Two-dimensional instantons with bosonization and physics of adjoint two-dimensional QCD

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We evaluate partition functions Z_I in topologically nontrivial (instanton) gauge sectors in the bosonized version of the Schwinger model and in a gauged WZNW model corresponding to two-dimensional QCD (QCD_2) with adjoint fermions. We show that the bosonized model is equivalent to the fermion model only if a particular form of the WZNW action with a gauge-invariant integrand is chosen. For the exact correspondence, it is necessary to integrate over the ways the gauge group $SU(N)/Z_N$ is embedded into the full $O(N^2-1)$ group for the bosonized matter field. For even N, one should also take into account the contributions of both disconnected components in $O(N^2-1)$. In that case, $Z_I \propto m^{n_0}$ for small fermion masses where $2n_0$ coincides with the number of fermion zero modes in a particular instanton background. The Taylor expansion of Z_I/m^{n_0} in mass involves only even powers of m, as it should. The physics of adjoint QCD_2 is discussed. We argue that, for odd N, the discrete chiral symmetry $Z_2 \otimes Z_2$ present in the action is broken spontaneously down to Z_2 and the fermion condensate $\langle \lambda\lambda \rangle_0$ is formed. The system undergoes a first order phase transition at $T_c=0$ so that the condensate is zero at an arbitrary small temperature. It is not yet quite clear what happens for even $N \geqslant 4$. [S0556-2821(96)00524-3]

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I. INTRODUCTION

It has been known for a long time that the Schwinger model involves topologically nontrivial gauge field configurations: the instantons (see [1] and references therein). The reason why they appear is the nontrivial $\pi_1[U(1)]=Z$. Instantons are characterized by an integer topological charge

$$\nu = \frac{1}{2\pi} \int d^2x F(x), \tag{1.1}$$

where $F=F_{01}=\partial_0 A_1 - \partial_1 A_0$. Their physics is rather similar to the physics of instantons in four-dimensional QCD (QCD_4) with one light quark flavor. In particular, the fermion condensate

$$|\langle \bar{\psi}\psi \rangle_0| = \frac{g}{2\pi^{3/2}} e^{\gamma} \tag{1.2}$$

is formed (g is the coupling and γ is the Euler constant). The path integral calculation of $|\langle \psi\psi \rangle_0|$ [2,3] follows closely the 't Hooft calculation of the instanton determinant in QCD_4. The condensate is formed due to the presence of one complex fermion zero mode for gauge field background configurations with unit topological charge ν. It was noted recently that topologically nontrivial configurations appear also in non-Abelian two-dimensional gauge theories with adjoint matter content [4,5]. In this paper, we will consider only the simplest nontrivial theory of this kind which involves a multiplet of adjoint real fermions λ^a. The Lagrangian of the model reads

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{i}{2} \{ \lambda^a_L [\delta^{ab}\partial_- - f^{abc} A^c_-] \lambda^b_L + \lambda^a_R [\delta^{ab}\partial_+ - f^{abc} A^c_+] \lambda^b_R \}, \tag{1.3}$$

where $\partial_\pm = \partial_0 \pm \partial_1$, $A^c_\pm = A^c_0 \pm A^c_1$, and $\lambda_{L,R} = \frac{1}{2}(1 \pm \gamma^5)\lambda$ are the left- and right-moving components of the Majorana fermion field (the Lagrangian is written in Minkowski space because Majorana fermions cannot be defined[1] in Euclidean space [6]). We will consider both massless model (1.3) and the model which includes a small mass term

$$m\overline{\lambda^a}\lambda^a = -2im\lambda^a_L \lambda^a_R, \tag{1.4}$$

$m \ll g$.

Adjointness of all fields in the Lagrangian is crucial for the instantons to appear: In the standard QCD_2 with fundamental quarks where the gauge group is $SU(N)$, $\pi_1[SU(N)]=0$ and topologically nontrivial configurations are absent. But in the theory with adjoint matter the true gauge group is $SU(N)/Z_N$ (the elements of the center act trivially on adjoint fields). $\pi_1[SU(N)/Z_N]=Z_N \neq 0$ and instantons appear.[2] It was found in [5] that these configurations involve fermion zero modes [which conforms with the analysis by Kogan [11] who showed that instantons do not con-

[1]Note, however, that although we cannot define the Euclidean counterpart of the Lagrangian (1.3), the Euclidean path integral can be easily defined as an analytic continuation of the Minkowski path integral. In Minkowski space, integration over Majorana fermions provides the factor which is the square root of the Dirac determinant. We can *define* the Euclidean path integral of the theory (1.3) as the integral over gauge fields involving the square root of the Euclidean Dirac determinant as a factor [7]. The extraction of square root presents no problem here as all eigenvalues of the Dirac operator for complex adjoint fermions are doubly degenerate [5,8].

[2]A nontrivial $\pi_1[SU(N)/Z_N]$ brings about topologically nontrivial configurations also in four-dimensional Yang-Mills theory without quarks. But here two extra transverse dimensions are present and these configurations are not localized and have infinite action. These planar instantons were obtained in Ref. [9] and misinterpreted as real ''walls between different Z_N phases.'' Actually, the instantons and planar instantons are essentially Euclidean configurations and do not exist as real physical objects in Minkowski space [10].

tribute in the partition function in the massless theory (1.3) in high temperature region]. For the simplest topologically nontrivial sector their number is $2(N-1)$. Instantons lead to physically observable effects (with an obvious reservation that we are discussing a model theory which is not found in nature). They are responsible, in particular, for finite string tension in fundamental Wilson loops, i.e., for confinement of heavy fundamental sources in a theory with nonzero mass of dynamic adjoint fermions (in massless theory, instantons decouple, string tension disappears, and the sources are not confined but screened) [12]. When $N=2$, instantons bring about a nonzero fermion condensate [5].

The latter follows also from semiheuristic arguments based on the bosonization approach. The bosonized version of QCD_2 with fermions in the adjoint representation of SU(2) is the gauged Wess-Zumino-Novikov-Witten (WZNW) model [13–18] with the matter fields presenting orthogonal matrices $h^{ab}(x)$, the elements of O(3). The theory involves only massive excitations, their mass being of the order of the coupling constant g. As a result, the matter field is ''frozen'' and a nonzero vacuum expectation value $\langle h^{aa}\rangle_0$ appears. In the fermion language, that means the appearance of nonzero $\langle \overline{\lambda^a}\lambda^a\rangle_0$ where λ^a are adjoint Majorana fermion fields. For $N\geqslant 3$, the situation is much more complicated and controversial. Instantons involve ''too many'' fermion zero modes and cannot generate a nonvanishing bilinear fermion condensate. On the other hand, the quoted bosonization arguments do not distinguish between different N. Say, for $N=3$, the matter fields present 8×8 adjoint SU(3) matrices and a nonzero

$$\langle \overline{\lambda^a}\lambda^a\rangle_0 = Cg\langle h^{aa}\rangle_0 \tag{1.5}$$

should appear. It is also known that the condensate is formed at infinite N [19].

This paradox, formulated in [5], is akin to a similar paradox which pops out in four-dimensional (4D) supersymmetric (SUSY) Yang-Mills theories with higher orthogonal groups [20] and is rather troublesome; it is not yet absolutely clear how it is resolved. It was the main motivation for the present study.

The main part of the paper is devoted to the analysis of Euclidean path integrals of the gauged WZNW model in the topologically nontrivial sectors. We show that the zero mode suppression factor $\propto m^{n_0}$ is reproduced indeed, but only if doing things with proper care.

The commonly used form of the gauged Euclidean WZNW action reads

$$\begin{aligned}
S_E[A,h] =& \frac{1}{4g^2}\int d^2x\, F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{16\pi}\int d^2x\, \mathrm{Tr}\{\partial_\mu h \partial_\mu h^{-1}\} - \frac{i}{24\pi}\int_Q d^3\xi\, \epsilon^{ijk}\mathrm{Tr}\{h^{-1}\partial_i h h^{-1}\partial_j h h^{-1}\partial_k h\} \\
&+ \frac{1}{8\pi}\int d^2x [\mathrm{Tr}\{A_+ h\partial_- h^{-1}\} + \mathrm{Tr}\{A_- h^{-1}\partial_+ h\} + \mathrm{Tr}\{A_+ h A_- h^{-1}\} - \mathrm{Tr}\{A_+A_-\}] \\
=& \frac{1}{2g^2}\int d^2x\, \mathrm{Tr}\, F^2_{\mu\nu} + N\Bigg\{\frac{1}{8\pi}\int d^2x\, \mathrm{Tr}\{\partial_\mu u\partial_\mu u^{-1}\} - \frac{i}{12\pi}\int_Q d^3\xi\, \epsilon^{ijk}\mathrm{Tr}\{u^{-1}\partial_i u u^{-1}\partial_j u u^{-1}\partial_k u\} \\
&+ \frac{1}{4\pi}\int d^2x [\mathrm{Tr}\{A_+ u\partial_- u^{-1}\} + \mathrm{Tr}\{A_- u^{-1}\partial_+ u\} + \mathrm{Tr}\{A_+ u A_- u^{-1}\} - \mathrm{Tr}\{A_+A_-\}]\Bigg\},
\end{aligned} \tag{1.6}$$

where h is the matrix $(N^2-1)\times(N^2-1)$ belonging to the adjoint representation of SU(N) and u is an associated unitary matrix $N\times N$,

$$h^{ab} = 2\,\mathrm{Tr}\{t^a u t^b u^{-1}\}, \tag{1.7}$$

A_μ are anti-Hermitian matrices $A_\mu = iA^a_\mu T^a$, T^a are the generators in a corresponding representation, $A_\pm = A_0 \pm iA_1$, $\partial_\pm = \partial_0 \pm i\partial_1$, and Q is a three-dimensional manifold with a two-dimensional boundary where the theory actually lives. Our statement is that, generally speaking, the action (1.6) is *wrong*. It is not gauge invariant and does not correspond to the original theory (1.3). One should rather choose the action in the form

$$\begin{aligned}
S_E(F,u) =& \frac{1}{2g^2}\int d^2x\, \mathrm{Tr}\, F^2_{\mu\nu} - \frac{N}{8\pi}\int d^2x\, \mathrm{Tr}\{u^{-1}\nabla_\mu u u^{-1}\nabla_\mu u\} - \frac{iN}{12\pi}\int_Q d^3\xi\, \epsilon^{ijk}\mathrm{Tr}\{u^{-1}\nabla_i u u^{-1}\nabla_j u u^{-1}\nabla_k u\} \\
&+ \frac{iN}{8\pi}\int_Q d^3\xi\, \epsilon^{ijk}\mathrm{Tr}\{F_{ij}(u^{-1}\nabla_k u + \nabla_k u u^{-1})\},
\end{aligned} \tag{1.8}$$

where

$$\nabla_i u = \partial_i u + [A_i, u], \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

The functional (1.8) was written earlier in [17]. The actions (1.6) and (1.8) differ by the integral of a total derivative. For topologically trivial configurations, this integral is zero and the actions (1.6) and (1.8) are equivalent, but in the instanton sectors they are not. Actually, the action (1.6) is not gauge invariant in the instanton sectors while the explicit invariance of the *integrand* in Eq. (1.8) under the gauge transformations

$$A_\mu \to \Omega^{-1}(A_\mu + \partial_\mu)\Omega,$$
$$u \to \Omega^{-1} u \Omega \tag{1.9}$$

is seen immediately.[3]

Adding the mass term

$$\propto m \ \mathrm{Tr} h = 2m \ \mathrm{Tr}\{u t^a u^{-1} t^a\} \tag{1.10}$$

in the action (1.8) and evaluating the path integral, we will show that the factor m^{n_0} is singled out where n_0 is half the number of fermion zero modes.[4]

Unfortunately, it is not yet the end of the story. We will see that the action (1.8) with the added mass term (1.10) *does* not exactly correspond to QCD_2 with massive adjoint fermions. Recall that the set of N^2-1 free adjoint fermion fields is habitually bosonized with the orthogonal matrices $h \in O(N^2-1)$ [16]. For the theory involving gauge fields $\in SU(N)/Z_N$, one should rather use bosonization with adjoint SU(N) matrices $h^{ab} \in SU(N)/Z_N \subset O(N^2-1)$ [21]. But there are many ways to choose a subgroup $SU(N)/Z_N$ within the large orthogonal group. It turns out that, in order to preserve all symmetries of the fermion Lagrangian and to get a correct mass dependence for the partition function in *topologically nontrivial sectors*, one has to average over all these ways. In other words, one has to write the mass term in the form

$$\propto m \ \mathrm{Tr}[h \in O(N^2-1)] = 2m R^{ab} \mathrm{Tr}\{u t^b u^{-1} t^a\} \tag{1.11}$$

and average over all R^{ab} belonging to the coset $O(N^2-1)/[SU(N)/Z_N]$.

The plan of the paper is the following. Before proceeding with our analysis of bosonized theories, we present in Sec. II a new derivation of zero mode counting rules in instanton sectors in the fermion language. Distinct topological sectors are labeled by an integer $k=0,1,\ldots,N-1$. In Ref. [5] only the cases $k=1$ (the instanton) and $k=N-1$ (the anti-instanton) were analyzed. For an arbitrary k the result is

$$n_L^0 = n_R^0 = k(N-k). \tag{1.12}$$

Note that we are dealing here with an index theorem of new variety—the numbers of left- and right-handed zero modes coincide and the conventional Atiyah-Singer index vanishes.

In Sec. III we start our analysis of bosonized theories with a warm-up example of the Schwinger model. We will show that correct results for the partition function in the instanton sectors are reproduced indeed in bosonization language, but only if choosing the gauge-invariant form for the bosonized Lagrangian depending explicitly only on field strength F. We show that the partition function in the sector with topological charge ν involves a factor $m^{|\nu|}$, reflecting the presence of $|\nu|$ zero modes in the fermion description.

In Sec. IV we analyze gauged WZNW models with the action (1.8). We show that the contribution of the fields in the topological class k in the partition function involves the factor $m^{k(N-k)}$ in agreement with the fermion counting (1.12). It also involves, however, the factor $\mathcal{A}^{k(N-k)}$ where $\mathcal{A}$ is the total area of our manifold. That implies the constant asymptotics of the correlator of $k(N-k)$ scalar fermion currents at large distances and the existence of a nonzero fermion condensate which seems to be excluded by *other* arguments.

In the first place, these are the arguments based on the assumed extensive form of the partition function $Z \propto \exp\{-\epsilon_{\mathrm{vac}}\mathcal{A}\}$ discussed earlier in [5] and anew in the end of Sec. IV. Second, one can rigourously *prove* that the fermion condensate is absent in the high temperature region—this is the subject of Sec. V.

Possible ways to resolve the paradox are briefly discussed at the end of Sec. IV and, in more detail, in Sec. VI. In particular, an attractive possibility is that the fermion condensate appears at $T=0$ due to spontaneous breaking of $Z_2 \otimes Z_2$ symmetry which the Lagrangian (1.3) enjoys: The transformations

$$\lambda_L \to -\lambda_L,$$
$$\lambda_R \to -\lambda_R, \tag{1.13}$$

leave $\mathcal{L}$ invariant. This discrete $Z_2 \otimes Z_2$ symmetry is the remnant of U(1) chiral symmetry which would be effective in a theory with complex fermions. A mass term (1.4) would break this symmetry down to Z_2. And the appearance of the fermion condensate in massless theory breaks it spontaneously.

Spontaneous breaking of discrete symmetry would imply a first order phase transition at $T_c=0$ (so that the condensate is zero at any nonzero temperature), much like in a one-dimensional Ising model. This picture is very much probable at $N=3$ and at higher odd N, but the situation at even $N \geqslant 4$ is not yet clear—$Z_2 \otimes Z_2$ symmetry of the Lagrangian (1.3) is *anomalous* in this case, being broken explicitly by instanton effects.

In Sec. VII, we discuss the correspondence of the fermion and the bosonized versions of the theory in more details. We show that the correct behavior of the fermion partition functions in the instanton sector is reproduced only if integrating the bosonized partition function over the parameter $R \in O(N^2-1)/[SU(N)/Z_N]$ characterizing the way the

[3]The problem does not arise, of course, in SU(N) WZNW models which are the most studied ones. They do not involve instantons and the action (1.6) is perfectly acceptable.

[4]It is half the number, not just the number, because we are dealing here with Majorana fermions and the fermion path integral provides the factor which is the square root of the Euclidean Dirac determinant.

$SU(N)/Z_N$ subgroup is embedded in the larger $O(N^2-1)$ group. This is the only way to enforce the symmetry (1.13) for odd N in the bosonized version. For even N, one has to take into account the contributions of both disconnected components in $O(N^2-1)$.

Possible implications of our analysis for four-dimensional supersymmetric gauge theories are discussed in the last section.

II. INDEX THEOREM

Instantons present a nontrivial fiber bundle $A_\mu(x)$ of the gauge group $SU(N)/Z_N$ on the two-dimensional Euclidean manifold where the theory is defined. In Ref. [5] it was convenient to choose the manifold to be a torus. When the size of the torus in one of the Euclidean directions is small compared to g^{-1}, the quasiclassical approximation works and path integrals in the instanton sector are saturated by fields at the vicinity of a particular configuration in the instanton class, which has a very simple Abelian form. In the case of large spatial volume and small temporal size β (which physically corresponds to high temperature $T=\beta^{-1}\gg g$), the relevant saddle point configuration in the topological class $k=1$ is (the gauge $A_1=0$ is chosen)

$$A_0(x)=\frac{i}{N}\,\mathrm{diag}(1,1,\ldots,1-N)a(x-x_0), \tag{2.1}$$

where the profile function $a(x-x_0)$ has the same form as in the Schwinger model [1] and the corresponding field density $F=-\partial A_0/\partial x$ is localized at the vicinity of x_0, the instanton center. With the solution (2.1) at hand, path integrals can be explicitly calculated and, for example, the fermion condensate in the high temperature limit can be found [5,22]. In [5] we explicitly solved the Dirac equation in the background (2.1) and found $N-1$ left-handed and $N-1$ right-handed fermion zero modes. We also showed that the eigenvalues do not shift from zero when perturbing the background (2.1) in every order of perturbation theory. This reasoning was convincing enough, but did not have the rank of a rigorous proof—one could, in principle, contemplate the presence of field configurations in the instanton class at some distance in Hilbert space from the Abelian instanton (2.1) where the eigenvalue is shifted from zero by nonperturbative effects. The main problem here is that a standard Atiyah-Singer index theorem says nothing about the presence or absence of these zero modes. The Atiyah-Singer index is just zero here:

$$I^{\text{Atiyah-Singer}}=n_0^L-n_0^R\sim\mathrm{Tr}\int F_{\mu\nu}\epsilon_{\mu\nu}d^2x=0. \tag{2.2}$$

A proof was constructed in [22] where the theory was studied on a finite spatial circle at zero temperature in a Hamiltonian approach. In that case, the gauge $A_0=0$ can be chosen and the dynamic variable is $A_1(x,t)$. The point is that the Hamiltonian has N classical vacua corresponding to shifting A_1 from zero by particular finite constant matrices belonging to a Cartan subalgebra (see Sec. V for some more details). The Hamiltonian has a symmetry which guarantees that the energy spectrum of the Dirac operator in all classical vacua is identical. When A_1 interpolates smoothly between adjacent vacua, exactly $N-1$ left-handed levels with positive energy cross zero and go down into the Dirac sea. Likewise, $N-1$ right-handed levels from the sea cross zero and appear in the physical spectrum.[5] The level crossing phenomenon guarantees that the Euclidean Dirac operator has $N-1$ right-handed and $N-1$ complex conjugated left-handed zero modes on any background which interpolates in Euclidean time between classical vacua, i.e., on any background belonging to the instanton topological class.

Both discussed proofs are somewhat indirect, and we believe it is worthwhile to give a *direct* proof with explicit construction of the zero mode solution. Let us first derive the gauge field topological classification more accurately. Topologically nontrivial configurations exist only on compact Euclidean manifolds. There are two convenient choices: a torus as in [5,22] or a sphere. We will return on torus in Sec. V, but currently we are moving onto sphere and will stay there for a while. A sphere geometry appears when one considers the gauge fields living on the Euclidean plane which tend to a pure gauge at infinity:

$$A_\mu(x)\xrightarrow{r\to\infty}\Omega^{-1}(\theta)\partial_\mu\Omega(\theta), \tag{2.3}$$

with $\Omega(\theta)\in SU(N)/Z_N$. The matrix $\Omega(\theta)$ defines a loop in the group space. Topologically nontrivial configurations are described by noncontractible loops. The topological invariant distinguishing different classes is

$$W(C)=\frac{1}{N}\,\mathrm{Tr}\exp\left\{\oint_C A_\mu dx_\mu\right\}=\exp\left\{\frac{2\pi ik}{N}\right\}, \tag{2.4}$$

where the contour C goes around the Euclidean infinity, and $k=0,\ldots,N-1$. It is the same standard construction as for the four-dimensional Yang-Mills instantons. The difference is that in the latter case the topological invariant

$$I^{d=4}\sim\int_{S^3}K_\mu n_\mu$$

can be written as a four-dimensional integral of the local topological charge density $\partial_\mu K_\mu\sim\mathrm{Tr}\{F_{\mu\nu}\tilde F_{\mu\nu}\}$. On the other hand, the invariant (2.4) is inherently nonlocal and cannot be presented as a two-dimensional integral of a local density. Let us now choose a particular representative in each topological class. A convenient choice is

$$A_\mu^{(0)k}=\frac{i}{N}\,\mathrm{diag}(\underbrace{k,\ldots,k}_{N-k},\underbrace{k-N,\ldots,k-N}_{k})\,\frac{\epsilon_{\mu\nu}x_\nu}{(x_\mu^2+\rho^2)}, \tag{2.5}$$

where we want to choose $\rho\sim g^{-1}$. This is a configuration belonging to the class (2.4) with localized field density and finite action. For $k=1$, the color structure of Eq. (2.5) is the same as in Eq. (2.1).

We emphasize that Eq. (2.5) is *not* a solution to the classical equations of motion—such a solution exists and has the

[5]Which levels—left handed or right handed—go down into the sea and which go out of it depends, of course, on convention and on the direction in which A_1 is changed.

same color structure as Eq. (2.5), but is delocalized: The field density is constant on S^2 and very small, $F \sim 1/\mathcal{A}$ ($\mathcal{A}$ is the area of the sphere). Mathematically, this delocalized configuration is as good a reference point as the configuration (2.5). The configuration (2.5) is, however, preferable from the physical viewpoint. Considering classical solutions makes sense only in the case when a quasiclassical description holds and characteristic fields in path integrals are in the vicinity of classical saddle points. However, QCD_2 at low temperature and large spatial volume is a nontrivial nonlinear theory with strong coupling and the quasiclassical description is not adequate. An analysis of the path integral in the instanton sector shows that characteristic field configurations are actually localized at distances of order of the correlation length $\sim g^{-1}$ and resemble Eq. (2.5) in this respect[6] [1].

The field (2.5) is defined on Euclidean plane and is singular at infinity. To define an instanton on the compact S^2 manifold, one should either to use stereographic coordinates in which case the field would be singular at the north pole of the sphere or to go over in the singular gauge:

$$A_\mu^{(0)k} = -\frac{i}{N}\,\mathrm{diag}(\underbrace{k,\ldots,k}_{N-k},\underbrace{k-N,\ldots,k-N}_{k})\,\frac{\rho^2 \epsilon_{\mu\nu} x_\nu}{x_\mu^2 (x_\mu^2+\rho^2)} \tag{2.6}$$

(the size of the sphere R is assumed to be much larger than ρ). The field (2.6) has the same field strength F as Eq. (2.5), is regular at infinity, and involves a Dirac string singularity at $x=0$. Obviously, a gauge where the Dirac string is placed at any other point x_* on the sphere can be chosen.

Let us now solve the Dirac equation

$$\gamma_\mu^E \{\partial_\mu \lambda_n + [A_\mu, \lambda_n]\} = \mu_n \lambda_n, \tag{2.7}$$

with $\gamma_0^E = i\sigma^2$ and $\gamma_1^E = i\sigma^1$, μ_n being the eigenvalue corresponding to the eigenmode λ_n, on the background (2.6). Consider the matrix $\lambda^a t^a$. In Euclidean space, Majorana fermions cannot be defined, and the fermion fields should be assumed to be complex. It is convenient to choose the complex basis $\{t^a\}$ for the Lee algebra with $N-1$ standard diagonal matrices and $N(N-1)/2 + N(N-1)/2$ off-diagonal matrices having only one nonzero component. In this basis, the Dirac operator with Abelian background (2.6) does not mix the components λ^a with different a so that each component can be treated separately. For some components, the commutator of the corresponding t^a with the diagonal color matrix in Eq. (2.6) is zero, these components do not feel a background gauge field at all, and the spectrum is the same as for free fermions. An example of the component which *does* feel the background is

$$(T_*^+)_{ij} = \left(\begin{array}{c|c|ccc} i\backslash j & N-k & & k & \\ \hline & & 1 & \cdots & 0 \\ N-k & 0 & \cdots & \cdots & \cdots \\ & & 0 & \cdots & 0 \\ \hline k & 0 & & 0 & \end{array}\right). \tag{2.8}$$

The Dirac equation for this component looks the same as the Dirac equation in Schwinger model for the charged fermion field in the background with unit Abelian topological charge (1.1). The standard Atiyah-Singer theorem dictates the presence of a left-handed zero mode. Its particular form is

$$\lambda_{*L}^{(0)}(x) = T_*^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathrm{spin}} \frac{x_+}{\sqrt{x_\mu^2 (x_\mu^2+\rho^2)}}. \tag{2.9}$$

There are $k(N-k)$ color matrices of the form (2.8) and, correspondingly, $k(N-k)$ left-handed zero modes. Also, there are $k(N-k)$ right-handed zero modes

$$\lambda_{*R}^{(0)}(x) = T_*^- \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathrm{spin}} \frac{x_-}{\sqrt{x_\mu^2 (x_\mu^2+\rho^2)}}, \tag{2.10}$$

where

$$(T_*^-)_{ij} = \left(\begin{array}{c|ccc|c} i\backslash j & & N-k & & k \\ \hline N-k & & 0 & & 0 \\ \hline & 1 & \cdots & 0 & \\ k & \cdots & \cdots & \cdots & 0 \\ & 0 & \cdots & 0 & \end{array}\right), \tag{2.11}$$

etc. Up to now, we have just adapted the derivation of [5] for the case when fields live on a sphere and generalized them to arbitrary k. In order to show explicitly the presence of $k(N-k)+k(N-k)$ zero modes on *any* topologically nontrivial background, we use the fact that any field belonging to the class k can be written as

$$A_- = g^{-1}(\partial_- + A_-^{(0)})g,$$

$$A_+ = g^\dagger(\partial_+ + A_+^{(0)})(g^\dagger)^{-1}, \tag{2.12}$$

where g is a general complex $N\times N$ matrix. For a unitary g it is just a gauge transformation. For a Hermitian g it is a nontrivial non-Abelian field with a different field density, but with the same invariant (2.4). We restrict, however, g to be *unitary* at the point x_* where the Dirac string is placed. To be quite precise, it is sufficient to require that the product $gg^\dagger$ commute with the matrix marking out the color direction of the Dirac string. Otherwise, the transformed field (2.12) is not a fiber bundle on S^2

The decomposition (2.12) is widely known for topologically trivial fields [18]. It is a direct non-Abelian analog of the decomposition

$$A_\mu = A_\mu^{(0)} + \epsilon_{\mu\nu}\partial_\nu \phi + \partial_\mu \chi \tag{2.13}$$

of a topologically nontrivial field in the Schwinger model on S^2 [2]. Substituting Eq. (2.12) in the Dirac equation (2.7), one can easily find the explicit expression for the zero modes.

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[6] At high temperature $T \gg g$ quasiclassical analysis becomes possible which allows one to determine the value of the fermion condensate for $N=2$ [5,22]. The saddle point field configuration of a high-T path integral in the instanton sector presents the solution of *effective* equations of motion with account of the fermion determinant. It has the form (2.1) and is localized [1].

$$(\lambda_L^{(0)})_g = g^{-1}\lambda_L^{(0)} g,$$

$$(\lambda_R^{(0)})_g = g^{\dagger}\lambda_R^{(0)}(g^{\dagger})^{-1}, \tag{2.14}$$

where $\lambda_{L,R}^{(0)}$ are the zero modes (2.9) and (2.10) for the instanton representative (2.6).

III. INSTANTONS IN BOSONIZED SCHWINGER MODEL

Our main goal is to reproduce the zero mode counting of the previous section in bosonization approach. Of course, there is no trace of fermion zero modes in the bosonized theory. The proper question to ask is how the contribution to the partition function coming from instanton sectors depends on a small (smaller than any other relevant scale) fermion mass m. In the original theory with fermions, the behavior is $Z_k \sim m^{k(N-k)}$. And the same should be true in the bosonized WZNW model—the bosonized version of QCD_2. As a warm-up, consider first the Abelian theory where the calculations can be carried out explicitly until the very end. The usual way to bosonize the Schwinger model is to establish the correspondences [23]

$$i\bar{\psi}\partial_\mu\gamma_\mu\psi \rightarrow \tfrac{1}{2}(\partial_\mu\phi)^2,$$

$$\bar{\psi}\gamma_\mu\psi \rightarrow \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\phi,$$

$$\bar{\psi}\psi \rightarrow -\frac{e^{\gamma}}{2\pi^{3/2}} g\cos(\sqrt{4\pi}\phi), \tag{3.1}$$

where γ is the Euler constant. Then the Euclidean action of the bosonized Schwinger model is

$$S_E = \int d^2x\left[\frac{1}{2g^2}F^2 + \frac{1}{2}(\partial_\mu\phi)^2 + A_\mu\frac{i}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\phi - mg\frac{e^{\gamma}}{2\pi^{3/2}}\cos(\sqrt{4\pi}\phi)\right], \tag{3.2}$$

where $F=\epsilon_{\mu\nu}\partial_\mu A_\nu$ and ϕ is a real scalar field. Adding a full derivative to Eq. (3.2), one can rewrite it in the form

$$S_E = \int d^2x\left[\frac{1}{2g^2}F^2 + \frac{1}{2}(\partial_\mu\phi)^2 + iF\phi\frac{1}{\sqrt{\pi}} - mg\frac{e^{\gamma}}{2\pi^{3/2}}\cos(\sqrt{4\pi}\phi)\right]. \tag{3.3}$$

Our remark is that the transformation from Eq. (3.2) to Eq. (3.3) is innocent *only* in the topologically trivial gauge sector. In instanton sectors, the integral of a full derivative produces a surface term which contributes in the action and *cannot* be disregarded. To see that, it is convenient to think of an instanton on S^2 as of a monopole. The flux (1.1) is then associated with the flux of the monopole magnetic field through a sphere surrounding the magnetic charge in a fictitious three-dimensional space, i.e., with the magnetic charge itself. The potential $A_\mu(x)$ of our instanton on monopole should involve a singularity (the Dirac string) at some point x_* on S^2. The surface term appears just due to this Dirac string singularity and produces the term $-i\sqrt{4\pi}\phi(x_*)$ in the action. Whenever this matters, it is the action (3.3) which should be used, not Eq. (3.2). Actually, the action (3.2) is not gauge invariant in topologically nontrivial sectors. The term $-i\sqrt{4\pi}\phi(x_*)$ by which Eq. (3.2) differs from the explicitly invariant action (3.3) depends on the position of the Dirac string singularity, i.e., on the gauge.[7]

A traditional way to handle the bosonized theory is to do first the Gaussian integration over $\prod dF$ to obtain

$$S_\phi = \int d^2x\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{g^2}{2\pi}\phi^2 - mg\frac{e^{\gamma}}{2\pi^{3/2}}\cos(\sqrt{4\pi}\phi)\right]. \tag{3.4}$$

It is acceptable as far as we are not interested in the contribution of a particular gauge topological sector. In the latter case, one should proceed more accurately. Let us consider the theory on a compact two-dimensional Euclidean manifold which we choose to be S^2 with large but finite area $\mathcal{A}$. To single out the contribution of a particular instanton sector, we impose the condition (1.1). The topological charge ν is an integer. In the original fermion theory, this follows from the necessity to define the Dirac operator on the compact manifold in a background gauge field. The eigenfunctions and spectrum exist only for integer ν. In the bosonized language, quantization of ν follows from an additional requirement that the action (3.3) be invariant under the shift $\phi \rightarrow \phi + \sqrt{\pi}$. $\sqrt{\pi}$ is just the period of the cosine in Eq. (3.3). We will shortly see that even if we would allow for noninteger ν's, the contribution of such fields in the partition function is zero.

Let us now expand the fields $F(x)$ and $\phi(x)$ in the series over spherical harmonics

$$F(x) = \sum_{lm} F_{lm}Y_{lm}(\theta,\varphi),$$

$$\phi(x) = \sum_{lm} \phi_{lm}Y_{lm}(\theta,\varphi). \tag{3.5}$$

The zero harmonic $F_0 = 2\pi\nu/\mathcal{A}$ is fixed due to Eq. (1.1). Integrating out all other harmonics of the gauge field, we obtain

$$Z_\nu = e^{-2\pi^2\nu^2/\mathcal{A}g^2}\int_{-\sqrt{\pi}/2}^{\sqrt{\pi}/2} d\phi_0 e^{i\nu\sqrt{4\pi}\phi_0}\int\prod d\tilde{\phi}(x) \times \exp\left[-\int_{S^2} d^2x\left(\frac{1}{2}(\partial_\mu\tilde{\phi})^2 + \frac{g^2}{2\pi}\tilde{\phi}^2 - mg\frac{e^{\gamma}}{2\pi^{3/2}}\cos[\sqrt{4\pi}(\phi_0+\tilde{\phi})]\right)\right], \tag{3.6}$$

where ϕ_0 is the zero harmonic of the matter field and $\tilde{\phi}(x)$ is the sum of all the rest. The interval of integration over ϕ_0 is

[7]Obviously, one can repeat this reasoning without invoking the Dirac string, but describing the instanton fiber bundle with a couple of maps which is more accurate from the mathematical viewpoint. The physical conclusion, however, is the same.

restricted due to the periodicity of the integrand. It is instructive to see what happens if we sum over ν. Using the dual representation of the Θ function, we obtain

$$Z=\sum_{\nu} Z_{\nu}\propto\int_{-\sqrt{\pi}/2}^{\sqrt{\pi}/2}d\phi_0\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{g^2\mathcal{A}}{2\pi}(\phi_0-k\sqrt{\pi})^2\right\}\times\int\prod d\tilde{\phi}(x)\exp\left\{-\int_{S^2}d^2x\left[\frac{1}{2}(\partial_\mu\tilde{\phi})^2+\frac{g^2}{2\pi}\tilde{\phi}^2-mg\frac{e^{\gamma}}{2\pi^{3/2}}\cos[\sqrt{4\pi}(\phi_0+\tilde{\phi})]\right]\right\}. \tag{3.7}$$

In the thermodynamic limit $g^2\mathcal{A}\to\infty$ only one term of the sum (3.7) survives, ϕ_0 is frozen at zero, and we reproduce the result (3.4). It is not difficult also to calculate the partition function in the theory with a particular nonzero vacuum angle θ:

$$Z(\theta)=\sum_{\nu} Z_{\nu}e^{i\nu\theta}. \tag{3.8}$$

Performing the same dual transformation for this sum as for $Z(0)$, we arrive at the same expression (3.7) but with the shift $\phi_0\to\phi_0+\theta/\sqrt{4\pi}$. In that case ϕ_0 freezes at the value $\phi_0=-\theta/\sqrt{4\pi}$.

Now let us look at Eq. (3.6). Note first of all that although the bosonized *action* (3.3) is complex, the path integral for Z_ν is real as it should be. Second, we see immediately that in the massless case $m=0$, $Z_\nu=0$ when $\nu\neq0$. But for small but nonzero m, Z_ν is nonzero too. A finite result is obtained when pulling down the mass term in Eq. (3.6) ν times. If we would try to calculate Z_ν for a fractional ν, the integral over ϕ_0 would run from $-\infty$ to ∞, the oscillating factor $\exp\{i\nu\sqrt{4\pi}\phi_0\}$ could not be compensated in any order in m, and we would get zero for any value of mass. This is the real reason for the topological charge to be quantized: Fractional topological charges just do not contribute here in the partition function.[8] In the limit $mg\mathcal{A}\ll1$, only the leading term in mass expansion survives (see [8,1] for a detailed discussion) and we obtain

$$Z_\nu=C_\nu(mg\mathcal{A})^\nu, \tag{3.9}$$

with a calculable coefficient. This is exactly what we also get in the fermion language. For $\nu=\pm1$, the coefficient $C_1=C_{-1}=e^{\gamma}/(4\pi^{3/2})$ just gives the value of the fermion condensate (1.2).

IV. INSTANTONS IN GAUGED WZNW MODEL

We have already mentioned that in topologically nontrivial sectors it is the action (1.8) which should be used, not Eq. (1.6). The action (1.8) relates to the action (1.6) exactly in the same way as the action (3.3) to (3.2). The following identity holds:

$$S_E^{\text{Eq. (1.8)}}[A_\mu,u]=\frac{1}{2g^2}\int\mathrm{Tr}\{F_{\mu\nu}^2\}-\frac{N}{8\pi}\int d^2x\,\mathrm{Tr}\{u^{-1}\nabla_\mu u u^{-1}\nabla_\mu u\}-\frac{iN}{12\pi}\int_Q d^3\xi\,\epsilon^{ijk}\mathrm{Tr}\{u^{-1}\partial_i u u^{-1}\partial_j u u^{-1}\partial_k u\}+\frac{iN}{4\pi}\int_Q d^3\xi\,\epsilon^{ijk}\partial_i\mathrm{Tr}\{uA_ju^{-1}A_k+A_j(u^{-1}\partial_k u+\partial_k u u^{-1})\}. \tag{4.1}$$

Let us assume that the gauge fields have only two components A_0 and A_1 and depend only on the physical coordinates $x_\mu\equiv\tau,x$. The matter field $u(x_\mu,\alpha)$ is smooth on Q and depends on the third coordinate $\alpha\in[0,1]$ in such a way that $u(x_\mu,0)=1$ and $u(x_\mu,1)$ is the field living on our physical two-dimensional Euclidean manifold $\mathcal{M}$: the boundary of Q. One can choose, for example, $u(x_\mu,\alpha)=\exp\{\alpha\phi(x_\mu)\}$ with anti-Hermitean ϕ.

For topologically trivial gauge fields which are regular on $\mathcal{M}$, the integral of the full derivative is reduced to two surface terms at $\alpha=0$ and $\alpha=1$ and produces, together with other terms, the standard form of the action (1.6). But in instanton sectors, fields involve Dirac string singularities on $\mathcal{M}$, which results in an additional contribution in the full derivative integral. For example, for $N=2$, the relation

$$S^{\text{Eq. (1.6)}}=S^{\text{Eq. (1.8)}}+2\,\mathrm{Tr}\{\phi(x_*)n^at^a\} \tag{4.2}$$

holds. Here x_* is the position of the Dirac string and $\mathbf{n}$ is its direction in the color space. Obviously, the extra term in Eq. (4.2) is gauge dependent.

Let us now estimate the contribution of the instanton sectors in the partition function using the correct gauge-invariant expression (1.8) for the action. An experience with Schwinger model teaches us that the relevant factors in the path integral appear due to integration over the zero harmonic of the matter field. Thus we assume

$$u(x_\mu,\alpha)=\exp\{\alpha\beta\}, \tag{4.3}$$

where $\beta=i\beta^at^a$ is a constant anti-Hermitean matrix.

Consider first the simplest case $N=2$. The field has a Dirac string singularity at some point x_* on S^2. We choose a gauge with $x_*=0$ and direct the Dirac string along the third isotopic axis. The singularity at small x can be inferred from Eq. (2.6):

[8]We hasten to comment that, in some theories like the twisted multiflavor Schwinger model [24] or four-dimensional Yang-Mills theory involving only adjoint color fields [25,8], fractional topological charges *do* contribute. In each particular theory, a particular study of this question is required.

$$A_\mu^{\rm sing}(x) = -it^3\,\frac{\epsilon_{\mu\nu}x_\nu}{x_\mu^2}. \qquad (4.4)$$

A look at Eq. (1.8) shows that the second and third terms in the action may provide a divergent contribution $\propto\int d^2x/x^2$ in the action. Actually, the integral

$$\propto\int_Q d^3\xi\ \epsilon^{ijk}{\rm Tr}\{u^{-1}\nabla_i u u^{-1}\nabla_j u u^{-1}\nabla_k u\}$$

is *not* divergent due to the fact that $\epsilon_{\mu\nu}A_\mu^{\rm sing}A_\nu^{\rm sing}=0$. But the integral

$$\propto\int d^2x\ {\rm Tr}\{u^{-1}\nabla_\mu u u^{-1}\nabla_\mu u\}$$
$$=\int d^2x\ {\rm Tr}\{u^{-1}[A_\mu,u]u^{-1}[A_\mu u]\} \qquad (4.5)$$

is singular provided $[A_\mu,u]\neq 0$. It would give an infinite contribution in the action, and the corresponding contribution in the partition function is suppressed. Thus we should restrict ourselves with the constant (x_μ independent) matrices (4.3) aligned in the same color direction as the Dirac string in a chosen gauge. For such u, the only nonzero contribution in the action comes from the last term in Eq. (1.8). We have

$$S_E = -2\,{\rm Tr}\{\beta t^3\} = -i\beta_3. \qquad (4.6)$$

The instanton contribution in the partition function is

$$Z_I \propto \int_0^{2\pi} d\beta_3 \exp\{-i\beta_3\} = 0,$$

as it should be in the massless case. [The range of β_3 is restricted to be $[0,2\pi]$ because changing β_3 from 0 to 2π multiplies u by the element of the center -1, and we arrive at the same associated orthogonal matrix (1.7).] If the fermion mass is not zero, the action involves an additional term

$$S_m \propto mg\int d^2x\ {\rm Tr}\ h(x) \propto mg\mathcal{A}[|{\rm Tr}\ u|^2-1]$$
$$= mg\mathcal{A}(2\cos\beta_3+1), \qquad (4.7)$$

where $\mathcal{A}$ is the area of the manifold. Pulling the mass term down, we get in the leading order in m

$$Z_I \propto mg\mathcal{A}\int_0^{2\pi} d\beta_3 \exp\{-i\beta_3\}(2\cos\beta_3+1) = Cmg\mathcal{A}, \qquad (4.8)$$

with a nonzero constant C. That agrees well with the results of the analysis in the fermion language: A couple of fermion zero modes provides a factor $\propto m$ in the partition function. Differentiating Eq. (4.8) over mass gives the fermion condensate [5].

Consider now the general color group SU(N) and the field configuration of the type (2.6) belonging to the topological class k. For any configuration in this class, a gauge can be chosen where the Dirac string is aligned in the direction

$$T^* = \frac{1}{\sqrt{2Nk(N-k)}}\,{\rm diag}(\underbrace{k,\ldots,k}_{N-k},\underbrace{k-N,\ldots,k-N}_{k}) \qquad (4.9)$$

in the color space. As earlier, we must require that the constant mode of the matter field u_0 commute with T^*—otherwise, the second term in Eq. (1.8) would give an infinite contribution to the action. A general $u_0(\alpha=1)$ satisfying this restriction has the form

$$u_0(1) = \exp\{i\beta^* T^*\}\begin{pmatrix} u^{(N-k)} & 0 \\ 0 & u^{(k)}\end{pmatrix}, \qquad (4.10)$$

where $u^{(N-k)}\in{\rm SU}(N-k)$ and $u^{(k)}\in{\rm SU}(k)$. We assume $u_0(\alpha)=[u_0(1)]^\alpha$ so that $u_0(0)=1$. The parameter β^* varies within the limits $\beta^*\in[0,2\pi\sqrt{2k(N-k)/N}]$—the shift of β^* by $2\pi\sqrt{2k(N-k)/N}$ multiplies $u_0(1)$ by an element of the center $\exp\{2\pi ik/N\}$, which results in the same adjoint matrix h. In the massless case, the only contribution in the action comes from the last term in Eq. (1.8). It does not depend on $u^{(k)}$ and $u^{(N-k)}$, but only on β^* and we have

$$Z_I^k \propto \int_0^{2\pi\sqrt{2k(N-k)/N}} d\beta^* \exp\left\{-i\beta^*\sqrt{\frac{Nk(N-k)}{2}}\right\} = 0.$$

Note that the phase factor winds by $2\pi\ k(N-k)$ times in the range of the integration.

The action in the massive theory involves the term

$$S_m \propto mg\int d^2x\ {\rm Tr}\ h(x) \propto mg\mathcal{A}[|{\rm Tr}\ u_0|^2-1] = mg\mathcal{A}\left[|{\rm Tr}\ u^{(k)}|^2 + |{\rm Tr}\ u^{(N-k)}|^2 + 2\,{\rm Re}\left({\rm Tr}\ u^{(k)}({\rm Tr}\ u^{(N-k)})^* \right.\right.$$
$$\left.\left.\times\exp\left\{-i\beta^*\sqrt{\frac{N}{2k(N-k)}}\right\}\right)-1\right]. \qquad (4.11)$$

To provide a nonzero contribution in the path integral for Z_I^k, the mass term should be pulled down at least $k(N-k)$ times—otherwise, the integral over β^* gives zero. Note that not only $\int d\beta^*$, but also group integrals over $u^{(k)}$ and $u^{(N-k)}$ provide here nonzero factors. Thus we get an estimate

$$Z_I^k \sim (mg\mathcal{A})^{k(N-k)}\int_0^{2\pi\sqrt{2k(N-k)/N}}\int du^{(k)}({\rm Tr}\ u^{(k)*})^{k(N-k)}\int du^{(N-k)}({\rm Tr}\ u^{(N-k)})^{k(N-k)} \sim (mg\mathcal{A})^{k(N-k)} \qquad (4.12)$$

for small $mg\mathcal{A}$.

The factor $m^{k(N-k)}$ appears also in the fermion approach—$k(N-k)$ is just the number of the fermion zero mode pairs.[9] What is, however, new and could not be figured out in the fermion approach is the total area dependence $\propto\mathcal{A}^{k(N-k)}$. Consider, e.g., the case $N=3$. The instanton partition function can be written as

$$Z_I^{N=3}\sim(mg)^2\int d^2x\,d^2y\langle\overline{\lambda^a}\lambda^a(x)\overline{\lambda^a}\lambda^a(y)\rangle\sim(mg\mathcal{A})^2. \tag{4.13}$$

The appearance of the factor $\mathcal{A}^2$ in this expression means that the correlator $\langle\lambda^a\lambda^a(x)\lambda^a\lambda^a(y)\rangle$ tends to a nonzero constant at large Euclidean distances $|x-y|$, i.e., that the fermion condensate $\langle\lambda^a\lambda^a\rangle$ is formed.

Thus a bosonization estimate for Z_I^k presented in this section has confirmed the existence of $k(N-k)+k(N-k)$ fermion zero modes in the path integral and, on the other hand, confirmed the appearance of the fermion condensate for any N, which also follows from simplistic bosonization arguments of Ref. [5]. This is rather remarkable, but unfortunately does not mean yet that the physical situation is now absolutely clear and a final resolution of the paradox mentioned in [5] [the conflicting results of the bosonized analysis and the fermion analysis of the theory (1.3) for higher gauge groups] is achieved.

The paradox displays itself if recalling the fact that the spectrum of the theory (1.3) does not involve massless particles. That means that in the limit $\mathcal{A}g^2\gg1$, when the size of the Euclidean box is much larger than the characteristic inverse mass scale $\sim g^{-1}$, the partition function must enjoy the extensive property

$$Z\propto\exp\{-\epsilon_{\rm vac}(m,g)\mathcal{A}\} \tag{4.14}$$

and the finite volume corrections (the boundary effects) should be exponentially suppressed [26]. At small $m\ll g$, $\epsilon_{\rm vac}(m,g)$ should involve the linear in mass term—the corresponding coefficient just gives the fermion condensate $=-1/\mathcal{A}\partial/\partial m\ \ln Z$, the existence of which is dictated by the estimates (4.12) and (4.13) for the instanton contribution in the partition function.[10]

The property (4.14) should hold both in the true thermodynamic limit $mg\mathcal{A}\gg1$ and also in the region $mg\mathcal{A}\ll1$ provided the condition $\mathcal{A}g^2\gg1$ is fulfilled. But, on the other hand, for $N\geq3$, no known contribution in the partition function involves the linear term $\propto mg\mathcal{A}$ and the expansion of Z in small $mg\mathcal{A}$ starts with the term $\sim m^{N-1}$.

There are only two ways out of this obvious contradiction.

(1) Perhaps, for some reason, topological classification does not hold in this case and, besides instantons, there are some *other* contributions in the partition function which involve a linear in mass term and would be responsible for the formation of the fermion condensate in the limit $mg\mathcal{A}\ll1$. These nondescript contributions would play the same role as the toron (or meron or fracton) contributions which are responsible for the formation of the gluino condensate in SU(N) supersymmetric 4D Yang-Mills theory [25] and the formation of the fermion condensate in multiflavor Schwinger model in finite volume with twisted boundary conditions [24]. This is the possibility advocated for in [5].

(2) Another possibility is that the topological classification is good, the ''fracton'' contributions are absent, and the partition function *does* not have an extensive form (4.14) for small $mg\mathcal{A}$. But that necessarily implies the existence of massless states in the spectrum. As there are no massless *particles,* the only choice is that the vacuum state involves a discrete degeneracy which is lifted by a small fermion mass. Then the *physical* partition function presents the sum of two extensive exponentials

$$Z\sim\exp\{-[\epsilon_0-Cmg+O(m^2)]\mathcal{A}\}+\exp\{-[\epsilon_0+Cmg+O(m^2)]\mathcal{A}\} \tag{4.15}$$

and the linear in mass term cancels out.

At present, we do not know what the answer is. We will discuss these two options in detail in Sec. VI and in the last section. But before that, let us discuss the physics of the theory (1.3) at finite temperatures where *definite* conclusions can be done.

V. ADJOINT QCD$_2$ AT HIGH TEMPERATURE

The main subject of this paper is analyzing the dynamics of adjoint QCD$_2$ in the bosonization approach. However, it is difficult to do at finite temperature. The reason is that, in contrast to S^2, a torus where a finite temperature theory is defined does not present a simply connected manifold, there are no smooth three-dimensional manifolds parametrized by a parameter $\alpha\in[0,1]$ such that the value $\alpha=0$ corresponds to a single point on the manifold, and the value $\alpha=1$ corresponds to the boundary, which is torus. That brings about problems with defining the Wess-Zumino term [15]. Thus we have to use the original fermion language.

The dynamics of the theory (1.3) at high temperature $T\gg g$ for $N=2,3$ was discussed at length in [5]. In [22] the same theory was studied at $T=0$, but on a small spatial circle $L\ll g^{-1}$ in the Hamiltonian approach. In the Euclidean approach the first theory is defined on a cylinder $0\leq\tau\leq\beta=T^{-1}$, $-\infty<x<\infty$ (for the theory to be completely regularized in the infrared, one may restrict also the range of x, $-L\leq x\leq L$, but the length of the box L should be assumed to be very large, larger than any relevant physical parameter), while the second theory is defined on a cylinder $-\infty<\tau<\infty$, $0\leq x\leq L$. Obviously, both cases are completely equivalent up to the interchange $x\leftrightarrow\tau$.

Let us briefly summarize the results of these studies. We will use mainly the Hamiltonian finite spatial circle language, which is a little more transparent physically. Eventually, however, we are going to translate the results obtained in the finite temperature language.

Consider first the simplest case $N=2$. Choose the gauge $A_0=0$. The dynamic variables are $A_1(x)$. In finite spatial volume, the zero Fourier mode $A_1^{(0)}$ of the field $A_1(x)$ plays

[9]As has already been mentioned in the Introduction, the bosonized theory with the action (1.8) still does not exactly correspond to the original fermion theory. It is convenient for us to postpone the discussion of this issue until Sec. VII.

[10]For a related discussion in QCD$_4$, see [8].

a crucial role. Actually, in the limit $gL \ll 1$, all other components and the fermion fields present the ''fast variables'' in the Born-Oppenheimer approach, which have high characteristic excitation energies and can be integrated out. We are left with the effective potential $V^{\rm eff}(A_1^{(0)})$ depending on the slow variable $A_1^{(0)}$. $V^{\rm eff}$ does not depend on isotopic orientation of $A_1^{(0)}$. For definiteness, we may direct it along the third isotopic axis: $A_1^{(0)} = iA_1^3 t^3$. The effective potential has the form [27,11]

$$V^{\rm eff}(A_1^3) = \frac{L}{2\pi}\left[\left(A_1^3 + \frac{\pi}{L}\right)_{\rm mod\ 2\pi/L} - \frac{\pi}{L}\right]^2. \qquad (5.1)$$

It is periodic in A_1^3 with the period $2\pi/L$ and has minima at $A_1^3 = 2\pi n/L$ with integer n. The points $A_1^3 = 0$ and $A_1^3 = 2\pi/L$ can be related by a gauge transformation

$$i\,\frac{2\pi}{L}\,t^3 = \Omega^\dagger(x)\partial_x\Omega(x), \quad \Omega(x) = \exp\left\{\frac{2\pi i x}{L}\,t^3\right\}. \qquad (5.2)$$

The unitary matrix $\Omega(x)$ is changed from $\Omega(0)=1$ to $\Omega(L) = -1$. The associated adjoint matrix $\in SO(3)$ [recall that for the theory involving only adjoint fields the true gauge group is $SU(2)/Z_2$ rather than just SU(2)] makes a closed loop in the group which cannot be contracted to zero. Thus Eq. (5.2) is a *large* gauge transformation which cannot be continuously deformed to zero and the point $A_1^3 = 2\pi/L$ presents a topologically nontrivial classical vacuum. Note that the configuration $A_1^3 = 4\pi/L$ corresponds to a gauge transformation $\Omega(x) = \exp\{4\pi i x t^3/L\}$, which can be continuously deformed to zero and is a *trivial* gauge copy of $A_1^3 = 0$.

The physical picture is very much similar to the vacuum structure in QCD_4 [28]. The only difference is that here we have not infinitely many, but just two topologically distinct vacua. An Euclidean field configuration which interpolates smoothly between $A_1^3 = 0$ at $\tau = -\infty$ to $A_1^3 = 2\pi/L$ at $\tau = \infty$ presents the instanton we were talking about before. It has one left-handed and one right-handed fermion zero mode, which give rise to a nonvanishing fermion condensate. An accurate calculation [5,22] gives

$$|\langle \overline{\lambda^a}\lambda^a\rangle| = \frac{8\pi^{3/2}}{gL^2}\exp\left\{-\frac{\pi^{3/2}}{gL}\right\}. \qquad (5.3)$$

This explicit formula is valid in the region $gL \ll 1$ when the Euclidean tunneling trajectory in the potential (5.1) has large action $\pi^{3/2}/gL$ and the quasiclassical approximation works. But a nonvanishing fermion condensate exists at any L (at any temperature). At $L = \infty$ $(T=0)$ it is estimated to be of order g. The condensate depends smoothly on L (on T), and there is no phase transition.

The large gauge transformation presents an extra discrete symmetry of the Hamiltonian. Like in QCD_4, the proper way of handling the theory is to impose a superselection rule and divide the Hilbert space of the systems into two sectors involving the states which are symmetric under such a transformation and the states which are antisymmetric. The partition functions in these sectors are

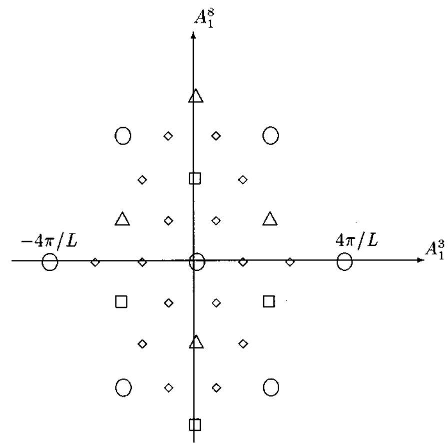

FIG. 1. Pattern of global minima.

$$Z_+ = Z_{\rm triv} + Z_I,$$
$$Z_- = Z_{\rm triv} - Z_I. \qquad (5.4)$$

This is quite analogous to choosing a particular value of θ in QCD_4; only in this case, with only two classical vacuum states, the parameter θ can acquire only two discrete values $\theta = 0$ and $\theta = \pi$. The fermion condensate has opposite sign in these two sectors. Let us turn now to the simplest paradoxical theory with $N=3$. Again, in the limit $gL \ll 1$, the low energy dynamics of the theory can be described by the effective potential $V^{\rm eff}(A_1^{(0)})$. The constant mode $A_1^{(0)}$ can be chosen to be a diagonal matrix

$$A_1 = i\ {\rm diag}(a_1, a_2, a_3), \quad \sum_i a_i = 0. \qquad (5.5)$$

The potential has the form [27,11]

$$V^{\rm eff}(a_i) = \frac{L}{2\pi}\sum_{i>j}^{3}\left[\left(a_i - a_j + \frac{\pi}{L}\right)_{\rm mod\ 2\pi/L} - \frac{\pi}{L}\right]^2. \qquad (5.6)$$

The pattern of its minima is shown in Fig. 1. First, there are global minima divided in three topological classes (they are marked out by circles, boxes, and triangles in Fig. 1). Each circle is gauge equivalent to any other circle with a topologically trivial gauge transformation. The same is true for boxes and triangles. The minima of different types are also gauge equivalent, but with a topologically nontrivial *large* gauge transformation. A Euclidean field configuration interpolating, say, from $A_1 = 0$ at $\tau = -\infty$ to $A_1 = (2\pi i/3L){\rm diag}(1,1,-2)$ at $\tau = \infty$ presents an instanton. It has two left-handed and two right-handed zero modes, which is too much for the fermion condensate to be formed. Like in the case $N=2$ and like in QCD_4, the Hilbert state of the system can be separated now in *three* sectors:

$$Z(\theta) = \sum_{k=0,1,2} Z_k \exp\{ik\theta\}, \qquad (5.7)$$

where $\theta = 2\pi n/3$ and $n = 0,1,2$. (Generally, there are N sectors with $\theta_n = 2\pi n/N$, $n = 0,1,\ldots,N-1$.)

It was observed in [22] that, besides global minima, the potential (5.6) has also *local* minima marked out with diamonds in Fig. 1. The value of the potential at diamond points is

$$V_{\diamond}=\frac{2\pi}{3L}. \tag{5.8}$$

We will shortly see that this value has some physical relevance.

The conclusions about the fermion condensate (its presence at $N=2$ and its absence at $N\geqslant 3$) can be also reached in a slightly different way of reasoning which does not invoke instantons at all. Consider the correlator

$$C(\tau)=\langle \overline{\lambda^a}\lambda^a(\tau)\ \overline{\lambda^a}\lambda^a(0)\rangle_L \tag{5.9}$$

at very large τ. In the limit of large $g^2\mathcal{A}$ and for $N\neq 3$, it is given by the path integral where gauge fields are *topologically trivial* (see [2] for a detailed related discussion in the Schwinger model). For $N=3$, also instanton sectors contribute in the correlator. But as we will shortly see, the instanton contributions can be analyzed along the same lines as the topologically trivial contribution, and their behavior is also the same. Consider first the case $N=2$. At small gL the quasiclassical approximation is valid and the correlator is mainly determined by the saddle point of the path integral. This saddle point presents an *Abelian* configuration

$$A_1(\tau')=if(\tau')t^3. \tag{5.10}$$

(It can, of course, be also rotated by a global gauge transformation, and it is important to take into account in a precise calculation, but for our purposes it is irrelevant.) The prime is put to distinguish the running argument τ' of the profile function from the point τ where the second fermion scalar current is defined and on which the correlator (5.9) depends. The calculation of the correlator (5.9) on Abelian background (5.10) is a simple problem. The point is that the component λ^3 does not feel the background and the term $\langle \lambda^3\lambda^3(\tau)\ \lambda^3\lambda^3(0)\rangle$ in the correlator (5.9) is just the free fermion correlator. In a finite box it decays exponentially[11] at large τ,

$$C_{\rm free}(\tau)\propto \exp\left\{-\frac{2\pi\tau}{L}\right\}, \tag{5.11}$$

and is irrelevant at large τ. The components λ^1 and λ^2 behave as a real and imaginary parts of the Dirac fermion field having the Abelian charge g in an Abelian gauge field background $A_1(\tau')=f(\tau')$. Thus the problem is reduced to the Abelian Schwinger model problem. The behavior of the fermion correlator in the Schwinger model on the circle is well known. At large τ, it tends to a constant. By cluster decomposition, one can infer from this that a fermion condensate is formed both in the Schwinger model[12] and in adjoint QCD_2.

Bearing in mind the generalizations which follow shortly, let us give a brief sketch how the result about the constant asymptotics of the fermion correlator in the Schwinger model is obtained (see, e.g., [1–3] for more details). Use the decomposition (2.13). In the topologically trivial sector, $A_\mu^{(0)}=0$. The gauge-dependent part $\partial_\mu\chi$ is also irrelevant. The field $\phi(x)$ is a nontrivial gauge-independent degree of freedom and is called a *prepotential*. In two dimensions, there exists an exact formula for the fermion Green's function in an arbitrary background $\phi(x)$:

$$S_\phi(x,y)=\langle \overline{\psi(x)}\,\psi(y)\rangle =\exp\{-g\gamma^5\phi(x)\}S_0(x-y)\exp\{-g\gamma^5\phi(y)\}, \tag{5.12}$$

where $S_0(x-y)$ is the free fermion Green's function. Using Eq. (5.12), we get

$$C^{\rm SM}(x)=\langle \overline{\psi}\psi(x)\,\overline{\psi}\psi(0)\rangle \propto C_{\rm free}(x)\prod d\phi \times\exp\left\{-\frac{1}{2}\int \phi(\Delta^2-\mu^2\Delta)\phi d^2y\right\} \times\cosh\{2g[\phi(x)-\phi(0)]\}, \tag{5.13}$$

where $\mu^2=g^2/\pi$ is the mass of the physical scalar particle in the spectrum (which may also be called heavy photon). Performing the Gaussian integration over $\Pi d\phi(x)$, we obtain, for the correlator at large Euclidean time τ in the theory defined on a cylinder with small spatial size L,

$$C^{\rm SM}(\tau)=C_{\rm free}(\tau)\exp\{4g^2[\mathcal{G}(0)-\mathcal{G}(\tau)]\}, \tag{5.14}$$

where $\mathcal{G}(x)$ is the Green's function of the operator $\Delta^2-\mu^2\Delta$ on a cylinder. The free correlator falls down exponentially at large τ according to Eq. (5.11), while the second factor rises:

$$\exp\{4g^2[\mathcal{G}(0)-\mathcal{G}(\tau)]\}\propto \exp\left\{\frac{2g^2}{\mu^2}\ \tau/L\right\}=\exp\{2\pi\tau/L\}. \tag{5.15}$$

We see that the exponential decay of the free correlator is exactly compensated by the rising factor (5.15) and the correlator tends to a constant at large τ.

Consider now the correlator (5.9) in the theory with $N=3$. Again, for small gL the quasiclassical approximation works and the path integral for the correlator is saturated by its saddle point, which is Abelian. A global SU(3) rotation brings the potential $A_1(\tau)$ in a diagonal-color-matrix form. Saddle points appear in different color directions, which are actually just the symmetry axes of the effective potential (5.6) and can be easily inferred from Fig. 1. Two essentially different options are $A_1^{\rm saddle}(\tau')=if(\tau')t^3$ and $A_1^{\rm saddle}(\tau')=ig(\tau')t^8$.

Consider first the second case. If the gauge field is directed along the eighth color axis, the fermion components $\lambda^{1,2,3,8}$ do not feel the field at all and the corresponding correlator has asymptotics (5.11) and is suppressed compared to the contribution of other components. The components $\lambda^{4\pm i5}$

[11] If going over in the finite temperature interpretation τ is substituted by x and the factor $2\pi/L\equiv 2\pi T$ in the exponent is just twice the lowest fermion Matsubara frequency.

[12] It is exactly the way the expression (1.2) for the fermion condensate in the Schwinger model was originally derived [29].

and $\lambda^{6\pm i7}$ interact with the background $A^8(\tau)$ as two complex fermions of charge $g\sqrt{3}/2$ with an Abelian gauge field background. Thus the correlator $C(\tau)$ behaves at large τ exactly in the same way as the fermion correlator in the Schwinger model with two flavors of equal charge. The behavior of the latter is also well known. Again, the expressions (5.14) and (5.15) are valid with the only difference that now we have $\mu^2=2(g\sqrt{3}/2)^2/\pi$—the two flavor loops contribute in the heavy photon mass on equal footing [and the parameter g in Eqs. (5.12)–(5.15) should, of course, be substituted by $g\sqrt{3}/2$]. We see that the rising factor (5.15) now compensates the exponential falloff of the free correlator only partially, and we have

$$C_8(\tau)\propto\exp\{-\pi\tau/L\}, \tag{5.16}$$

where the subscript 8 indicates the chosen color direction of the gauge field background.

Consider now the case when the gauge field is directed along the third color axis. The components $\lambda^{3,8}$ are free, and the components $\lambda^{1\pm i2}$, $\lambda^{4\pm i5}$, and $\lambda^{6\pm i7}$ behave in the same way as complex fermions of charge g, $g/2$, and $g/2$, correspondingly. The problem is reduced to the Schwinger model with three flavors of inequal charge. Consider the correlator $\langle\lambda^1\lambda^1(\tau)\,\lambda^1\lambda^1(0)\rangle$. It has the same form as before; only the factor $2g^2/\mu^2$ acquires now the value

$$\frac{2g^2}{g^2/\pi+(g/2)^2/\pi+(g/2)^2/\pi}=\frac{4\pi}{3},$$

rather than 2π as in the standard Schwinger model or π as in the Schwinger model with two flavors of equal charge. We have

$$C_3(\tau)=\exp\left\{-\frac{2\pi\tau}{3L}\right\}. \tag{5.17}$$

For the components $\lambda^{4\pm i5}$ and $\lambda^{6\pm i7}$, the correlator decays faster $\propto\exp\{-5\pi\tau/(3L)\}$ and their contribution in the correlator (5.9) can be safely neglected.[13] Let us compare now the contributions (5.16) and (5.17). Both decay exponentially at large τ, but the value of exponent in Eq. (5.17) is smaller than that in Eq. (5.16) and, at large τ, the leading asymptotics of the correlator (5.9) is determined by the gauge field background aligned along the third color axis and is given by Eq. (5.17).

Notice now that the same result could be obtained from the Hamiltonian analysis of Ref. [22]: Equation (5.17) can be interpreted as

$$C(\tau)=\exp\{-V_\diamond\tau\},$$

where $V_\diamond$ is the energy (5.8) of the fourth local minimum of the potential (5.6) discussed before. Indeed, an accurate treatment shows that the profile function $f(\tau')$ defined in Eq. (5.10) for the saddle point field configuration saturating the path integral for Eq. (5.9) rises from 0 to $4\pi/(3L)$ in the region $\tau'\sim 0$ (the width of this region is of order g^{-1}), stays at this value for a while until τ' approaches the point τ, and goes down to zero in the region $\tau'\sim\tau$. But the point

$$A_1^3=\frac{4\pi}{3L},\quad A_1^8=0$$

is exactly the point where one of the diamond minima of the potential sits. We have, for large τ,

$$C(\tau)=|\langle o|\overline{\lambda^a}\lambda^a|\diamond\rangle|^2\ \exp\{-V_\diamond\tau\}, \tag{5.18}$$

which coincides with Eq. (5.17).

The instanton (anti-instanton) contribution to the correlator $C(\tau)$ has the same asymptotic behavior. The relevant saddle point configuration starts from the central circle in Fig. 1 at $\tau'=-\infty$. Then at $\tau'\sim 0$ the field rises in, say, the t^3 color direction to the diamond point, stays there for a while, and that provides the exponent $V_\diamond\tau$ in the asymptotics of the correlator, after which it does not go back to origin at $\tau'\sim\tau$ as in the topologically trivial case, but moves farther to the closest triangle or box along the color direction $(1,0,-1)$ or $(0,-1,1)$ (the symmetry axes of the effective potential which are equivalent to t^3 and correspond to other roots of Lee algebra).

The advantage of the method suggested here is that it can be easily generalized for higher $N\geqslant 4$ where, working in the Hamiltonian approach, we should have studied an intricate multidimensional structure of the effective potential.[14] It turns out that for any N the leading asymptotics of the correlator (5.9) is due to the Abelian saddle point field configuration (5.10). In this background, off-diagonal components λ^a are "organized" in a complex fermion field $\lambda^{1\pm i2}$ of the charge g and $2(N-2)$ complex fermion fields of the charge $g/2$. The correlator of $\lambda^{1\pm i2}$ components gives the factor

$$\frac{2g^2}{\pi\mu^2}=\frac{4}{N}$$

[13]The behavior of the correlator on the Euclidean plane can be found along the same lines. In the Schwinger model with several flavors of arbitrary charges, the factor (5.15) rises as a *power* at large distances. That compensates partially the falloff $\sim x^{-2}$ of the free correlator, and the full correlator for the fermion with charge g_i behaves as $x^{-2\Delta_i}$ with $\Delta_i=1-g_i^2/(\pi\mu^2)=1-g_i^2/\Sigma_i g_i^2$. The term $g_i^2/(\pi\mu^2)$ is nothing else but the anomalous dimension of the operator $\bar\psi_i\psi_i$. A corresponding conformal theory where this operator appears naturally as a primary field can be formulated. For example, for two flavors of equal charge, it is the primary operator $\cos(\sqrt{2\pi}\chi)$ (or, better to say, a couple of operators $\exp\{\pm i\sqrt{2\pi}\chi\}$ corresponding to $\bar\psi_{1L}\psi_{1R}$, $\bar\psi_{1R}\psi_{1L}$ or, equivalently, to $\bar\psi_{2R}\psi_{2L}$, $\bar\psi_{2L}\psi_{2R}$) in the conformal theory of the real massless scalar field χ [30]. It is no wonder thereby that one and the same factor Δ determines the power asymptotics of the correlator on the Euclidean plane and the exponential asymptotics of the correlator on the cylinder. One can map the complex plane on the stripe $0\leqslant x\leqslant L$, $-\infty<\tau<\infty$, by a conformal transformation, after which the power behavior of the correlator at large distances is transformed to the exponential one (see, e.g., [31]).

[14]We have performed such a study for $N=4$. The pattern of the minima of the effective potential presents an interesting three-dimensional lattice akin to the lattice of diamond. But as it bears little relevance for the main question studied in this paper, we will not distract ourselves here for this issue.

in the exponent, and the correlator decays as

$$C_N(\tau)\propto \exp\left\{-\frac{2(N-2)}{N}\frac{\pi\tau}{L}\right\}. \tag{5.19}$$

Rotating the cylinder where the theory is defined by $\pi/2$, we arrive at the conclusion that for $N\geqslant 3$ at high temperatures $T\gg g$, the spatial correlator

$$C(x)=\langle \overline{\lambda^a}\lambda^a(x)\overline{\lambda^a}\lambda^a(0)\rangle_T \tag{5.20}$$

decays exponentially at large distances. By cluster decomposition, that certainly implies that

$$\langle \overline{\lambda^a}\lambda^a\rangle_{T\gg g, N\geqslant 3}=0. \tag{5.21}$$

VI. PHASE TRANSITION

The bosonization analysis of Sec. IV suggests the presence of the fermion condensate in the theory (1.3) at $T=0$ for any N in the thermodynamic limit $\mathcal{A}\to\infty$, while the fermion mass m is kept small but fixed. On the other hand, as $\ln Z$ does not involve a linear term in mass expansion, the condensate is zero in the chiral limit $m\to 0$ when the total area of the manifold, $\mathcal{A}$, is kept large but fixed. Also, we have seen in the previous section that for $N\geqslant 3$ the condensate is absent at high temperature $T\gg g$ even in the limit when the length of the spatial box L is sent to infinity in the first place. Two options to resolve this controversy were mentioned at the end of Sec. IV.

One of them postulates the relevance of some nontopological field configurations which have only a pair of fermion zero modes and provide for a nonzero fermion condensate in the chiral limit. It is a possible way out, but it has two obvious weak points. First, we have no idea on what these nontopological field configurations are. Second, assuming their existence, we do not understand why they disappear at finite temperature.

Another option is that the condensate appears at $T=0$ as an order parameter of a spontaneously broken symmetry. In that case, the limits (i) $\mathcal{A}\to\infty$, m fixed and (ii) $m\to 0$, $\mathcal{A}$ fixed need not commute. The partition function presents the sum of two exponentials (4.15), and the linear term $\propto mg\mathcal{A}$ in the expansion of $Z(m)$ cancels out.

We will argue now that, at least for odd N, this second possibility is rather probable, indeed. First, there is a discrete symmetry (1.13) to be broken. It remains the exact symmetry of the Lagrangian also on the quantum level because instantons involve a couple of left-right pairs of zero modes, and the induced 't Hooft term in the effective Lagrangian $\sim(\lambda_L^a\lambda_R^a)^2$ (we will first consider the simplest case $N=3$) respects the symmetry (1.13).

Spontaneous breaking of *continuous* symmetries is excluded in (1+1)-dimensional systems due to the Coleman theorem [32], but a *discrete* symmetry can well be broken spontaneously. The only important restriction is that the symmetry should be restored at any finite temperature. Really, a physical picture of spontaneously broken discrete symmetry involves the presence of the domain walls between two different ordered phases. If only one spatial dimension is there, these ''walls'' present solitons; the corresponding quantum states have a finite energy. It is obvious that at finite temperature, however small it is, the heat bath involves some number of these ''walls.'' And that exactly means that the vacuum is disordered.

A classical example of a theory involving spontaneous breaking of Z_2 symmetry is one-dimensional Ising model[15] [33]. The theory has the Hamiltonian

$$H=-J\sum_{i=-N}^{N}\sigma_i\sigma_{i+1}, \tag{6.1}$$

$N\to\infty$ in the thermodynamic limit. The vacuum state of Eq. (6.1) is doubly degenerate: $\langle\sigma\rangle=1$ or $\langle\sigma\rangle=-1$. At any nonzero temperature, the domain walls (the states with $\sigma_i=-1$ at $i\leqslant n_0$ and $\sigma_i=1$ at $i>n_0$) appear in the heat bath. Their characteristic density is $\sim\exp\{-J/T\}$. Thereby, the state is not ordered anymore and the correlator $\langle\sigma_i\sigma_{i+M}\rangle$ tends to zero at $M\to\infty$, although the spatial correlation length (a characteristic value of M when the spin correlator starts to die away) is exponentially large $\sim\exp\{J/T\}$ when the temperature is small. The system has a first order phase transition at $T=0$.[16]

Our suggestion is that the same happens in adjoint QCD_2 at $N=3$, the fermion condensate $\langle\lambda_L^a\lambda_R^a\rangle$ being the order parameter of the symmetry (1.13) and playing the role of $\langle\sigma\rangle$. A number of nontrivial physical consequencies follow from this assumption.

First, it implies that the correlation length l of Eq. (5.20) rapidly grows as the temperature goes down and becomes exponentially large $\sim\exp\{g/T\}$ in the region $T\ll g$. No analytic calculation in the region $T\ll g$ is possible. It would be rather interesting, however, working still in the region $T\gg g$ where the quasiclassical approximation applies, to find out what are the *corrections* to the leading Born-Oppenheimer result [cf. Eq. (5.17)].

$$l_{T\gg g}=\frac{3}{2\pi T}. \tag{6.2}$$

If the first nonleading correction turns out to be positive, it could serve as an argument in favor of the scenario that the corrections become overwhelmingly large at $T\ll g$.

The second very interesting corollary is that the spectrum of the Hamiltonian should involve ''walls,'' the states interpolating from the vacuum with negative $\langle\lambda^a\lambda^a\rangle$ on the left to the vacuum with positive $\langle\lambda^a\lambda^a\rangle$ on the right. If the wall states do not exist, but only the states presenting excitations over the vacuum with $\langle\lambda^a\lambda^a\rangle>0$ or the excitations over the vacuum with $\langle\lambda^a\lambda^a\rangle<0$, we cannot talk about spontaneous symmetry breaking in the physical meaning of the word. The whole Hilbert state of the system would be separated into two subspaces which do not know of each other, and a su-

[15]One-dimensional statistical systems correspond to (1+1)-dimensional field theories.

[16]Note that second order phase transitions at $T=0$ associated with would-be spontaneous breaking of a continuous symmetry are also possible in (1+1)-dimensional systems. It is exactly what happens in multiflavor Schwinger model [34]. But as the order parameter is zero at the phase transition point and, if $T_c=0$, there is nothing below, the Coleman theorem is not violated.

perselection rule singling out one of these subspaces could be imposed. The situation would be the same as with instantons in QCD_4 [28] or as with adjoint QCD_2 at $N=2$ [5]. It would imply the presence of ‘‘fractons’’ like in [24] and, as was mentioned, it would be difficult to explain where the condensate is gone at $T\neq 0$.

Presently, we do *not* know whether such wall states exist. The spectrum of adjoint QCD_2 was studied with some care only in the limit $N\to\infty$ [35], but not at finite $N=3$.[17]

The reasoning of this section can be relatively easily generalized for higher odd N. The common point is that when N is *odd*, the number of zero modes (1.12) is always *even*, the symmetry (1.13) is not anomalous, and can be broken spontaneously at $T=0$. The partition function presents a sum (4.15) of two extensive exponentials as before. A little bit troublesome point, however, is that, say, for $N=5$, the instanton contributions first show up only in the quartic term of the expansion of Eq. (4.15) in $mg\mathcal{A}$. For $N=137$, they first appear in the term $\sim(mg\mathcal{A})^{136}$. The terms $\sim(mg\mathcal{A})^2,\dots,\sim(mg\mathcal{A})^{134}$ should come from the path integral in the topologically trivial sector. Well, it is somewhat unusual, but at least not paradoxical.

The situation with even $N\geqslant 4$ is more complicated. The matter is that in this case the symmetry (1.13) *is* anomalous. For example, for $N=4$, the field configurations in the topological class $k=1$ involve three pairs of zero modes, and the corresponding ’t Hooft effective Lagrangian $\sim(\lambda_L^a\lambda_R^a)^3$ is odd under the transformation (1.13). Generally, the partition function in the topological sector k acquires the factor $(-1)^k$. If there is no symmetry, one cannot talk about its spontaneous breaking. There should be *a unique* physical vacuum state (in a sector with a particular value of discrete θ brought about by instantons) and Eq. (4.15) cannot be written. Thus the physics of the theory with odd $N\geqslant 3$ differs essentially from the theory with even $N\geqslant 4$ (cf. [36]). In the first case, the hypothesis about spontaneous Z_2 symmetry breaking resolves the paradox rather satisfactory (with all reservations given). For large even N, the paradox is still there, and, at the current level of understanding, we do not dare to speculate more in this direction.

VII. $O(N^2-1)$ AND DISCONNECTED COMPONENTS

As far as odd N are concerned, the suggested picture looks rather self-consistent and nice, and I am ready to believe that Z_2 symmetry in the theory with $N=3,5,\dots$ is broken spontaneously, indeed. There is, however, a theoretical problem which is not yet fully understood and we are in a position to discuss it.

The arguments in favor of the existence of fermion condensates at $T=0$ come from the bosonization analysis. We have interpreted the condensate as the order parameter of the spontaneously broken symmetry (1.13). The symmetry (1.13) clearly displays itself in the fermion language. In bosonization language, the corresponding symmetry is

$$h^{ab}\to -h^{ab}. \tag{7.1}$$

At first sight, the action (1.8) is invariant under the transformation (7.1), indeed. The problem is, however, that the matrix $-h^{ab}$ does not belong to the adjoint representation of $SU(N)$ if the matrix h^{ab} does. In particular, the equation

$$-\delta^{ab}=2\,\mathrm{Tr}\{ut^au^\dagger t^b\}$$

has no solution (it is best seen using the identity $\mathrm{Tr}\,h=|\mathrm{Tr}\,u|^2-1\geqslant -1$). Notice now that the symmetry (7.1) could be reinforced if assuming $h\in O(N^2-1)$ (as Witten originally suggested for *free* fermions). If bosonizing the theory with h belonging to the adjoint representation of the gauge group $SU(N)/Z_N$, the transformation (7.1) relates not the variables in one and the same bosonized theory, but relates different theories corresponding to different subgroups of $O(N^2-1)$. But we may equally well multiply h by any matrix of the coset $O(N^2-1)/[SU(N)/Z_N]$. All such theories come on equal footing. We are thus arriving at Eq. (1.11): The partition function of QCD_2 with massive Majorana fermions is equal to the sum (the integral) of the partition functions $Z(R)$ in all possible bosonized theories characterized by a matrix R.

We cannot *prove* now the validity of this recipe. However, we can *show* that the bosonized partition function with a particular R has wrong analytic properties as a function of mass. If summing over all R with a particular sign prescription (see below), the correct analytic properties are reproduced.

Consider first the theory with $N=3$. Let us concentrate on the instanton sector and put $R^{ab}=\delta^{ab}$ at first. We have seen in Sec. IV that the leading term in the mass expansion of Z_I is $\sim(mg\mathcal{A})^2$. Consider now the next term $\propto m^3$. It appears when pulling down the mass term in the action thrice. Proceeding along the same lines as in Sec. IV (i.e., taking into account only the zero Fourier harmonic u_0 and imposing the requirement $[u_0,T^*]=0$), we obtain

$$Z_I^{N=3}=C_2(mg\mathcal{A})^2+C_3(mg\mathcal{A})^3\int du^{(2)}|\mathrm{Tr}\,u^{(2)}|^2(\mathrm{Tr}\,u^{(2)})^2 + O(m^4). \tag{7.2}$$

The group integral in Eq. (7.2) is nonzero, and we get a nonzero cubic term in the expansion of Z_I in mass.

However, the cubic term is absent in the original fermion theory. Really, the mass dependence comes from the fermion determinant

[17] However, it is not a hopeless problem to study the spectrum of the theory on lattices. The ‘‘lattice experimental evidence’’ in favor or against our hypothesis is highly desirable. Actually, two-dimensional systems are a lot simpler than four-dimensional QCD where the efforts of lattice people are mostly applied. One can only express a wish that the fashion would change some time and more lattice works on two-dimensional systems including fermions would be done. The field involves many unsolved but easily solvable problems for the experts. In the first place, a number of exact nontrivial results in the Abelian theory (see [34] and references therein) should be checked. If theoretical predictions for the spectrum and correlators are reproduced in the Abelian case, one could proceed with two-dimensional non-Abelian theories. Also, if numerical lattice calculations would reproduce the exact theoretical results in two dimensions, there would be more trust in lattice calculations in QCD_4 with dynamic fermions.

$$\mathrm{Det}^{N=3}_{\mathrm{Majorana}}\|i\mathcal{D}+m\| = [\mathrm{Det}^{N=3}_{\mathrm{Dirac}}\|i\mathcal{D}+m\|]^{1/2} \sim m^2 \prod_n{}' (m^2+\lambda_n^2), \qquad (7.3)$$

where the product runs over all nonzero eigenvalues of the Euclidean Dirac operator, only one eigenvalue of each doubly degenerate pair being taken into account [5]. The determinant (7.3) involves only even powers of m.

It is easy to see that, if allowing for an arbitrary $R \in \mathrm{O}(8)/[\mathrm{SU}(3)/Z_3]$ and integrating over R, the expression

$$Z_I^{\mathrm{true}} = \int dR\ Z_I(R) \qquad (7.4)$$

also involves only even powers. For each R, the theory with $R' = -R$ also contributes in the integral. But the mass terms (1.11) in these two theories have opposite sign.

Consider now a theory with even N. The case $N=2$ is already nontrivial. The symmetry (7.1) is realized on the full O(3) group involving two disconnected components SO(3) where the bosonized theory (1.8) is formulated. We have to take into account the contributions of both components in the partition function. But in contrast to $N=3$, it would be incorrect just to sum up the corresponding contributions. Speaking precisely, it is correct in the topological trivial sector, but not in the instanton sector.

The contribution of the component with $\mathrm{Det}\|h\|=1$ in the partition function in the instanton sector is

$$Z_I^{N=2}(+) = C_1 mg\mathcal{A} + C_2 (mg\mathcal{A})^2 + O(m^3), \qquad (7.5)$$

with a nonzero C_2 given by the integral

$$C_2 \propto \int_0^{2\pi} d\beta_3 \exp\{-i\beta_3\}(2\cos\beta_3+1)^2 \neq 0.$$

Like in the previous case, it has wrong analytic properties involving both odd and even powers of mass. The mass dependence of Z_I in the fermion theory comes from the Majorana fermion determinant, which involves, for $N=2$, only odd powers,

$$\mathrm{Det}^{N=2}_{\mathrm{Majorana}}\|i\mathcal{D}+m\| \sim m \prod_n{}' (m^2+\lambda_n^2).$$

To reproduce this behavior, we have to *subtract* the contribution $Z_I^{N=2}(-)$ of the odd SO(3) component with $\mathrm{Det}\|h\| = -1$ from Eq. (7.5). The corresponding theory differs from the theory of the even SO(3) component only by the sign of the mass term (1.11). The expansion of $Z_I^{N=2}(-)$ in mass has exactly the same form as Eq. (7.5) up to the opposite sign of odd powers. We are defining now

$$Z_I^{N=2}(\mathrm{true}) = Z_I^{N=2}(+) - Z_I^{N=2}(-). \qquad (7.6)$$

$Z_I^{N=2}(\mathrm{true})$ involves only odd powers of mass. Our hypothesis is that it exactly corresponds to the instanton partition function of the fermion theory.

Consider now a general case. Let first N be odd. The number of zero mode pairs $k(N-k)$ is even for any k, and the expansion of the partition function in mass in the topological sector k starts with $m^{k(N-k)}$ and involves only even powers of m. The expansion of the partition function Z_k in the bosonized theory (1.8) with the mass term (1.10) also starts with $m^{k(N-k)}$ [see Eq. (4.12)], but includes both even and odd powers. For odd N, the group $\mathrm{O}(N^2-1)$ includes only one connected component. The same arguments as for the case $N=3$ considered before show that the odd powers of mass cancel out in the integral (7.4) over the theories with different R. This integral should correspond to the partition function Z_k in the original fermion theory.

Let now N be even. The value $k(N-k)$ may be odd or even depending on k. For example, for $N=4$, the sectors $k=1,3$ involve three pairs of fermion zero modes, and the sector $k=2$ involves four such pairs. In the former case, the expansion of Z_k^{ferm} involves only odd powers of mass and in the latter case only even powers. On the other hand, the mass expansion of Z_k in the bosonized theory with the mass term (1.10) includes both even and odd powers for any k. Note now that the group $\mathrm{O}(N^2-1)$ includes two disconnected components for even N. Our recipe reads

$$Z_k^{N\ \mathrm{even}}(\mathrm{true}) = \int dR_+ Z_k^{N\ \mathrm{even}}(R_+) + (-1)^k \int dR_- Z_k^{N\ \mathrm{even}}(R_-). \qquad (7.7)$$

The odd (even) powers of mass cancel out in the integrated partition function (7.7) with even (odd) k and the correct analytic properties of Z_k are reproduced.

Again, we see the distinction between odd and even N. Obviously, there is a relation between the existence of two disconnected components in $\mathrm{O}(N^2-1)$ for even N and the fact that the symmetry (1.13) is anomalous. Indeed, the partition function (7.7) is invariant over the bosonic counterpart of this symmetry, the transformation $h \to -h$, for even k, but not for odd k.

VIII. DISCUSSION

The main physical signature of the suggested scenario with spontaneous breaking of discrete Z_2 symmetry is the presence of the domain wall solitons—the states which interpolate between different vacua—in the spectrum of the theory. If the domain walls are absent, different vacua are completely unrelated to each other and belong to the different sectors of Hilbert space. In that case, a superselection rule which selects a particular sector once and forever in the whole physical space should be imposed. Then there is no spontaneous symmetry breaking in the physical meaning of this word. This is the situation in standard QCD_4 [the vacuum involves a continuous degeneracy in θ, but one cannot talk of the spontaneous breaking of U(1) symmetry because the physical signature of this breaking—the massless Goldstone boson which is singlet in flavor—is absent]. This is also a situation in pure Yang-Mills theory at high temperature where the physical domain walls interpolating between different Z_N ‘‘phases’’ are absent and one cannot talk about spontaneous breaking of Z_N discrete symmetry [10]. And this is the situation in adjoint QCD_2 with $N=2$ where two sectors (5.4) are not physically related and there are no walls.

The fact that we cannot at present establish the existence of domain walls in adjoint QCD_2 with $N \geqslant 3$ explicitly is the main reason why we are still talking about the possibility of spontaneous breaking of Z_2 symmetry in this theory [even for odd N where the symmetry (1.13) to be broken is retained on the quantum level] without complete certainty.

The two-dimensional model considered in this paper presents an interest on its own, but the main point of interest is the lessons one can learn from the analysis of this model for four-dimensional supersymmetric Yang-Mills theories. These theories attracted recently a considerable attention after appearance of the paper of Witten and Seiberg who calculated exactly the spectrum of physical states in $\mathcal{N}=2$ supersymmetric Yang-Mills theory [37].

There is a long-standing unresolved problem in a more simple $\mathcal{N}=1$ supersymmetric Yang-Mills theory involving only gluons and gluinos. Supersymmetric Ward identities display the constant (x-independent) behavior of the fermion correlator

$$\langle \lambda^a_\alpha \lambda^{a\alpha}(x_1) \cdots \lambda^a_\alpha \lambda^{a\alpha}(x_N) \rangle = \text{const} \tag{8.1}$$

[for SU(N) gauge group]. Instanton calculations (which are valid at small $|x_i - x_j|$) show that this constant is nonzero [38]. That implies the presence of gluino condensate. However, standard instantons involve $2N$ fermion zero modes and, assuming that only instantons contribute and the extensive form (4.14) of the physical partition function with only one physical vacuum state is valid, we are led to the same contradiction as in adjoint QCD_2 at $N \geqslant 3$ considered in this paper, that the linear in mass term in the Taylor expansion of the partition function, which should be there due to the presence of a nonzero linear term in the Taylor expansion of $\epsilon_{\text{vac}}(m) \equiv$ the fermion condensate, cannot be reproduced.

Just as in adjoint QCD_2, there are only two ways out. Either we should assume that Z_{2N} symmetry in the super Yang-Mills (SYM) Lagrangian [a remnant of U(1) symmetry after taking anomaly into account] is broken spontaneously down to Z_2 or that an additional superselection rule should be imposed. It amounts to allowing the θ parameter to vary within the interval

$$\theta \in (0, 2\pi N). \tag{8.2}$$

In the first case, the physical domain walls separating different Z_N phases should be present in the theory. In the second case, the ''phases'' should be completely unrelated and the domain walls must be absent.

As far as SYM theory with SU(N) gauge group is concerned, we favor more the second possibility. After all, at least in toroidal geometry, the Euclidean configurations with fractional topological charge $\propto 1/N$ appear on an equal footing with instantons [25] and an additional superselection rule with respect to a large gauge transformation changing the Chern-Simons number by $1/N$ arises quite naturally. Actually, one can explicitly calculate the toron contribution in the partition function of the theory at finite volume [8]. There are also additional arguments coming from the analysis of the pure Yang-Mills theory in large-N limit. If no fermions are there, the partition function is a nontrivial function of θ. At large N, a smooth θ dependence of the partition function can be achieved only if allowing θ to vary within the interval (8.2) [8]. All together that makes us believe that the superselection rule leading to the classification (8.2) should be imposed, and that there are no walls and no spontaneous symmetry breaking.

For a proper balance, we should also mention counterarguments to this scenario.

(1) Toron configurations can be written in a finite toroidal box but not in S^4 or $S^3 \times R$ geometry. If we do not restrict ourselves to fiber bundles on compact manifolds, meron solutions with fractional topological charge which live in R^4 and have a singular field strength at one point can be written [39]. They have an infinite action, but still may be relevant for physics [40]. Torons on tori are not similar to merons in flat space and to the absence of anything on a sphere. The physics, however, should not depend on boundary conditions if the box is large enough.

(2) In contrast to instantons, toron configurations are delocalized. Again, we cannot visualize at present how these delocalized configurations manage to contribute in local physical quantities.[18]

(3) An argument in favor of existence of the walls in SU(N) theory can be put forward if considering the $\mathcal{N}=1$ theory with matter fields (supersymmetric QCD). When the mass of quarks and squarks is small, the theory is in weak coupling Higgs phase (see e.g., [41]). The different Z_N phases are associated with different values of the Higgs average and the domain wall solitons with finite energy density interpolating between different Higgs phases probably exist. One can send then the mass of matter fields to infinity after which they decouple. A renormalization group analysis seems to show that the energy density of these walls remains finite also in this limit which means the existence of physical walls also in pure SYM theory [42].

As I already mentioned, my own guess is that the arguments *pro* overweigh in this case the arguments *contra* and the walls are not really there in SU(N) theory. But this guess does not have the rank of a statement. Obviously, more study of the question is necessary.

The situation is, however, different in theories with higher orthogonal and exceptional gauge groups. Again, supersymmetric Ward identities and instanton calculations imply that the d-point function of several fermion scalar densities like Eq. (8.1) (d is the Dynkin index of the group; for higher orthogonal groups SO($N \geqslant 5$), $d = N-2$) is a nonzero constant [20]. That implies the presence of the fermion condensate, but in contrast to theories with unitary groups, no toron configurations with fractional topological charge which could generate the condensate explicitly are known. In that case, the option involving spontaneous breaking of Z_d symmetry looks much more probable. The domain walls should exist.

[18]A counterargument to this counterargument can also be suggested. Really, *classical* instanton solutions in the Schwinger model are also delocalized, but still instantons contribute to local observables like the fermion condensate [1–3]. Anyway, we understand the mechanism of that in the Schwinger model—after taking into account the fermion determinant, a relevant saddle point of the corresponding path integral presents a localized vortex like configuration [1] [cf. Eq. (2.5) and the discussion thereafter]. But we do not understand it in the four-dimensional (SYM) theory which we would like to.

We think that the further study of adjoint QCD_2 for $N\geqslant3$ would make a lot of sense. This 2D theory is much simpler than 4D SYM theories. One can hope that a definite answer to the question whether domain walls exist in two dimensions (we believe they do) would be obtained reasonably soon. The resolution of this question could provide crucial insight into what happens in four dimensions.

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[1] A. V. Smilga, Phys. Rev. D **49**, 5480 (1994).
[2] C. Jayewardena, Helv. Phys. Acta **61**, 636 (1988).
[3] I. Sachs and A. Wipf, Helv. Phys. Acta **65**, 371 (1992).
[4] E. Witten, Nuovo Cimento A **51**, 325 (1979).
[5] A. V. Smilga, Phys. Rev. D **49**, 6836 (1994).
[6] P. Ramon, *Field Theory: A Modern Primer* (Benjamin/Cunnings, Reading, MA, 1981).
[7] A. I. Vainshtein and V. I. Zakharov, JETP Lett. **35**, 323 (1982).
[8] H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607 (1992).
[9] T. Bhattacharya, A. Gocksch, R. D. Pisarski, and C. Korthals-Altes, Nucl. Phys. **B383**, 497 (1992).
[10] A. V. Smilga, Ann. Phys. (N.Y.) **238**, 1 (1994).
[11] I. I. Kogan, Phys. Rev. D **49**, 6799 (1994).
[12] D. Gross, I. Klebanov, A. Smilga, and A. Matitsyn, Nucl. Phys. **B461**, 109 (1996).
[13] J. Wess and B. Zumino, Phys. Lett. **B37**, 95 (1971).
[14] S. Novikov, Russ. Math. Surv. **37**, 5 (1982).
[15] E. Witten, Nucl. Phys. **B223**, 422 (1983).
[16] E. Witten, Commun. Math. Phys. **92**, 455 (1984).
[17] L. D. Faddeev, Lett. Math. Phys. **1**, 289 (1976); A. Losev, G. Moore, N. Nekrasov, and S. Shatashvili, Report No. PUPT-1568, hep-th/9511185 (unpublished).
[18] P. Di Vecchia, B. Durhuus, and J. L. Petersen, Phys. Lett. **144B**, 245 (1984); D. Gonzalez and A. Redlich, *ibid.* **147B**, 150 (1984).
[19] I. I. Kogan and A. R. Zhitnitsky, Nucl. Phys. **B465**, 99 (1996).
[20] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. **B296**, 445 (1988); A. Morozov, M. Olshanetsky, and M. Shifman, *ibid.* **B304**, 291 (1988).
[21] L. S. Brown and R. I. Nepomechie, Phys. Rev. D **35**, 3239 (1987).
[22] F. Lenz, M. Shifman, and M. Thies, Phys. Rev. D **51**, 12 (1995).
[23] S. Coleman, Phys. Rev. D **11**, 2088 (1975); S. Mandelstam, *ibid.* **11**, 3026 (1975).
[24] M. A. Shifman and A. V. Smilga, Phys. Rev. D **50**, 7659 (1994).
[25] G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981); E. Cohen and C. Gomez, Phys. Rev. Lett. **52**, 237 (1984); A. R. Zhitnitsky, Nucl. Phys. **B340**, 56 (1990); A. Gonzalez-Arroyo and A. Montero, Report No. FTUAM/96-13, hep-th/9604017 (unpublished).
[26] C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952).
[27] D. Kutasov, Nucl. Phys. **B414**, 33 (1994).
[28] C. G. Callan, R. Dashen, and D. J. Gross, Phys. Lett. **63B**, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[29] E. Marinary, G. Parisi, and C. Rebbi, Nucl. Phys. **B190**, 734 (1981); J. Hetrick and Y. Hosotani, Phys. Rev. D **38**, 2621 (1988); A. V. Smilga, Phys. Lett. B **278**, 371 (1992).
[30] S. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976).
[31] V. S. Dotsenko, in *Proceedings of the Symposium ''Conformal Field Theory and Solvable Lattice Models''*, Kyoto, 1986, edited by M. Jimbo, T. Miwa, and A. Tsuchiya (Academic Press, Boston, 1988), p. 123.
[32] S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
[33] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[34] A. Smilga, and J. J. M. Verbaarschot, Phys. Rev. D **54**, 1087 (1996); A. V. Smilga, Report No. ITEP-TM-27-96, hep-th/9607154 (unpublished).
[35] S. Dalley and I. R. Klebanov, Phys. Rev. D **47**, 2517 (1993); G. Bhanot, K. Demeterfi, and I. R. Klebanov, *ibid.* **48**, 4980 (1993).
[36] P. Huet, R. Narayana, and H. Neuberger, Phys. Lett. B **380**, 291 (1996).
[37] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994).
[38] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov Nucl. Phys. **B229**, 407 (1983); G. Rossi and G. Veneziano, Phys. Lett. **138B**, 195 (1984).
[39] V. de Alfaro, S. Fubini, and G. Furlan, Phys. Lett. **65B**, 163 (1976).
[40] C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D **19**, 1826 (1979).
[41] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Sov. Phys. Usp. **28**, 709 (1985).
[42] This argument belongs to M. A. Shifman (private communication).