

with $\nu = \frac{1}{2}[1+(3p^2\hbar^2/\pi G)]^{1/2}$, and from Eq. (2.8), $\tau = \int dt/a(t)$, and where

$$B(\phi,a) = (1/\sqrt{2\pi})\exp(ip\phi)R_{nl}(\sqrt{\mu}a). \quad (3.20)$$

The Friedmann universe with massless scalar fields, therefore, provides an example of an exactly path-integrable minisuperspace model.

IV. CONCLUSIONS

In this paper, a path-integral treatment of the Friedmann minisuperspace model with scalar fields was presented. The treatment differs from those normally found in the literature where a semiclassical approximation is employed [3–5], or the corresponding differential equation of the path integral is resorted to [2,3,6] when handling the Friedmann universe with matter fields. Moreover, the present approach also differs from the one found for example in Ref. [3] in two other ways: (a) We did not use a Euclidean action for gravity, but employed instead a Minkowski signature action as advocated in Ref. [4]; and (b) the model discussed here has a scalar field that does not completely separate from the gravitational part.

We also note that since an integral equation such as Eq.

(3.2) includes a boundary condition, it appears to be more powerful than the differential equation, i.e., the Wheeler-DeWitt equation, which arises from the constraints in the canonical quantization approach. General investigations on the relation between the path-integral approach and the Wheeler-DeWitt equation, however, has been made [3,10] and a further examination of this connection will be addressed elsewhere.

Lastly, exact path integration of the variables to be quantized was made possible in this paper by taking advantage of its similarity to the path-integral formalism for quantum relativistic particles in curved spacetime where exact path integration can also be carried out [11]. One hopes that the techniques developed to path integrate relativistic particles in curved spacetime would also be useful in solving other minisuperspace models. In particular, the path-integral treatment of the Friedmann universe with $m\neq 0$ for the scalar field is currently under investigation.

ACKNOWLEDGMENTS

A research grant provided by the University of the Philippines is gratefully acknowledged.

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