Multi-instanton calculus in $N=2$ supersymmetric gauge theory. II. Coupling to matter

Nicholas Dorey*
Physics Department, University College of Swansea, Swansea SA2 8PP, United Kingdom

Valentin V. Khoze†
Department of Physics, Centre for Particle Theory, University of Durham, Durham DH1 3LE, United Kingdom

Michael P. Mattis‡
Theoretical Division T-8, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

(Received 26 July 1996)

We further discuss the $N=2$ superinstantons in SU(2) gauge theory, obtained from the general self-dual solutions of topological charge n constructed by Atiyah, Drinfeld, Hitchin, and Manin (ADHM). We realize the $N=2$ supersymmetry algebra as actions on the superinstanton moduli. This allows us to recast in concise superfield notation our previously obtained expression for the exact classical interaction between n ADHM superinstantons mediated by the adjoint Higgs bosons, and, moreover, to incorporate N_F flavors of hypermultiplets. We perform explicit one- and two-instanton checks of the Seiberg-Witten prepotentials for all N_F and arbitrary hypermultiplet masses. Our results for the low-energy couplings are all in precise agreement with the predictions of Seiberg and Witten except for $N_F=4$, where we find a finite renormalization of the coupling which is absent in the proposed solution. [S0556-2821(96)03124-4]

PACS number(s): 11.30.Pb, 11.15.Tk

I. INTRODUCTION

A. Recent background

The low-energy dynamics of $N=2$ supersymmetric gauge theory in the Coulomb phase is determined by a single holomorphic function: the prepotential $\mathcal{F}$. In the case of $N=2$ supersymmetric Yang-Mills (SYM) theory with gauge group SU(2), an exact solution for $\mathcal{F}$ has been obtained by Seiberg and Witten [1]. In Ref. [2], these authors have generalized their analysis to include the coupling to N_F flavors of matter hypermultiplets in the fundamental representation of the gauge group. Their analysis relies on an elegant physical interpretation of the singularities of $\mathcal{F}$, as points at which the theory admits a weakly coupled dual description in terms of massless monopoles and dyons.

An important feature of the Seiberg-Witten analysis is that it comprises a complete set of predictions for all multi-instanton contributions to the long-distance physics. In principle, these predictions can be compared with the results of supersymmetric instanton calculus at weak coupling. Semiclassical instanton methods rely neither on duality, nor on any subtle assumptions about the number or nature of the singularities of $\mathcal{F}$ at strong coupling. As such, they provide independent tests of the proposed exact results of [1,2], and consequently, of the electric-magnetic duality on which they are grounded.

This instanton program has been carried out in the one- and two-instanton sectors of $N=2$ SYM theory in Refs. [3] and [4], respectively. [Another approach is that of Ref. [5]; also see Ref. [6] for higher gauge groups than SU(2).] A new feature in the presence of massless matter hypermultiplets is that only even numbers of instantons contribute, due to an anomalous discrete symmetry [2]. Recently we have extended our two-instanton analysis to this case as well [7], focusing on the four-fermion vertex in the low-energy effective action. In a parallel calculation, the authors of Ref. [8] have extracted the two-instanton contribution to the expectation value of the quantum modulus $u=\langle \mathrm{Tr}A^2\rangle$, with A the adjoint Higgs field. The generalization to the case of massive hypermultiplets was also briefly described in [7]. So far, virtually all the instanton calculations described above have precisely confirmed the predictions of Seiberg and Witten. The sole exception has been a discrepancy [8] in the two-instanton contribution to u in the model with $N_F=3$. In the following, we will also find an interesting discrepancy in the case $N_F=4$.

In the absence of matter, it has also been possible to make some progress for arbitrary instanton number n. The relevant field configurations are constrained supersymmetric instantons based on the general solutions of the self-dual Yang-Mills equation obtained by Atiyah, Drinfeld, Hitchin, and Manin (ADHM) [9]. In Ref. [4], we solved for the large- and short-distance behavior of all the component fields in the self-dual background. This enabled us to construct the exact classical interaction between n ADHM instantons mediated by the adjoint Higgs bosons, both in the pure bosonic as well as in the $N=2$ SYM theory.

Unfortunately, the problem of specifying the multi-instanton measure for integration over the ADHM moduli remains unsolved for $n>2$; this is the principal obstruction to an all-orders check of the prepotential. Nevertheless, it is still possible to verify with our methods certain general features of the proposed exact solution in the n-instanton sector. An example is the nonperturbative relation between the vacuum modulus u and the prepotential $\mathcal{F}$. While this relation was

*Electronic address: n.dorey@swansea.ac.uk
†Electronic address: valya.khoze@durham.ac.uk
‡Electronic address: mattis@pion.lanl.gov

originally derived from the Seiberg-Witten solution by Matone [10], it turns out to be true on much broader grounds; in fact, it is built into the instanton approach. This was shown in Ref. [11], which extends to all n the observation of Ref. [12].

The first goal of this paper is to generalize the multi-instanton SYM results of Ref. [4] to allow for N_F fundamental hypermultiplets with arbitrary masses. We will construct the superinstanton action in this larger class of models, extending to all n the two-instanton formula used in Refs. [7, 8]. To accomplish this, we adopt a method which was originally developed in the early papers on supersymmetric instantons by Novikov, Shifman, Vainshtein, and Zakharov [13]. As the relevant field configurations obey equations of motion which are manifestly supersymmetric, any nonvanishing action of the $N=2$ supersymmetry generators on a particular solution necessarily yields another solution. It follows that the supersymmetry transformations of the fields are equivalent (up to a gauge transformation) to certain transformations of the bosonic and fermionic collective coordinates of the superinstanton solution. Physically relevant quantities such as the saddle-point action of the superinstanton must be constructed out of supersymmetric invariant combinations of the collective coordinates.

An especially attractive feature of the ADHM construction is that the various constraints on the parameters of the bosonic and fermionic fields are automatically supersymmetric. This means that the $N=2$ supersymmetry algebra can be realized directly as transformations of the highly overcomplete (order n^2) set of collective coordinates which appear explicitly in the ADHM construction.[1] In fact, these parameters assemble naturally into a single space-time-constant $N=2$ chiral superfield. This superfield notation systematizes the construction of supersymmetric invariant combinations of the collective coordinates. Thus we demonstrate, in retrospect, the invariance of the $N=2$ SYM superinstanton action obtained by component methods in [4]. The incorporation of N_F massive hypermultiplets into this action is then straightforward.

Subsequently, we apply these general formulas to some explicit calculations in the one- and two-instanton sectors. These sections provide a more detailed account of the results presented in our recent paper [7]. As stated there our results agree with the predictions of Seiberg and Witten for the four-antifermion correlator $\langle \bar\lambda(x_1)\bar\lambda(x_2)\bar\psi(x_3)\bar\psi(x_4)\rangle$ which is proportional to $\partial^4\mathcal{F}/\partial v^4$, where v is the classical vacuum expectation value (VEV). However, we now also extract the effective low-energy gauge coupling $\tau_{\rm eff}$ which involves the second derivative of the prepotential:[2]

$$\tau_{\rm eff}=2\,\partial^2\mathcal{F}/\partial v^2. \tag{1.1}$$

The difference between these two quantities as tests of the proposed exact solutions is purely academic, except for the case $N_F=4$ which we now discuss.

B. The case $N_F=4$

For $N_F=4$, both the perturbative β function and the $U(1)_R$ anomaly vanish. This theory is parametrized by the dimensionless gauge coupling g^2 and the vacuum angle θ which cannot be rotated away; these are combined to form a single complex parameter $\tau_{\rm cl}$. In this model, Seiberg and Witten propose an exact electric-magnetic duality which relates theories characterized by different values of $\tau_{\rm cl}$. For this duality to hold it is necessary that the massless $N_F=4$ theory be exactly conformally invariant. In other words, the conformal anomaly which vanishes to all orders in perturbation theory must remain zero when non-perturbative effects are included. These authors also make the stronger assumption that the massless $N_F=4$ theory is classically exact, which means that the low-energy effective coupling (1.1) is simply equal to its classical counterpart:

$$\tau_{\rm eff}\equiv\tau_{\rm cl}\,. \tag{1.2}$$

In the massive $N_F=4$ theory the low-energy correlators have, instead, an infinite expansion in the dimensionless one-instanton factor q:

$$q=\exp(i\pi\tau_{\rm cl}). \tag{1.3}$$

As mentioned above, certain general features of the multi-instanton contributions can be deduced from the general form of the superinstanton action. In particular, we will obtain a representation for the prepotential itself as an integral over the superinstanton moduli, generalizing to $N_F>0$ a result of Ref. [11]. Although for $n>2$ the measure of integration is not known, its only dependence on v is through the superinstanton action for which we have obtained an exact formula. By dimensional analysis, each term in the multi-instanton series for $\mathcal{F}$ is seen to scale like v^2 when $N_F=4$, so that the vanishing of the β function [which is related to $\mathcal{F}'''(v)$] is essentially built into the instanton approach.[3]

However, we should also comment on a result we have obtained for the $N_F=4$ theory with massless hypermultiplets which appears to differ from the predictions of Seiberg and Witten. We have calculated the two-instanton contribution to the low-energy effective coupling $\tau_{\rm eff}$ (the odd-instanton contributions being zero as noted above). We find that $\tau_{\rm eff}$ receives finite corrections of the form

$$\tau_{\rm eff}=\tau_{\rm cl}+\frac{i}{\pi}\sum_{n=2,4,6,\ldots} c_n q^n, \tag{1.4}$$

in contrast with the proposed classical exactness (1.2). Specifically we extract the dimensionless number c_2 in Sec. VIII below and find that it is nonzero, $c_2=-7/(2^6 3^5)$. From our expression for the prepotential we expect all the c_n to be generically nonzero as well. Note that the ''translation'' half

[1]This feature of the ADHM construction has been noticed by other authors [14–16].

[2]The factor of 2 on the right-hand side of this equation is introduced in order to normalize $\tau_{\rm cl}$ to the notation of [2], $\tau_{\rm cl}=8\pi i/g^2+\theta/\pi$. We perform no further rescalings of the parameters of $N=2$ SYM theory when the hypermultiplets are added. See Ref. [4] for a complete list of our conventions.

[3]A caveat to this is that the integrals over the superinstanton moduli must presumably be finite.

of the modular group is preserved by such a series: $\tau_{cl}\to\tau_{cl}+2$ implies $\tau_{eff}\to\tau_{eff}+2$. It is natural to conjecture that the ''inversion'' half of the modular group is realized as well [albeit in a more complicated way than for Eq. (1.2)] as some SL(2,**Z**) transformation $\tau_{eff}\to(a+b\tau_{eff})/(c+d\tau_{eff})$, with the value of c_2 providing a helpful clue.

Finally we should mention the case of massless $N=4$ supersymmetric gauge theory. In this case the instanton has eight fermion zero modes which are protected by $N=4$ supersymmetry and cannot be lifted [17]. It follows that instantons do not contribute to the prepotential and in particular corrections of the form (1.4) do not occur.

C. The plan of this paper

This paper is organized as follows. In Sec. II, after a brief review of our ADHM conventions from [4], we implement the $N=2$ supersymmetry algebra directly as an action on the overcomplete set of bosonic and fermionic ADHM parameters. The results of this exercise are collected in Eq. (2.28). In Sec. III we revisit the $N=2$ SYM multi-instanton action from Ref. [4], and demonstrate that it is in fact a supersymmetry invariant. To make this manifest we assemble the ADHM parameters into a single $N=2$ space-time-constant ''superfield,'' and recast the SYM action as an $N=2$ ''F term.'' In Sec. IV we show that the various ADHM bosonic and fermionic constraints likewise assemble into a single $N=2$ supermultiplet.

Hypermultiplets are introduced starting in Sec. V. We incorporate them into the general multi-instanton SYM action using invariance arguments. The final expression for the action, Eq. (5.20), is discussed further in Sec. VI.

In Sec. VII we discuss the prepotential $\mathcal{F}$. General aspects of the prepotential that emerge from the Seiberg-Witten formalism are reviewed in Sec. VII A. Alternatively, a formal representation, Eq. (7.20), of the prepotential as an integral over the multi-instanton supermoduli space is derived in Sec. VII B. Finally the explicit one- and two-instanton tests discussed above of the proposed exact solutions are performed in Sec. VIII. Numerical values of the two-instanton contributions to the prepotentials for $N_F=0,1,2,3,4$ are given in Eqs. (8.21) and (8.22).

The paper also contains three Appendixes. In particular, in Appendix C we explain an important difference in the numbers of fermion zero modes lifted by the VEV of an adjoint Higgs field as opposed to a fundamental Higgs field; in the latter case the different topological sectors do not interfere.

II. $N=2$ SUPERSYMMETRY ALGEBRA ON THE INSTANTON MODULI

A. ADHM preliminaries

The basic object in the ADHM construction [9] of self-dual SU(2) gauge fields of topological number n is an $(n+1)\times n$ quaternion-valued matrix $\Delta_{\lambda l}(x)$, which is a linear function of the space-time variable x:[4]

$$\Delta_{\lambda l}=a_{\lambda l}+b_{\lambda l}x, \quad 0\leqslant\lambda\leqslant n,\ 1\leqslant l\leqslant n. \tag{2.1}$$

The gauge field $v_m(x)$ is then given by (displaying color indices)

$$v_m{}^{\dot\alpha}{}_{\dot\beta}=\bar{U}_\lambda^{\dot\alpha\alpha}\partial_m U_{\lambda\alpha\dot\beta}, \tag{2.2}$$

where the quaternion-valued vector U_λ lives in the $\perp$ space of Δ:

$$\bar{\Delta}_{l\lambda}U_\lambda=\bar{U}_\lambda\Delta_{\lambda l}=0, \tag{2.3a}$$

$$\bar{U}_\lambda U_\lambda=1. \tag{2.3b}$$

It is easy to show that self-duality of the field strength v_{mn} is equivalent to the quaternionic condition $\bar{\Delta}^{\dot\beta\beta}_{k\lambda}\Delta_{\lambda l\beta\dot\alpha}=(f^{-1})_{kl}\delta^{\dot\beta}{}_{\dot\alpha}$; Taylor expanding in x then gives

$$\bar{a}a=(\bar{a}a)^T\propto\delta^{\dot\beta}{}_{\dot\alpha}, \tag{2.4a}$$

$$\bar{b}a=(\bar{b}a)^T, \tag{2.4b}$$

$$\bar{b}b=(\bar{b}b)^T\propto\delta_\alpha{}^\beta, \tag{2.4c}$$

where the T stands for transpose in the ADHM indices (λ, l, etc.) only.

In a supersymmetric theory there is also the gaugino

$$(\lambda_\alpha)^{\dot\beta}{}_{\dot\gamma}=\bar{U}^{\dot\beta\gamma}\mathcal{M}_\gamma f\bar{b}U_{\alpha\dot\gamma}-\bar{U}^{\dot\beta}{}_\alpha bf\mathcal{M}^{\gamma T}U_{\gamma\dot\gamma}. \tag{2.5}$$

We suppress ADHM indices but exhibit color (dotted) and Weyl (undotted) indices for clarity. The condition that λ be considered the superpartner of the self-dual gauge field v_m is simply that it satisfy the two-component Dirac equation in the ADHM background [13], $\bar{\mathcal{D}}^{\dot\alpha\alpha}\lambda_\alpha=0$. This, in turn, is equivalent to the following linear constraints on the $(n+1)\times n$ constant Grassmann matrix $\mathcal{M}_\gamma$ [19]:

$$\bar{a}^{\dot\alpha\gamma}\mathcal{M}_\gamma=-\mathcal{M}^{\gamma T}a_\gamma{}^{\dot\alpha}, \tag{2.6a}$$

$$\bar{b}_\alpha{}^\gamma\mathcal{M}_\gamma=\mathcal{M}^{\gamma T}b_{\gamma\alpha}. \tag{2.6b}$$

In the $N=2$ theory the fermion zero modes (2.5) are reduplicated by the Higgsino ψ as well, to which we associate the matrix $\mathcal{N}_\gamma$; in addition there is the adjoint Higgs field A to be discussed shortly.

Without loss of generality we can restrict to the following canonical forms for the $(n+1)\times n$ matrices $a_{\alpha\dot\alpha}$, $b_\alpha{}^\beta$, $\mathcal{M}^\gamma$, and $\mathcal{N}^\gamma$:

[4]We use quaternionic notation $x=x_{\alpha\dot\alpha}=x_n\sigma^n_{\alpha\dot\alpha}$, $\bar{a}=\bar{a}^{\dot\alpha\alpha}=\bar{a}^n\bar\sigma_n^{\dot\alpha\alpha}$, $b=b_\alpha{}^\beta$, etc., where σ^n and $\bar\sigma^n$ are the spin matrices of Wess and Bagger [18]. See the reposted/published version of [4] for a self-contained introduction to the ADHM construction including a full account of our conventions and a set of useful identities used throughout the present paper. We also set $g=1$ throughout, except, for clarity, in the Yang-Mills instanton action $8\pi^2n/g^2$.

$$a_{\alpha\dot\alpha}=\begin{pmatrix} w_{1\alpha\dot\alpha} & \cdots & w_{n\alpha\dot\alpha} \\ & a'_{\alpha\dot\alpha} & \end{pmatrix},\quad b_{\alpha}{}^{\beta}=\begin{pmatrix} 0 & \cdots & 0 \\ \delta_{\alpha}{}^{\beta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_{\alpha}{}^{\beta}\end{pmatrix},$$

$$\mathcal{M}^{\gamma}=\begin{pmatrix} \mu_1^{\gamma} & \cdots & \mu_n^{\gamma} \\ & \mathcal{M}'^{\gamma} & \end{pmatrix},\quad \mathcal{N}^{\gamma}=\begin{pmatrix} \nu_1^{\gamma} & \cdots & \nu_n^{\gamma} \\ & \mathcal{N}'^{\gamma} & \end{pmatrix}. \tag{2.7}$$

Thanks to this simple form for b, the constraint (2.4c) is now automatically satisfied, while Eqs. (2.4b) and (2.6b) reduce to the symmetry conditions on the $n\times n$ submatrices:

$$a'=a'^{T},\quad \mathcal{M}'=\mathcal{M}'^{T},\quad \mathcal{N}'=\mathcal{N}'^{T}. \tag{2.8}$$

Let us pause to count the number of superinstanton collective coordinates. We expect there to be $8n$ bosonic degrees of freedom in the matrix a ($4n$ instanton positions, n scale sizes, and $3n$ iso-orientations in the far-separated limit), matched by $4n$ Grassmann degrees of freedom in each of $\mathcal{M}$ and $\mathcal{N}$. From Eqs. (2.8), (2.6a), and (2.4a) we see that the fermionic count is correct; however there are far too many bosonic variables, in fact, order n^2. These necessarily unphysical redundancies reflect the existence of the remaining x-independent $SU(2)\times O(n)$ symmetries which preserve all ADHM constraints as well as the canonical form of b given above:

$$\Delta\to\begin{pmatrix} \Omega & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R^{T} & \\ 0 & & & \end{pmatrix}\cdot\Delta\cdot R. \tag{2.9}$$

Here $R\in O(n)$ and carries no SU(2) indices; in contrast $\Omega_{\alpha}{}^{\beta}$ is a unit-normalized quaternion which acts on these indices. While the Ω degrees of freedom merely double count the global gauge rotations, the action of the $O(n)$ poses a bigger technical problem; in general it must be eliminated from the path integral with a Faddeev-Popov prescription [20,4].

B. $N=2$ SUSY algebra

We can now construct the $N=2$ supersymmetry variation of the collective coordinate matrix a. We start with the usual transformation law[5] for the gauge field under $\sum_{i=1,2}\xi_i Q_i+\bar\xi_i\bar Q_i$:

$$\delta v_m=\bar\xi_1\bar\sigma_m\lambda+\bar\xi_2\bar\sigma_m\psi-\bar\lambda\bar\sigma_m\xi_1-\bar\psi\bar\sigma_m\xi_2. \tag{2.10}$$

In the present case the first two terms on the right-hand side are obtained from Eq. (2.5); the final two terms vanish, since the antifermions are zero at the classical level (they are down by one power of the coupling). Following [13], the strategy is to trade an *active* transformation on the fields such as Eq. (2.10), for an equivalent *passive* transformation on the collective coordinates. In order to do so, we must first understand [21] how to relate a variation δU_λ (hence δv_m) to an underlying variation δa of the collective coordinate matrix, assuming that we restrict attention to variations that preserve the various ADHM conditions and constraints. Varying Eq. (2.3a) gives $\delta U\Delta=-U\delta\Delta$, or equivalently $\delta U\,(1-\mathcal{P})=-U\delta\Delta f\bar\Delta$, where $\mathcal{P}$ is the usual ADHM projection operator

$$\mathcal{P}_{\lambda\kappa}=U_\lambda\bar U_\kappa=\delta_{\lambda\kappa}-\Delta_{\lambda l}f_{lk}\bar\Delta_{k\kappa}. \tag{2.11}$$

By inspection, the general solution is

$$\delta\bar U=-\bar U\delta\Delta f\bar\Delta+\Sigma(x)\bar U. \tag{2.12}$$

Here $\Sigma(x)^{\dot\alpha}{}_{\dot\beta}$ is arbitrary, save for the condition (2.3b) which forces $0=\bar\Sigma+\Sigma\equiv\mathrm{tr}_2\,\Sigma$; consequently $\Sigma(x)$ is precisely an infinitesimal local SU(2) gauge transformation. From Eq. (2.2) we obtain, finally [21],

$$\begin{aligned}\delta v_m&=\delta\bar U\partial_m U+\bar U\partial_m\delta U\\&=\bar U\delta a f\bar\sigma_m\bar b U-\bar U\sigma_m b f\delta\bar a U-\mathcal{D}_m\Sigma.\end{aligned} \tag{2.13}$$

In the last rewrite we have used Eq. (2.3a), Eq. (2.1), and an integration by parts; we have also set $\delta\Delta=\delta a$ as we are holding b fixed as per Eq. (2.7). Comparing the active transformation (2.10) and (2.5) with the passive transformation (2.13), we extract the simple rule

$$\delta a_{\alpha\dot\alpha}=\bar\xi_{1\dot\alpha}\mathcal{M}_\alpha+\bar\xi_{2\dot\alpha}\mathcal{N}_\alpha \tag{2.14}$$

Notice that the local gauge transformation represented by the last term of Eq. (2.13) proved unnecessary; one can set $\Sigma=0$.

Next we turn to the subtler case of the fermions, whose active supersymmetry transformation is given by

$$\delta\lambda=i\sqrt{2}\bar\xi_2\bar{\mathcal{D}}A-i\xi_1\sigma^{mn}v_{mn}, \tag{2.15a}$$

$$\delta\psi=-i\sqrt{2}\bar\xi_1\bar{\mathcal{D}}A-i\xi_2\sigma^{mn}v_{mn}. \tag{2.15b}$$

Here the adjoint Higgs component A of the superinstanton is defined by the Euler-Lagrange equation[6]

$$\mathcal{D}^2A=\sqrt{2}i[\lambda,\psi]. \tag{2.16}$$

(In contrast, the antiboson $A^\dagger$ obeys the homogeneous equation

$$\mathcal{D}^2A^\dagger=0 \tag{2.17}$$

when $N_F=0$; the superinstanton breaks the conjugation symmetry between A and $A^\dagger$.) The construction of the solution of Eq. (2.16) for general n is one of the principal results of [4]. In brief, the answer has the additive form $A=A^{(1)}+A^{(2)}$, where

[5]We follow the supersymmetry conventions of Appendix A of [4], with the exception that $v_m\to iv_m$ due to our conventional use of anti-Hermitian ADHM gauge fields. The relation to the supersymmetry parameters of [7] is given by $\xi=\xi_1$ and $\xi'=-\xi_2$.

[6]In both Eqs. (2.15) and (2.16), and elsewhere in this paper, we ignore the auxiliary fields F and D which only turn on at a higher order in the coupling.

$$iA^{(1)\dot\alpha}{}_{\dot\beta}=\frac{1}{2\sqrt{2}}\,\overline{U^{\dot\alpha\alpha}}(\mathcal{N}_\alpha f\mathcal{M}^{\beta T}-\mathcal{M}_\alpha f\mathcal{N}^{\beta T})U_{\beta\dot\beta}, \tag{2.18}$$

and $iA^{(2)\dot\alpha}{}_{\dot\beta}=\overline{U^{\dot\alpha\alpha}}\mathcal{A}_\alpha{}^\beta U_{\beta\dot\beta}$, with $\mathcal{A}$ a block-diagonal constant matrix,

$$\mathcal{A}_\alpha{}^\beta=\begin{pmatrix}\mathcal{A}_{00\alpha}{}^\beta & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & \mathcal{A}_{\rm tot}\delta_\alpha{}^\beta & \\ 0 & & & \end{pmatrix}. \tag{2.19}$$

$\mathcal{A}_{00}$ is related in a trivial way to the VEV (which we point in the τ^3 direction),

$$\mathcal{A}_{00\alpha}{}^\beta=\frac{i}{2}\,v\,\tau^3{}_\alpha{}^\beta,\quad \overline{\mathcal{A}_{00\alpha}}{}^\beta=-\frac{i}{2}\,\overline{v\,\tau^3}{}_\alpha{}^\beta, \tag{2.20}$$

while the $n\times n$ antisymmetric matrix $\mathcal{A}_{\rm tot}$ is defined as the solution to an inhomogeneous linear matrix equation [4], namely Eq. (A1) in Appendix A below. (In the language of Sec. VII of [4], $\mathcal{A}_{\rm tot}$ is the sum

$$\mathcal{A}_{\rm tot}=\mathcal{A}'+\mathcal{A}'_f, \tag{2.21}$$

where $\mathcal{A}'$ is purely bosonic while $\mathcal{A}'_f$ is a fermion bilinear.)

As above we need to equate the active transformation (2.15a) with the passive transformation derived from Eq. (2.5):

$$\delta\lambda=\bar U\delta\mathcal{M}fb\bar U+\delta\bar U\mathcal{M}fb\bar U+\bar U\mathcal{M}fb\,\delta U+\bar U\mathcal{M}\delta fb\bar U-\text{H.c.} \tag{2.22}$$

The first term on the right-hand side contains the unknown $\delta\mathcal{M}$ that we wish to determine; the second and third terms are already fixed by Eqs. (2.12) and (2.14); the fourth term, too, is a known entity, since

$$\delta f=-f\,\delta(\bar\Delta\Delta)f=-f(\delta\bar a\Delta+\bar\Delta\delta a)f. \tag{2.23}$$

A lengthy but straightforward calculation yields a welcome simplification: the second, third, and fourth terms, taken together, cancel precisely against the piece $i\sqrt{2}\,\overline{\xi_2}\,\overline{\mathcal{D}}A^{(1)}$ from Eq. (2.18) that enters the right-hand side of Eq. (2.15a). Equating what remains gives the defining condition for $\delta\mathcal{M}$:

$$\overrightarrow{U^{\beta\gamma}}\delta\mathcal{M}_\gamma f\bar b\bar U_{\alpha\dot\gamma}-\text{H.c.}$$
$$=i\sqrt{2}\,\overline{\xi_2}\,\overline{\mathcal{D}}A^{(2)}-i\xi_1\sigma^{mn}v_{mn}$$
$$=\overrightarrow{U^{\beta\gamma}}(-4ib\,\xi_{1\gamma}-2\sqrt{2}\mathcal{C}_{\gamma\dot\alpha}\overline{\xi_2^{\dot\alpha}})f\bar b\bar U_{\alpha\dot\gamma}-\text{H.c.} \tag{2.24}$$

The final rewrite makes use of the well-known form of the ADHM field strength,

$$v_{mn}{}^{\dot\alpha}{}_{\dot\beta}=(v_{mn}{}^{\dot\alpha}{}_{\dot\beta})^{\rm dual}=4\overline{U^{\dot\alpha\alpha}}b\,\sigma_{mn\,\alpha}{}^\beta f\bar b\bar U_{\beta\dot\beta}, \tag{2.25}$$

as well as identities (7.8) and (C1) from [4]. The $(n+1)\times n$ quaternion-valued constant matrix $\mathcal{C}$ is defined as

$$\mathcal{C}=\begin{pmatrix}\mathcal{A}_{00}w_1-w_k\mathcal{A}_{{\rm tot}\,k1} & \cdots & \mathcal{A}_{00}w_n-w_k\mathcal{A}_{{\rm tot}\,kn}\\ & [\mathcal{A}_{\rm tot},a'] & \end{pmatrix}. \tag{2.26}$$

It follows that

$$\delta\mathcal{M}_\gamma=-4ib\,\xi_{1\gamma}-2\sqrt{2}\mathcal{C}_{\gamma\dot\alpha}\overline{\xi_2^{\dot\alpha}} \tag{2.27a}$$

and likewise

$$\delta\mathcal{N}_\gamma=-4ib\,\xi_{2\gamma}+2\sqrt{2}\mathcal{C}_{\gamma\dot\alpha}\overline{\xi_1^{\dot\alpha}}. \tag{2.27b}$$

The final ingredient needed is the $N=2$ transformation law for $\mathcal{A}_{\rm tot}$ itself. As shown in Appendix A, it is a singlet: $\delta\mathcal{A}_{\rm tot}=0$. This equation together with Eqs. (2.14) and (2.27) are the sought-after realization of the $N=2$ supersymmetry algebra on the collective coordinates of the ADHM superinstanton. When hypermultiplets are included, these equations are supplemented by Eq. (5.18) below, where $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are the Grassmann collective coordinates associated with the fundamental fermions. For ease of reference we assemble them all here:

$$\delta a_{\alpha\dot\alpha}=\overline{\xi_{1\dot\alpha}}\mathcal{M}_\alpha+\overline{\xi_{2\dot\alpha}}\mathcal{N}_\alpha, \tag{2.28a}$$

$$\delta\mathcal{M}_\gamma=-4ib\,\xi_{1\gamma}-2\sqrt{2}\mathcal{C}_{\gamma\dot\alpha}\overline{\xi_2^{\dot\alpha}}, \tag{2.28b}$$

$$\delta\mathcal{N}_\gamma=-4ib\,\xi_{2\gamma}+2\sqrt{2}\mathcal{C}_{\gamma\dot\alpha}\overline{\xi_1^{\dot\alpha}}, \tag{2.28c}$$

$$\delta\mathcal{A}_{\rm tot}=0, \tag{2.28d}$$

$$\delta\mathcal{K}_i=0, \tag{2.28e}$$

$$\delta\tilde{\mathcal{K}}_i=0. \tag{2.28f}$$

The careful reader will notice, however, that the $N=2$ algebra is not precisely obeyed by the above. For instance, the anticommutator $\{Q_1,Q_2\}$, rather than vanishing when acting on a, $\mathcal{M}$, or $\mathcal{N}$, gives a residual symmetry transformation of the form (2.9). (This is analogous to naive realizations of supersymmetry that fail to commute with Wess-Zumino gauge fixing, for example.) For present purposes this poses no problem, as we are always ultimately concerned with singlets under Eq. (2.9); otherwise one would have to covariantize the supersymmetry transformations with respect to Eq. (2.9) in the standard way.

III. MULTI-INSTANTON ACTION IN PURE $N=2$ SUPERSYMMETRIC GAUGE THEORY

Although, as we saw in the previous section, the superinstanton transforms under supersymmetry, its saddle-point action must be invariant. For a single instanton, in the presence of a Higgs field (fundamental or adjoint), the bosonic part of the action is proportional to $|v|^2\rho^2$. In $N=1$ models, such as those considered in [13,22], the squared instanton scale-size ρ^2 is augmented in the action by a fermion bilinear term to form a supersymmetric invariant combination $\rho^2_{\rm inv}$. We now check that the same property holds for the action in $N=2$ supersymmetric Yang-Mills theory, for arbitrary topological number n. In this case the action is given by [4]

$$S^0_{\rm inst}=\frac{8n\pi^2}{g^2}+16\pi^2|\mathcal{A}_{00}|^2\sum_k |w_k|^2-8\pi^2\overline{\Lambda}_{lk}\mathcal{A}_{{\rm tot}\,kl} +4\sqrt{2}\pi^2\mu^{\alpha}_k\overline{\mathcal{A}}_{00\,\alpha}{}^{\beta}\nu_{k\beta}, \tag{3.1}$$

where $\overline{\Lambda}$ is the $n\times n$ scalar-valued antisymmetric matrix

$$\overline{\Lambda}_{lk}=\overline{w}_l\overline{\mathcal{A}}_{00}w_k-\overline{w}_k\overline{\mathcal{A}}_{00}w_l, \quad \Lambda_{lk}=\overline{w}_l\mathcal{A}_{00}w_k-\overline{w}_k\mathcal{A}_{00}w_l. \tag{3.2}$$

The supersymmetric invariance of this expression under the transformations (2.28a)–(2.28d) is immediate: the second and third terms on the right-hand side give, respectively, $-16\pi^2|\mathcal{A}_{00}|^2(\mu_k w_k\xi_1+\nu_k w_k\xi_2)$ and $-16\pi^2(\xi_2\overline{w}_l\mathcal{A}_{00}\nu_k-\mu_l\mathcal{A}_{00}w_k\xi_1)\mathcal{A}_{{\rm tot}\,kl}$ which are canceled precisely by the variation of the last term.

Despite the simplicity of this last calculation, it is illuminating to reformulate the action (3.1) in a more concise form in which the supersymmetry is manifest. To this end, we promote the ADHM collective coordinate matrix a to a space-time-constant ''superfield'' $a(\theta_i)$ in an obvious way:[7]

$$\begin{aligned} a_{\alpha\dot\alpha}\to a_{\alpha\dot\alpha}(\overline{\theta}_i)&=e^{\overline{\theta}_2\overline{Q}_2}\times e^{\overline{\theta}_1\overline{Q}_1}\times a_{\alpha\dot\alpha}\\ &=a_{\alpha\dot\alpha}+\overline{\theta}_{1\dot\alpha}\mathcal{M}_\alpha+\overline{\theta}_{2\dot\alpha}\mathcal{N}_\alpha+2\sqrt{2}\mathcal{C}_{\alpha\dot\beta}\overline{\theta}_2^{\dot\beta}\overline{\theta}_{1\dot\alpha}\\ &\quad+\sqrt{2}\,\overline{\theta}_{1\dot\alpha}\overline{\theta}_2^2\mathcal{C}_{\mathcal{N}\,\alpha}, \end{aligned} \tag{3.3}$$

where the Grassmann matrix $\mathcal{C}_{\mathcal{N}}$ is defined in analogy with $\mathcal{C}$,

$$\mathcal{C}_{\mathcal{N}}=\begin{pmatrix} \mathcal{A}_{00}\nu_1-\nu_k\mathcal{A}_{{\rm tot}k1} & \cdots & \mathcal{A}_{00}\nu_n-\nu_k\mathcal{A}_{{\rm tot}\,kn}\\ & [\mathcal{A}_{\rm tot},\mathcal{N}\,'] & \end{pmatrix}. \tag{3.4}$$

A short calculation making use of the defining equation (A1) for $\mathcal{A}_{\rm tot}$ gives the desired rewrite of the action (3.1) as a manifestly supersymmetric $N=2$ ''F term:''

$$S^0_{\rm inst}=\frac{8n\pi^2}{g^2}-\pi^2{\rm Tr}\overline{a(\theta)}(\mathcal{P}_\infty+1)a(\theta)\Big|_{\overline{\theta}_1^2\overline{\theta}_2^2}. \tag{3.5}$$

Here the capitalized ''Tr'' indicates a trace over both ADHM and SU(2) indices, ${\rm Tr}={\rm Tr}_n\circ{\rm tr}_2$, and $\mathcal{P}_\infty$ is the $(n+1)\times(n+1)$ matrix

$$\mathcal{P}_\infty=\lim_{r\to\infty}\mathcal{P}=1-b\overline{b}=\delta_{\lambda 0}\delta_{\kappa 0}. \tag{3.6}$$

Note the following.

(1) The intermediate expression (3.3) is not symmetric in θ_1 and θ_2. This merely reflects the point noted earlier, that we have only realized the $N=2$ algebra up to transformations of the type (2.9); therefore Q_1 and Q_2 do not actually anticommute. Nevertheless the final expression (3.5) is a singlet under Eq. (2.9), so this poses no problems.

(2) The purely bosonic part of $S^0_{\rm inst}$ in Eq. (3.5) may be viewed in two ostensibly different ways. On the one hand, in the above construction, it comes entirely from the square of the fourth term on the right-hand side of Eq. (3.3). On the other hand, we also know [4] that it comes entirely from the Higgs kinetic energy term in the component Lagrangian (where only the bosonic part of A is taken). To reconcile these two statements, note that the bosonic part of $\mathcal{C}$ from Eq. (2.26) is precisely Eq. (C1) in [4]. The bosonic action therefore corresponds to the expression (B5) of [4] for the overlap of two vector zero modes $\mathcal{D}_nA$, which is indeed the Higgs kinetic energy (see Appendixes B and C of [4] for details). The form of Eq. (3.5) was therefore inevitable.

$$\begin{array}{ccc} & v_m & \\ \lambda & & \psi \\ & A & \end{array} \quad\Longleftrightarrow\quad \begin{array}{ccc} & 1 & \\ \bar\theta_1 & & \bar\theta_2 \\ & \bar\theta_1\bar\theta_2 & \end{array}$$

FIG. 1. The $N=2$ supermultiplet, and the corresponding elements of the super-ADHM constraints (4.1).

IV. SUPERSYMMETRIC REFORMULATION OF THE CONSTRAINT EQUATIONS

The component fields of the $N=2$ superinstanton, and their respective moduli, follow a suggestive pattern.

(i) The gauge field v_m obeys a nonlinear homogeneous differential equation (the Yang-Mills equation). The associated collective coordinates a obey a nonlinear homogeneous constraint (2.4a). This condition imposes $\frac{3}{2}n(n-1)$ constraints on the upper-triangular traceless quaternionic elements of $\overline{a}a$.

(ii) The fermions λ and ψ obey linear homogeneous differential equations (the covariant Dirac equation). Their associated moduli $\mathcal{M}$ and $\mathcal{N}$ obey the linear homogeneous constraint (2.6a). This imposes $n(n-1)$ conditions on each of $\mathcal{M}$ and $\mathcal{N}$.

(iii) Finally the Higgs field A is the solution to an inhomogeneous linear differential equation (the covariant Klein-Gordon equation with a Yukawa source term). Correspondingly, the matrix $\mathcal{A}_{\rm tot}$ satisfies an inhomogeneous linear ''constraint equation,'' namely Eq. (A1) below. This equation determines the $\frac{1}{2}n(n-1)$ scalar degrees of freedom in the $n\times n$ antisymmetric matrix $\mathcal{A}_{\rm tot}$.

Notice that the total number of bosonic and fermionic constraints are each $2n(n-1)$. This balancing between bosonic and fermionic degrees of freedom suggests that the set of constraints (2.4a), (2.6a), and (A1) might naturally be combined into an $N=2$ ''super-multiplet'' of constraints. Here we show that this is in fact the case.

In light of the ''superfield'' $a(\theta)$ constructed above, the obvious ansatz for this super-multiplet of constraints is to introduce θ dependence into the original ADHM condition (2.4a):

$$\overline{a(\theta)}a(\overline{\theta})=(\overline{a(\theta)}a(\overline{\theta}))^T\propto\delta^{\dot\beta}{}_{\dot\alpha}. \tag{4.1}$$

[Note that Eq. (2.4b) is automatically satisfied for $a\to a(\overline{\theta})$ thanks to the canonical choices (2.7)–(2.8).] The first few terms in the Taylor expansion of Eq. (4.1) look promising (see Fig. 1): The bosonic component is just Eq. (2.4a)

[7]From now on we ignore the action of the Q_i which act in a trivial way, and focus exclusively on the $\overline{Q}_i$. For a related construction in a model without a VEV, see [14].

itself, while the $\overline{\theta_1}$ and $\overline{\theta_2}$ components indeed reproduce the zero-mode condition (2.6a) for $\mathcal{M}$ and $\mathcal{N}$, respectively.

Less obvious is the $\theta_1\times\theta_2$ component of Eq. (4.1), which we rewrite as the triplet of conditions

$$\mathrm{tr}_2\,\tau^k\overline{a(\theta)}a(\overline{\theta})=0, \quad k=1,2,3, \tag{4.2}$$

where τ^k is a Pauli matrix. Extracting the $\overline{\theta_{2\dot\beta}}\overline{\theta_1^{\dot\alpha}}$ component of Eq. (4.2) after some index rearrangement gives

$$0=\tau^{k\dot\beta}{}_{\dot\alpha}\Lambda_f+\tau^k_{\dot\gamma\dot\alpha}((\overline{a\mathcal{C}})^{\dot\gamma\dot\beta}+(\overline{\mathcal{C}a})^{\dot\beta\dot\gamma}), \tag{4.3}$$

where in the notation of [4],

$$\Lambda_f=-\Lambda_f^T=\frac{1}{2\sqrt{2}}\,(\mathcal{M}^{\beta T}\mathcal{N}_\beta-\mathcal{N}^{\beta T}\mathcal{M}_\beta). \tag{4.4}$$

This equation is analyzed as follows. Tracing on color indices tells us that $\overline{a\mathcal{C}}+\overline{\mathcal{C}a}\propto\delta^{\dot\alpha}{}_{\dot\beta}$. So we plug

$$(\overline{\mathcal{C}a})^{\dot\alpha}{}_{\dot\beta}=X\delta^{\dot\alpha}{}_{\dot\beta}-(\overline{a\mathcal{C}})^{\dot\alpha}{}_{\dot\beta} \tag{4.5}$$

into Eq. (4.3), the $n\times n$ matrix X being the unknown, and deduce $X=\Lambda_f+\mathrm{tr}_2\overline{a\mathcal{C}}$. Equation (4.5) then becomes

$$\Lambda_f=\overline{\mathcal{C}a}+(\overline{a\mathcal{C}}-\mathrm{tr}_2\;\overline{a\mathcal{C}})=\overline{\mathcal{C}a}-(\overline{\mathcal{C}a})^T. \tag{4.6}$$

Up to this point the manipulations have been valid for arbitrary $\mathcal{C}$. But if one substitutes the explicit expression (2.26) for $\mathcal{C}$ in terms of $\mathcal{A}_{\mathrm{tot}}$, Eq. (4.6) does in fact become the defining linear equation (A1) for $\mathcal{A}_{\mathrm{tot}}$, expressed in especially concise form. With Eqs. (2.26) and (2.4a) one also confirms that $\overline{a\mathcal{C}}+\overline{\mathcal{C}a}$ is pure trace in the SU(2) space; thus all tensor components of Eq. (4.3) have properly been accounted for. The remaining θ components of Eq. (4.1) turn out to be ''auxiliary'' as they contain no new information. Some are satisfied trivially, while others boil down to the earlier relations (2.4a), (2.6a), or (A1).

V. MULTI-INSTANTON ACTION IN $N=2$ SUPERSYMMETRIC QCD

Following [2] we now turn our attention to the richer class of models in which the $N=2$ supersymmetric Yang-Mills action is augmented by N_F matter hypermultiplets which transform in the fundamental representation of SU(2). Each $N=2$ hypermultiplet corresponds to a pair of $N=1$ chiral multiplets, Q_i and $\tilde{Q}_i$ where $i=1,2,\ldots,N_F$, which contain scalar quarks (squarks) q_i and $\tilde{q}_i$, respectively, and fermionic partners χ_i and $\tilde\chi_i$. We will restrict our attention to the Coulomb branch of the theory where the squarks do not acquire a VEV. In the $N=1$ language, the matter fields couple to the gauge multiplet via a superpotential,

$$W=\sum_{i=1}^{N_F}\sqrt{2}\tilde{Q}_i\Phi Q_i+m_i\tilde{Q}_iQ_i \tag{5.1}$$

suppressing color indices. The second term is an $N=2$ invariant mass term.

As reviewed above, the component fields of the superinstanton which reside in the adjoint representation of SU(2) have the generic form $\bar{U}XU$. Similarly, those in the fundamental representation have the structure $\bar{U}X$. The solution of the coupled Euler-Lagrange equations for each of these fields is simplified by the use of the differentiation identities in the ADHM background:

$$\mathcal{D}_n(\bar{U}X)=-\bar{U}\partial_n\Delta f\bar{\Delta}X+\bar{U}\partial_nX, \tag{5.2a}$$

$$\mathcal{D}^2(\bar{U}X)=\bar{U}\partial^2X-2\bar{U}b\sigma^n f\bar{\Delta}\partial_nX+4\bar{U}bf\bar{b}X, \tag{5.2b}$$

$$\mathcal{D}_n(\bar{U}XU)=-\bar{U}\partial_n\Delta f\bar{\Delta}XU-\bar{U}X\Delta f\partial_n\bar{\Delta}U+\bar{U}\partial_nXU, \tag{5.2c}$$

$$\begin{aligned}\mathcal{D}^2(\bar{U}XU)=&4\bar{U}\{bf\bar{b},X\}U-4\bar{U}bf\cdot\mathrm{tr}_2\;\bar{\Delta}X\Delta\cdot f\bar{b}U\\&+\bar{U}\partial^2XU-2\bar{U}bf\sigma_n\bar{\Delta}\partial^nXU\\&-2\bar{U}\partial^nX\Delta\overline{\sigma_n}f\bar{b}U.\end{aligned} \tag{5.2d}$$

As in [4], the construction of the short-distance superinstanton starts with the fermion zero modes in the ADHM background, then proceeds to the Higgs bosons in the presence of fermion-bilinear Yukawa source terms. The fundamental fermion zero modes χ_i and $\tilde\chi_i$, for $i=1,\ldots,N_F$, were constructed in [19,23]:

$$(\chi_i^\alpha)^{\dot\beta}=\bar{U}_\lambda^{\dot\beta\alpha}b_{\lambda k}f_{kl}\mathcal{K}_{li}, \quad (\tilde\chi_i^\alpha)^{\dot\beta}=\bar{U}_\lambda^{\dot\beta\alpha}b_{\lambda k}f_{kl}\tilde{\mathcal{K}}_{li}, \tag{5.3}$$

with α a Weyl and $\dot\beta$ an SU(2) color index. Using Eq. (5.2a) it is easily checked that these are annihilated by $\bar{\mathcal{D}}^{\dot\gamma}{}_\alpha$. Note that each $\mathcal{K}_{ki}$ and $\tilde{\mathcal{K}}_{ki}$ is a Grassmann number rather than a Grassmann spinor; there is no SU(2) index. The normalization matrix of these modes is given by [23]

$$\int d^4x(\chi_i^\alpha)^{\dot\beta}(\tilde\chi_{\alpha j})_{\dot\beta}=\pi^2\mathcal{K}_{li}\tilde{\mathcal{K}}_{lj}. \tag{5.4}$$

Next we consider the adjoint Higgs bosons. In the presence of the superpotential the Euler-Lagrange equation (2.16) for A is unchanged; however Eq. (2.17) for $A^\dagger$ now becomes

$$(\mathcal{D}^2A^\dagger)^{\dot\gamma}{}_{\dot\alpha}=\frac{1}{2\sqrt{2}}\sum_{i=1}^{N_F}(\chi_i^{\dot\gamma}\tilde\chi_{i\dot\alpha}+\tilde\chi_i^{\dot\gamma}\chi_{i\dot\alpha}), \tag{5.5}$$

displaying color and flavor but suppressing Weyl indices. The solution of Eq. (5.5) is similar to, but simpler than, that of Eq. (2.16). At the purely bosonic level, with all Grassmanns turned off, A and $A^\dagger$ must coincide, except for $v\to\bar{v}$. In contrast, the fermion bilinear contributions to A and to $A^\dagger$ in the path integral are to be treated as independent. This bilinear contribution to $A^\dagger$ is straightforwardly obtained from Eq. (5.5), using the identity (5.2d), together with the manipulations described in Sec. VII B of [4]. It has the form

$$-i\bar{U}^{\dot\alpha\alpha}\cdot\begin{pmatrix}0&\cdots&0\\ \vdots&\mathcal{A}_{\mathrm{hyp}}\delta_\alpha{}^\beta&\\ 0&&\end{pmatrix}\cdot U_{\beta\dot\beta}, \tag{5.6}$$

where the $n\times n$ antisymmetric matrix $\mathcal{A}_{\mathrm{hyp}}$ is defined as the solution to the inhomogeneous linear equation

$$\mathbf{L}\cdot\mathcal{A}_{\rm hyp}=\Lambda_{\rm hyp}. \tag{5.7}$$

Here **L** is the ubiquitous linear matrix operator reviewed in Appendix A, and the $n\times n$ antisymmetric matrix $\Lambda_{\rm hyp}$ is given by

$$(\Lambda_{\rm hyp})_{k,l}=\frac{i\sqrt{2}}{16}\sum_{i=1}^{N_F}(\mathcal{K}_{ki}\tilde{\mathcal{K}}_{li}+\tilde{\mathcal{K}}_{ki}\mathcal{K}_{li}). \tag{5.8}$$

Similarly, the squarks q_i satisfy the leading-order Euler-Lagrange equation

$$\mathcal{D}^2 q_i=-i\sqrt{2}\lambda\chi_i \tag{5.9}$$

and likewise for $\tilde{q}_i$. Using Eq. (5.2b) together with identities (7.9) and (C.3a) in [4] one easily derives

$$q_i^{\dot\beta}=\overline{\vec{U}}_\lambda^{\dot\beta\beta}\cdot\left(\delta_{\lambda 0}v_{i\beta}+\frac{i}{2\sqrt{2}}\mathcal{M}_{\lambda l\beta}f_{lk}\mathcal{K}_{ki}\right), \tag{5.10}$$

where v_i is the fundamental VEV of the ith hypermultiplet; in the Coulomb branch all the v_i are zero. All remaining adjoint and fundamental component fields of the superinstanton in $N=2$ supersymmetric QCD may be constructed by these methods. Fortunately, through judicious use of integrations by parts together with the equations of motion, the new expressions (5.3), (5.6), and (5.10) are all that are needed for our present goal of constructing the superinstanton action, $S_{\rm inst}^{N_F}$. By inspection of the component Lagrangian, one sees that this action consists of a sum of five types of terms: (i) purely bosonic terms, (ii) terms bilinear in the adjoint fermion collective coordinates $\mathcal{M}$ and $\mathcal{N}$, (iii) terms bilinear in the fundamental fermion collective coordinates $\mathcal{K}$ and $\tilde{\mathcal{K}}$, (iv) fermion quadrilinear terms, consisting of one parameter drawn from each of $\mathcal{M}$, $\mathcal{N}$, $\mathcal{K}$, and $\tilde{\mathcal{K}}$, and finally (v) the $N=2$ invariant hypermultiplet mass term. Let us consider each in turn.

The construction of (i), (ii), and (iii) proceeds precisely along the lines discussed in detail in Secs. IV C and VII D of [4]: the relevant bits of the component action are converted to a surface term, and are given by the coefficient of the $1/x^2$ falloff of the total adjoint Higgs field, including fermion bilinear contributions, as it approaches its VEV. In this way one immediately finds that the contributions (i) and (ii) to $S_{\rm inst}^{N_F}$ are still given by $S_{\rm inst}^0$, Eq. (3.1) or Eq. (3.5) above. By identical arguments, (iii) is given by

$$-8\pi^2\Lambda_{lk}\mathcal{A}_{{\rm hyp}\,kl}, \tag{5.11}$$

where Λ was defined in Eq. (3.2).

More subtle is the construction of the fermion quadrilinear term (iv). Our calculation of this term proceeds in three steps, summarized as follows. (1) Show that **L** is self-adjoint, and use this property to rewrite Eq. (5.11) as

$$-8\pi^2\Lambda_{{\rm hyp}lk}\mathcal{A}'_{kl}, \tag{5.12}$$

where $\mathcal{A}'$ was defined in Eq. (2.21) as the purely bosonic piece of $\mathcal{A}_{\rm tot}$; (2) show that $\Lambda_{\rm hyp}$ is a supersymmetric invariant; and finally (3) promote Eq. (5.12) to a supersymmetric invariant in the unique way. Here are the details.

(1) Let Ω and Ω' be two $n\times n$ antisymmetric matrices that are scalar valued (i.e., proportional to the identity in the quaternionic space). Let us define an inner product on the space of such matrices in the naive way, by

$$\langle\Omega'|\Omega\rangle={\rm Tr}_n\Omega'^T\Omega. \tag{5.13}$$

From the explicit expressions in Appendix A, it is elementary to show that **L** is self-adjoint with respect to the above metric:

$$\langle\Omega'|\mathbf{L}\cdot\Omega\rangle=\langle\Omega|\mathbf{L}\cdot\Omega'\rangle. \tag{5.14}$$

The claimed equality between Eqs. (5.11) and (5.12) then follows immediately from Eqs. (5.14) and (5.7) together with the defining equation for $\mathcal{A}'$ [Eq. (7.21) of [4]]:

$$\mathbf{L}\cdot\mathcal{A}'=\Lambda. \tag{5.15}$$

(2) Next we show that each individual $\mathcal{K}_{ki}$ and $\tilde{\mathcal{K}}_{ki}$, and hence $\Lambda_{\rm hyp}$, is a supersymmetric invariant. As in Sec. II above, we equate the active supersymmetry transformation,[8]

$$\delta(\chi_i^\alpha)^{\dot\beta}=-i\sqrt{2}\bar{\xi}_{1\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha\alpha}q_i^{\dot\beta}=-\frac{1}{2}\bar{\xi}_{1\dot\alpha}\bar{\sigma}^{n\dot\alpha\alpha}\overline{\vec{U}}^{\dot\beta\beta}(\mathcal{M}_\beta f\partial_n(\bar{\Delta}\Delta)+b\,\sigma_{n\beta\dot\gamma}f\bar{\Delta}^{\dot\gamma\gamma}\mathcal{M}_\gamma)f\mathcal{K}_i, \tag{5.16}$$

with the passive supersymmetry transformation

$$\delta(\chi_i^\alpha)^{\dot\beta}=\delta(\overline{\vec{U}}^{\dot\beta\alpha}bf\mathcal{K}_i)=\delta\overline{\vec{U}}^{\dot\beta\alpha}bf\mathcal{K}_i+\overline{\vec{U}}^{\dot\beta\alpha}b\,\delta f\mathcal{K}_i+\overline{\vec{U}}^{\dot\beta\alpha}bf\,\delta\mathcal{K}_i. \tag{5.17}$$

Remembering Eqs. (2.12), (2.23), and (2.28a), one finds that the first two terms on the right-hand side of Eq. (5.17) equal the first two terms on the right-hand side of Eq. (5.16), respectively; this leaves

$$0=\delta\mathcal{K}_i=\delta\tilde{\mathcal{K}}_i, \tag{5.18}$$

as claimed.

(3) Since $\delta\Lambda_{\rm hyp}=0$, supersymmetrizing Eq. (5.12) simply means promoting $\mathcal{A}'\rightarrow\mathcal{A}_{\rm tot}$, as per Eqs. (2.21) and (2.28d). Since the difference between them consists of fermion bilinears, this step introduces the promised fermion quadrilinears, and restores supersymmetry invariance.

Finally we turn to (v), the hypermultiplet mass term given in Eq. (5.1). In the Coulomb branch this reduces to a mass term for the fundamental fermions only, up to higher-order corrections in the coupling constant. From the normalization condition (5.4) one derives

$$S_{\rm mass}=\pi^2\sum_{i=1}^{N_F}m_i\mathcal{K}_{li}\tilde{\mathcal{K}}_{li}. \tag{5.19}$$

Putting these pieces together gives the general ADHM superinstanton action for $N=2$ supersymmetric QCD with gauge group SU(2):

[8]To avoid clutter we restrict ourselves here to the first supersymmetry.

$$S_{\rm inst}^{N_F} = S_{\rm inst}^0 - 8\pi^2 \Lambda_{{\rm hyp}\, lk} \mathcal{A}_{{\rm tot}\ kl} + S_{\rm mass}. \tag{5.20}$$

This is the generalization to all n of the two-instanton action presented recently in [7,8].

VI. GENERAL FEATURES OF THE SUPER-MULTI-INSTANTON ACTION

From the form of the action (5.20) we can immediately make several observations of a general nature.

(1) The self-adjointness property of **L** noted above allows us to reexpress this action in a variety of equivalent ways. For instance we can rewrite

$$8\pi^2 \Lambda_{{\rm hyp}\, lk} \mathcal{A}_{{\rm tot}\ kl} = 8\pi^2 (\Lambda_{lk} + \Lambda_{f\ lk}) \mathcal{A}_{{\rm hyp}\ kl}, \tag{6.1}$$

where the antisymmetric matrix Λ_f introduced in Eq. (4.4) is the Yukawa source term for $\mathcal{A}'_f$ [Eq. (7.29) of [4]]:

$$\mathbf{L} \cdot \mathcal{A}'_f = \Lambda_f. \tag{6.2}$$

(2) Let us isolate $S_{\rm mass}$ from the action (5.20) and assign it to the n-instanton collective coordinate integration measure $d\mu_{\rm hyp}$ for the fundamental fermions:

$$\int d\mu_{\rm hyp} = \frac{1}{\pi^{2nN_F}} \int \prod_{i=1}^{N_F} d\mathcal{K}_{1i} \cdots d\mathcal{K}_{ni} d\tilde{\mathcal{K}}_{1i} \cdots d\tilde{\mathcal{K}}_{ni} \times \exp(-S_{\rm mass}), \tag{6.3}$$

where the normalization constant in front has been read from Eq. (5.4). Consider this expression in the chiral limit, $S_{\rm mass}=0$. In this limit, for fixed flavor index i, the Grassmann measure is obviously even or odd under the discrete symmetry

$$\mathcal{K}_{li} \leftrightarrow \tilde{\mathcal{K}}_{li}, \tag{6.4}$$

depending on whether n itself is even or odd. On the other hand, the term $-8\pi^2 \Lambda_{{\rm hyp}\, lk} \mathcal{A}_{{\rm tot}\ kl}$ in the action (5.20) is always even under this symmetry, as follows from Eq. (5.8). Therefore, for $N_F>0$, only the even-instanton sectors $n=0,2,4...$ can contribute in the chiral limit[9] (recall that when $N_F=0$, all instanton numbers contribute). This selection rule was already noted by Seiberg and Witten in Sec. III of [2]; it is not surprising to find that it is built into the instanton calculus. It is violated when masses are turned on, since $\mathcal{K}_{li}\tilde{\mathcal{K}}_{li}$ is odd under this symmetry.

(3) In the even-instanton sectors (and in the odd-instanton sectors as well when masses are turned on), the number of exact fermionic modes, i.e., those modes which do not appear in the action, remains the same for all n and for all N_F. Just as in pure Yang-Mills theory [4], the unbroken modes are the four adjoint fermionic modes associated with the four supersymmetry generators that act nontrivially on the self-adjoint ADHM gauge field.[10] These are the supersymmetric gaugino and Higgsino zero modes generated by $\xi_1 Q_1$ and $\xi_2 Q_2$, respectively. Explicitly, they are given by Eq. (2.5), with

$$\mathcal{M}_\gamma = 4\xi_{1\gamma} b \quad \text{and} \quad \mathcal{N}_\gamma = 4\xi_{2\gamma} b. \tag{6.5}$$

(4) Following the strategy originally established by Affleck, Dine, and Seiberg [24], in the explicit calculations to follow we will saturate these four unbroken modes by suitable insertions of long-distance fields. These are the components of the superinstanton that are parallel to the adjoint VEV and hence have power-law falloff; in comparison, the components orthogonal to the VEV decay exponentially as $\exp(-M_W r)$, and can be ignored. As we saw in pure Yang-Mills theory [4], for any n the structure of these long-distance fields can be read off directly from the superinstanton action itself. In that theory, the long-distance anti-Higgsino and antigaugino components satisfy [4]

$$\overline{\psi}_{\dot\alpha}(x) = i\sqrt{2} v^{-1} S_{\rm Higgs} \xi_1^\alpha S_{\alpha\dot\alpha}(x,x_0) \tag{6.6a}$$

and

$$\overline{\lambda}_{\dot\alpha}(x) = -i\sqrt{2} v^{-1} S_{\rm Higgs} \xi_2^\alpha S_{\alpha\dot\alpha}(x,x_0), \tag{6.6b}$$

respectively. Here $S_{\rm Higgs}$ is the purely bosonic part of the superinstanton action, x_0 is the position of the multi-instanton, and $S(x,x_0)$ is the Weyl spinor propagator:

$$S(x,x_0) = \partial\!\!\!/ G(x,x_0), \quad G(x,x_0) = \frac{1}{4\pi^2 (x-x_0)^2}. \tag{6.7}$$

In [11] we found it helpful to rewrite Eq. (6.6) in a slightly different way, as

$$\overline{\psi}_{\dot\alpha}(x) = i\sqrt{2} \frac{\partial S_{\rm inst}^0}{\partial v} \xi_1^\alpha S_{\alpha\dot\alpha}(x,x_0) \tag{6.8a}$$

and

$$\overline{\lambda}_{\dot\alpha}(x) = -i\sqrt{2} \frac{\partial S_{\rm inst}^0}{\partial v} \xi_2^\alpha S_{\alpha\dot\alpha}(x,x_0), \tag{6.8b}$$

[9] The absence of a one-instanton contribution is particularly easy to see since $\Lambda_{\rm hyp}$ vanishes identically for $n=1$, and so the $\mathcal{K}$ and $\tilde{\mathcal{K}}$ Grassmann integrations are unsaturated.

[10] This mode counting contrasts sharply with that of $N=1$ theories with only fundamental Higgs bosons; see Appendix C for a brief discussion of those types of models.

where in performing these derivatives we distinguish between v and $\bar{v}$. Of course, in the pure Yang-Mills case, the expressions (6.6) and (6.8) are identical. This is because $S_{\rm Higgs}$ is linear in v (bilinear in v and $\bar{v}$), whereas the fermion-bilinear contribution to $S^0_{\rm inst}$ depends only on $\bar{v}$ [see Eqs. (3.1) and (2.20) above]. However, for $N_F>0$, this equality no longer holds, since the new term in the action, $-8\pi^2\Lambda_{{\rm hyp}lk}\mathcal{A}_{{\rm tot}\,kl}$, depends on v, not $\bar{v}$ [see Eqs. (6.1), (3.2), and (2.20)]. It turns out that the correct generalization to $N_F>0$ is given, not by Eq. (6.6), but by the differential representation (6.8) with $S^0_{\rm inst}$ replaced by $S^{N_F}_{\rm inst}$. As these long-distance expressions enter pervasively in the calculations below, we should make this point especially clear; this is done in Appendix B.

Also needed below is the piece of the long-distance Abelian field strength v_{mn} that is bilinear in ξ_1 and ξ_2. The relevant expression is Eq. (5.13) of [4] which we likewise rewrite as a derivative:

$$\sqrt{2}\,\frac{\partial S^0_{\rm inst}}{\partial v}\,\xi_1\sigma^{kl}\xi_2 G_{mn,kl}(x,x_0), \tag{6.9}$$

where $G_{mn,kl}$ is the gauge-invariant propagator of U(1) field strengths,

$$G_{mn,kl}(x,x_0)=(\eta_{nl}\partial_m\partial_k-\eta_{nk}\partial_m\partial_l-\eta_{ml}\partial_n\partial_k+\eta_{mk}\partial_n\partial_l)\times G(x,x_0). \tag{6.10}$$

As explained in Appendix B, this expression, too, generalizes immediately to $N_F>0$, with the substitution $S^0_{\rm inst}\to S^{N_F}_{\rm inst}$.

VII. THE PREPOTENTIAL

In this section we discuss some general features of the prepotential $\mathcal{F}^{(N_F)}$ for $N=2$ supersymmetric QCD. In Sec. VII A, which is restricted to the cases $N_F<4$ unless otherwise stated, we review the predictions of Seiberg and Witten [2] (see also Refs. [25] and [26]). Alternatively, in Sec. VII B, we derive a formal expression, valid for $N_F\leqslant 4$, for the prepotential in terms of the multi-instanton measure, extending a result given in [11] to incorporate hypermultiplets. Explicit numerical comparisons in the one-instanton and two-instanton sectors will be given in Sec. VIII below.

A. Seiberg-Witten predictions for the prepotential

In $N=2$ SQCD with $N_F<4$ massless hypermultiplets in the fundamental representation, the restrictions imposed by holomorphy, renormalization group invariance, and the anomaly imply that the prepotential has the following expansion at weak coupling:

$$\mathcal{F}^{(N_F)}(v)=i\,\frac{(4-N_F)}{8\pi}\,v^2\,\ln\!\left(\frac{Cv^2}{\Lambda^2_{N_F}}\right)-\frac{i}{\pi}\sum_{n=1}^{\infty}\mathcal{F}^{(N_F)}_n\times\left(\frac{\Lambda_{N_F}}{v}\right)^{n(4-N_F)}v^2. \tag{7.1}$$

Λ_{N_F} is the dynamically generated scale of the theory[11] and C is a numerical constant. The logarithm comes from the classical result combined with one-loop perturbation theory while the remaining terms correspond to an infinite series of instanton corrections. As discussed above, for $N_F>0$ the discrete symmetry (6.4) ensures that only even numbers of instantons contribute: hence $\mathcal{F}^{(N_F)}_{2k+1}=0$. Each nonzero coefficient $\mathcal{F}^{(N_F)}_{2k}$ is a pure number characterizing the leading semiclassical contribution of instantons of topological charge $2k$. In Sec. VIII below, we carry out an explicit two-instanton computation of $\mathcal{F}^{(N_F)}_2$ for $N_F\leqslant 4$. For the special case of $N_F=4$ massless hypermultiplets the β function is zero and we expect the following expression for the prepotential:

$$\mathcal{F}^{(4)}(v)=\frac{1}{4}\,\tau_{\rm cl}v^2-\frac{i}{\pi}\sum_{n=2,4,6,\ldots}\mathcal{F}^{(4)}_n q^n v^2,\quad q=e^{i\pi\tau_{\rm cl}}. \tag{7.2}$$

Furthermore, Seiberg and Witten propose that the massless $N_F=4$ theory is classically exact, Eq. (1.2), which implies $\mathcal{F}^{(4)}_n=0$ for all n. Instead, in Sec. VIII we will obtain a nonzero value for $\mathcal{F}^{(4)}_2$.

The general description of the low-energy theory, Eqs. (7.1) and (7.2), is modified in two ways by the introduction of masses for the hypermultiplets. First, as noted earlier, the mass terms explicitly break the discrete symmetry (6.4); hence the contribution of odd numbers of instantons becomes nonzero. Second, each term in the instanton expansion will itself be a polynomial in the dimensionless ratios m_i/v. As we will see below, these polynomials can be obtained from the exact solution of the low energy theory proposed by Seiberg and Witten.

For each N_F, the exact behavior of the low-energy theory is characterized by an elliptic curve of the form

$$y^2=x^3+Bx^2+Cx+D\equiv(x-e_1)(x-e_2)(x-e_3). \tag{7.3}$$

For $N_F<4$, where the theory is asymptotically free, the coefficients B, C, and D (and hence the roots e_i) are functions of the modulus u, the dynamical scale Λ_{N_F}, and the masses m_i. The exact solution for the VEV v and its dual v_D as functions of the modulus u is given in terms of the periods of the elliptic curve (7.3). In the region of parameter space where the roots are real and $e_1\geqslant e_2\geqslant e_3$, we have the explicit formulas (in the conventions of [3])

[11]The numerical values of the constants $\mathcal{F}^{(N_F)}_{2k}$ depend on the definition of the scale Λ_{N_F}. In this paper, as in [4], we adopt the Λ parameter of the Pauli-Villars regularization scheme which is appropriate for instanton calculations, and corresponds to 't Hooft conventions for the collective coordinate measure [27]. In this scheme the renormalization group matching conditions are most straightforward since the threshold factors are unity [3]. The Λ of the Pauli-Villars scheme is related to the Λ parameter of [1,2] as $4\Lambda^{4-N_F}_{N_F}=(\Lambda^{\rm SW}_{N_F})^{4-N_F}$ for $0\leqslant N_F<4$.

$$\frac{\partial v}{\partial u}=\frac{cK(k)}{\sqrt{e_1-e_3}},\quad \frac{\partial v_D}{\partial u}=\frac{icK'(k)}{\sqrt{e_1-e_3}}. \tag{7.4}$$

Here K and K' are elliptic functions of the first kind, $k=\sqrt{(e_2-e_3)/(e_1-e_3)}$, and c is a numerical constant fixed by demanding the asymptotic behavior $v=\sqrt{2u}+\cdots$ in the weakly-coupled regime of large u. The second derivative of the prepotential is then given as

$$\frac{\partial^2\mathcal{F}}{\partial v^2}=\frac{1}{2}\frac{\partial v_D}{\partial v}=\frac{iK'(k)}{2K(k)}. \tag{7.5}$$

This equation, together with Eq. (7.4), determines the prepotential corresponding to a particular elliptic curve up to irrelevant constants of integration.

With the definition of the Λ parameter given above, the elliptic curve for the $N_F=0$ theory is simply

$$y_{(0)}^2=x^2(x-u)+\Lambda_0^4 x. \tag{7.6}$$

For $0<N_F<4$, the curves can be written in terms of the following set of symmetric polynomials in the masses m_i:

$$M_0^{(N_F)}=1,$$

$$M_1^{(N_F)}=\sum_{i=1}^{N_F} m_i^2,$$

$$M_2^{(N_F)}=\sum_{i<j}^{N_F} m_i^2 m_j^2,$$

$$\vdots \quad \vdots$$

$$M_{N_F}^{(N_F)}=\prod_{j=1}^{N_F} m_j^2. \tag{7.7}$$

The curves are then given by

$$y_{(N_F)}^2=x^2(x-u)+\sqrt{M_{N_F}^{(N_F)}}\Lambda_{N_F}^{4-N_F}x-\frac{1}{4}\Lambda_{N_F}^{2(4-N_F)}\times\sum_{\delta=0}^{N_F-1} M_\delta^{(N_F)}(x-u)^{N_F-1-\delta}. \tag{7.8}$$

Given these explicit forms, it is straightforward to expand Eqs. (7.4) and (7.5) as a power series in $\Lambda_{N_F}^{4-N_F}$ and extract the first few terms in the instanton expansion of the prepotential. However, as we will see below, several features of the expansion can be deduced without further calculation.

Let us order the masses so that they satisfy $m_{N_F}\geqslant m_{N_F-1}\geqslant\cdots\geqslant m_1\geqslant 0$. An important restriction on the form of the elliptic curves comes from the scaling limit $m_{N_F}\to\infty$, $\Lambda_{N_F}\to 0$ with $m_{N_F}\Lambda_{N_F}^{4-N_F}$ held fixed. In this limit one of the flavors becomes infinitely massive and decouples, leaving an effective theory described by $N=2$ SQCD with N_F-1 flavors. In the chosen regularization scheme, the Λ parameters for different numbers of flavors are related as

$$m_{N_F}\Lambda_{N_F}^{4-N_F}=\Lambda_{N_F-1}^{5-N_F}. \tag{7.9}$$

It is easy to check that the curves (7.8) have the required property $y_{(N_F)}^2\to y_{(N_F-1)}^2$ in the decoupling limit. This property is then inherited by the prepotential itself:

$$\mathcal{F}^{(N_F)}(v;\{m_i\},\Lambda_{N_F})\to\mathcal{F}^{(N_F-1)}(v;\{m_i,i<N_F\},\Lambda_{N_F-1}). \tag{7.10}$$

Moreover, this relation must hold order by order in the instanton expansion.

The single-instanton factor $\Lambda_{N_F}^{4-N_F}$ appears in Eq. (7.8) multiplied by the product $m_1 m_2\cdots m_{N_F}$. Obviously this is the only term in Eq. (7.8) capable of generating odd powers of $\Lambda_{N_F}^{4-N_F}$. It follows that the odd terms in the instanton expansion of the prepotential vanish unless all of the masses are nonzero, as expected from the discrete symmetry (6.4).

The one-instanton contribution has the form

$$\mathcal{F}^{(N_F)}(v;\{m_i\},\Lambda_{N_F})|_{n=1}=-\frac{i}{\pi}\frac{\Lambda_{N_F}^{4-N_F}}{v^2}\mathcal{F}_1^{(0)}\prod_{j=1}^{N_F} m_j. \tag{7.11}$$

This clearly obeys the decoupling relation (7.10). The numerical coefficient $\mathcal{F}_1^{(0)}$ can be extracted from the instanton expansion of the prepotential of the $N_F=0$ theory [3,4]: $\mathcal{F}_1^{(0)}=1/2$.

The remaining terms in Eq. (7.8) are proportional to $\Lambda_{N_F}^{2(4-N_F)}$ and can therefore be thought of as a two-instanton effect. In particular, note that the term proportional to $(x-u)^{N_F-1}$ remains nonzero in the massless limit. Every term of order $\Lambda_{N_F}^{2(4-N_F)}$ which can be formed from the coefficients of the elliptic curve is proportional to one of the polynomials $M_\delta^{(N_F)}$ defined above. Hence, by dimensional analysis, the two-instanton contribution to the prepotential must have the form

$$\mathcal{F}^{(N_F)}(v;\{m_i\},\Lambda_{N_F})|_{n=2}=-\frac{i}{\pi}v^2\left(\frac{\Lambda_{N_F}}{v}\right)^{2(4-N_F)}\sum_{\delta=0}^{N_F} f_\delta^{(N_F)}\left(\frac{M_\delta^{(N_F)}}{v^{2\delta}}\right). \tag{7.12}$$

This expression may be constrained further by considering various limits of the masses. In the chiral limit, $m_i\to 0$, we recover the coefficients of Eq. (7.1); this forces $f_0^{(N_F)}=\mathcal{F}_2^{(N_F)}$. In the opposite limit $m_{N_F}\to\infty$, the decoupling relation (7.10) implies that the numerical coefficients $f_\delta^{(N_F)}$ are not independent, but obey

$$f_\delta^{(N_F)}=f_{\delta-1}^{(N_F-1)}=\cdots=f_0^{(N_F-\delta)}. \tag{7.13}$$

It follows that the constant $f_\delta^{(N_F)}$ is equal to the coefficient $\mathcal{F}_2^{(N_F-\delta)}$ of the massless case. An explicit calculation using Eqs. (7.4) and (7.5) yields the values $\mathcal{F}_2^{(0)}=5/2^4$, $\mathcal{F}_2^{(1)}=-3/2^5$, and $\mathcal{F}_2^{(2)}=1/2^6$. The coefficient $\mathcal{F}_2^{(3)}$ corresponds to an additive constant in the prepotential which does not contribute to the low-energy effective Lagrangian and is irrelevant for our purposes.

In component form, for any number of flavors, the low-energy effective Lagrangian is defined in terms of the prepotential as follows (the superscript SD stands for ''self-dual''):

$$\mathcal{L}_{\rm eff}=\frac{1}{4\pi}\,{\rm Im}\Big[-\mathcal{F}''(A)(\partial_m A^\dagger\partial^m A+i\psi\partial\!\!\!/\bar\psi+i\lambda\partial\!\!\!/\bar\lambda+\tfrac12(v^{\rm SD}_{mn})^2)+\frac{1}{\sqrt2}\mathcal{F}'''(A)\lambda\sigma^{mn}\psi v_{mn}+\tfrac14\mathcal{F}''''(A)\psi^2\lambda^2\Big]. \qquad (7.14)$$

As usual we ignore auxiliary fields as they are subleading in the coupling constant. The last three terms in the above Lagrangian yield nonvanishing tree-level contributions to the following three Green's functions [4]:

$$\langle v_{mn}(x_1)v_{kl}(x_2)\rangle=\frac{1}{16\pi i}\frac{\partial^2\mathcal{F}}{\partial v^2}\int d^4x_0\ {\rm tr}_2\sigma^{pq}\sigma^{rs}\times G_{mn,pq}(x_1,x_0)G_{kl,rs}(x_2,x_0), \qquad (7.15a)$$

$$\langle v_{mn}(x_1)\bar\lambda_{\dot\alpha}(x_2)\bar\psi_{\dot\beta}(x_3)\rangle=\frac{1}{8\sqrt2\pi i}\frac{\partial^3\mathcal{F}}{\partial v^3}\int d^4x_0\sigma^{kl\alpha\beta}G_{mn,kl}(x_1,x_0)\times S_{\alpha\dot\alpha}(x_2,x_0)S_{\beta\dot\beta}(x_3,x_0), \qquad (7.15b)$$

$$\langle\bar\lambda_{\dot\alpha}(x_1)\bar\lambda_{\dot\beta}(x_2)\bar\psi_{\dot\gamma}(x_3)\bar\psi_{\dot\delta}(x_4)\rangle=\frac{1}{8\pi i}\frac{\partial^4\mathcal{F}}{\partial v^4}\int d^4x_0\epsilon^{\alpha\beta}S_{\alpha\dot\alpha}(x_1,x_0)S_{\beta\dot\beta}(x_2,x_0)\times\epsilon^{\gamma\delta}S_{\gamma\dot\gamma}(x_3,x_0)S_{\delta\dot\delta}(x_4,x_0), \qquad (7.15c)$$

where the Weyl and field-strength propagators $S(x,x_0)$ and $G_{mn,kl}(x,x_0)$ were defined in Eqs. (6.7) and (6.10) above. We now discuss how these correlation functions may be calculated from first principles, using instanton methods.

B. The prepotential in the instanton approach

Our strategy for determining $\mathcal{F}$ is to calculate the leading semiclassical contributions to the Green functions (7.15) in the large distance limit. The first step is to replace each of the fields ψ, λ, and v_{mn} with the long-distance ''tail'' of the corresponding component of the superinstanton. These expressions, which we denote $\psi^{\rm LD}$, $\lambda^{\rm LD}$, and $v^{\rm LD}_{mn}$, were given above in Eqs. (6.8a), (6.8b), and (6.9), respectively, with the substitution $S^0_{\rm inst}\to S^{N_F}_{\rm inst}$.

One also needs the superinstanton measure. In the n-instanton sector, the integration runs over $8n$ bosonic and $8n+2nN_F$ fermionic collective coordinates, which we denote generically as X_i and χ_i, respectively. At a purely formal level, the measure for this integration can be expressed as

$$\int d\mu_n^{(N_F)}=\frac{1}{\mathcal{S}_n}\int\Bigg(\prod_{i=1}^{8n}dX_i\prod_{j=1}^{8n+2nN_F}d\chi_j\Bigg)\times(J_{\rm Bose}/J_{\rm Fermi})^{1/2}\exp(-S^{N_F}_{\rm inst}(n)). \qquad (7.16)$$

Here $J_{\rm Bose}$ and $J_{\rm Fermi}$ are the collective coordinate Jacobians for the bosonic and fermionic parameters, respectively, and $\mathcal{S}_n$ is a symmetry factor.

As reviewed in [4], it is only possible to solve the ADHM constraints and find an explicit formula for the measure for $n<3$. However, for the following, it suffices to know that the only dependence on the VEV v in Eq. (7.16) is that of the action $S^{N_F}_{\rm inst}$ which is separately linear in both v and $\bar v$. In addition, as discussed in Sec. VI, we know that all fermionic zero modes are lifted by the action except for the four supersymmetric zero modes (6.5) parametrized by $\xi_{1\alpha}$ and $\xi_{2\alpha}$. It is convenient to separate out from the measure these unbroken modes together with their bosonic partner, the translational degrees of freedom, x_0:

$$\int d\mu_n^{(N_F)}=\int d^4x_0d^2\xi_1d^2\xi_2\int d\tilde\mu_n^{(N_F)}. \qquad (7.17)$$

We will refer to $d\tilde\mu_n^{(N_F)}$ as the ''reduced measure.''

Putting the pieces together, one finds for the n-instanton contribution to the Green's function (7.15c):

$$\langle\bar\lambda_{\dot\alpha}(x_1)\bar\lambda_{\dot\beta}(x_2)\bar\psi_{\dot\gamma}(x_3)\bar\psi_{\dot\delta}(x_4)\rangle=\int d\mu_n^{(N_F)}\bar\lambda^{\rm LD}_{\dot\alpha}(x_1)\bar\lambda^{\rm LD}_{\dot\beta}(x_2)\bar\psi^{\rm LD}_{\dot\gamma}(x_3)\bar\psi^{\rm LD}_{\dot\delta}(x_4), \qquad (7.18)$$

with similar expressions for the other two Green's functions. Following [11], we substitute the expressions (6.8)–(6.9) into the right-hand side, and perform the trivial integration over $\xi_{1\alpha}$ and $\xi_{2\alpha}$. This leaves

$$\langle v_{mn}(x_1)v_{kl}(x_2)\rangle=\frac12\frac{\partial^2}{\partial v^2}\int d\tilde\mu_n^{(N_F)}\int d^4x_0\ {\rm tr}_2\sigma^{pq}\sigma^{rs}\times G_{mn,pq}(x_1,x_0)G_{kl,rs}(x_2,x_0), \qquad (7.19a)$$

$$\langle v_{mn}(x_1)\bar\lambda_{\dot\alpha}(x_2)\bar\psi_{\dot\beta}(x_3)\rangle=\frac{1}{\sqrt2}\frac{\partial^3}{\partial v^3}\int d\tilde\mu_n^{(N_F)}\int d^4x_0\sigma^{kl\alpha\beta}G_{mn,kl}(x_1,x_0)\times S_{\alpha\dot\alpha}(x_2,x_0)S_{\beta\dot\beta}(x_3,x_0), \qquad (7.19b)$$

$$\langle\bar\lambda_{\dot\alpha}(x_1)\bar\lambda_{\dot\beta}(x_2)\bar\psi_{\dot\gamma}(x_3)\bar\psi_{\dot\delta}(x_4)\rangle=\frac{\partial^4}{\partial v^4}\int d\tilde\mu_n^{(N_F)}\int d^4x_0\epsilon^{\alpha\beta}\times S_{\alpha\dot\alpha}(x_1,x_0)S_{\beta\dot\beta}(x_2,x_0)\epsilon^{\gamma\delta}S_{\gamma\dot\gamma}(x_3,x_0)S_{\delta\dot\delta}(x_4,x_0). \qquad (7.19c)$$

The linearity of $S^{N_F}_{\rm inst}$ in v has allowed us to pull the v differentiation outside the collective coordinate integral. Comparing the semiclassical expressions (7.19) with their exact counterparts (7.15), we deduce

$$\mathcal{F}^{(N_F)}(v;\{m_i\},\Lambda_{N_F})|_{n\text{-inst}}=8\pi i\int d\tilde{\mu}_n^{(N_F)}+Av+B, \tag{7.20}$$

where A and B are undetermined constants of integration. As these constants do not contribute to the low-energy effective Lagrangian, we are free to set $A=B=0$ for convenience.

Equation (7.20) is the desired expression for the prepotential as a formal integral over the superinstanton moduli. It is the obvious generalization to $N_F>0$ of the analogous SYM formula (21) in [11]. Note that all memory of the long-distance field insertions has disappeared from this equation. In hindsight, these insertions were merely a convenient bookkeeping device for extracting the appropriate derivatives of $\mathcal{F}$ dictated by the low-energy Lagrangian (7.14). Henceforth we will drop all reference to the ''tail'' of the superinstanton, and focus directly on the concise expression (7.20). Importantly, by absorbing a factor of $\sqrt{v}$ into a, $\mathcal{M}$, and $\mathcal{N}$, one sees that $\mathcal{F}$ depends only on v, and not on $\bar{v}$, so that holomorphicity of the prepotential (7.20) is built into the instanton calculus.

VIII. EXPLICIT CALCULATIONS FOR $n=1$ AND $n=2$

In this section we will use the action derived above to calculate the one- and two-instanton contributions to the prepotential for $N_F\leq 4$. In addition we include nonzero masses for the hypermultiplets. This section supplies additional details for the calculation of $\langle\psi\psi\lambda\lambda\rangle\sim\mathcal{F}''''$ presented in our recent paper [7]. See also [8] for a related calculation of a different quantity which is not simply given by a derivative of $\mathcal{F}$. By focusing on $\mathcal{F}$ itself using Eq. (7.20), we also extract information about the special case $N_F=4$.

A. The one-instanton contribution

In ADHM language, the bosonic and fermionic parameters of a single $N=2$ superinstanton are contained in three 2×1 matrices of unconstrained parameters:

$$a=\begin{pmatrix} w \\ X\end{pmatrix},\quad \mathcal{M}_\gamma=\begin{pmatrix}\mu_\gamma \\ M_\gamma\end{pmatrix},\quad \mathcal{N}_\gamma=\begin{pmatrix}\nu_\gamma\\ N_\gamma\end{pmatrix}. \tag{8.1}$$

In addition there are $2N_F$ Grassmann variables $\mathcal{K}_i$ and $\tilde{\mathcal{K}}_i$ which parametrize the fundamental zero modes (5.3). The reduced measure (7.17) is given by

$$\int d\tilde{\mu}_1^{(N_F)}=\frac{2^7\Lambda_{N_F}^{4-N_F}}{\pi^{4+2N_F}}\int d^4w\,d^2\mu\,d^2\nu \times\prod_{i=1}^{N_F} d\mathcal{K}_i d\tilde{\mathcal{K}}_i\,\exp[-\tilde{S}_{\rm inst}^{N_F}(n=1)], \tag{8.2}$$

where the single-superinstanton action is easily read from the general expression (5.20):[12]

[12]We place a tilde over the action to indicate that the Maxwell piece $8\pi^2n/g^2$ has been subtracted out.

$$\tilde{S}_{\rm inst}^{N_F}(1)=16\pi^2|w|^2|\mathcal{A}_{00}|^2+4\sqrt{2}\pi^2\mu\overline{\mathcal{A}_{00}}\nu+\pi^2\sum_{i=1}^{N_F} m_i\mathcal{K}_i\tilde{\mathcal{K}}_i\,. \tag{8.3}$$

[For the special case $N_F=4$, where the β function vanishes, the factor $\Lambda_{N_F}^{4-N_F}$ should simply be replaced by q from Eq. (1.3).] Notice that the only dependence on $\mathcal{K}_i$ and $\tilde{\mathcal{K}}_i$ comes from the mass term $S_{\rm mass}$ (this is because $\Lambda_{\rm hyp}$ vanishes identically for $n=1$, as do Λ and Λ_f). The corresponding Grassmann integrations can only be saturated by bringing down N_F powers of $S_{\rm mass}$; as expected from the discrete symmetry (6.4), the result is nonzero only when all the m_i are nonzero. The remaining integration is identical to the case of $N=2$ SYM theory [3,4]. Using Eq. (7.20) one finds after a simple calculation:

$$\mathcal{F}^{(N_F)}(v;\{m_i\},\Lambda_{N_F})|_{n=1}=-\frac{i}{\pi}\frac{\Lambda_{N_F}^{4-N_F}}{v^2}\mathcal{F}_1^{(0)}\prod_{j=1}^{N_F} m_j\,, \tag{8.4}$$

where $\mathcal{F}_1^{(0)}=1/2$, in agreement with the Seiberg-Witten prediction.

B. The two-instanton contribution

Next we consider the two-instanton contribution to the prepotential. The notation and the various changes of integration variables in the present calculation closely parallel the simpler case of $N=2$ SYM theory worked out in detail in Sec. VIII of [4]. Because the calculation for $N_F>0$ is so similar, we will chiefly stress those points where they differ.

The parameters of the $n=2$ ADHM superinstanton are contained in the following 3×2 matrices:

$$a=\begin{pmatrix} w_1 & w_2\\ x_0+a_3 & a_1\\ a_1 & x_0-a_3\end{pmatrix}, \tag{8.5a}$$

$$\mathcal{M}_\gamma=\begin{pmatrix}\mu_{1\gamma} & \mu_{2\gamma}\\ 4\xi_{1\gamma}+\mathcal{M}_{3\gamma} & \mathcal{M}_{1\gamma}\\ \mathcal{M}_{1\gamma} & 4\xi_{1\gamma}-\mathcal{M}_{3\gamma}\end{pmatrix}, \tag{8.5b}$$

$$\mathcal{N}_\gamma=\begin{pmatrix}\nu_{1\gamma} & \nu_{2\gamma}\\ 4\xi_{2\gamma}+\mathcal{N}_{3\gamma} & \mathcal{N}_{1\gamma}\\ \mathcal{N}_{1\gamma} & 4\xi_{2\gamma}-\mathcal{N}_{3\gamma}\end{pmatrix}. \tag{8.5c}$$

In addition, there are now $4N_F$ fundamental zero modes (5.3) parametrized by the Grassmann numbers $\mathcal{K}_{li}$ and $\tilde{\mathcal{K}}_{li}$ with $l=1,2$. We also define the following frequently occurring combinations of these collective coordinates:

$$L=|w_1|^2+|w_2|^2, \tag{8.6}$$

$$H=|w_1|^2+|w_2|^2+4|a_1|^2+4|a_3|^2,$$

$$\Omega=w_1\bar{w}_2-w_2\bar{w}_1,$$

$$\omega=\bar{w}_2\mathcal{A}_{00}w_1-\bar{w}_1\mathcal{A}_{00}w_2=\tfrac{1}{2}{\rm tr}_2\Omega\mathcal{A}_{00}=-\Lambda_{1,2}\,,$$

$$Y=\mu_1\nu_2-\nu_1\mu_2+2\mathcal{M}_3\mathcal{N}_1-2\mathcal{N}_3\mathcal{M}_1=2\sqrt{2}(\Lambda_f)_{1,2}\,,$$

$$Z=\sum_{i=1}^{N_F}\mathcal{K}_{ki}\epsilon_{kl}\tilde{\mathcal{K}}_{li}=-8\sqrt{2}i(\Lambda_{\rm hyp})_{1,2}\,.$$

From Eq. (5.20) we write down the following expression for the two-instanton action $\tilde{S}^{N_F}_{\rm inst}(n=2)$:

$$\tilde{S}^{N_F}_{\rm inst}(2)=\tilde{S}^{0}_{\rm inst}(2)-8\pi^2\Lambda_{{\rm hyp}lk}\mathcal{A}_{{\rm tot}kl}+S_{\rm mass}. \quad (8.7)$$

The hypermultiplet mass term $S_{\rm mass}$ is given in Eq. (5.19) above, whereas $\tilde{S}^{0}_{\rm inst}(2)$ was evaluated in [4], and is concisely expressed in terms of the quantities (8.6):

$$\tilde{S}^{0}_{\rm inst}(2)=16\pi^2L|\mathcal{A}_{00}|^2+4\sqrt{2}\pi^2\mu_k\overline{\mathcal{A}_{00}}\nu_k - \frac{16\pi^2\bar{\omega}}{H}\left(\omega-\frac{Y}{2\sqrt{2}}\right). \quad (8.8)$$

The remaining term in the action, $-8\pi^2\Lambda_{{\rm hyp}lk}\mathcal{A}_{{\rm tot}kl}$, is easily extracted from Eq. (5.8), together with the defining equation for $\mathcal{A}_{\rm tot}$, namely Eq. (A1) below. The linear operator **L** that enters that expression is a $[\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)]$-dimensional map from the space of $n\times n$ antisymmetric matrices onto itself. When $n=2$ this space is one-dimensional, and **L** reduces to ordinary multiplication by the quantity H. From Eqs. (3.2), (4.4), and (8.6), one therefore finds for this term:

$$-\frac{i\sqrt{2}\pi^2Z}{H}\left(\omega-\frac{Y}{2\sqrt{2}}\right). \quad (8.9)$$

Note that Eq. (8.9) may be absorbed into the SYM action (8.8) with the simple substitution

$$\bar{\omega}\rightarrow\bar{\omega}+iZ/8\sqrt{2}. \quad (8.10)$$

Given the action, the next step is the construction of the two-instanton measure. We begin by eliminating the redundant degrees of freedom from Eq. (8.5). A convenient resolution of the ADHM constraints (2.4a) and (2.6a) is to eliminate the off-diagonal elements a_1, $\mathcal{M}_{1\gamma}$, and $\mathcal{N}_{1\gamma}$, as follows:

$$a_1=\frac{1}{4|a_3|^2}\,a_3(\overline{w_2}w_1-\overline{w_1}w_2), \quad (8.11a)$$

$$\mathcal{M}_1=\frac{1}{2|a_3|^2}\,a_3(2\overline{a_1}\mathcal{M}_3+\overline{w_2}\mu_1-\overline{w_1}\mu_2), \quad (8.11b)$$

and

$$\mathcal{N}_1=\frac{1}{2|a_3|^2}\,a_3(2\overline{a_1}\mathcal{N}_3+\overline{w_2}\nu_1-\overline{w_1}\nu_2). \quad (8.11c)$$

The remaining degrees of freedom are unconstrained, and appear as integration variables in the measure. It is helpful to factor the reduced measure $d\tilde{\mu}_2^{N_F}$ into three parts $d\tilde{\mu}_b$, $d\tilde{\mu}_f$, and $d\mu_{\rm hyp}$ corresponding to the bosonic, adjoint fermionic, and fundamental fermionic parameters, respectively:

$$\int d\tilde{\mu}_2^{(N_F)}=\int d\mu_{\rm hyp}d\tilde{\mu}_b d\tilde{\mu}_f. \quad (8.12)$$

Here $d\mu_{\rm hyp}$ was defined in Eq. (6.3) above,

$$\int d\mu_{\rm hyp}=\frac{1}{\pi^{4N_F}}\int\prod_{i=1}^{N_F}d\mathcal{K}_{1i}d\mathcal{K}_{2i}d\tilde{\mathcal{K}}_{1i}d\tilde{\mathcal{K}}_{2i}\,\exp(-S_{\rm mass}), \quad (8.13)$$

while $d\tilde{\mu}_b$ may be read off from [4], subject to the replacement (8.10), and to the appropriate redefinition of the dynamically generated scale parameter:

$$\int d\tilde{\mu}_b=\frac{2^{10}\Lambda_{N_F}^{2(4-N_F)}}{\pi^8\mathcal{S}_2}\int d^4a_3d^4w_1d^4w_2\,\frac{||a_3|^2-|a_1|^2|}{H}\times\exp\left(-16\pi^2\left[L|\mathcal{A}_{00}|^2-\frac{\omega}{H}\,(\bar{\omega}+iZ/8\sqrt{2})\right]\right). \quad (8.14)$$

The symmetry factor $\mathcal{S}_2=16$ is associated with a discrete redundancy in the chosen parametrization (8.11a) of the two-instanton solution [4,20]. For the special case $N_F=4$, where the β function vanishes, the factor $\Lambda_{N_F}^{2(4-N_F)}$ should simply be replaced by q^2. The third piece of the measure comprises the remaining terms in the action:

$$\int d\tilde{\mu}_f=\int d^2\mathcal{M}_3d^2\mu_1d^2\mu_2d^2\mathcal{N}_3d^2\nu_1d^2\nu_2\times\exp\left(-4\sqrt{2}\pi^2\left[\mu_k\overline{\mathcal{A}_{00}}\nu_k+\frac{Y}{H}\,(\bar{\omega}+iZ/8\sqrt{2})\right]\right). \quad (8.15)$$

Performing the Grassmann integration over the parameters of the adjoint zero modes is a straightforward exercise; one finds

$$\int d\tilde{\mu}_f=-\left(\frac{16\sqrt{2}\pi^6(\bar{\omega}+iZ/8\sqrt{2})}{|a_3|^2H}\right)^2\left[\frac{1}{16}\overline{v^4}|\Omega|^2+\frac{L}{2H}\,\overline{v^2}(\bar{\omega}+iZ/8\sqrt{2})\bar{\omega}+\frac{1}{H^2}\,(\bar{\omega}+iZ/8\sqrt{2})^2\times(\tfrac{1}{4}\overline{v^2}(L^2-|\Omega|^2)+\bar{\omega}^2)\right]. \quad (8.16)$$

This is the generalization to $N_F>0$ of the Yukawa determinant given in Eq. (8.13) of [4]. The next step is to integrate over the fundamental fermionic coordinates using the identity

$$\int d\mu_{\rm hyp}G(Z)=\sum_{k=0}^{N_F}\frac{M^{(N_F)}_{N_F-k}}{\pi^{4k}}\,\frac{\partial^{2k}G}{\partial Z^{2k}}\bigg|_{Z=0}, \quad (8.17)$$

where the $M^{(N_F)}_{\delta}$ are the polynomials defined in Eq. (7.7) above. This is the only new feature involved for $N_F>0$.

Finally we turn to the remaining integration over the bosonic moduli. Following [4], it is convenient to change variables in the bosonic measure from $\{a_3,w_1,w_2\}$ to the new set $\{H,L,\Omega\}$. The relevant formulas are

$$\int_{-\infty}^{\infty}d^4a_3\,\frac{||a_3|^2-|a_1|^2|}{|a_3|^4}\rightarrow\frac{\pi^2}{2}\int_{L+2|\Omega|}^{\infty}dH \quad (8.18)$$

and

$$\int_{-\infty}^{\infty} d^4w_1 d^4w_2 \to \frac{\pi^3}{8}\int_0^{\infty} dL \int_{|\Omega|\le L} d^3\Omega. \tag{8.19}$$

The numerator and denominator in the left-hand side of Eq. (8.18) are supplied by Eqs. (8.14) and (8.16), respectively. In addition we introduce rescaled variables $\Omega = L\Omega'$, $H = LH'$, and $\omega = L\omega'$. Following [4] we now carry out the trivial integration over L. Thanks to Eq. (8.17), at this stage in the calculation the renormalization group decoupling property (7.12) and (7.13) is manifest; this is another example of a general feature of the hyperelliptic curves being built into the instanton calculus.

Finally we switch to spherical polar coordinates,

$$d^3\Omega' \to 2\pi \int_{-1}^{1} d(\cos\theta)\int_0^1 |\Omega'|^2 d|\Omega'|, \tag{8.20}$$

where the polar angle is defined by $|\omega'| = |\Omega'||\mathcal{A}_{00}|\cos\theta = \frac{1}{2}|\Omega'||v|\cos\theta$. This leaves an ordinary three-dimensional scaleless integral over the remaining variables H', $\cos\theta$, and $|\Omega'|$ which is the precise analog of Eq. (8.19) in [4]. Performing this elementary integral with the help of a standard symbolic manipulation routine gives

$$\mathcal{F}_2^{(0)} = 5/2^4, \quad \mathcal{F}_2^{(1)} = -3/2^5, \quad \mathcal{F}_2^{(2)} = 1/2^6,$$

$$\mathcal{F}_2^{(3)} = -5/(2^7 3^3). \tag{8.21}$$

These values of $\mathcal{F}_2^{(N_F)}$ with $N_F = 0,1,2$ agree with the predictions extracted from [1,2]. $\mathcal{F}_2^{(3)}$ corresponds to a constant shift in the prepotential, which does not affect the low-energy Lagrangian (7.14). These numbers are the input parameters for Eqs. (7.1) and (7.12). Finally, for the conformally invariant case $N_F = 4$ we likewise find a nonvanishing result,

$$\mathcal{F}_2^{(4)} = 7/(2^8 3^5), \tag{8.22}$$

which is associated with the series (7.2). This implies Eq. (1.4) which is in contradiction with the classical exactness (1.2) proposed in [2].

Note added in proof. We propose a resolution of the $N_F = 4$ discrepancy in N. Dorey, V. Khoze, and M. Mattis, ''On $N = 2$ Supersymmetric QCD with 4 Flavors,'' Report No. hep-th/9611016 (unpublished).

ACKNOWLEDGMENTS

We acknowledge useful discussions with S. Elitzur, T. Harano, M. Matone, M. Peskin, M. Sato, and A. Yung. The work of N.D. was supported in part by the PPARC Advanced Research Program; both N.D. and V.V.K. were supported in part by the Nuffield Foundation; M.M. was supported by the Department of Energy. V.V.K. thanks the CERN theory group for hospitality.

APPENDIX A: SUPERSYMMETRIC INVARIANCE OF $\mathcal{A}_{\rm tot}$

The purpose of this appendix is to demonstrate that the $n \times n$ antisymmetric matrix $\mathcal{A}_{\rm tot}$ is in fact a supersymmetric invariant, $\delta\mathcal{A}_{\rm tot} = 0$, given the transformation laws (2.28a)–(2.28c). Our starting point is the defining equation for $\mathcal{A}_{\rm tot}$:

$$\mathbf{L}\cdot\mathcal{A}_{\rm tot} = \Lambda_{\rm tot}, \tag{A1}$$

where $\Lambda_{\rm tot} = \Lambda + \Lambda_f$, while $\mathcal{A}_{\rm tot} = \mathcal{A}' + \mathcal{A}'_f$ is the quantity under examination. Note that this equation is the sum of Eqs. (5.15) and (6.2) that we used in the text. $\mathbf{L}$ is a linear operator that maps the space of $n \times n$ scalar-valued antisymmetric matrices onto itself. Explicitly, if Ω is such a matrix, then $\mathbf{L}$ is defined as [4]

$$\mathbf{L}\cdot\Omega = \tfrac{1}{2}\{\Omega, W\} - \tfrac{1}{2}\,\mathrm{tr}_2([\overline{a'},\Omega]a' - \overline{a'}[a',\Omega]), \tag{A2}$$

where a' was defined in Eqs. (2.7) and (2.8) above, and W is the symmetric scalar-valued $n \times n$ matrix $W_{kl} = \bar{w}_k w_l + \bar{w}_l w_k$.

Applying a general $N = 2$ supersymmetry variation to Eq. (A1) gives

$$\mathbf{L}\cdot\delta\mathcal{A}_{\rm tot} = \delta\Lambda_{\rm tot} - \delta\mathbf{L}\cdot\mathcal{A}_{\rm tot}. \tag{A3}$$

Since $\mathbf{L}$ is generically invertible, it suffices to show that the right-hand side vanishes. To minimize clutter we restrict the variation to $\xi_2 Q_2$, as the calculation with $Q_2 \rightleftharpoons Q_1$ proceeds identically, while the claim for Q_1 and Q_2 is a trivial consequence of Eq. (2.6b). We define the n-vectors

$$\vec{\nu} = (\nu_1,\ldots,\nu_n), \quad \vec{w} = (w_1,\ldots,w_n), \quad \overline{\vec{w}} = (\bar{w}_1,\ldots,\bar{w}_n). \tag{A4}$$

Starting with the most complicated term on the right-hand side of Eq. (A3), one finds

$$\delta\Lambda_f = (\vec{\nu}^T\mathcal{A}_{00}\vec{w}\bar{\xi}_2 - \bar{\xi}_2\overline{\vec{w}}^T\mathcal{A}_{00}\vec{\nu}) - (\vec{\nu}^T\vec{w}\cdot\mathcal{A}_{\rm tot}\bar{\xi}_2 - \bar{\xi}_2\mathcal{A}_{\rm tot}\cdot\overline{\vec{w}}^T\vec{\nu})$$
$$+ (\mathcal{N}'^T[\mathcal{A}_{\rm tot}, a']\bar{\xi}_2 + \bar{\xi}_2[\mathcal{A}_{\rm tot}, \overline{a'}]\mathcal{N}') \tag{A5}$$

using Eq. (2.28b). The first term in big parentheses on the right-hand side precisely cancels $\delta\Lambda$; the second term in big parentheses groups together with $-\frac{1}{2}\{\mathcal{A}_{\rm tot},\ \delta W\}$ to give $-\frac{1}{2}[\vec{\nu}^T\vec{w}\bar{\xi}_2 + \bar{\xi}_2\overline{\vec{w}}^T\vec{\nu},\ \mathcal{A}_{\rm tot}]$; the third term in big parentheses combines with the remaining terms on the right-hand side of Eq. (A3) to give $-\frac{1}{2}[\mathcal{N}'^T a'\bar{\xi}_2 + \bar{\xi}_2\overline{a'}\mathcal{N}',\ \mathcal{A}_{\rm tot}]$. Since $\overline{\vec{w}}^T\vec{\nu} + \overline{a'}\mathcal{N}' = \bar{a}\mathcal{N}$ these two commutators add to

$$-\frac{1}{2}[\mathcal{N}^T a\bar{\xi}_2 + \bar{\xi}_2\bar{a}\mathcal{N},\ \mathcal{A}_{\rm tot}] \tag{A6}$$

which vanishes by virtue of Eq. (2.6a). Q.E.D.

APPENDIX B: LONG-DISTANCE FIELDS IN $N = 2$ SUPERSYMMETRIC QCD

In this appendix we justify the differential expressions (6.8) and (6.9) for the long-distance ''tail'' of the superinstanton, specifically in the background of the supersymmetric adjoint zero modes (6.5). As stated in Sec. VI, for the case $N_F=0$ these expressions are equivalent rewritings of the formulas in [4] [e.g., Eq. (6.6) above], but for $N_F>0$ they differ by $\xi\mathcal{K}\mathcal{K}$ Grassmann trilinears.

From the explicit component Lagrangian including the superpotential (5.1), together with the identities (5.2), one can of course solve the Euler-Lagrange equations for these fields explicitly. But it is easier to exploit the $N=2$ supersymmetry algebra itself to generate these solutions automatically. As in Sec. IV of [4], we use a construction due to Refs. [13] and [22]: One starts with a ''reference'' superinstanton $\Psi^{(0)}$ comprising a convenient initial choice of component fields (bosonic and fermionic, adjoint and fundamental), and generates the desired configuration by acting on it with the appropriate symmetry generators.

For present purposes $\Psi^{(0)}$ is specified as follows. In the hypermultiplets Q_i and $\tilde{Q}_i$, fill only the fundamental fermion zero modes (5.3); the remaining fundamental fields are turned off. In contrast, in the adjoint sector, fill only the bosonic components initially. Thus the gauge field is the usual ADHM configuration, while the Higgs bosons $A^\dagger$ and A satisfy, respectively, Eq. (5.5), and the homogeneous equation $\mathcal{D}^2A=0$. All antifermions are initially zero. Note that the components of $\Psi^{(0)}$ correctly obey the leading-order coupled Euler-Lagrange equations.

Next one acts on $\Psi^{(0)}$ infinitesimally with the $N=2$ generators $\Sigma_{i=1,2}\xi_iQ_i$. This action generates the desired supersymmetry modes (6.5) in the adjoint gaugino and Higgsino components. At the same time it produces nonzero antigaugino and anti-Higgsino fields that automatically satisfy their respective Euler-Lagrange equations in this background. The $N=2$ algebra gives

$$\bar{\lambda}=i\sqrt{2}\xi_2\not{\mathcal{D}}A^\dagger,\quad \bar{\psi}=-i\sqrt{2}\xi_1\not{\mathcal{D}}A^\dagger, \tag{B1}$$

where $A^\dagger$ obeys Eq. (5.5) as stated above. In particular $A^\dagger$ has not only a pure bosonic part but a part bilinear in $\mathcal{K}\tilde{\mathcal{K}}$ as well, which implies trilinear Grassmann contributions to $\bar{\lambda}$ and $\bar{\psi}$.

Now think about the behavior of Eq. (B1) far away from the center of the multi-instanton. On the one hand (in generalized singular gauge [4]), the x dependence and the tensor structure of the right-hand side approaches a spinor propagator (6.7), up to $O(x^{-5})$ corrections. On the other hand, from the divergence theorem, we actually know the coefficient of the leading $1/x^3$ falloff as well. Specifically, this coefficient can be equated to certain terms in the superinstanton action. This follows from an integration by parts in the Higgs kinetic energy term in the component Lagrangian, together with the Higgs equation of motion to cancel the Yukawa term; see Secs. IV C and VII D of [4] for details. In the case of $N=2$ SYM theory the entire superinstanton action $S^0_{\rm inst}$ may be read off from the residue in this way. In the models at hand, with $N_F>0$, this particular surface integral only accounts for two of the five separate pieces of $S^{N_F}_{\rm inst}$ listed in Sec. V above, namely those labeled (i) and (iii); these are precisely the pieces proportional to v. Combining these two observations, about the tensor structure and about the residue, gives Eq. (6.8); for $N_F>0$ one needs to substitute $S^0_{\rm inst}\to S^{N_F}_{\rm inst}$ as stated.

In the same way, the desired $\xi_1\xi_2$ bilinear piece of the field strength v_{mn} may be generated from $\Psi^{(0)}$ by acting with $\exp(\xi_2Q_2)\exp(\xi_1Q_1)$ and keeping the cross term. Under Q_1 one has $\delta v_{mn}=\xi_1\sigma_{[n}\mathcal{D}_{m]}\bar{\lambda}$, which is followed by the replacement (B1) under the action of Q_2. The remaining steps in the argument proceed as before (see Sec. V of [4]), and are left to the reader.

APPENDIX C: $N=1$ THEORIES WITH FUNDAMENTAL HIGGS BOSONS

In this appendix we touch on certain basic features of the $N=1$ supersymmetric theories of the type considered long ago in [13,24,22], in which all Higgs bosons live in the fundamental representation of SU(2). We can easily construct the superinstanton action by the methods of [4], as follows. The two relevant terms of the component Lagrangian, the Higgs kinetic energy and the Yukawa interaction, are turned into a surface term with an integration by parts in the former together with the Euler-Lagrange equation for the fundamental scalar, q. As per the divergence theorem, the action may then be extracted from the $1/x^3$ falloff of $\mathcal{D}_\perp q$, where the normal covariant derivative $\mathcal{D}_\perp$ is defined as $(x^m/\sqrt{|x|^2})\mathcal{D}_m$. The generic form of q including fermion bilinear contributions was given in Eq. (5.10) above:

$$q^{\dot\beta}=\bar{U}^{\dot\beta\beta}_\lambda\cdot\left(\delta_{\lambda0}v_\beta+\frac{i}{2\sqrt{2}}\mathcal{M}_{\lambda l\beta}f_{lk}\mathcal{K}_k\right), \tag{C1}$$

ignoring flavor indices from now on. As in the text the $\mathcal{K}$ are the Grassmann parameters associated with the fundamental fermion zero modes (5.3).

Using Eq. (5.2a) together with the asymptotics of the various ADHM quantities listed in Sec. VI B of [4], one easily derives

$$\mathcal{D}_\perp q^{\dot\beta}\overset{|x|\to\infty}{\longrightarrow}\frac{\bar{\sigma}_0^{\dot\beta\beta}}{|x|^3}\sum_{k=1}^{n}\left(|w_k|^2v_\beta-\frac{i}{\sqrt{2}}\mu_{k\beta}\mathcal{K}_k\right) \tag{C2}$$

and hence

$$S_{\rm inst}\propto\sum_{k=1}^{n}\left(|w_k|^2|v|^2-\frac{i}{\sqrt{2}}\bar{v}^\beta\mu_{k\beta}\mathcal{K}_k\right) \tag{C3}$$

using the notation of Eq. (2.7).

This supersymmetric multi-instanton action (originally derived by Yung [28] by different means) differs from that of the $N=2$ theory discussed herein, in two important ways. First, it has the form of a disconnected sum of n single instantons; in these coordinates there is no interaction between them. Second, the only gaugino modes that are lifted are those associated with the top-row elements μ^γ_k of the collective coordinate matrix $\mathcal{M}^\gamma$. This leaves $2n$ unlifted modes [after one implements the constraints (2.6)], which are naturally associated with the diagonal entries of the $n\times n$ submatrix $\mathcal{M}'_\gamma$. This counting contrasts sharply with the $N=2$ theory in which the number of unlifted modes is independent of the winding number.

Saturating these modes with anti-Higgsinos as per Affleck, Dine, and Seiberg [24], one therefore needs

$\langle \overline{\psi(x_1)}\,\overline{\psi(x_2)}\rangle$ in the one-instanton sector, and in general, $\langle \psi(x_1)\cdots\psi(x_{2n})\rangle$ in the n-instanton sector—unlike the $N=2$ theory, the sectors of different topological number do not interfere with one another. For completeness we write down the generic form of these antifermions, which satisfy the inhomogeneous equation

$$(\mathcal{D}_{\alpha\dot{\alpha}}\overline{\psi^{\dot{\alpha}}})^{\dot{\beta}}=c(\lambda_{\alpha})^{\dot{\beta}}{}_{\dot{\gamma}}q^{\dagger\dot{\gamma}}, \tag{C4}$$

where c is a normalization constant. Using Eq. (5.2a) once again, one easily finds

$$(\overline{\psi^{\dot{\alpha}}})^{\dot{\beta}}=-(c/2)\overline{\vec{U}^{\dot{\beta}\beta}}\mathcal{M}_{\beta}f\overline{\vec{w}^{\dot{\alpha}\gamma}}\overline{v_{\gamma}}. \tag{C5}$$

Here $\dot{\beta}$ and $\dot{\alpha}$ are color and Weyl indices, respectively; also Eq. (C5) is only valid when the top row of $\mathcal{M}$ (i.e., the lifted modes) consists entirely of zeros.

[1] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B430**, 485(E) (1994).
[2] N. Seiberg and E. Witten, Nucl. Phys. **B431**, 484 (1994).
[3] D. Finnell and P. Pouliot, Nucl. Phys. **B453**, 225 (1995).
[4] N. Dorey, V. V. Khoze, and M. P. Mattis, Phys. Rev. D **54**, 2921 (1996).
[5] A. Yung, ''Instanton-induced effective Lagrangian in the Seiberg-Witten model,'' Report No. hep-th/9605096 (unpublished).
[6] K. Ito and N. Sasakura, Phys. Lett. B **382**, 95 (1996).
[7] N. Dorey, V. V. Khoze, and M. P. Mattis, Phys. Lett. B (to be published).
[8] H. Aoyama, T. Harano, M. Sato, and S. Wada, ''Multi-instanton calculus in $N=2$ supersymmetric QCD,'' Report No. hep-th/9607076 (unpublished).
[9] M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin, and Yu. I. Manin, Phys. Lett. **65A**, 185 (1978).
[10] M. Matone, Phys. Lett. B **357**, 342 (1995).
[11] N. Dorey, V. V. Khoze, and M. P. Mattis, Phys. Lett. B (to be published).
[12] F. Fucito and G. Travaglini, Phys. Rev. D (to be published).
[13] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B229**, 394 (1983); **B229**, 407 (1983); **B260**, 157 (1985).
[14] A. Semikhatov, Phys. Lett. **120B**, 171 (1982).
[15] A. Galperin and E. Sokatchev, Nucl. Phys. **B460**, 431 (1995); S. J. Gates and L. Rana, Phys. Lett. B **345**, 233 (1995); E. Witten, J. Geom. Phys. **15**, 215 (1995).
[16] W. Siegel, Phys. Rev. D **52**, 1042 (1995).
[17] N. Seiberg, Phys. Lett. B **206**, 75 (1988).
[18] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, NJ, 1992).
[19] E. Corrigan, P. Goddard, and S. Templeton, Nucl. Phys. **B151**, 93 (1979); E. Corrigan, D. Fairlie, P. Goddard, and S. Templeton, *ibid.* **B140**, 31 (1978).
[20] H. Osborn, Ann. Phys. (N.Y.) **135**, 373 (1981).
[21] E. Corrigan, P. Goddard, H. Osborn, and S. Templeton, Nucl. Phys. **B159**, 469 (1979).
[22] J. Fuchs and M. G. Schmidt, Z. Phys. C **30**, 161 (1986).
[23] H. Osborn, Nucl. Phys. **B140**, 45 (1978).
[24] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B241**, 493 (1984); **B256**, 557 (1985).
[25] Y. Ohta, ''Prepotentials of $N=2$ SU(2) Yang-Mills gauge theory coupled with a massive matter multiplet,'' Report No. hep-th/9604051 (unpublished); ''Prepotentials of $N=2$ SU(2) Yang-Mills theories coupled with massive matter multiplets,'' Report No. hep-th/9604059 (unpublished).
[26] K. Ito and S. K. Yang, in Proceedings of ''Frontiers of quantum field theory,'' Toyonaka, Japan, 1995 (unpublished), Report No. hep-th/9603073 (unpublished).
[27] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976); **18**, 2199 (1978).
[28] A. Yung, Nucl. Phys. **B344**, 73 (1990).