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Topological aspects of gauge-fixing Yang-Mills theory on S_4

Laurent Baulieu*
LPTHE Paris VI-VII, Boite 126, Tour 16, 1^er^ étage, 4 place Jussieu, F-75252 Paris CEDEX 05, France
and Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan

Alexander Rozenberg† and Martin Schaden‡
Physics Department, New York University, 4 Washington Place, New York, New York 10003
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For an S_4 space-time manifold global aspects of gauge fixing are investigated using the relation to topological quantum field theory (TQFT) on the gauge group. The partition function of this TQFT is shown to compute the regularized Euler character of a suitably defined space of gauge transformations. Topological properties of the space of solutions to a covariant gauge conditon on the orbit of a particular instanton are found using the SO(5) isometry group of the S_4 base manifold. We obtain that the Euler character of this space differs from that of an orbit in the topologically trivial sector. This result implies that an orbit with a Pontryagin number $\kappa=\pm 1$ in covariant gauges on S_4 contributes to physical correlation functions with a different multiplicity factor due to the Gribov copies than an orbit in the trivial $\kappa=0$ sector. Similar topological arguments show that there is no contribution from the topologically trivial sector to physical correlation functions in gauges defined by a nondegenerate background connection. We discuss the possible physical implications of the global gauge dependence of Yang-Mills theory. [S0556-2821(96)06024-9]

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I. INTRODUCTION

Since the pioneering work of Gribov [1] and Singer [2] it is known that the gauge condition

$$\mathcal{F}(A)=0 \tag{1.1}$$

generally does not determine the representative connection A of a gauge orbit uniquely for non-Abelian gauge groups such as $\mathrm{SU}(n)$. It is, therefore, of interest to study the space of gauge equivalent solutions to Eq. (1.1)

$$\mathcal{E}_{\mathcal{F}}[A]=\{\mathrm{U}:\mathrm{U}(x)\in \mathrm{SU}(n),\mathcal{F}(A^{\mathrm{U}})=0\}, \tag{1.2}$$

on the gauge orbit $\{A^{\mathrm{U}}\}$ with representative connection A:

$$A^{\mathrm{U}}=\mathrm{U}^{\dagger}A\mathrm{U}+\mathrm{U}^{\dagger}d\mathrm{U}. \tag{1.3}$$

It was recently pointed out [3] that gauge fixing of Yang-Mills theory defined on a compact space-time manifold amounts to the construction of a certain topological quantum field theory (TQFT) on the space of gauge transformations, whose partition function computes some topological number of Eq. (1.2). This relation to TQFT enables us to investigate global aspects of the gauge fixing procedure in greater detail than was previously possible. We will show in Sec. II that the partition function of the TQFT proposed in Ref. [3] is proportional to the Euler characteristic $\chi(\mathcal{E}_{\partial\cdot A})$ of the moduli space

$$\mathcal{E}_{\partial\cdot A}[A]=\{\mathrm{U}:\mathrm{U}(x)\in \mathrm{SU}(n),\partial\cdot A^{\mathrm{U}}=0\}/\mathrm{SU}(n). \tag{1.4}$$

Note that the Euler character of the space of solutions to the covariant (Landau) gauge condition $\partial\cdot A^{\mathrm{U}}=0$ vanishes due to the isometry with respect to right multiplication by constant group elements. This isometry group has been factored out in Eq. (1.4) and it was shown [3] that $\chi(\mathcal{E}_{\partial\cdot A})=\mathrm{odd}\neq 0$ in the vicinity of flat connections $A\simeq \mathrm{U}^{\dagger}d\mathrm{U}$ for an SU(2) gauge group defined on any compact space-time manifold.

Since the topological properties of the moduli space (1.4) do not change under continuous deformations, one is guaranteed that $\chi(\mathcal{E}_{\mathcal{F}}[A])$ is constant within a topologically connected sector of gauge orbits. The Euler character could, however, depend on (i) the topological sector of the gauge orbit and/or (ii) the gauge fixing condition (1.1).

The two possibilities are intimately related, since the existence of topologically disconnected sectors in the orbit space implies that background gauge conditions of the type

$$\mathcal{F}(A)=D^{B}\cdot A-\partial\cdot B=0 \tag{1.5}$$

also cannot be deformed into each other for background connections B belonging to different topological sectors.

In this paper we will investigate both possibilities by considering an SU(2) gauge theory defined on S_4. Disconnected sectors of the orbit space with different Pontryagin number $\kappa[A]\in\mathbf{Z}$,

$$\frac{1}{4\pi^2}\int_{S_4}\mathrm{Tr}\,F(A)\wedge F(A)=\kappa[A],\quad F(A)=dA+A\wedge A, \tag{1.6}$$

exist in this case. For a given gauge condition $\mathcal{F}(A)$, the Euler character $\chi(\mathcal{E}_{\mathcal{F}})$ of the moduli space $\mathcal{E}_{\mathcal{F}}$ associated with an orbit can therefore in general depend on the Pontrya-

*Electronic address: Baulieu@lpthe.jussieu.fr
†Electronic address: sasha@acf2.nyu.edu
‡Electronic address: schaden@mafalda.physics.nyu.edu—research offers are welcome.

gin number of the orbit $\chi(\mathcal{E}_{\mathcal{F}}[A])=\chi_{\mathcal{F}}(\kappa[A])$. Since an S_4 manifold has only trivial 1-cycles, the result presented in the appendix of [3] actually implies that $\chi_{\mathcal{F}}(\kappa[A]=0)=1$ for an SU(2) theory in any covariant gauge which can be deformed to Landau gauge $\partial\cdot A=0$. In Sec. III we discuss the moduli space $\mathcal{E}_{\mathrm{cov}}$ of such covariant gauges for a particular instanton orbit with Pontryagin number $\kappa[A]=1$. In Sec. IV we construct this space explicitly in Landau gauge. On the other hand, we see in Sec. V that the Euler character of the moduli space $\mathcal{E}_{\mathcal{F}}[A]$ vanishes in the topologically trivial sector $\kappa[A]=0$ for gauges (1.5) defined by any nondegenerate background B. These results show that global characteristics of the gauge theory can depend on the gauge-fixing condition employed. We conclude by discussing the possible relevance of this dependence for the quantization of a gauge theory.

II. GAUGE FIXING AND TQFT

Gauge fixing an SU(n) YM theory with the gauge condition (1.1) amounts to the statement that the partition function

$$\mathcal{Z}[A]=\int[d\mathrm{U}][dc][d\overline{c}][db]\exp(S_A) \tag{2.1}$$

with the action

$$\begin{aligned}S_A&=-2\int dx\,s\,\mathrm{Tr}\,[\overline{c(x)}\mathcal{F}(A^{\mathrm{U}})(x)]\\&=-2\int dx\,\mathrm{Tr}\Bigg(b(x)\mathcal{F}(A^{\mathrm{U}})(x)\\&\quad-\int dy\,\overline{c(x)}\,\frac{\delta\mathcal{F}(A^{\mathrm{U}})(x)}{\delta\mathrm{U}(y)}c(y)\Bigg)\end{aligned} \tag{2.2}$$

does not vanish and is independent of the orbit which the connection A represents. In this case, the partition function (2.1) could be inserted in the gauge-invariant measure of the Yang-Mills (YM) theory and the change of variables $A^{\mathrm{U}}\to A$ would yield the usual gauge-fixed action proposed by Faddeev and Popov. The so far rather formal functional integral in Eq. (2.1) extends over the gauge-group elements $\mathrm{U}(x)\in\mathrm{SU}(n)$, the Lie-algebra-valued Nakanishi-Lautrup field $b(x)$ as well as the Lie-algebra valued anticommuting Faddeev-Popov ghosts and antighosts $c(x)$, $\overline{c(x)}$. The functional derivative in (2.2) is computed with respect to right multiplication of the group.

In Ref. [3] it was observed that Eq. (2.1) with action (2.2) can be regarded as the partition function of a TQFT in the gauge group with the Becchi-Rouet-Stora-Tyutin (BRST) algebra,

$$\begin{aligned}s\mathrm{U}(x)&=\mathrm{U}(x)c(x),\\ sc(x)&=-\frac{1}{2}[c(x),c(x)],\\ s\overline{c(x)}&=b(x),\quad sb(x)=0.\end{aligned} \tag{2.3}$$

Note that the connection A is a parameter of the TQFT ($sA=0$). The nilpotent BRST algebra (2.3) suggests [4] that one may identify the ghost fields as Maurer-Cartan forms on the infinite dimensional space of gauge transformations. This geometrical interpretation of $c(x)$ as a basis for the cotangent space at U becomes particularly convincing in the vicinity of $\mathrm{U}=\mathbf{1}$. The BRST operator can be identified as the coboundary operator of the Lie algebra cohomology and Eqs. (2.3) are regarded as the explicit representation of its action at an arbitrary point $\mathrm{U}(x)$ in the space of gauge transformations. General arguments (for a review see, e.g., [5]) suggest that the TQFT with the s-exact action (2.2) computes the Euler characteristic of the space of gauge transformations. One, however, first has to make sense of the Euler character of an infinite dimensional space.

There are essentially two ways to compute the Euler characteristic of a manifold. One is by use of the Gauss-Bonnet theorem, which gives the Euler characteristic as an integral of the Euler class over the manifold. The other makes use of the Poincaré-Hopf theorem, which amounts to counting with signs the number of isolated zeros of some vector field $\mathcal{F}=\vec{\nabla}V$ generated by a potential V on the manifold. The signs are determined by the sign of the Hessian at these isolated points. More generally, when the zero locus of the vector field $\mathcal{F}$ is finite dimensional, the regularized Euler characteristic of a manifold $\mathcal{M}$ is

$$\chi(\mathcal{M})=\sum_{X:\mathcal{F}|_X=0}(\pm1)\chi(X), \tag{2.4}$$

where the signs depend on the orientation of the fixed point manifolds X embedded in M. The latter method can be generalized to the case of infinite dimensional manifolds, where Eq. (2.4) is regarded as the *definition* of the regularized Euler characteristic of the manifold [6].

The usual argument that the TQFT with BRST algebra (2.3) computes the right-hand side of Eq. (2.4) is based on the fact that the saddle point approximation is apparently an exact evaluation of the functional integral. To see this one can for example modify the action (2.2) by the s-exact term $(\alpha/2)b^2$:

$$\begin{aligned}S_A'&=-2\int dx\,s\,\mathrm{Tr}\left[\overline{c(x)}\left(\mathcal{F}(A^{\mathrm{U}})-\frac{\alpha}{2}b(x)\right)\right]\\&=2\int dx\,\mathrm{Tr}\Bigg(\frac{\alpha}{2}b^2(x)-b(x)\partial\cdot A^{\mathrm{U}}(x)\\&\quad+\int dy\,\overline{c(x)}\,\frac{\delta\mathcal{F}(A^{\mathrm{U}})(x)}{\delta\mathrm{U}(y)}c(y)\Bigg).\end{aligned} \tag{2.5}$$

Because of the topological nature of the theory, the partition function should be independent of α. Gaussian integration over the multiplier field b in the functional integral with action S_A' shows that

$$\begin{aligned}\mathcal{Z}[A]\propto\int[d\mathrm{U}][dc][d\overline{c}]\exp\Bigg[2\int dx\Bigg(&-\frac{1}{2\alpha}(\mathcal{F}(A^{\mathrm{U}}))^2\\&+\int dy\,\overline{c(x)}\,\frac{\delta\mathcal{F}(A^{\mathrm{U}})(x)}{\delta\mathrm{U}(y)}c(y)\Bigg)\Bigg].\end{aligned} \tag{2.6}$$

In the $\alpha\to0$ limit only fluctuations around U's satisfying the fixed point equation $\mathcal{F}(A^{\mathrm{U}})=0$ will contribute to the functional integral. With the correct (α-dependent) normalization the contribution of an isolated fixed point to the partition function (2.1) is ± 1 depending on the sign of the ''Hessian'' at that point. To evaluate the contribution from a finite dimensional subspace of fixed points, one introduces local coordinates and restricts the BRST algebra (2.3) to that space. This procedure generally induces curvature terms [7]. One may then use the Gauss-Bonnet theorem to find that the contribution to the partition function of the TQFT is the Euler characteristic of the submanifold [8]. In the $\alpha\to 0$ limit, the partition function of the TQFT is thus seen to reproduce the right-hand side of Eq. (2.4).

It could appear that the space of gauge transformations and therefore its Euler characteristic can be defined independently of the connection A and that the TQFT (2.1) does not depend on A at all. Let us stress, however, that the Euler characteristic computed via the Poincaré-Hopf theorem is actually that of the *domain* $D:=\{x\in\mathcal{M},|V(x)|<\infty\}$, rather than the Euler characteristic of the whole manifold $\mathcal{M}$. Thus the Euler characteristic defined by Eq. (2.4) coincides with that of the original manifold $\mathcal{M}$ only if the potential is finite everywhere.

If the TQFT on the gauge group is to be employed as a gauge-fixing device, the gauge condition (1.1) will only depend on the orbit $\{A^{\mathrm{U}}\}$ rather than U itself. Any background gauge condition of the form (1.5) is the gradient of an associated potential, introduced in[1] [9]

$$V[U]=||A^{\mathrm{U}}-B||^2=\frac{1}{2}\int dx\mathrm{Tr}(A^{\mathrm{U}}-B)\cdot(A^{\mathrm{U}}-B)$$
$$=\frac{1}{2}\int dx\mathrm{Tr}(A-B^{\mathrm{U}^\dagger})\cdot(A-B^{\mathrm{U}^\dagger}). \tag{2.7}$$

Indeed, the fixed points of this potential $\delta V[\mathrm{U}]=0$,

$$\frac{\delta V}{\delta \mathrm{U}(x)}=-\mathcal{F}(A^{\mathrm{U}})(x),$$

are just the solutions to the gauge condition (1.5). The domain of the potential (2.7) in the sense of the Poincaré-Hopf theorem consists only of those gauge transformations $\{\mathrm{U}\}$ for which the connection $A^{\mathrm{U}}-B$ is square integrable. This domain does not depend on continuous deformations of the connections A and B but may (and we will indirectly see that it does) depend on their topological characteristics.

Completing the space of C^∞ connections in the L^2 norm $||\cdot||$ was shown [10] to naturally extend to considering the space of C^∞ gauge transformations completed in the Sobolev norm $||\mathrm{U}||_1=||\mathrm{U}||+||\partial\mathrm{U}||$. For the TQFT it is important that this is a topological space of gauge transformations which furthermore fully describes the gauge orbit $\overline{\{A^{\mathrm{U}}\}}=\{A^{\overline{\{\mathrm{U}\}}}\}$ [10]. We know of no other space of gauge transformations where this important property holds and will therefore work in this space. Note that in order to span the space of gauge transformations in the neighborhood of a particular $\mathrm{U}(x)$ one has to consider all fluctuations $\delta\mathrm{U}$ which are normalizable in the L^2-norm $||\delta\mathrm{U}||$. To preserve invariance under infinitesimal isometry transformations $x\to x+\varepsilon(x)$ of the base manifold, $\mathrm{U}(x)$ as well as $\mathrm{U}[x+\varepsilon(x)]$ have to belong to the space of allowed gauge transformations. This implies that one has to account for fluctuations $\delta\mathrm{U}=\mathrm{U}[x+\varepsilon(x)]-\mathrm{U}(x)$, which in general are only normalizable in the L^2, norm $||\cdot||=||\cdot||_0$, if U is normalizable in $||\cdot||_1$. These considerations determine the functional space we should consider in the TQFT with BRST algebra (2.3): $\mathrm{U}(x)\in C^\infty$ completed in the norm $||\cdot||_1$; $b(x)$, $c(x)$, and $\overline{c(x)}$ in C^∞ completed in the norm $||\cdot||=||\cdot||_0$.

Our previous argument indicates that the *regularized* Euler characteristic so obtained does not necessarily coincide with the Euler characteristic of the full space of gauge transformations and may depend on topological properties of the gauge condition (1.1). The purpose of this paper is to investigate this possibility.

For certain gauge conditions (1.1) it is easy to see that the regularized Euler characteristic computed in this way vanishes. This is for instance the case, whenever the Euler characteristic of each subspace of fixed points vanishes individually due to a group isometry. One encounters this situation in any background gauge (1.5) defined by a degenerate orbit B and in particular in covariant gauges, which correspond to choosing the degenerate background orbit $B=0$. The associated potential V is invariant with respect to right multiplication by a certain group in this case and the fixed point spaces therefore possess an isometry generated by the group action. Since this isometry has no fixed points, the Euler characteristic of each subspace vanishes.

The problem can be circumvented by an equivariant BRST construction [3] which divides out this group manifold. In covariant gauges one considers $X/\mathrm{SU}(n)$ rather than the space of fixed points X itself and the TQFT based on the equivariant cohomology computes the Euler characteristic $\chi[X/\mathrm{SU}(n)]$. For details on the equivariant BRST construction in covariant gauges and the associated TQFT we refer to [3].

III. INSTANTONS ON S_4 IN COVARIANT GAUGES

The main reason for employing covariant gauges in Minkowski or Euclidean space-time is that they allow a manifestly relativistically invariant formulation of the quantum field theory (QFT). The natural generalization of this distinction to other spacetime manifolds like S_4, is that a covariant gauge condition preserves *all* isometries of the base manifold. Covariant gauges preserve the homogeneity and isotropy of space-time and do not select a preferred point and/or direction in the gauge-fixed theory. This could be particularly important for defining the thermodynamic limit unambiguously, especially since color forces are expected to be strong and of infinite range.

By formulating a massless theory such as unbroken YM on compact Euclidean space-time, one can avoid infrared divergences while preserving the gauge (or rather BRST) symmetry. One hopes that the thermodynamic limit of physical correlation functions can be obtained by rescaling the

[1]In contrast with [9] we here however do not select a representative connection by the absolute minimum of the potential (2.7) on the gauge orbit, but rather sum over all its relative minima in the sense of the Poincaré-Hopf theorem.

compact base manifold and that this limit is independent of the manifold used to formulate the theory. The thermodynamic limit, if it exists, can at most depend on topological characteristics of the base manifold.

In the case of Yang-Mills theory on compact space-time with the topology of an S_4 it is well known that the space of gauge orbits is disconnected. The possible importance of sectors with Pontryagin number $\kappa[A]\neq 0$ has been recognized long ago [11] and 't Hooft's semiclassical calculation [12] indicates that this could resolve the $\mathrm{U}_A(1)$ problem [13]. We wish to stress that the $\mathrm{U}_A(1)$ problem is intimately related to the existence of covariant gauges, since one makes use of the Goldstone theorem to formulate it [13]. Due to an anomalous contribution, the conserved $\mathrm{U}_A(1)$ current is gauge dependent. In covariant gauges the Ward identities nevertheless imply the existence of a Goldstone pole in correlation functions of this current with quark multilinears, if the $\mathrm{U}_A(1)$ symmetry is spontaneously broken [14]. It is therefore important to verify that gauge orbits with Pontryagin number $\kappa[A]=0$ contribute to physical correlation functions in *covariant* gauges.

We adopt conformal coordinates to parametrize the S_4 and use its diameter $2R=1$ as the unit of length. In these coordinates the metric is diagonal ($x^2=\Sigma_\mu x_\mu x_\mu$),

$$g_{\mu\nu}(x)=g(x)\,\delta_{\mu\nu}=(1+x^2)^{-2}\delta_{\mu\nu}, \tag{3.1}$$

and the invariant volume element of the S_4 is simply $dx=d^4xg^2(x)$.

The SO(5) isometry group of an S_4 is generated by the coordinate transformations

$$x'_\mu=x_\mu-\epsilon_\mu(x;a,\omega)=x_\mu-\omega_{\mu\nu}x_\nu+a_\mu(x^2-1)-2x_\mu x\cdot a, \tag{3.2}$$

depending on ten infinitesimal parameters a_μ and $\omega_{\mu\nu}=-\omega_{\nu\mu}$.

A scalar density $s(x)$ transforms as

$$\delta s(x)=s_{,\mu}(x)\epsilon_\mu(x;a,\omega) \tag{3.3}$$

under the isometry group. Together with the variation of vector fields such as a connection $A_\mu(x)$,

$$\delta A_\mu(x)=A_{\mu,\nu}\epsilon_\nu(x;a,\omega)+A_\nu(x)\epsilon_{\nu,\mu}(x;a,\omega), \tag{3.4}$$

Eq. (3.3) determines the transformation properties of all higher rank tensors. The isometry generators (3.2) of course do not change the conformal metric (3.1):

$$\begin{aligned}\delta g_{\mu\nu}(x)=0=g(x)[\epsilon_{\mu,\nu}(x;a,\omega)+\epsilon_{\nu,\mu}(x;a,\omega)]\\+\delta_{\mu\nu}g_{,\sigma}(x)\epsilon_\sigma(x;a,\omega).\end{aligned} \tag{3.5}$$

The change of a pseudoscalar density $p(x)$ such as the YM Lagrangian

$$\mathcal{L}(x)=\frac{1}{2g^2}\sqrt{\det(g_{\mu\nu})}\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}g^{-1}_{\mu\rho}g^{-1}_{\nu\sigma}, \tag{3.6}$$

or the Pontryagin density

$$\mathcal{P}(x)=\frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma} \tag{3.7}$$

under conformal transformations

$$\delta p(x)=p_{,\mu}(x)\epsilon_\mu(x;a,\omega)+p(x)\epsilon_{\mu,\mu}(x;a,\omega), \tag{3.8}$$

will also be useful in the following.

The prototype of a covariant gauge condition is the Landau gauge, which on an S_4 in conformal coordinates takes the form

$$\begin{aligned}\mathcal{F}(A)=\partial\cdot A=(\det g)^{-1/2}\partial_\mu(\det g)^{1/2}g^{-1}_{\mu\nu}A_\nu(x)\\=g^{-2}(x)\partial_\mu g(x)A_\mu(x)=0.\end{aligned} \tag{3.9}$$

The following considerations are, however, not particular to Landau gauge and only make use of the fact that the gauge condition $\mathcal{F}(A)$ is covariant. To investigate topological characteristics of the space of solutions (1.2) to a covariant gauge condition in the sector with Pontryagin number $\kappa=1$, we will choose a particular orbit in that sector.

As discussed in the previous section the space of solutions (1.2) to a covariant gauge condition like Eq. (3.9) for general orbits only possesses an isometry with respect to right multiplication by constant gauge group elements. The space of solutions (1.2) of certain orbits can, however, have a larger isometry. Covariant gauges are invariant under the SO(5) isometry group of the S_4 by definition, and we will find a particular connection $A^{(s)}$ in the $\kappa=1$ sector for which isometry transformations (3.4) of the base space are equivalent to infinitesimal gauge transformations

$$\begin{aligned}\delta A^{(s)}_\mu(x)=A^{(s)}_{\mu,\nu}\epsilon_\nu(x;a,\omega)+A^{(s)}_\nu(x)\epsilon_{\nu,\mu}(x;a,\omega)\\=D^{A^{(s)}}\theta(x;a,\omega).\end{aligned} \tag{3.10}$$

The equivalence (3.10) between isometry and gauge transformations is only possible for an orbit whose classical YM Lagrangian (3.6) and Pontryagin density (3.7) are both invariant under the SO(5) isometry group. The SO(4) invariance of $\mathcal{L}(x)$ and $\mathcal{P}(x)$ implies that these pseudoscalar densities are only functions of $t=x^2$. Using Eqs. (3.8) and (3.2) the required invariance with respect to the SO(5)/SO(4) coset generators gives the differential equation

$$\left((1+t)\frac{d}{dt}+4\right)\begin{cases}\mathcal{P}(t)\\\mathcal{L}(t)\end{cases}=0, \tag{3.11}$$

determining $\mathcal{P}(t)$ and $\mathcal{L}(t)$ up to a normalization. The normalization of $\mathcal{P}(t)$ is fixed by the Pontryagin number $\kappa=1$ and for $\mathcal{L}(x)$ can be absorbed in the definition of the coupling constant. The solution of Eq. (3.11) is thus seen to be just the familiar pseudoscalar density of a BPST instanton [15] with scale $\rho=2R=1$ located at the "origin" $x=0$

$$\frac{g^2}{8\pi^2}\mathcal{L}(x)=\mathcal{P}(x)=\frac{6}{\pi^2}(1+x^2)^{-4}=\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,F^{(s)}_{\mu\nu}F^{(s)}_{\rho\sigma}. \tag{3.12}$$

A self-dual configuration with Pontryagin number $\kappa=1$ will be called a *standard* instanton in what follows if its Pontryagin density is invariant under the SO(5) isometry group of the S_4. In conformal coordinates, its Pontryagin density is given by Eq. (3.12).

Jackiw and Rebbi [16] studied the Belavin-Polyakov-Schwarz-Tyupkin (BPST) instanton [15] solutions to the classical equations of motion of an SU(2) Yang-Mills theory on a Euclidean S_4. The space of these solutions forms an SO(5,1)/SO(5) modulo gauge transformations, which tallies with the fact that the moduli space in the $\kappa=\pm 1$ sector of an SU(2) theory is five dimensional [17]. They found that a particular connection of the standard instanton is invariant under the SO(5) isometry group of the S_4 modulo gauge transformations. Actually this does not depend on the point on the gauge orbit of the standard instanton, but is true on the whole orbit. This slight generalization of their result enables us also to study the infinitely many Gribov copies of the connection considered in [16] without the need to explicitly construct them.

The generalization is possible because the moduli space of an SU(2) instanton is only five dimensional [17] and very well known. Any variation of $A^{(s)}$ which does not change the YM action is either a gauge transformation or would have to dilate or translate the standard instanton and therefore show up as a change of its Pontryagin density. There are no *hidden* moduli parameters in the $\kappa=1$ sector on which the Pontryagin density of an instanton does not depend. Since the Pontryagin density of a standard instanton is invariant under isometry transformations, we are assured that this transformation does not move in the moduli space of the instanton and must be a gauge transformation only. The equivalence (3.10) therefore holds at every point on the gauge orbit of a standard instanton.

We still have to determine the space these gauge modes of the standard instanton actually span. Only the trivial configuration $A=0$ is invariant under the full SO(5) isometry group. Vector fields which are invariant under an SO(4) subgroup of SO(5) are pure gauge—the corresponding antisymmetric field strength tensor, an $(1,0)\oplus(0,1)$ representation of SO(4), has to vanish since only the null vector is invariant under SO(3) rotations.[2] The best one can achieve in the $\kappa\neq 0$ sectors is invariance of the connection under an SO(3) subgroup of SO(5). The corresponding field strength tensor is then (anti-)self-dual. We therefore conclude that:

A BPST instanton in the $\kappa=1$ topological sector is changed by (or breaks) the seven generators of the SO(5)/SO(3) coset space of the isometry group of an S_4. The broken generators form an $(\frac{1}{2},\frac{1}{2})\oplus(1\,,\,0)$ representation of SO(4).

We already know that isometry transformations of the *standard* instanton are equivalent to gauge transformations. The isometries of an SO(5)/SO(3) coset space therefore correspond to *nontrivial* gauge transformations of the standard instanton. The three broken generators of the $(1\,,\,0)$ representation obviously generate constant gauge transformations. The other four broken generators in the $(\frac{1}{2},\frac{1}{2})$ representation must generate gauge modes of the standard instanton which are *linearly independent* of these—simply because they transform according to different representations of the SO(4) subgroup.

To summarize: isometry transformations of an S_4 are mapped to an $\mathrm{SO}(5)/\mathrm{SO}(3)\simeq S_7$ subspace of the automorphisms of the gauge orbit of a standard instanton. Modulo global SU(2) transformations, the space of solutions to a covariant gauge condition on the orbit of a standard instanton has the topological structure

$$\mathcal{E}_{\mathrm{cov}}[A^{(s)}]\simeq\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)\times\mathrm{SU}(2)}\times\mathcal{B}\simeq S_4\times\mathcal{B}. \quad (3.13)$$

The space $\mathcal{B}$ was not determined and could depend on the covariant gauge condition employed. The result (3.13) shows that the gauge orbit of a standard instanton is *on* the Gribov horizon in any covariant gauge.

Our arguments did not make use of any *explicit* form for the connection of a standard instanton. They only rely on the special property (3.10) of the orbit of a standard instanton and apply to any covariant gauge.

IV. THE EULER CHARACTERISTIC OF THE SPACE OF SOLUTIONS TO COVARIANT GAUGE CONDITIONS IN THE $\kappa=\pm 1$ SECTORS

In Landau gauge two explicit standard instanton connections have been studied extensively. In conformal coordinates on an S_4 they are

$$A_\mu^{(1)}(x)=\frac{x^2u^\dagger\partial_\mu u}{1+x^2}=\frac{2\,\eta_{\mu\nu}x_\nu}{1+x^2}$$

and

$$A_\mu^{(2)}(x)=A_\mu^{(1)u^\dagger}(x)=\frac{u\,\partial_\mu u^\dagger}{1+x^2}=\frac{2\,\overline{\eta_{\mu\nu}}x_\nu}{x^2(1+x^2)}. \quad (4.1)$$

$(\overline{\eta_{\mu\nu}}),\eta_{\mu\nu}$ are the Lie-algebra-valued (anti-)self-dual tensors introduced by 't Hooft [12]. As indicated in Eq. (4.1) $u(x)=(x^4\mathbf{1}+i\vec{\sigma}\vec{x})/\sqrt{x^2}$ is the gauge transformation relating these two connections. Both connections (4.1) have finite L^2 norm $||\cdot||$ and the gauge transformation $u(x)$ relating them is normalizable in $||\cdot||_1$. In the functional space we are considering, the connections (4.1) therefore belong to the *same* gauge orbit and are *both* instantons with Pontryagin number $\kappa=+1$.

In fact they are just two points of an S_4 of gauge equivalent instantons parametrized by $b\in\mathcal{R}^4$:

$$A_\mu^{(b)}(x)=A_\mu^{(1)u^\dagger(x+b)}(x)=u(x+b)A_\mu^{(1)}(x)u^\dagger(x+b)$$
$$+u(x+b)\partial_\mu u^\dagger(x+b). \quad (4.2)$$

Using

$$u^\dagger(x)\partial_\mu u(x)=\frac{2\,\eta_{\mu\nu}x_\nu}{x^2} \quad (4.3)$$

and the su(2) algebra of the self-dual tensors $\eta_{\mu\nu}$ it is straightforward to verify that any connection (4.2) satisfies the Landau gauge condition (3.9). Modulo constant gauge transformations $A^{(b\to\infty)}\simeq A^{(1)}$ and $A^{(b=0)}=A^{(2)}$. Points at infinity in the parameter space correspond to connections

[2]A more pedestrian proof of this statement is obtained by using conformal coordinates adapted to the SO(4) in question. The invariant vector field in these coordinates is of the form $A_\mu(x)=x_\mu\Phi(x^2)$ and easily seen to be pure gauge.

$A^{(b)}$ which are equivalent modulo constant gauge transformations and are identified in equivariant TQFT. Thus the parameters b can be considered projective coordinates on S_4 of gauge equivalent standard instanton connections satisfying the Landau gauge condition. The connections (4.1) are those at the ''north'' and ''south'' poles of this S_4. One may also verify explicitly that an infinitesimal variation of the parameters b in $A^{(b)}$ corresponds to an isometry transformation (3.2).

The construction in Landau gauge shows that there is no further identification of connections on the S_4 since *all* constant gauge transforms of $A^{(1)}$ are identified with a single point on the S_4. The space of solutions to the Landau gauge condition on the orbit of a standard instanton therefore indeed has the structure (3.13) and we conclude from $\chi(S_4)=2$ that

$$\chi(\mathcal{E}_{\partial\cdot A}[A^{(s)}])=\text{even}. \tag{4.4}$$

The Euler characteristic of the space should not depend on continuous deformations of the orbit or the gauge condition. Hence for a generic *covariant* gauge condition which can be continuously deformed to Landau gauge, the Euler characteristic of the space of solutions on orbits in the $\kappa=\pm 1$ sectors is *even*:

$$\chi_{\text{cov}}(\kappa[A]=\pm 1)=\text{even}. \tag{4.5}$$

To determine the Euler characteristic more accurately than in Eq. (4.5) would require a better understanding of the so far undetermined space $\mathcal{B}$ in Eq. (3.13). Together with the previous result of Ref. [3] for the $\kappa=0$ sector on S_4,

$$\chi_{\text{cov}}(\kappa[A]=0)=1. \tag{4.6}$$

Equation (4.5), however, already shows that the partition function of the TQFT *depends* on the topological sector of the orbit in covariant gauges.

V. GENERAL BACKGROUND GAUGES AND POSSIBLE PHYSICAL IMPLICATIONS OF GLOBAL GAUGE DEPENDENCE

In the previous section we observed that the partition function of the TQFT associated with a covariant gauge-fixing condition depends on the topological sector of the orbit. Although we cannot construct the space of solutions for an arbitrary background gauge (1.5), we will see that its Euler characteristic vanishes for orbits in the trivial topological sector $\kappa=0$ when the background connection B is not degenerate. Thus the partition function of the TQFT either vanishes identically in this case, or depends on the topological sector also in these gauges.

Consider for instance a gauge condition (1.5) where the background connection B is the instanton connection $A^{(1)}$. Since the orbit of B is nondegenerate,[3] the associated potential (2.7) in this case has a unique absolute minimum at $\mathrm{U}=\mathbf{1}$ for $A=B$. There is no degenerate subspace of solutions to divide out in this gauge, and the equivariant construction of Ref. [3] cannot be employed. 't Hooft used this gauge for his semiclassical calculation [12]. The Faddeev-Popov operator in this case is $D^B\cdot D^{A^{\mathrm{U}}}$. At the absolute minimum of the potential at $A^{\mathrm{U}}=B=A^{(1)}$ this operator is positive definite. Although we do not have much control over the Gribov copies in this gauge, the $\kappa=1$ sector may very well contribute to the partition function of the TQFT.

On the other hand this background gauge condition is degenerate for the flat orbit $A=\mathrm{U}^\dagger d\mathrm{U}$. The Faddeev-Popov operator $D^B\cdot D^{\mathrm{U}^\dagger d\mathrm{U}}$ on this orbit apparently has zero modes corresponding to left multiplication of U by global gauge transformations and the space of solutions to this gauge condition on the flat orbit has the topological structure

$$\mathcal{E}_{D^B\cdot A}[A=0]=\{\mathrm{U}:D^B(\mathrm{U}^\dagger\partial\mathrm{U})=0\}\simeq \mathrm{SU}(n)\times\overline{\mathcal{B}}. \tag{5.1}$$

Since the Euler character of a group manifold vanishes, the Euler characteristic of Eq. (5.1) vanishes irrespective of the space $\overline{\mathcal{B}}$. We thus see that orbits in the $\kappa=0$ sector cannot contribute to the partition function of the TQFT in this gauge. The above argument is easily extended to any nondegenerate background connection B—the space of solutions to the background gauge condition on the orbit of flat connections has vanishing Euler characteristic in this case and it is impossible to remove this degeneracy with an equivariant construction analogous to that of [3].

We have shown for an S_4 that an orbit with Pontryagin number $\kappa=\pm 1$ contributes to physical correlation functions in covariant gauges with a different multiplicity factor due to Gribov copies, than an orbit in the trivial topological sector and that nondegenerate background gauges generally annihilate the $\kappa=0$ sector of the theory altogether. This confirms the claim of Ref. [3] that global properties of the *quantized* gauge theory may depend on the gauge fixing. Global properties of a YM theory therefore will in general depend on the gauge.[4]

Although this ambiguity was found in the continuum formulation of a gauge-fixed YM theory, it could also arise in the thermodynamic limit of a lattice gauge formulation. From this point of view it is perhaps less surprising that topologically nontrivial configurations contribute mainly in lattice gauge theories with noncovariant boundary conditions.

Our observation can have implications for the $\mathrm{U}_A(1)$- and strong CP-violation problem, if it turns out that topologically nontrivial sectors do not contribute in covariant gauges. Our topological arguments are not sufficiently refined to actually determine the regularized Euler character in the $\kappa\neq 0$ sectors of the theory in covariant gauges. They only indicate that it is *even* and therefore differs from that of the $\kappa=0$ sector. It is obviously desirable to improve on this result. Our investigation nevertheless shows that the well known global dependence of a YM theory on the topology of compact space-time may also extend to the gauge condition.

[3] $D^B\omega=0$ only has the trivial solution $\omega=0$ when $B=A^{(1)}$ of Eq. (4.1).

[4] Since perturbation theory only accounts for fluctuations around a *single* solution to the gauge condition, it is not sensitive to the global issues discussed here.

Physical correlation functions are gauge invariant and any gauge-dependent answer is usually attributed to a badly performed gauge fixing. This, however, presumes that the quantization of a classical gauge theory is unique. There may be many different quantized gauge theories which correspond to a single classical theory with a certain *infinitesimal* gauge invariance. There is no *a priori* reason that so quantized ‘‘gauge theories’’ are identical, since the extension of this procedure to the whole orbit space will not be unique in general. Of course, only *one* of these extensions can (at best) describe physical reality and physical criteria have to be used to select this model. The phenomenon is very similar to the spontaneous breakdown of a symmetry where the realistic model is selected by choosing one of the degenerate vacua.

Our investigation only addressed theories with a BRST symmetry and not the orbit space of gauge theory *per se*. We adopted BRST symmetry as the guiding principle for constructing a gauge theory. The relation of the usual gauge-fixing procedure to a certain TQFT in the gauge group observed in [3] allowed us to determine global characteristics of such a theory. We believe that our definition of the partition function based on a complete BRST approach will allow us to investigate the Gribov problem in more detail. We speculate that physical properties such as the apparent absence of strong CP violation may select a preferred class of ‘‘gauges.’’

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