Improved Cauchy horizon stability conjecture

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An improved stability conjecture for Cauchy horizons is presented. The conjecture predicts the stability of Cauchy horizons based upon the behavior of test fields, and in the case of instability it also predicts the nature of the singularities produced. The results for Cauchy horizons in Reissner-Nordström, Kerr, Reissner–Nordström–de Sitter, Kerr–de Sitter, anti–de Sitter, and a type V LRS spacetime are reviewed. A new prediction is made for scalar fields in anti–de Sitter spacetime. The improved conjecture agrees with the stability and singularity types in all cases for which exact back reaction solutions have been found. [S0556-2821(96)03424-8]

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I. INTRODUCTION

A stability conjecture for Cauchy horizons was recently shown to be incomplete [1]. Although the conjecture correctly predicts the occurrence of singularities when fields are added to a number of spacetimes containing Cauchy horizons, it fails in a single case [anti–de Sitter] to predict the *type* of singularity formed. In this paper we modify the conjecture to remove this discrepancy. We also use scalar fields to further investigate the anti–de Sitter case.

We use a singularity classification scheme based on one devised by Ellis and Schmidt [2]. They classified singularities in maximal spacetimes into three basic types: quasiregular, nonscalar curvature, and scalar curvature. The mildest singularity is quasiregular, and the strongest is scalar curvature. Physical quantities such as energy density and tidal forces diverge for all observers who approach a scalar curvature singularity. No observers see physical quantities diverge as they approach a quasiregular singularity, even though their world lines end at the singularity in a finite proper time. Finally, some but not all observers feel infinite tidal forces as they approach a nonscalar curvature singularity, even though no physical scalars diverge.

The classification scheme can be expressed mathematically. Start with a maximal spacetime with incomplete geodesics. A singular point q is a C^k (or C^{k-}) *quasiregular* singularity ($k \geqslant 0$) if all components and appropriate derivatives of the Riemann tensor $R_{abcd;e_1 e_2 \cdots e_k}$ evaluated in an orthonormal (ON) frame parallel propagated (PP) along an incomplete geodesic ending at q are C_0 (or C^{0-}). In other words, the Riemann-tensor components and derivatives tend to finite limits (or are bounded). On the other hand, a singular point q is a C^k (or C^{k-}) *curvature* singularity if some components or derivatives are not bounded in this way. If all scalars in g_{ab}, the antisymmetric tensor η_{abcd}, and $R_{abcd;e_1 e_2 \cdots e_k}$ nevertheless tend to a finite limit (or are bounded), the singularity is *nonscalar*, but if any scalar is unbounded, the point q is a *scalar* curvature singularity.

II. ORIGINAL SINGULARITY CONJECTURE

In previous papers [3–5], we have used a stability conjecture for Cauchy horizons (CH's) which states the following.

Original conjecture. For all maximally extended spacetimes with CH's, the back reaction due to a field (whose test-field stress-energy tensor is $T_{\mu\nu}$) will affect the horizon in the following manner: (1) If both $T^{\mu}{}_{\mu}$ and $T_{\mu\nu}T^{\mu\nu}$ are finite and if the stress-energy tensor $T_{(ab)}$ in all PPON frames is finite, then the CH will remain nonsingular; (2) if both $T^{\mu}{}_{\mu}$ and $T_{\mu\nu}T^{\mu\nu}$ are finite, but $T_{(ab)}$ diverges in some PPON frame, then a nonscalar curvature singularity will be formed at the CH; (3) if either $T^{\mu}{}_{\mu}$ or $T_{\mu\nu}T^{\mu\nu}$ diverges, then a scalar curvature singularity will be formed at the CH.

This conjecture has been applied to CH's in Reissner-Nordström, Kerr, Reissner–Nordström–de Sitter, Kerr–de Sitter, anti–de Sitter, and a type-V LRS spacetime. Only in the type-V LRS spacetime is the CH predicted to be completely stable [3]; all points on the CH are stable under perturbation by all modes of a massless scalar field. However, in that case no exact back reaction solution, in which the field is allowed to influence the geometry, has been carried out with which to compare the prediction. In the other spacetimes at least one point on the CH is generally predicted to be unstable to the formation of singularities. Here and in the following we use the word ''predict'' to indicate the application of the test-field conjecture to a particular spacetime, whether or not an exact back reaction calculation has already been carried out.

In the Reissner-Nordström spacetime [4] the conjecture

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predicts that the addition of infalling null dust with a power-law tail produces a nonscalar curvature singularity at the CH, in agreement with the exact Reissner-Nordström-Vaidya spacetime already studied by Hiscock [6]. The conjecture also predicts that a combination of infalling and outgoing null dust produces a scalar curvature singularity at the CH. This result agrees with the mass-inflation results of Poisson and Israel [7–9] and Ori [10], and the numerical calculations of Brady and Smith [11]. Finally, with the addition of infalling scalar or electromagnetic waves, the conjecture predicts a scalar curvature singularity at the CH; we have found no exact solutions with which to verify the conjecture in these latter cases.

The Kerr spacetime [5] predictions are similar to the Reissner-Nordström case. We have shown that under the addition of the lowest-mode electromagnetic field, the CH should remain nonsingular, in agreement with the back reaction calculation embodied in the Kerr-Newman solution. With infalling null dust, a nonscalar curvature singularity is predicted unless the density falls off like e^{-2av} (where a is the rotation parameter and v a null coordinate), in which case the CH remains nonsingular. Adding any outgoing radiation is sufficient to turn the nonscalar curvature singularity into a scalar curvature singularity. Although no exact solutions exist with which to verify these predictions, an analysis of the nonlinear instability of Kerr-type Cauchy horizons has been carried out by Brady and Chambers [12]; their results agree with those of the conjecture.

Cai and Su [13] have used the conjecture to investigate Reissner–Nordström–de Sitter black holes [14–16] when infalling and both infalling and outgoing null dust are added. For purely infalling null dust, the conjecture predicts that the Cauchy horizon is stable if $k_i \leq k_c$, where k_i and k_c represent, respectively, the surface gravities of the Cauchy horizon and the cosmological horizon. If $k_i > k_c$, however, the conjecture predicts that a nonscalar curvature singularity forms at the CH. Cai and Su show that this agrees with an exact solution [13]. With both infalling and outgoing null dust, the conjecture predicts that when $k_i \leq k_c$, the Cauchy horizon is completely stable, but a scalar curvature singularity forms when $k_i > k_c$. This agrees with an exact back reaction calculation of Brady, Nunez, and Sinha [16].

Cai and Su have also considered Kerr–de Sitter black holes [13]. The predictions they make are similar to those in the Reissner–Nordström–de Sitter case. Using the conjecture, they predict that the CH of Kerr–de Sitter black holes is unstable for $k_i > k_c$ and stable for $k_i \leq k_c$. As in the Kerr case, no exact back reaction calculations are available, but recent linear perturbation calculations are in complete agreement [17].

Finally, we have also considered the universal covering space of anti–de Sitter spacetime (AdS) [1]. Null infinity is timelike, and so the spacetime contains no global Cauchy surfaces [18]. If one places initial data on a spacelike surface, one cannot predict beyond its Cauchy development; there are Cauchy horizons. Since AdS is maximally symmetric, if a different spacelike surface were chosen, different Cauchy horizons would be formed. Adding radially infalling null dust, the scalars $T^{\mu}{}_{\mu}$ and $T^{\mu\nu}T_{\mu\nu}$ both vanish everywhere, but in a PPON frame some components $T_{(ab)} \rightarrow \infty$ at $r=0$: A nonscalar curvature singularity is therefore predicted to form. However, examination of an exact anti–de Sitter–Vaidya spacetime [1] shows that in fact a scalar curvature singularity forms at $r=0$ in the case of purely infalling null dust. The conjecture correctly predicts a singularity, but it predicts the wrong kind. In the presence of both infalling and outgoing null dust, the conjecture predicts a scalar curvature singularity at $r=0$; there is, however, no exact solution with which to compare in this case.

III. IMPROVED STABILITY CONJECTURE

The original stability conjecture correctly predicts the stability or instability of CH's in all verifiable cases; however, in anti-de Sitter spacetime the conjecture fails to predict the correct singularity type at $r=0$ in the presence of infalling null dust. Further study of scalars in the Riemann, Ricci, and Weyl tensors show that it is the Weyl portion of the curvature which is causing the divergence at $r=0$. Because the conjecture inspects test-field stress-energy tensors, which can be related to the Ricci tensor through the field equations, it is not surprising that the conjecture can only predict divergences in the Ricci tensor and not the Weyl tensor portion of the curvature.

The conjecture must be altered. In the anti–de Sitter case, it is easy to see that a finite-density shell of null dust achieves infinite density (an infinite concentration of dust) when it reaches $r=0$. Thus, by examining a physical scalar quantity in the test field (i.e., the dust density), one can correctly predict the type of singularity in the anti–de Sitter case.

The improved version of the stability conjecture therefore reads as follows.

Improved conjecture. For all maximally extended spacetimes with CH's, the back reaction due to a field (whose test-field stress-energy tensor is $T_{\mu\nu}$) will affect the horizon in the following manner: (1) If both $T^{\mu}{}_{\mu}$ and $T^{\mu\nu}T_{\mu\nu}$ are finite, if no scalar physical field or matter quantity (e.g., density) diverges, and if the stress-energy tensor $T_{(ab)}$ in all PPON frames is finite, then the CH will remain nonsingular. (2) If $T^{\mu}{}_{\mu}$, $T^{\mu\nu}T_{\mu\nu}$, and physical scalar quantities are finite, but $T_{(ab)}$ diverges in some PPON frame, then a nonscalar curvature singularity will be formed at the CH. (3) If one of $T^{\mu}{}_{\mu}$, $T^{\mu\nu}T_{\mu\nu}$, or a physical scalar diverges, then a scalar curvature singularity will be formed at the CH.

This improved conjecture works in all of the cases tested, in particular the anti-de Sitter case for which the original conjecture fails.

IV. SCALAR WAVES ON ANTI–de SITTER SPACETIME

It is useful to study the interesting anti–de Sitter case more fully. Here we look at the effect of massless scalar waves on an anti–de Sitter background and study the predictions of the conjecture.

In Einstein universe coordinates, the metric for anti–de Sitter spacetime is [1]

$$ds^2=(\alpha\cos\psi)^{-2}(-dt^2+d\psi^2+\sin^2\psi d\Omega^2), \qquad (1)$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$, $0\leq\psi<\pi/2$, $0\leq\theta\leq\pi$, and $0\leq\phi<2\pi$. In ordinary anti–de Sitter space, t is restricted to $-\pi\leq t\leq\pi$, the topology is $S^1\times R^3$, and there are closed timelike lines. In the universal covering space AdS (which we consider here), the space is unwrapped S^1 to R, giving an R^{24} topology with $-\infty<t<\infty$ and no closed timelike lines.

Massless minimally coupled scalar waves obey the equation [19]

$$\Box\Phi=g^{\mu\nu}\Phi,_{\mu\nu}+g^{-1/2}(g^{1/2}g^{\mu\nu}),_{\nu}\Phi,_{\mu}=0. \tag{2}$$

From Eqs. (1) and (2) we find

$$(-\Phi_{,00}+\Phi_{,11})+\csc^2\psi(\Phi_{,22}+\csc^2\theta\Phi_{,33})+2\csc\Phi\sec\psi\Phi_{,1} + \csc^2\psi\cot\theta\Phi_{,2}=0, \tag{3}$$

where (0, 1, 2, 3) $=(t,\psi,\theta\phi)$. Mode solutions have the form

$$\Phi(t,\psi,\theta,\phi)=\Psi(\psi)Y_{\ell m}(\theta,\phi)e^{i\omega t} \tag{4}$$

in terms of spherical harmonics $Y_{\ell m}$, where $\Psi(\psi)$ obeys the equation

$$(\sin^2\psi)\Psi''+2\tan\psi\Psi'+[\omega^2\sin^2\psi-\ell(\ell+1)]\Psi=0. \tag{5}$$

We try solutions of the form

$$\Psi=(\sin\psi)^{\ell}(\cos\psi)^{\lambda}P(x), \tag{6}$$

where $x=\cos\beta\psi$, with β a constant, and $P(x)$ obeys

$$(1-x^2)P'' + \left(-x-\frac{2}{\beta}\sqrt{1-x^2}(\ell\cot\psi-\lambda\tan\psi+\sec\psi\csc\psi)\right)P' + \frac{1}{\beta^2}[-\ell^2-2\lambda\ell+\lambda^2\tan^2\psi-3\lambda\sec^2\psi+\omega^2]P=0, \tag{7}$$

where the derivatives are with respect to x. With the choice $\beta=2$, $\lambda=3$, the equation becomes

$$(1-x^2)P''+[1-\ell-x(\ell+4)]P'+\tfrac{1}{4}[-(\ell+3)^2+\omega^2]P =0, \tag{8}$$

which is the equation for Jacobi functions [13]. Solutions are the Jacobi polynomials $P_n^{\ell+1/2,3/2}(x)$ and the Jacobi functions of the second kind $Q_n^{\ell+1/2,3/2}(x)$, where the index $n=1/2(-\ell-3\pm\omega)$. We must discard the Jacobi functions of the second kind to ensure finite data at all points on an initial spacelike slice. The Jacobi polynomials form a complete set of functions, and so any finite initial data can be expressed in terms of them.

The scalar wave mode solutions are thus

$$\Phi_{\omega\ell m}=(\sin\psi)^{\ell}(\cos\psi)^3P_n^{\ell+1/2,3/2}(\cos2\psi)Y_{\ell m}(\theta,\phi)e^{i\omega t}, \tag{9}$$

where $n=0,1,2,\ldots$, and $\omega=\pm(2n+\ell+3)$. Each mode is finite for $0\leqslant\psi<\pi/2$, the full range of ψ. The total solution is the sum

$$\Phi=\sum_{\omega,\ell,m}N_{\omega\ell m}\Phi_{\omega\ell m} \tag{10}$$

in terms of constants $N_{\omega\ell m}$, which are arbitrary except that those constants with equal but opposite ω's are related to keep Φ real.

The stress-energy tensor for a massless minimally coupled scalar field is

$$T_{\mu\nu}=\frac{1}{4\pi}[\Phi,_{\mu}\Phi,_{\nu}-\tfrac{1}{2}g_{\mu\nu}S], \tag{11}$$

where $S=g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}$. The scalars are

$$T^{\mu}{}_{\mu}=-S, \tag{12}$$

$$T_{\mu\nu}T^{\mu\nu}=S^2, \tag{13}$$

where

$$S=-\alpha^2\cos^2\psi\Phi_{,t}\Phi_{,t}+\alpha^2\cos^2\psi\Phi_{,\psi}\Phi_{,\psi}+\alpha^2\cot^2\psi\Phi_{,\theta}\Phi_{,\theta} +\alpha^2\cot^2\psi\csc^2\theta\Phi_{,\phi}\Phi_{,\phi}. \tag{14}$$

The only term in S which can give a possible divergence is the second term, which involves $\Phi_{,\psi}$. The ω,ℓ,m mode of Φ has the derivative

$$\Phi_{,\psi}=\Bigg\{[\ell(\sin\psi)^{\ell-1}(\cos\psi)^4 -3(\sin\psi)^{\ell+1}(\cos\psi)^2]P_n^{\ell+1/2,3/2}(\cos2\psi) -\frac{(\sin\psi)^{\ell-1}(\cos\psi)^2}{[2n+\ell+2]} \times[n(\ell-1)-(2n+\ell+2)\cos2\psi]P_n^{\ell+1/2,3/2}(\cos2\psi) +2(n+\ell+1/2)(n+3/2)P_{n-1}^{\ell+1/2,3/2}(\cos2\psi)\Bigg\} \times Y_{\ell m}(\theta,\phi)e^{i\omega t}. \tag{15}$$

This quantity diverges at $\psi=0$ only if $\ell=0$. That is, only the spherically symmetric $(\ell=0)$ modes cause the stress-energy scalars to diverge at $\psi=0$.

Zeros of the radial coordinate $r=\alpha^{-1}\tan\psi$ occur at $\psi=0$, and so the spherically symmetric scalar field modes diverge at $r=0$. The conjecture thus predicts that $r=0$ on the CH (and in fact the entire $r=0$ world line) becomes a scalar curvature singularity if any $\ell=0$ modes are introduced. On the other hand, if only $\ell\neq0$ modes are added, no singularity will from on the CH (or, in fact, anywhere in the spacetime).

This result agrees with the null dust results reviewed in Sec. II: Spherically symmetric infalling and outgoing null dust was predicted to produce a scalar curvature singularity at $r=0$. This scalar field case is interesting because it clearly points out the necessity of the spherically symmetric constraint. Fields which are not spherically symmetric are not focused onto $r=0$ and do not lead to any singularity formation.

It is interesting to note that these results really have nothing to do with the existence or nonexistence of CH's in a spacetime. Consider a spherically symmetric shell of infalling dust in Minkowski space. It too will form a scalar curvature singularity at $r=0$. The exact back reaction solution which illustrates this shell-focusing singularity is the Vaidya solution (albeit with infalling, rather than outgoing, radiation). However, since the CH in the anti–de Sitter spacetime contains an $r=0$ point, any Cauchy horizon stability conjecture must account for these results. Our improved Cauchy horizon stability conjecture does so.

V. CONCLUSION

An improved Cauchy horizon stability conjecture holds in all the special cases examined. Why is the conjecture important? Its usefulness arises from the fact that while we need to understand the stability of Cauchy horizons, it is seldom possible to find exact back reaction solutions. The conjecture is often relatively simple to apply, at the very least suggesting what we expect to happen in a given case. Our hope is that eventually the conjecture can be proved; it could then be used to definitely test the stability of Cauchy horizons and determine the resulting singularity types.

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