# Path integral for the loop representation of lattice gauge theories

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We show how the Hamiltonian lattice *loop representation* can be cast straightforwardly in the path integral formalism. The procedure is general for any gauge theory. Here we present in detail the simplest case: pure compact QED. The lattice loop path integral approach allows us to knit together the power of statistical algorithms with the transparency of the gauge-invariant loop description. The results produced by numerical simulations with the loop classical action for different lattice models are discussed. We also analyze the lattice path integral in terms of loops for the non-Abelian theory. [S0556-2821(96)05924-3]

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## I. INTRODUCTION

The loop approach to Abelian quantum gauge theories was introduced in the early 1980s [1]. Later it was generalized to the non-Abelian Yang-Mills gauge theory [2]. This Hamiltonian method allows us to formulate gauge theories in terms of their gauge-invariant physical excitations: the loops. The original aim of this general description of gauge theories was to avoid gauge redundancy by working directly in the space of the gauge invariant excitations. However soon it was realized that the loop formalism goes far beyond a simple gauge-invariant description. The introduction by Ashtekar [3] of a new set of variables that cast general relativity in the same language as gauge theories allowed us to apply loop techniques as a natural nonperturbative description of Einstein's theory. In particular, the loop representation appeared as the most appealing application of the loop techniques to this problem [4]. The covariant world sheet formulation of quantum gravity corresponding to the canonical loop representation is still unknown. Though some progress in that direction has been reported recently [5]. This version of loop quantization would make the general four-dimensional diffeomorphism invariance manifest and it will be probably more suitable to tackle some crucial problems which require a covariant description (as for instance the black hole dynamics).

The Hamiltonian analytical computation techniques for gauge theories have been developed during the last decade and they provide qualitatively good results for several lattice models [6–8]. On the other hand, a Lagrangian approach in terms of loops has been elusive, due mainly to the noncanonical character of the loop algebra. This feature forbids the possibility of performing a Legendre transformation as a straightforward way to obtain the Lagrangian from the Hamiltonian. A path integral loop formulation will allow us to employ the more powerful statistical computation techniques. Gauge-invariant actions corresponding to the Villain form of the U(1) model were proposed recently [9] and generalized to include matter fields [10].

These actions have been used as a computational tool. A Metropolis Monte Carlo algorithm was implemented fixing the acceptance ratio, as it is usual in random surfaces analysis. The U(1) model was studied for different lattice sizes [9], imposing periodic boundary conditions. Simple thermal cycles showed the presence of a phase transition in the neighborhood of $\beta_V = 0.639$. In the case of the U(1) Higgs model, the $\beta$-$\gamma$ phase diagram was mapped out [10]. In both models, by virtue of the gauge invariance of this description, the equilibrium configurations were reached faster than with the ordinary gauge-variant descriptions. Furthermore, the typical strong metastability of Monte Carlo analysis for these models was not observed using this description. This is another advantage of the method because it makes easiest the numerical analysis of the phase transition critical exponents.

In this paper we want to address the issue of the explicit correspondence between these actions and the Hamiltonian loop representation using the transfer matrix techniques.

This paper is organized as follows. In Sec. II we show how the loop description, introduced originally in the Hamiltonian formalism, can be cast in the lattice path integral formalism. We illustrate this in detail for the Abelian case (compact electromagnetism). The path integral of lattice U(1) theory is expressed as a sum of the world sheets of electric loops. We discuss the connection of this classical loop action with the Nambu string action. In Sec. III we consider the extension to the path integral loop formalism to the case of non-Abelian Yang-Mills fields. In Sec. IV we conclude with some remarks.

## II. THE LATTICE PATH INTEGRAL IN TERMS OF LOOPS: ELECTROMAGNETISM

The loop based approach of Ref. [1] describes the quantum electrodynamics in terms of the gauge-invariant holonomy (Wilson loop)

$$\hat{W}(\gamma) = \exp[ie\oint_\gamma \hat{A}_a(y)dy^a], \tag{1}$$

where $\gamma$ is a spatial loop at constant time $t$. $\hat{E}^a(x)$ is the conjugate electric field operator. They obey the commutation relations

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$$[\hat{E}^a(x),\hat{W}(\gamma)]=e\int_\gamma \delta(x-y)dy^a\hat{W}(\gamma). \tag{2}$$

These operators act on a space of loop dependent wave functions $\psi(\gamma)$ that may be expressed in terms of the transform

$$\psi(\gamma)=\int d_\mu[A]\langle\gamma|A\rangle\langle A|\psi\rangle=\int d_\mu[A]\psi[A]$$
$$\times\exp[-ie\oint_\gamma A_a dy^a]. \tag{3}$$

This loop representation has very appealing features. (i) It allows us to do away with the first class constraints of gauge theories and, therefore, the Gauss law is satisfied automatically. The formalism only involves gauge-invariant objects, i.e., no gauge redundancy. (ii) All the gauge-invariant operators have a transparent geometrical meaning when they are realized in the loop space.

When this loop representation is implemented in a lattice of spacing $a$, the Eqs. (1) and (2) become

$$\hat{W}(\gamma)=\prod_{\ell\in\gamma}\hat{U}(\ell), \tag{4}$$

$$[\hat{E}_\ell,\hat{W}(\gamma)]=N_\ell(\gamma)\hat{W}(\gamma), \tag{5}$$

where $\ell\equiv(x,n)$ denotes a lattice link leaving the lattice site $x$ and pointing along the unit vector $n$, $U_\ell=e^{ieaA_n(x)}$ is an element of the gauge group [U(1) in the present case], $\hat{E}_\ell$ is the lattice electric field operator, and $N_\ell(\gamma)$ is the number of times that the link $\ell$ appears in the closed path $\gamma$.

In this loop representation, the Wilson loop acts as the loop creation operator:

$$\hat{W}(\gamma')|\gamma\rangle=|\gamma'\cdot\gamma\rangle. \tag{6}$$

The product of $\gamma$ and $\gamma'$ is the reduced composition of their irreducible chains and one can show that the loops form a group.[1]

The physical meaning of a loop may be deduced from Eqs. (5) and (6), in fact,

$$\hat{E}_\ell|\gamma\rangle=N_\ell(\gamma)|\gamma\rangle, \tag{7}$$

which implies that $|\gamma\rangle$ is an eigenstate of the electric field. The corresponding eigenvalue is different from zero if the link $l$ belongs to $\gamma$. Thus $\gamma$ represents a confined line of electric flux.

The U(1) lattice Hamiltonian can be written in terms of both previous fundamental operators as

$$\hat{H}=\frac{g^2}{2}\sum_l \hat{E}_\ell^2-\frac{1}{2g^2}\sum_p(\hat{W}_p+\hat{W}_p^\dagger). \tag{8}$$

Now, starting from the path integral for the Wilson U(1) lattice action, let us show how to set up the loop path integral:[2]

$$Z_W=\int_{-\pi}^{\pi}[d\theta_\ell]\exp\left(\frac{\beta}{2}\sum_p\cos\theta_p\right), \tag{9}$$

where the subscripts $\ell$ and $p$ stand for the lattice links and plaquettes, respectively, $\beta=1/e^2$ ($e$ is the electric charge of the electron) and $\theta_\ell$ is a compact variable $\in[-\pi,\pi]$ attached to the links and $\theta_p\equiv\theta_{\mu\nu}(x)\equiv\theta_\nu(x+a\hat{\mu})-\theta_\mu(x+a\hat{\nu})$ is its "discrete curl." Fourier expanding the $\exp[\beta/2\cos\theta]$ we get

$$Z_W=\int_{-\pi}^{\pi}[d\theta_\ell]\prod_p\sum_{n_p}I_{n_p}(\beta)e^{in_p\theta_p}, \tag{10}$$

where the $I_n$ are modified Bessel functions. $Z_W$ can be written as

$$Z_W=\sum_{\{n_p\}}\int_{-\pi}^{\pi}[d\theta_\ell]\exp\left(\sum_p \ln I_{n_p}(\beta)\right)e^{i\langle n,d\theta_\ell\rangle}, \tag{11}$$

where we use the notation of the calculus of differential forms on the lattice developed in Ref. [11]. In the above expression: $\theta$ is a real compact one-form defined in each link of the lattice $d$ is the coboundary operator—which maps $k$-forms into $(k+1)$-forms—, $n_p$ are integer two-forms defined at the lattice plaquettes and we have used the scalar product of $p$-forms $\langle f|g\rangle=\Sigma_{c_k}f(c)g(c)$ where the sum runs over the $k$-cells $c_k$ of the lattice ($c_0$ sites, $c_1$ links, and so on). Under this product the $d$ operator is adjoint to the border operator $\delta$ which maps $k$-forms onto ($k$-1)-forms and which corresponds to minus times the usual divergence operator: i.e.,

$$\langle f|dg\rangle=\langle\delta f|g\rangle, \tag{12}$$

$$\langle df|g\rangle=\langle f|\delta g\rangle. \tag{13}$$

Using Eq. (13) and integrating over $\theta_\ell$ we get a $\delta(\delta n_p)$. Then, we arrive at the following expression of $Z_W$ in terms of the integer two-forms $n$:

$$Z_W\propto\sum_{\{n_p;\delta n_p=0\}}\exp\left(\sum_p \ln I_{n_p}(\beta)\right), \tag{14}$$

the constraint $\delta n_p=0$ means that the sum is restricted to *closed* two-forms. Thus the sum runs over collections of plaquettes constituting closed surfaces.[3]

[1] In order to define loops from a chain of lattice links which form closed curves, a reduction process was introduced. A reduced chain was obtained by the elimination of opposite successive vectors. Once a couple of vectors is removed, new collinear opposite vectors may appear and must be eliminated. The process is repeated until one gets an irreducible chain. A loop $\gamma$ is an irreducible closed chain of vectors starting at $n_0$. For more details see Ref. [1].

[2] Next we summarize the main steps to get the description of the lattice path integral in terms of loops described in detail in Ref. [9].

[3] Notice that due to the Abelian character of the U(1) gauge group, the sum actually runs over equivalence classes of surfaces with different routing at self-intersections.

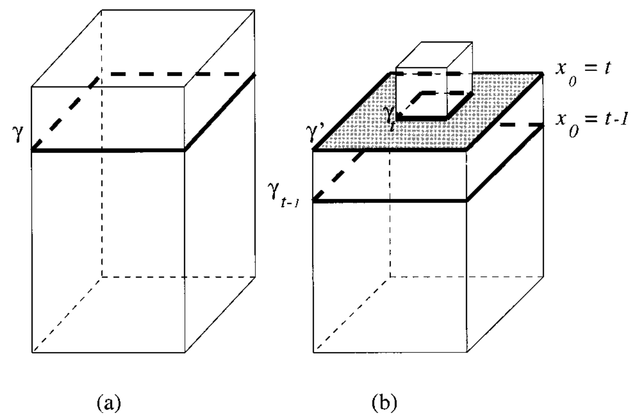


FIG. 1. (a) A spatial closed path or loop. (b) An open surface connecting the loop $\gamma_t'$.

An alternative and more easy-to-handle lattice action than the Wilson form is the Villain form. The partition function of that form is given by

$$Z_V=\int [d\theta]\sum_s \exp\left(-\frac{\beta_V}{2}\left|\left|d\theta-2\pi s\right|\right|^2\right), \tag{15}$$

where $||\dots||^2=\langle \dots,\dots\rangle$. If we use the Poisson summation formula,

$$\sum_s f(s)=\sum_n \int_{-\infty}^{\infty} d\phi f(\phi)e^{2\pi i\phi n}$$

and, as we integrate the continuum $\phi$ variables, we get

$$Z_V=(2\pi\beta_V)^{-N_p/2}\int [d\theta]\sum_n \exp\left(-\frac{1}{2\beta_V}\langle n,n\rangle+i\langle n,d\theta\rangle\right), \tag{16}$$

where $N_p$ in the number of plaquettes of the lattice. Again, we can use the equality: $\langle n,d\theta\rangle=\langle\delta n,\theta\rangle$ and integrating over $\theta$ we obtain a $\delta(\delta n)$. Then, we get

$$Z_V^{\text{loops}}=(2\pi\beta_V)^{-N_p/2}\sum_{\{n;\delta n=0\}} \exp\left(-\frac{1}{2\beta_V}\langle n,n\rangle\right). \tag{17}$$

Let us first discuss the (2+1)-dimensional case. When intersecting one of the previous surfaces with a $t=$ const plane, we get either loops or surfaces. In Fig. 1 we show two possible situations: (i) a spatial closed path or loop [Fig. 1(a)] and (ii) an open surface connecting the loop $\gamma_t'$ obtained by translating the loop $\gamma_{t-1}$ by one temporal lattice unit, and $\gamma_t$ [the shaded surface in Fig. 1(b)]. A loop $\gamma_t$ living on the $t$ slice is specified properly by the temporal plaquettes which leave this slice. This is equivalent to saying that in two spatial dimensions, given the loops at time $t$ and time $t+a_0$, the surface which connect them is defined unambiguously.

In more than three space-time dimensions, the situation is different; the loops $\gamma_{t-1}$ and $\gamma_t$ do not define a unique world sheet connecting them. Thus if we consider the intersection of one of the world-surfaces with a $t=$ const plane, we can get (i) a loop $\gamma_t$, (ii) an open surface connecting the loop $\gamma_t'$ and $\gamma_t$, and (iii) a closed spatial surface. The situation is completely analogous to the path integral of a free particle on a (2+1)-dimensional lattice, i.e., the world lines connecting points at different times intersected with a $t=$ const plane give (i) points or (ii) open paths connecting the intersecting point of the timelike link at $p_{t-1}$ with surface $t=$ const and the point $p_t$ and (iii) loops associated with different choices of the previously mentioned open paths.

Now we will show that Eq. (17) is really a possible expression of the path integral $Z_V$ in terms of loop variables.[4] First, it is easy to prove that the creation operator of these loops is just the creation operator of the loop representation, namely, the Wilson loop operator. Repeating the steps from Eq. (15) to Eq. (17) we get, for $\langle\hat{W}(\gamma_t)\rangle$,

$$\langle W(\gamma_t)\rangle=\frac{1}{Z}(2\pi\beta_V)^{-N_p/2}\sum_{\substack{n\\(\delta n=\gamma_t)}} \exp\left(-\frac{1}{2\beta_V}\langle n,n\rangle\right). \tag{18}$$

This is a sum over all surfaces bounded by the loop $\gamma_t$, i.e., over loop world sheets. Secondly, by means of the transfer matrix method, let us show that we reobtain the Hamiltonian (8) from Eq. (17). As we wish to consider the continuous time limit of the previous lattice Euclidean space-time theory, we introduce a different lattice spacing $a_0$ for the time direction. The couplings on timelike and spacelike plaquettes are no longer equal in the action, i.e., we have two coupling constants: $\beta_0$ and $\beta_s$. The temporal coupling constant $\beta_0$ decreases with $a_0$, whilst the spatial coupling constant $\beta_s$ increases with $a_0$. We wish to find an operator $\hat{T}$ over the Hilbert space of loops $\{|\gamma\rangle\}$ such that it is related with $Z_V^{\text{loops}}$ by

$$Z_V^{\text{loops}}=\sum_{\{\gamma\}}\prod_t \langle\gamma_{t+a_0}|\hat{T}|\gamma_t\rangle.$$

$\hat{T}$ is related, when $a_0$ is small, with the Hamiltonian $\hat{H}$ by

$$\hat{T}\propto e^{-a_0\hat{H}+O(a_0^2)}. \tag{19}$$

The transfer matrix between times $t$ and $t+a_0$ in the loop representation of kets $|\gamma\rangle$ can be written as

$$\langle\gamma_{t+a_0}|\hat{T}|\gamma_t\rangle=\sum_{\{\Delta S_{t+a_0}\}} \exp\left(\frac{1}{2\beta_0}\sum_{p_t} n_p^2-\frac{1}{2\beta_s}\sum_{p_s\in\Delta S_{t+a_0}} n_p^2\right), \tag{20}$$

where $p_t$ denotes temporal plaquettes, i.e., plaquettes with a couple of temporal links, $\Delta S_{t+a_0}=S_{t+a_0}-S_{t+a_0}'$, $S_{t+a_0}$, and $S_{t+a_0}'$ are the minimal surfaces enclosed by the loops $\gamma_{t+a_0}$ and $\gamma_{t+a_0}'$ (the loop $\gamma_t$ translated $a_0$ along the temporal direction), respectively. Due to the fact that the surfaces are closed, the integers $n_p$ of the temporal plaquettes which depart from the loop $\gamma_t$ at time $t$ and arrive to the loop $\gamma_{t+a_0}$ at time $t+a_0$ are equal to the number of times the spatial link

[4]Another possible path integral in terms of loops is given by Eq. (14).

$\ell_t$ appears in the loop $\gamma_t$, $N_\ell$. $\{\Delta S_{t+a_0}\}$ are all the possible modifications between the surfaces enclosed by the loops $\gamma_{t+a_0}$ and $\gamma'_{t+a_0}$, i.e., it denotes the set of possible configurations $\{n_p\}$ of integers attached to the plaquettes $p \in \Delta S_{t+a_0}$. Therefore the Eq. (20) can be rewritten as

$$\langle \gamma_{t+a_0}|\hat{T}|\gamma_t\rangle = \sum_{\{\Delta S_{t+a_0}\}} \exp\left( -\frac{1}{2\beta_0}\sum_{\ell} N_\ell^2(\gamma_t) - \frac{1}{2\beta_s}\sum_{p \in \Delta S_{t+a_0}} n_p^2 \right). \tag{21}$$

The kets $|\gamma_{t+a_0}\rangle$ and $|\gamma_t\rangle$ are connected by

$$|\gamma_{t+a_0}\rangle = \prod_{p \in \Delta S_{t+a_0}} \hat{W}_p^{n_p}|\gamma_t\rangle. \tag{22}$$

Using Eqs. (6), (7), (22), and (25), we get

$$\hat{T} = \sum_{\{\Delta S_{t+a_0}\}} \prod_{p \in \Delta S_{t+a_0}} \hat{W}_p^{n_p} \exp\left( -\frac{1}{2\beta_0}\sum_{\ell} \hat{E}_\ell^2 - \frac{1}{2\beta_s} n_p^2 \right). \tag{23}$$

To obtain a proper continuum limit, we should take

$$\beta_0 = \frac{a}{g^2 a_0}, \tag{24}$$

$$\beta_s = \frac{1}{2}\,\frac{1}{\ln(2g^2 a/a_0)}, \tag{25}$$

where $a$ continues to denote the spacelike spacing. This implies that for $a_0$ small, the operator $\hat{T}$ is given by

$$\hat{T} = \exp\left\{ -a_0\left[ \frac{g^2}{2a}\sum_{\ell} \hat{E}_\ell^2 + \frac{1}{2ag^2}\sum_p (\hat{W}_p + \hat{W}_p^\dagger) \right] + O(a_0^2) \right\}, \tag{26}$$

i.e., we recover the Hamiltonian (8). This confirms definitely that, Eq. (17) is the expression of the path integral of compact electrodynamics in terms of the world sheets of loops: the *loop* (Lagrangian) representation.

From Eq. (17) we can observe that the loop action is proportional to the *quadratic area* $A_2$:

$$S_L = -\frac{1}{\beta_V}A_2 = -\frac{1}{\beta_V}\sum_{p \in \mathcal{S}} n_p^2 = -\frac{1}{\beta_V}\langle n,n\rangle, \tag{27}$$

i.e., the sum of the squares of the multiplicities $n_p$ of plaquettes which constitute the loop's world sheet $\mathcal{S}$. It is interesting to note the similarity of this action with the continuous Nambu action or its lattice version, the Weingarten action [12] which are proportional to the area swept out by the bosonic string.[5]

[5]The relation between the surfaces of the Wilson action and those of Weingarten action has been analyzed by Kazakov *et al.* in Ref. [13].

## III. THE NON-ABELIAN LOOP ACTION

Let us see how the path integral loop description can be extended to the non-Abelian case of Yang-Mills theory. The path integral for the Wilson action for a general non-Abelian compact gauge group $G$ is given by

$$Z_W = \int [dU_\ell]\exp\left[ \beta\sum_p \mathrm{Re}(\mathrm{tr}U_p) \right], \tag{28}$$

where the $U_\ell \in G$ and $U_p = \Pi_{\ell \in p} U_\ell$. Equation (28) reduces to Eq. (9) for the case $G \equiv$ U(1). The analogous of the Fourier expansion for the non-Abelian case is the *character* expansion. The characters $\chi_r(U)$ of the irreducible (unitary) representation $r$ of dimension $d_r$, defined as the traces of these representations, are an orthonormal basis for the *class* functions of the group: i.e., [14],

$$\int dU \chi_r(U)\chi_s^*(U) = \delta_{rs}, \tag{29}$$

$$\sum_r d_r\chi_r(UV^{-1}) = \delta(U,V). \tag{30}$$

In particular, as a useful consequence we have

$$d_r\int dU\chi_s(U)\chi_r(UV^{-1}) = \delta_{rs}\chi_r(V). \tag{31}$$

By means of the character expansion, we can express

$$\exp\left\{ \beta\sum_p \mathrm{Re}[\chi(U_p)] \right\} = \prod_p \sum_r c_r\chi_r(U_p), \tag{32}$$

with

$$c_r = \int dU\chi_r^*(U)\exp(\beta\chi(U)). \tag{33}$$

For instance, in the case of $G \equiv$ SU(2) the gauge fields can be parametrized as

$$U = \cos\tfrac{1}{2}\theta + i\sigma_a n_a \sin\tfrac{1}{2}\theta, \quad 0 \leqslant \theta \leqslant 4\pi,$$

and the corresponding irreducible representations are classified by a non-negative integer or half-integer spin $j$, i.e., $r \equiv j$ and the characters are given by

$$\chi_j(U) = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \tag{34}$$

A direct application of Eq. (33) yields the $c_j$ in terms of modified Bessel functions, and, therefore, we can express Eq. (28) as

$$Z_W=\int [d\theta_\ell]\prod_p \left( \sum_{j_p} 2(2j_p+1) \times\frac{I_{2j_p+1}(\beta)}{\beta}\frac{\sin(j+\frac{1}{2})\theta_p}{\sin\frac{\theta_p}{2}}\right) =\sum_{\{j_p\}}\prod_p \left(2(2j_p+1)\frac{I_{2j_p+1}(\beta)}{\beta}\right)\int [dU_\ell]\prod_p \times\frac{\sin(j_p+\frac{1}{2})\theta_p}{\sin\frac{\theta_p}{2}}. \tag{35}$$

A given subset of plaquettes carrying $j_p\neq 0$ is homeomorphic to a simple surface if any link bounds at most two plaquettes of this subset. The links bounding exactly one plaquette make up the boundary of this surface (homeomorphic to a set of simple closed curves). Any configuration of plaquettes can be decomposed as a set of maximal simple surfaces by cutting it along the links bounding more than two plaquettes. In principle, there are two possibilities for the boundary curves: (a) either a true free boundary, bounding only one simple surface or (b) a singular branch line along which more than two simple surfaces meet. In fact, relation (29) forbids the existence of free boundaries for nontrivial configurations contributing to the path integral (the integration over the gauge group on the links belonging to the free boundaries gives a vanishing contribution).

The integration over the internal links of the simple surfaces is performed using Eq. (31). Note that the plaquettes of a simple surface component should carry the same group representation. After integrating over all the inner links of the simple components, one gets an expression involving only the links of the boundary, i.e., something proportional to

$$\prod_{\text{boundaries}} \chi_r(U_{\text{boundary}}).$$

What follows is the integration of gauge fields along the singular branches which gives rise to the Clebsch-Gordan coefficients coupling the different representations of the considered gauge group $G$. For instance, imagine that there is only one singular closed branch line which is the common boundary shared by $n$ simple surfaces with representations $r_1,r_2,\ldots,r_n$. The integration over gauge fields produces a factor $N_{r_1,\ldots,r_n}$, which counts the number of times the trivial representation is contained in the product $r_1\otimes r_2\otimes\cdots\otimes r_p$.

Cases in which different singular branch lines meet in a point are considered in Ref. [5]. Each point of intersection involves a Racah-Wigner symbol. Thus one can see that the Hamiltonian formulation associated to this action will be given in terms of a spin network of colored loops [15,16]. A rigorous proof of this fact using the transfer matrix technique is still required. This is not straightforward because one still needs to develop the lattice Hamiltonian formulation of Yang-Mills theory in terms of spin networks.

We also can generalize the Villain form of the action for any gauge group using the *heat kernel* action [17,18]:

$$\exp(\beta S_{\text{HK}})=\prod_p \sum_r d_r\chi_r(U_p)\exp[-C_r(2)/N\beta], \tag{36}$$

where $C_r(2)$ is the quadratic Casimir invariant for the representation $r$. For $G\equiv$ SU(2) the heat kernel action reads

$$\exp(\beta S_{HK})=\prod_p \sum_{j=0,1/2,\ldots}(2j+1)\frac{\sin(j+\frac{1}{2})\theta_p}{\sin\frac{\theta_p}{2}} \times\exp[-j(j+1)/2\beta], \tag{37}$$

while only integers values of $j$ are used for the SO(3) group.

## IV. CONCLUSIONS

As it was mentioned, the loop space provides a common scenario for a nonlocal description of gauge theories and quantum gravity. The loop approach is no more exclusively Hamiltonian; its Lagrangian counterpart is now available. A path integral action for the Yang-Mills theory in terms of loop variables is very valuable because it combines the geometrical transparency and economy of the loop description with the versatility to perform calculus. We have presented the state of the art in that program, which still is an open issue.

The path integral approach to quantum gravity has very appealing features. In particular, it may provide a more suitable framework for the development of useful approximation schemes for the study of black hole physics and it may allow us to analyze issues such as the computation of the probabilities for a change of the spatial topology that seem to be very difficult to formulate in the canonical approach. Even though the connection of the canonical loop representation of quantum gravity and the path integral approach is still an open problem, the determination of the explicit form of the loop actions in gauge theories is an important step in this direction. An important remark is that the lattice framework seems to be unavoidable in order to have well-defined loop actions. In fact, in spite of the similarities with the Nambu actions, the loop actions for gauge theories involve quadratic surface elements that are not well defined in the continuum.

Finally, concerning the lattice loop action as a computational tool, we mentioned already that the results produced by numerical simulations for different models are very encouraging. In Ref. [9], the loop action (27) corresponding to Villain form of U(1) model was considered. The extension of the Lagrangian *loop* description in such a way to include matter fields also was simulated [10]. The lattice path integral of U(1) Higgs model is expressed as a sum over closed as much as open surfaces. These surfaces correspond to world sheets of looplike pure electric flux excitations and open electric flux tubes carrying matter fields at their ends. This representation is connected by a duality transformation with the *topological* representation of the path integral (in terms of world sheets of Nielsen-Olesen strings [19] both closed and open connecting pairs of magnetic monopoles). Simulating numerically, the loop action corresponding to the Villain form, the two-coupling phase diagram of this model was mapped out. The gauge invariance of the loop description bears the advantages of economy in computational time and the absence of the strong metastability observed previously in the ordinary Monte Carlo analysis.

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