Radial propagators and Wilson loops

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We present a relation which connects the propagator in the radial (Fock-Schwinger) gauge with a gauge-invariant Wilson loop. It is closely related to the well-known field strength formula and can be used to calculate the radial gauge propagator. The result is shown to diverge in four-dimensional space even for free fields; its singular nature is, however, naturally explained using the renormalization properties of Wilson loops with cusps and self-intersections. Using this observation we provide a consistent regularization scheme to facilitate loop calculations. Finally, we compare our results with previous approaches to derive a propagator in Fock-Schwinger gauge. [S0556-2821(96)05922-X]

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I. INTRODUCTION

While perturbation theory for gauge fields formulated in covariant gauges is very well established [1] many aspects of noncovariant gauges are still under discussion. In principle, one expects physical quantities to be independent of the chosen gauge. However, this might lead to the naive conclusion that a quantum theory in an arbitrary gauge is simply obtained by inserting the respective gauge-fixing term and the appropriate Faddeev-Popov ghosts in the path integral representation and reading off the Feynman rules. Unfortunately, it is not so easy to obtain the correct Feynman rules, i.e., a set of rules yielding the same results for observable quantities as calculations in covariant gauges. Prominent examples are formulations in temporal and axial gauges. Such gauge choices are considered since one expects the Faddeev-Popov ghosts to decouple. However, problems even start with the determination of the appropriate free gauge propagators. Temporal and axial gauge choices yield propagators plagued by gauge poles in their momentum space representation. These are caused by the fact that such gauge conditions are insufficient to *completely* remove the gauge degrees of freedom. The correct treatment of such poles can cause ghost fields to reappear [2], can break translational invariance [3], or both [4]. While these problems seem to be ''restricted'' to the evaluation of the correct gauge propagators and ghost fields, the necessity of introducing even new multigluon vertices appears in the Coulomb gauge [5]. These additional vertices are due to operator-ordering problems which are difficult to handle in the familiar path integral approach. They give rise to anomalous interaction terms at the two-loop level [6] and cause still unsolved problems with renormalization at the three-loop level [7].

In this article we are interested in the radial (Fock-Schwinger) gauge condition

$$x_\mu A^\mu(x)=0. \tag{1.1}$$

It found widespread use in the context of QCD sum rules (e.g., [8]). There, it is used as being more or less synonymous with the important field strength formula

$$A_\mu^{\rm rad}(x)=\int_0^1 ds\, s x^\nu F_{\nu\mu}(sx) \tag{1.2}$$

which enormously simplifies the task of organizing the operator product expansion of QCD n-point functions in terms of gauge-invariant quantities by expressing the gauge potential via the gauge-covariant field strength tensor (concerning field strength formulas see also [9]). It was introduced long ago [10,11] and rediscovered several times (e.g., [12]).

Only few efforts have been made to establish perturbation theory for radial gauge. The main reason for this is that the gauge condition breaks translational invariance since the origin [in general, an arbitrary but fixed point z, cf. Eq. (1.6)] is singled out by the gauge condition. Thus, perturbation theory cannot be formulated in momentum space as usual but must be set up in coordinate space.

The first attempt to evaluate the free radial propagator was performed in [13]. Later, however, the function $\Gamma_{\mu\nu}(x,y)$ presented there was shown to be not symmetric [14]. Moreover, it could not be symmetrized by adding $\Gamma_{\nu\mu}(y,x)$ since the latter is not a solution of the free Dyson equation. By examining the general form of the homogeneous and inhomogeneous solutions of the equation of motion for the free radial propagator it was even suspected that it might be impossible to find a symmetric solution of this equation in four-dimensional space. In addition, it was shown in [14] that one obtains a singular expression when one uses the field strength formula to derive a free radial propagator. Indeed, we agree with this statement in principle, but we will present an explanation for this problem and a way to bypass it. Other approaches to define a radial gauge propagator try to circumvent the problem (e.g., [15]) by sacrificing the field strength formula as given in Eq. (1.2) which was one of the main reasons the gauge became popular in

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nonperturbative QCD sum-rule calculations [8] in the first place. If we are not prepared to do so we are forced to understand the origin of the divergences that plague most of the attempts to define even free propagators in radial gauges and see whether they can be dealt with in a satisfying manner.

In Sec. II we will make the first and decisive step in this direction by exploring the completeness of the gauge condition (1.1) and its relation to the field strength formula and developing a new representation of the gauge potentials via link operators.

In Sec. III we use this information to relate the divergences encountered in some of the attempts to define radial propagators to the renormalization properties of link operators. We find that even free propagators in radial gauge may feel remnants of the renormalization properties of closed, gauge-invariant Wilson loops. Surprising as this seems to be superficially, it is not impossible, however, if we recall that the inhomogeneous term in the gauge transformation has an explicit $1/g$ factor in it. As a result we are able to define a regularized radial propagator using the field strength formula and established regularization procedures for link operators.

Section IV will be devoted to demonstrate the consistency of our approach by calculating a closed Wilson loop using our propagator and relating the steps to the equivalent calculation in Feynman gauge.

In Sec. V we summarize and compare our results to other approaches in the literature and briefly discuss the next steps in the program of establishing a new perturbative framework in radial gauges which, although, the steps to be performed are quite straightforward, we will postpone for a future publication.

In the following we work in a D-dimensional Euclidean space. The vector potentials are given by

$$A_\mu(x) \equiv A_\mu^a(x) t_a , \tag{1.3}$$

where t_a denotes the generators of an SU(N) group in the fundamental representation obeying

$$[t_a, t_b] = i f_{abc} t^c \tag{1.4}$$

and

$$\mathrm{tr}(t_a t_b) = \frac{1}{2} \delta_{ab} . \tag{1.5}$$

In general, the radial gauge condition with respect to z reads

$$(x-z)_\mu A^\mu(x) = 0. \tag{1.6}$$

For simplicity, we will set $z=0$ in most expressions. Generalization to arbitrary z should be obvious at any rate.

II. THE GAUGE CONDITION REVISITED

Before we can go ahead and tackle the problem of divergences in the radial gauge propagator we have to establish a clearer picture of the situation at hand. Clearly, the many problems encountered in earlier attempts show that there are unexpected and yet unclarified features of the radial gauge problem. Surprising as this may be, to our knowledge there has been no thorough discussion of the one textbook question that will immediately arise when encountering infinities in the inversion of the *free* differential operator in a gauge theory: Is there still a zero eigenvalue of the differential operator in the space on which we try to perform the inversion?

This of course is simply questioning the completeness of the gauge condition one is about to implement. This point has initiated a yet unresolved debate in the case of axial gauges (e.g., [4]) but is only briefly mentioned in the context of radial gauges (e.g., [22]). We will see that the gauge condition (1.6) is *not* complete, at least if it is not supplemented by other constraints such as for instance, carefully implemented boundary conditions. This might explain part of the problems encountered in previous attempts. The methods used in Sec. II A. are suited to analyze this question, we have not done so in any detail since they also reveal that the field strength formula (1.2) is only valid under such more restricted circumstances and *does* correspond to a completely fixed gauge. The infinities encountered in applying Eq. (1.2) to the problem at hand as in [14], therefore, require a quite different explanation which will be the main focus of this paper. Section II B. will provide the key tool to reach this goal.

A. The field strength formula and complete gauge fixing

To clarify whether the gauge condition (1.6) is sufficient to completely fix the gauge degrees of freedom, we have to catalogue the gauge transformations $U[B](x)$ which transform an arbitrary vector potential B into the field A satisfying Eq. (1.6). A gauge condition is complete if $U[B](x)$ is uniquely determined up to a global gauge transformation. In other words, we want to find all solutions of

$$(x-z)_\mu U[B](x)\left[B^\mu(x) - \frac{1}{ig}\partial^\mu\right]U[B]^{-1}(x) = 0. \tag{2.1}$$

It is easily checked that we have an infinite family of such solutions which can all be cast in the form of a product of two gauge transformations of the form

$$U[B](x) = V(z(x))U[B](z(x),x). \tag{2.2}$$

Here,

$$U[B](z(x),x) = \mathcal{P}\exp\left(ig\int_x^{z(x)} d\omega_\mu B^\mu(\omega)\right) \tag{2.3}$$

is a link operator whose geometric ingredients are parametrized via its end points x and $z(x)$ and the straight line path ω between them, and $\mathcal{P}$ denotes path ordering.

In particular, $z(x)$ is the point where a straight line from z through x and a given closed hypersurface around z intersect. Since there is a unique relation between these points and the hypersurface we will also refer to the hypersurface itself by $z(x)$. This geometry is illustrated in Fig. 1. Both the detailed forms of $z(x)$ and the local gauge transformation $V(x)$ are completely unconstrained as long as $(x-z)\cdot\partial^x z(x) = 0$. In short, they parametrize the residual gauge freedom not eliminated by Eq. (1.1). Note that while $V(x)$ is completely arbitrary the solutions (2.2) ask only for its behavior at the given hypersurface $z(x)$. The simplest and most intuitive choice for $z(x)$ is a spherical hypersurface

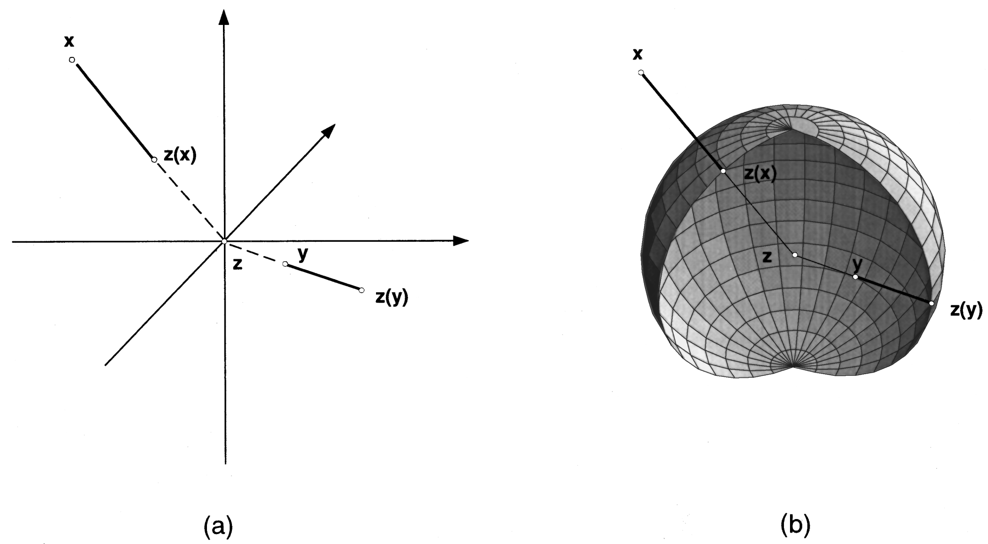

FIG. 1. (a) Straight-line path in the links for two points x and y. (b) Example of the same for a spherical hypersurface $z(x)$.

around z. Introducing the appropriate spherical coordinates, it becomes obvious that $V(z(x))$ parametrizes gauge transformations which purely depend on the angles. Clearly, the radial gauge condition (1.6) cannot fix the angular dependence of any gauge transformation in Eq. (2.1).

To eliminate the residual gauge freedom one has to impose a condition which is stronger than Eq. (1.6) and it suffices to pin down $V(x)$ up to a global transformation. A possible choice for such a gauge fixing would be the condition

$$\Box\left[\int_{z(x)}^{x} d\omega\cdot A(\omega)+\int d^4y\frac{1}{\Box}(z(x),y)\partial^y\cdot A(y)\right]\equiv 0 \tag{2.4}$$

which, in addition to the vanishing of the radial component of the gauge potential, also implements a covariant gauge on the hypersurface $z(x)$. Such a gauge for arbitrary $z(x)$ would immediately force us to introduce ghosts into the path integral. Moreover, the field strength formula would also be lost as we will illustrate below.

There is one exception to these unwanted modifications however, which may be implemented by contracting the closed surface $z(x)$ to the point z. Then the influence of $V(x)$ becomes degenerate with a global transformation and the gauge is completely fixed. Incidentally, this is also the only case which entails the field strength formula. To see this we use

$$\delta U(x,z)=ig\Bigg\{A_\mu(x)U(x,z)dx^\mu-U(x,z)A_\mu(z)dz^\mu - \int_0^1 ds[U(x,w_x)F_{\mu\nu}(w_x)U(w_x,z)] \times\frac{dw_x^\mu}{ds}\left(\frac{dw_x^\nu}{dx^\alpha}dx^\alpha+\frac{dw_x^\nu}{dz^\alpha}dz^\alpha\right)\Bigg\}, \tag{2.5}$$

(see, e.g., [16,17]) to differentiate the link operators in the gauge transformation (2.2) in order to find an expression for the radial gauge field:

$$\begin{aligned}A_\mu^{\rm rad}(x)&=U[A](z,x)\left[A_\mu(x)-\frac{1}{ig}\partial_\mu^x\right]U[A](x,z)\\&=\int_0^1 dss\frac{d\omega^\nu}{ds}U[A](z,\omega)F_{\nu\mu}(\omega)U[A](\omega,z)\\&=\int_0^1 dss\frac{d\omega^\nu}{ds}F_{\nu\mu}^{\rm rad}(\omega).\end{aligned} \tag{2.6}$$

This is nothing but Eq. (1.2) for arbitrary z [note that in this case $\omega=\omega(s)$ is simply given by $\omega(s)=z+(x-z)s$.] This simple result is only true since $\partial_\mu z(x)\equiv\partial_\mu z=0$. For general $z(x)$ there would be an additional term in the above formula reflecting the residual gauge freedom encoded in $V(z(x))$.

This sets the stage for a further exploration of the radial gauge in a context where we can be sure of having completely fixed the gauge in such a way that the field strength formula is guaranteed to be valid. Before we go on to study the consequences the above has for the implementation of propagators, we will introduce yet another representation of the gauge field in this particular complete radial gauge, this time solely in terms of link operators.

B. Representing the gauge potential via link operators

From now on we will assume the reference point z to be the origin, but it will always be straightforward to recover the general case without any ambiguities. We will also suppress the explicit functional dependence of link operators on the gauge potential A for brevity.

Let us start with a link operator along a straight line path

$$U(x,x')=\mathcal{P}\exp\left[ig\int_0^1 d\omega_\mu A^\mu(\omega)\right], \tag{2.7}$$

where now $\omega(s):=x'+s(x-x')$. According to Eq. (2.5) we have

$$\partial_\mu^x U(x,x') = ig\left[A_\mu(x) - \int_0^1 dss\frac{d\omega^\nu}{ds}U(x,\omega)F_{\nu\mu}(\omega)U(\omega,x)\right]U(x,x') \tag{2.8}$$

which can be used to express the vector potential in terms of the link operator

$$\lim_{x'\to x}\partial_\mu^x U(x,x') = igA_\mu(x). \tag{2.9}$$

In the case at hand the fact that $U(0,x)=1$ in any of the $x\cdot A(x)=0$ gauges allows us to introduce a new gauge-covariant representation

$$A_\mu^{\rm rad}(x) = \frac{1}{ig}\lim_{x'\to x}\partial_\mu^x[U(0,x)U(x,x')U(x',0)] \tag{2.10}$$

for the Fock-Schwinger gauge field. It is easy to see that this is indeed equivalent to the field strength formula as given in Eq. (1.2) and consequently, satisfies the same complete gauge fixing condition [i.e., Eq. (2.4) for $z(x)\to z$]:

$$\begin{aligned}A_\mu^{\rm rad}(x) &= \frac{1}{ig}\lim_{x'\to x}\partial_\mu^x[U(0,x)U(x,x')U(x',0)]\\ &= \frac{1}{ig}\lim_{x'\to x}[\partial_\mu^x U(0,x)U(x,x')U(x',0) + U(0,x)\partial_\mu^x U(x,x')U(x',0)]\\ &= \frac{1}{ig}\partial_\mu^x U(0,x)U(x,0) + U(0,x)A_\mu(x)U(x,0)\\ &= \int_0^1 dssx^\nu F_{\nu\mu}^{\rm rad}(sx),\end{aligned} \tag{2.11}$$

where the last step uses Eq. (2.8), mirroring the relations in Eq. (2.6) for $z=0$.

III. THE RADIAL GAUGE PROPAGATOR

With the type of radial gauge we are interested in uniquely specified and the corresponding representations for the gauge field derived above, it is now straightforward to devise expressions for the propagator as a two-point function. According to the above we know that

$$\begin{aligned}\langle A_\mu(x)\otimes A_\nu(y)\rangle_{\rm rad} &= \lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y\langle U(0,x)U(x,x')U(x',0)\otimes U(0,y)U(y,y')U(y',0)\rangle\\ &= \int_0^1 ds\int_0^1 dtsx^\alpha ty^\beta\langle U(0,sx)F_{\alpha\mu}(sx)\times U(sx,0)\otimes U(0,ty)F_{\beta\nu}(ty)U(ty,0)\rangle.\end{aligned} \tag{3.1}$$

Since we are in a fixed gauge it makes sense to perform a multiplet decomposition and, for instance, extract the singlet part of this propagator. The latter reduces to the free propagator in the limit $g\to 0$.

We define

$$\mathrm{tr}\,\langle A_\mu(x)A_\nu(y)\rangle = \mathrm{tr}(t_a t_b)\underbrace{\langle A_\mu^a(x)A_\nu^b(y)\rangle^{\rm singlet}}_{=:\,\delta^{ab}D_{\mu\nu}(x,y)} = \frac{N^2-1}{2}D_{\mu\nu}(x,y) \tag{3.2}$$

to extract

$$\begin{aligned}\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\rm rad}^{\rm singlet} &= \delta^{ab}\frac{2}{N^2-1}\mathrm{tr}\langle A_\mu(x)A_\nu(y)\rangle_{\rm rad}\\ &= \delta^{ab}\frac{2}{N^2-1}\frac{1}{(ig)^2}\lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y\mathrm{tr}\langle U(0,x)\times U(x,x')U(x',0)U(0,y)U(y,y')\times U(y',0)\rangle.\end{aligned} \tag{3.3}$$

Obviously,

$$W_1(x,x',y,y') := \frac{1}{N}\mathrm{tr}\langle U(0,x)U(x,x')U(x',0)U(0,y)\times U(y,y')U(y',0)\rangle \tag{3.4}$$

is a gauge-invariant Wilson loop. Its geometry is illustrated in Fig. 2.

On the other hand, using the second expression in Eq. (3.1) we have an equivalent representation for the singlet part of radial gauge propagator via the field strength formula:

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\rm rad}^{\rm singlet} = \delta^{ab}\frac{2}{N^2-1}\int_0^1 ds\int_0^1 dtsx^\alpha ty^\beta\times\mathrm{tr}\langle U(0,sx)F_{\alpha\mu}(sx)U(sx,0)U(0,ty)\times F_{\beta\nu}(ty)U(ty,0)\rangle. \tag{3.5}$$

Modanese [14] tried to calculate the free radial gauge propagator from Eq. (3.5) in a D-dimensional space-time.[1] Unfortunately, one gets a result which diverges in the limit $D\to 4$.

Let us briefly recapitulate how this divergence makes its appearance: Since the right-hand side of Eq. (3.5) is gauge invariant we can choose an arbitrary gauge to calculate it. For simplicity, we take the Feynman gauge with its free propagator

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\rm Feyn}=\delta^{ab}D_{\mu\nu}^{\rm Feyn}(x,y)$$
$$=-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}g_{\mu\nu}\delta^{ab}[(x-y)^2]^{1-D/2}. \tag{3.6}$$

Using the free field relations $U(a,b)=1$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, we get (for more details see Appendix A)

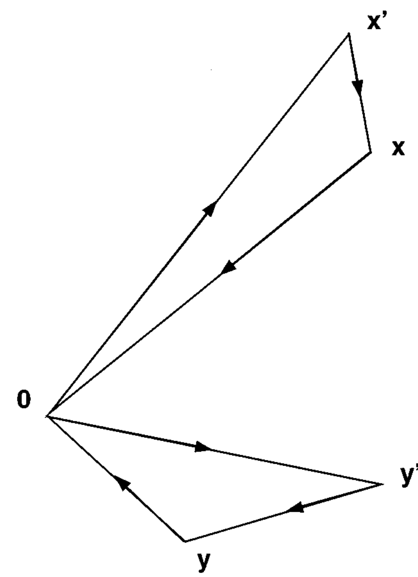

FIG. 2. The Wilson loop W_1 which is intimately connected with the radial gauge propagator according to Eqs. (3.3) and (3.4).

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\rm rad}^0$$
$$=-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\delta^{ab}\int_0^1 ds\int_0^1 dt\, sx^\alpha\, ty^\beta\left(g_{\mu\nu}\partial_\alpha^{sx}\partial_\beta^{ty}+g_{\alpha\beta}\partial_\mu^{sx}\partial_\nu^{ty}-g_{\alpha\nu}\partial_\mu^{sx}\partial_\beta^{ty}-g_{\mu\beta}\partial_\alpha^{sx}\partial_\nu^{ty}\right)[(sx-ty)^2]^{1-D/2}$$
$$=-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\delta^{ab}\left\{g_{\mu\nu}[(x-y)^2]^{1-D/2}-\partial_\mu^x\int_0^1 ds\, x_\nu\,[(sx-y)^2]^{1-D/2}-\partial_\nu^y\int_0^1 dt\, y_\mu\,[(x-ty)^2]^{1-D/2}\right.$$
$$\left.+\partial_\mu^x\partial_\nu^y\underbrace{\int_0^1 ds\int_0^1 dt\, x\cdot y\,[(sx-ty)^2]^{1-D/2}}_{\sim\frac{1}{4-D}}\right\}. \tag{3.7}$$

In the last term we encounter an ultraviolet divergence at the lower limits of the parameter integrals where $sx-ty$ vanishes for arbitrary pairs of x and y, with one remarkable exception: this whole term does not contribute if either $x=0$ or $y=0$. The latter being simply a consequence of our attempt to preserve the field strength formula (1.2) which forces the vector field to vanish at the origin (in general, at the reference point z). Careful analysis reveals another, mathematically distinct, type of singularity that emerges when x and y are aligned with respect to the origin. This singularity, appearing for special combinations of x and y only, is however, not as striking as the one observed above which is present for (nearly) arbitrary arguments. We stress that also this singularity is regularized by the techniques presented below and can be dealt with exactly the same way as the ones we will focus on.[2] Since one can tune which singularities are present simply by choosing appropriate external points, we will refrain from explicitly discussing this special case below in order to keep the argument compact and easier to follow. In summary, we will come to the conclusion that this attitude can also be maintained for more general n-point functions and in higher order perturbation theory.

The observation that the radial gauge propagator as calculated here diverges in four-dimensional space raises the question, whether it is perhaps impossible to formulate a quantum theory in radial gauge. This would suggest that the radial gauge condition, in the form that facilitates the field strength formula, is inherently inconsistent (''unphysical'')

[1]In fact, he discussed the Abelian case but this makes no difference for free fields.

[2]Indeed, one encounters singularities for special combinations of arguments in many gauges, not only in the radial gauge. The temporal component of the Coulomb gauge propagator might serve as an example. It is given by [6] $D_{00}^{\rm Coulomb}(x,y)=\int[d^4k/(2\pi)^4]e^{ik\cdot(x-y)}(1/\vec{k}^2)$ which is obviously singular for $x_0=y_0$.

in contrast with the general belief that it is ''very physical'' since it allows one to express gauge variant quantities such as the vector potential in terms of gauge-invariant ones. To answer this question we have to understand from where this divergence comes from. In the following we will see that for this purpose the complicated-looking Wilson loop representation (3.3) is much more useful than the field strength formula (3.5). [Note, however, that the result for the free propagator (3.7) of course will be the same.]

It is well known that Wilson loops need renormalization to make them well defined (see, e.g., [18] and references therein). The expansion of an arbitrary Wilson loop

$$W(C)=\frac{1}{N}\mathrm{tr}\langle\mathcal{P}\exp[ig\oint_C dx^\mu A_\mu(x)]\rangle \tag{3.8}$$

in powers of the coupling constant is given by

$$W(C)=1+\frac{1}{N}\sum_{n=2}^{\infty}(ig)^n\oint_C dx_1^{\mu_1}\ldots\oint_C dx_n^{\mu_n} \times\Theta_C(x_1>\cdots>x_n)\mathrm{tr}G_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n), \tag{3.9}$$

where $\Theta_C(x_1>\cdots>x_n)$ orders the points $x_1,\ldots,x_n$ along the contour C and

$$G_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n):=\langle A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\rangle \tag{3.10}$$

are the Green functions.

In general, Wilson loops show ultraviolet singularities in any order of the coupling constant. If the contour C is smooth (i.e., differentiable) and simple (i.e., without self-intersections), the conventional charge and wave-function renormalization, denoted by $\mathcal{R}$ in the following, is sufficient to make $W(C)$ finite. We refer the reader to [19] for more details about renormalization of regular (smooth and simple) loops.

However, new divergences appear if the contour C has cusps or self-intersections. The renormalization properties of such loops are discussed in [20,21]. While the singularities of regular loops appear at the two-loop level [order g^4 in Eq. (3.9)] cusps and cross points cause divergences even in leading (nontrivial) order g^2.

Since W_1 as given in Eq. (3.4) is indeed plagued by cusps and self-intersections, a second renormalization operation must be carried out to get a renormalized expression W_1^R from the bare one W_1. This observation has an important consequence for our radial gauge propagator as given in Eq. (3.3): Even the free propagator needs renormalization. This provides a natural explanation for the fact that a naive calculation of this object yields an ultraviolet divergent result [14]. Note that the usual divergences of Wilson loops which are removed by $\mathcal{R}$, such as e.g., vertex divergences, appear at $O(g^4)$ and thus do not contribute to the free part of the radial gauge propagator, while the cusp singularities indeed contribute since they appear at $o(g^2)$ and affect the free field case due to the factor $1/g^2$ in Eq. (3.3).

Now we are able to answer the question whether the radial gauge is ''unphysical'' or ''very physical.'' It is just its intimate relation to physical, i.e., gauge-invariant, quantities which makes the gauge propagator, even the free one, divergent. One might cast the answer in the following form: *The propagator diverges because of, and not contrary to, the fact that the radial gauge is ''very physical.''*

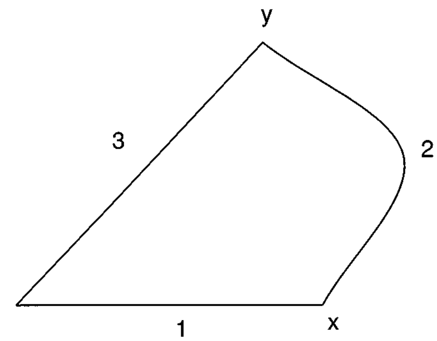

FIG. 3. A Wilson loop with two straight-line parts.

Consequently, the next questions are as follows: Is there any use for a divergent expression for the free propagator? Especially: Can we use it to perform (dimensionally regularized) loop calculations? Can one find a renormalization program which yields a finite radial gauge propagator?

In the next section we will answer these questions. First, we will perform a one-loop calculation of a Wilson loop using the radial gauge propagator (3.7) and compare the dimensionally regularized result with a calculation in Feynman gauge.

The answer to the second question turns out to be surprising. Indeed, the renormalization program developed for Wilson loops can be extended to the radial gauge propagator yielding a finite expression for the latter. Naively, one would expect this expression to be the correct input as the free propagator for perturbative calculations. However, as we shall see in the next section there is no correct *and* finite version of a free radial gauge propagator for the purpose of Feynman rules. The divergence in four-dimensional space turns out to be mandatory to perform loop calculations. For the sake of completeness we have nevertheless worked out a renormalization scheme for the propagator in Appendix B. It is shown there that the singularity is indeed removed by renormalizing the appropriate Wilson loop. In addition, the properties of the finite ''propagator'' are contrasted with the regularized version.

IV. CALCULATING A WILSON LOOP IN RADIAL GAUGE

We choose the path

$$\ell:z(\sigma)=\begin{cases}\sigma x & , \quad \sigma\in[0,1], x\in\mathbb{R}^D,\\ w(\sigma-1) & , \quad \sigma\in[1,2], w(0)=x, w(1)=y,\\ (3-\sigma)y & , \quad \sigma\in[2,3], y\in\mathbb{R}^D.\end{cases} \tag{4.1}$$

It is shown in Fig. 3. The line $w(\sigma-1)$ is supposed to be an arbitrary curve connecting x and y.

First, we will perform the calculation of this Wilson loop in Feynman gauge. Using Eq. (3.6) we get, in leading order of the coupling constant,

$$W(\ell)=\frac{1}{N}\,\mathrm{tr}\left\langle \mathcal{P}\exp\left[ig\oint_{\ell}dz^{\mu}A_{\mu}(z)\right]\right\rangle \approx 1+(ig)^2\frac{N^2-1}{2N}\int_0^3 d\sigma\int_0^3 d\tau\,\Theta(\sigma-\tau)\,\dot z^{\mu}(\sigma)\,\dot z^{\nu}(\tau)\,D^{\text{Feyn}}_{\mu\nu}\big(z(\sigma),z(\tau)\big)$$
$$=1+(ig)^2\frac{N^2-1}{2N}\frac{1}{2}\underbrace{\int_0^3 d\sigma\int_0^3 d\tau\,\dot z^{\mu}(\sigma)\,\dot z^{\nu}(\tau)\,D^{\text{Feyn}}_{\mu\nu}\big(z(\sigma),z(\tau)\big)}_{=:\,I_f} \tag{4.2}$$

To get rid of the Θ function we have exploited the symmetry property of two-point Green functions

$$D^{\text{Feyn}}_{\mu\nu}(x,y)=D^{\text{Feyn}}_{\nu\mu}(y,x). \tag{4.3}$$

Decomposing the contour ℓ according to Eq. (4.1) we find that the Feynman propagator in Eq. (4.2) connects each segment of ℓ with itself and with all the other segments. Thus, I_f is given by

$$I_f=\sum_{A=1}^{3}\sum_{B=1}^{3}(A,B), \tag{4.4}$$

where (A,B) denotes the contribution with propagators connecting loop segments A and B (cf. Fig. 3), e.g.,

$$(1,2)=\int_0^1 d\sigma\int_0^1 d\tau x^{\mu}\dot w^{\nu}(\tau)D^{\text{Feyn}}_{\mu\nu}(\sigma x,w(\tau))-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\int_0^1 d\sigma\int_0^1 d\tau x^{\mu}$$
$$\times\dot w_{\mu}(\tau)[(\sigma x-w(\tau))^2]^{1-D/2}. \tag{4.5}$$

Next, we will evaluate the same Wilson loop in radial gauge. Clearly, the first and the third part of the path do not contribute if the radial gauge condition $x_{\mu}A^{\mu}(x)=0$ holds. We insert the free propagator

$$\langle A^a_{\mu}(x)A^b_{\nu}(y)\rangle^0_{\text{rad}}=:\delta^{ab}D^0_{\mu\nu}(x,y) \tag{4.6}$$

from Eq. (3.7) into

$$W(\ell)=\frac{1}{N}\,\mathrm{tr}\left\langle \mathcal{P}\exp\left[ig\int_0^1 d\sigma\,\dot w_{\mu}(\sigma)A^{\mu}(w(\sigma))\right]\right\rangle$$
$$\approx 1+(ig)^2\frac{N^2-1}{2N}$$
$$\times\frac{1}{2}\underbrace{\int_0^1 d\sigma\int_0^1 d\tau\,\dot w^{\mu}(\sigma)\,\dot w^{\nu}(\tau)\,D^0_{\mu\nu}(w(\sigma),w(\tau))}_{=:\,I_r} \tag{4.7}$$

and observe that

$$\dot w_{\mu}(\sigma)\partial^{\mu}_{w(\sigma)}=\frac{d}{d\sigma}. \tag{4.8}$$

Thus, the integral in Eq. (4.7) reduces to

$$I_r = -\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\Bigg[\int_0^1 d\sigma\int_0^1 d\tau \dot{w}_\mu(\sigma)\dot{w}^\mu(\tau)[(w(\sigma)-w(\tau))^2]^{1-D/2}+\int_0^1 ds\int_0^1 dt(w_\mu(1)w^\mu(1)\{(sw(1)-tw(1))^2\}^{1-D/2}$$
$$+w_\mu(0)w^\mu(0)\{[sw(0)-tw(0)]^2\}^{1-D/2}-w_\mu(1)w^\mu(0)\{[sw(1)-tw(0)]^2\}^{1-D/2}-w_\mu(0)w^\mu(1)\{[sw(0)$$
$$-tw(1)]^2\}^{1-D/2})-\int_0^1 ds\int_0^1 d\tau \dot{w}_\mu(\tau)(w^\mu(1)\{[sw(1)-w(\tau)]^2\}^{1-D/2}-w^\mu(0)\{[sw(0)-w(\tau)]^2\}^{1-D/2})$$
$$-\int_0^1 dt\int_0^1 d\sigma \dot{w}_\mu(\sigma)(w^\mu(1)\{[w(\sigma)-tw(1)]^2\}^{1-D/2}-w^\mu(0)\{[w(\sigma)-tw(0)]^2\}^{1-D/2})\Bigg]$$
$$=(2,2)+(3,3)+(1,1)+(3,1)+(1,3)+(3,2)+(1,2)+(2,3)+(2,1). \tag{4.9}$$

A careful analysis of Eq. (4.9) shows that it exactly coincides with the Feynman gauge calculation. This is expressed in the last line where we have denoted which parts of the loop are connected by the Feynman gauge propagator to reproduce Eq. (4.9) term by term. Thus, using the radial gauge propagator as given in Eq. (3.7) yields the same result as the calculation in Feynman gauge. Finally, this regularized expression has to be renormalized. This can be performed without any problems according to [20]. Since we are not interested in the Wilson loop itself but in the comparison of the results obtained in radial and Feynman gauge, we will not calculate the renormalized expression for $W(\mathscr{C})$.

However, a qualitative discussion of the renormalization properties of $W(\mathscr{C})$ is illuminating. By construction $W(\mathscr{C})$ has at least a cusp at the origin. (Other cusps are possible at x or y or along the line parametrized by w, but are not important for our considerations.) To give the right behavior of the Wilson loop the calculation of $W(\mathscr{C})$ in an arbitrary gauge must reproduce the cusp singularity. Usually, the parameter integrals in the vicinity of the cusp do the job. For gauge choices where the propagator does not vanish in the vicinity of the origin, this is automatically achieved. Let us assume for a moment that it is possible to construct a *finite* radial gauge propagator obeying the field strength formula (1.2) while, due to the gauge condition, having trivial gauge factors along radial lines. Of course, this is nothing but saying that there are no contributions from parts 1 and 3 of the loop, i.e., in the vicinity of the origin. Since the propagator is assumed to be finite, there are no singular integrals corresponding to the cusp at the origin. Thus a finite radial gauge propagator cannot reproduce the correct behavior of the Wilson loop. In turn we conclude that *a singular radial gauge propagator is mandatory* to get the right renormalization properties of Wilson loops.

V. SUMMARY AND OUTLOOK

In this article we have shown how to calculate the radial gauge propagator in a D-dimensional space using Wilson loops. As discovered in [14] the free propagator diverges in four-dimensional space. We were able to explain this singular behavior by studying the properties of associated Wilson loops. Furthermore, we have shown that the free propagator, in spite of being divergent in four dimensions, can be used for perturbative calculations in a (dimensionally) regularized framework and that the result for a gauge-invariant quantity agrees with the calculation in Feynman gauge. Finally, we have pointed out that any version of the radial propagator which is finite in four-dimensional space at least cannot reproduce the correct renormalization properties of Wilson loops with cusps at the reference point z.

It is instructive to compare the radial gauge propagators as presented here with other approaches: As discussed in Sec. II the radial gauge condition (1.1) does not completely fix the gauge degrees of freedom. Thus, the field strength formula

$$A_\mu(x)=\int_0^1 ds s x^\nu F_{\nu\mu}(sx) \tag{5.1}$$

is not the only solution of the system of equations[3]

$$x_\mu A^\mu(x)=0, \quad F_{\mu\nu}(x)=\partial_\mu^x A_\nu(x)-\partial_\nu^x A_\mu(x). \tag{5.2}$$

One might add a function [14]

$$A_\mu^0(x)=\partial_\mu^x f(x) \tag{5.3}$$

to Eq. (5.1) where f is an arbitrary homogeneous function of degree 0. However, any $A_\mu^0(x)$ added in order to modify Eq. (5.1) is necessarily singular at the origin. Hence, regularity at the origin may be used as a uniqueness condition [12]. If we relax this boundary condition other solutions are possible, e.g.,

$$\overline{A_\mu}(x)=-\int_1^\infty ds s x^\nu F_{\nu\mu}(sx), \tag{5.4}$$

where we must assume that the field strength vanishes at infinity. While Eq. (5.1) is the only solution which is regular at the origin, Eq. (5.4) is regular at infinity. Ignoring boundary conditions for the moment one can construct a radial gauge propagator by [15]

$$\frac{1}{2}[G_{\mu\nu}(x,y)+G_{\nu\mu}(y,x)] \tag{5.5}$$

with

[3] For simplicity, we discuss the QED case here. Aiming at an expression for the free gauge propagator, this is no restriction of generality. For non-Abelian gauge groups cf. [22].

$$G_{\mu\nu}(x,y):=-\int_0^1 ds s x^{\alpha}\int_1^{\infty} dt t y^{\beta}\langle F_{\alpha\mu}(sx)F_{\beta\nu}(ty)\rangle. \tag{5.6}$$

It turns out that this propagator is finite in four dimensions. However, the price one has to pay is that boundary conditions are ignored and thus the object ''lives'' in the restricted space $\mathbb{R}^4\backslash\{0\}$ and not in $\mathbb{R}^4$ anymore. In our approach we insist on the field strength formula (5.1) widely used in operator product expansions [8] and on the regular behavior of vector potentials at the origin [12]. One might use the propagator (5.5) to calculate the g^2 contribution to the Wilson loop on the contour (4.1). It is easy to check that the result differs from those obtained in Eqs. (4.7) and (4.9). Clearly, this is due to the fact that Eq. (5.5) is ill defined at the origin. Especially, it is shown in Appendix C that there appears no divergence reflecting the cusp at the origin if one uses Eq. (5.5) instead of Eq. (3.7) for the calculation of the Wilson loop. This is an explicit example for our general statement that any finite version of the radial propagator cannot correctly reproduce cusp singularities at the origin.

In the above, all calculations were performed in Euclidean space. In Minkowski space Wilson loops show additional divergences if part of the contour coincides with the light cone [23]. Thus, we expect the appearance of new singularities also for the radial propagator, at least if one or both of its arguments are lightlike. Further investigation is required to work out the properties of the radial gauge propagator in Minkowski space.

To formulate perturbation theory in a specific gauge the knowledge of the correct free propagator is only the first step. In addition, one has to check the decoupling of Faddeev-Popov ghosts in radial gauge which is suggested by the algebraic nature of the gauge condition. However, the still continuing discussion about temporal and axial gauges might serve as a warning that the decoupling of ghosts for algebraic gauge conditions is far from being trivial (cf. [2,4] and references therein). To prove (or disprove) the decoupling of ghosts in radial gauge we expect that our Wilson loop representation of the propagator is of great advantage since it yields the possibility to calculate higher loop contributions in two distinct ways: On the one hand, one might use the Wilson loop representation to calculate the full radial propagator up to an arbitrary order in the coupling constant. The appropriate Wilson loop can be calculated in any gauge, e.g., in a covariant gauge. On the other hand, the radial propagator might be calculated according to Feynman rules. Since the results should coincide this might serve as a check for the validity and completeness of a set of radial gauge Feynman rules.

Indeed, the above philosophy allows for more than just algebraically checking the correctness of a particular calculation. It can be used to predict properties of perturbative quantities. Let us end with just one more example of how powerful these arguments can be by coming back to the one type of singularity we have put aside in our discussion up to now: the singularities that are present only if both end points and the radial reference point z are in a line. Since they constitute only a special case in the above procedures, there was no need to explicitly address them here. When we are going to do loop integrals however, there will be one issue one may worry about even in the case of ''safe'' combinations of external points: What happens in loop integrals if one has to integrate over such combinations of internal points? Here again, comparing to the Wilson loop representation combined with covariant gauge Feynman rules immediately tells us that it is only the location of external points that governs the types of link-related singularities. Hence, all the potentially worrisome internal points must have canceling contributions in the regularized expression.

All this of course illustrates clearly how much is still left to be done but it also shows the wealth of tools and cross-checks available on the way to a new perturbation theory in a radial gauge that respects the field strength formula.

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APPENDIX A: DERIVATION OF THE FREE RADIAL PROPAGATOR

The free radial propagator derived from the field strength formula shows a divergence in $D=4$, as already indicated in Sec. III, Eq. (3.7). Here, we give the details of the algebra leading to this conclusion.

The following relations summarize the steps carried out in the calculation below:

$$x_\mu \partial_x^\mu = |x|\partial_{|x|}, \tag{A1}$$

$$\begin{aligned} T_{\mu\nu}(x,y):=&x^\alpha y^\beta(g_{\mu\nu}\partial_\alpha^x\partial_\beta^y+g_{\alpha\beta}\partial_\mu^x\partial_\nu^y-g_{\alpha\nu}\partial_\mu^x\partial_\beta^y\\ &-g_{\mu\beta}\partial_\alpha^x\partial_\nu^y)=g_{\mu\nu}\partial_{|x|}\partial_{|y|}|x||y|-\partial_\mu^x x_\nu\partial_{|y|}|y|\\ &-\partial_\nu^y y_\mu\partial_{|x|}|x|+\partial_\mu^x\partial_\nu^y x\cdot y. \end{aligned} \tag{A2}$$

Introducing $\hat{x}:=x/|x|$ and $u=s|x|$, we have for arbitrary f:

$$s x_\alpha \partial_\beta^{sx} f(sx)=x_\alpha\partial_\beta^x f(sx), \tag{A3}$$

$$\begin{aligned} \partial_{|x|}\int_0^1 ds|x|f(sx)&=\partial_{|x|}\int_0^1 ds|x|f(s|x|\hat{x})=\partial_{|x|}\int_0^{|x|}du f(u\hat{x})\\ &=f(|x|\hat{x})=f(x). \end{aligned} \tag{A4}$$

We get

$$
\begin{aligned}
&-\frac{4\pi^{D/2}}{\Gamma(D/2-1)}\langle A^a_\mu(x)A^b_\nu(y)\rangle^0_{\rm rad}\\
&=\delta^{ab}\int_0^1 ds\int_0^1 dt\, T_{\mu\nu}(sx,ty)\,[(sx-ty)^2]^{1-D/2}\\
&=\delta^{ab}\,T_{\mu\nu}(x,y)\int_0^1 ds\int_0^1 dt\,[(sx-ty)^2]^{1-D/2}\\
&=\delta^{ab}\left(g_{\mu\nu}\partial_{|x|}\partial_{|y|}|x|\,|y|-\partial^x_\mu\,x_\nu\,\partial_{|y|}|y|-\partial^y_\nu\,y_\mu\,\partial_{|x|}|x|+\partial^x_\mu\partial^y_\nu\,x\cdot y\right)\int_0^1 ds\int_0^1 dt\,[(sx-ty)^2]^{1-D/2}\\
&=\delta^{ab}\Bigg(g_{\mu\nu}[(x-y)^2]^{1-D/2}-\partial^x_\mu\int_0^1 ds\,x_\nu\,[(sx-y)^2]^{1-D/2}-\partial^y_\nu\int_0^1 dt\,y_\mu\,[(x-ty)^2]^{1-D/2}\\
&\qquad+\partial^x_\mu\partial^y_\nu\underbrace{\int_0^1 ds\int_0^1 dt\,x\cdot y\,[(sx-ty)^2]^{1-D/2}}_{\sim\frac{1}{4-D}}\Bigg).
\end{aligned}
\tag{A5}
$$

The divergent part of the double integral in the last line can be found in Appendix D. At the moment, however, the exact form of the divergence is not important.

APPENDIX B: RENORMALIZATION PROGRAM FOR THE FREE PROPAGATOR

Here, we shall use the renormalization program developed for Wilson loops with cusps and self-intersections [20,21] to obtain a finite expression for the radial gauge propagator. This proves that the overall singularity of the propagator is indeed caused by the cusps of the Wilson loop (3.4). However, as we have argued above this finite expression cannot be used as an input for perturbation theory. In the following we concentrate on the free radial gauge propagator. The generalization to higher orders in perturbation theory is straightforward.

As a first step, we must apply the renormalization operation $\mathcal{R}$ to W_1 as given in Eq. (3.4). This yields

$$\widetilde{W}_1(x,x',y,y';g_R,\mu,D)=\mathcal{R}W_1(x,x',y,y';g,D), \tag{B1}$$

where $W_1(x,x',y,y';g,D)$ is a regularized expression calculated in D dimensions and μ is a subtraction point introduced by the renormalization procedure $\mathcal{R}$. As mentioned above this serves to perform the usual coupling constant and wave-function renormalization. For the purpose of the present work the only important relation is

$$g_R=\mu^{(D-4)/2}g+o(g^3). \tag{B2}$$

In a second step the cusps and self-intersections must be renormalized. According to [20] each cusp is multiplicatively renormalizable with a renormalization factor Z depending on the cusp angle. In our case we have four cusps with angles

$$\alpha:=\angle(x-x',-x), \tag{B3}$$

$$\alpha':=\angle(x',x-x'), \tag{B4}$$

$$\beta:=\angle(y-y',-y), \tag{B5}$$

$$\beta':=\angle(y',y-y'). \tag{B6}$$

The cross point at the origin introduces a mixing between W_1 and

$$W_2(x,x',y,y'):=\left\langle\frac{1}{N}\mathrm{tr}[U(0,x)U(x,x')U(x',0)]\frac{1}{N}\times\mathrm{tr}[U(0,y)U(y,y')U(y',0)]\right\rangle. \tag{B7}$$

Again, the divergences appearing here are functions of the angles

$$\left.\begin{array}{lcl}\gamma_{xx'}&:=&\angle(-x,x')\\ \gamma_{yy'}&:=&\angle(-y,y')\\ \gamma_{xy}&:=&\angle(-x,-y)\\ \gamma_{x'y'}&:=&\angle(x',y')\\ \gamma_{x'y}&:=&\angle(x',-y)\\ \gamma_{xy'}&:=&\angle(-x,y')\end{array}\right\}\vec{\gamma}. \tag{B8}$$

The renormalized Wilson loop W_1^R is given by

$$
\begin{aligned}
&W_1^R(x,x',y,y';g_R,\mu,\overline{C_\alpha},\overline{C_{\alpha'}},\overline{C_\beta},\overline{C_{\beta'}},\overline{C_{\vec\gamma}})\\
&=\lim_{D\to4}Z(\overline{C_\alpha},g_R,\mu;D)Z(\overline{C_{\alpha'}},g_R,\mu;D)\\
&\quad\times Z(\overline{C_\beta},g_R,\mu;D)Z(\overline{C_{\beta'}},g_R,\mu;D)\\
&\quad\times[Z_{11}(\overline{C_{\vec\gamma}},g_R,\mu;D)\widetilde{W}_1(x,x',y,y';g_R,\mu,D)\\
&\quad+Z_{12}(\overline{C_{\vec\gamma}},g_R,\mu;D)\widetilde{W}_2(x,x',y,y';g_R,\mu,D)\\
&=:\lim_{D\to4}\overline{W_1}(x,x',y,y';g_R,\mu,\overline{C_\alpha},\overline{C_{\alpha'}},\overline{C_\beta},\overline{C_{\beta'}},\\
&\quad\times\overline{C_{\vec\gamma}};D),
\end{aligned}
\tag{B9}
$$

where the second renormalization procedure introduces new subtraction points $\overline{C_\sigma}$ (cf. [18,20] for more details).

We define a renormalized expression for the radial gauge propagator by

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^{\text{singlet}} := \lim_{D\to 4}\delta^{ab}\frac{2N}{N^2-1}\frac{1}{(ig_R)^2}\mu^{D-4} \times \lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y\overline{W_1}(x,x',y,y';D), \tag{B10}$$

where we have suppressed most of the other variables on which W_1 depends [see Eq. (B9)].

From now on, we will concentrate on the calculation of a renormalized expression for the free propagator $\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0$. As we shall see only a few of the many possible renormalization constants will contribute to the final result.

Since in the relation between the propagator and the appropriate Wilson loop (3.3) a factor $1/g^2$ is involved, all quantities, especially all the Z's and W's of Eq. (B9), have to be calculated up to $O(g_R^2)$. We have

$$\widetilde{W}_i = 1+(ig_R)^2\delta\widetilde{W}_i + O(g_R^4)(i=1,2), \tag{B11}$$

$$Z = 1+(ig_R)^2\delta Z + O(g_R^4), \tag{B12}$$

$$Z_{11} = 1+(ig_R)^2\delta Z_{11} + O(g_R^4), \tag{B13}$$

$$Z_{12} = 0+(ig_R)^2\delta Z_{12} + O(g_R^4), \tag{B14}$$

yielding

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0 = \lim_{D\to 4}\delta^{ab}\frac{2N}{N^2-1}\mu^{D-4}\lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y[\delta Z(\overline{C_\alpha}) + \delta Z(\overline{C_{\alpha'}}) + \delta Z(\overline{C_\beta}) + \delta Z(\overline{C_{\beta'}}) + \delta Z_{11} + \delta Z_{12} + \delta\widetilde{W}_1]. \tag{B15}$$

Using the fact that up to $O(g_R^2)$ the two quantities W_1 and $\widetilde{W}_1$ are essentially the same,[4] we find

$$\lim_{D\to 4}\delta^{ab}\frac{2N}{N^2-1}\mu^{D-4}\lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y\delta\widetilde{W}_1$$
$$= \lim_{D\to 4}\delta^{ab}\frac{2N}{N^2-1}\frac{1}{(ig)^2}\lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y[1+(ig)^2\delta W_1]$$
$$= \lim_{D\to 4}\delta^{ab}\frac{2N}{N^2-1}\frac{1}{(ig)^2}\lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y W_1\big|_{g=0}$$
$$= \lim_{D\to 4}\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\text{rad}}^0. \tag{B16}$$

To get the δZ's we must calculate $\delta\widetilde{W}_1$ which is straightforward using Eqs. (3.4) and (B1). We only need the Feynman propagator (3.6) to get

$$\delta\widetilde{W}_1 = -\mu^{4-D}\frac{N^2-1}{2N}\frac{\Gamma(D/2-1)}{4\pi^{D/2}}[(|x'|^{4-D}+|x-x'|^{4-D}+|x|^{4-D}+|y'|^{4-D}+|y-y'|^{4-D}+|y|^{4-D})I_1 + I_2(x',x-x') + I_2(x-x',-x)+I_2(-x,x')+I_2(y',y-y')+I_2(y-y',-y)+I_2(-y,y')-I_2(x',-y')+I_2(x',-y)+I_2(y',-x) - I_2(x,-y)-I_3(y',-x',y-y')+I_3(y',-x,y-y')-I_3(x'-y',x-x',y'-y)-I_3(x',-y',x-x') + I_3(x',-y,x-x')] \tag{B17}$$

with

$$I_1 := \int_0^1 ds\int_0^1 dt\,\Theta(s-t)\frac{1}{[(s-t)^2]^{D/2-1}}, \tag{B18}$$

$$I_2(p,q) := \int_0^1 ds\int_0^1 dt\frac{p\cdot q}{[(sp+tq)^2]^{D/2-1}}, \tag{B19}$$

and

$$I_3(m,p,q) := \int_0^1 ds\int_0^1 dt\frac{p\cdot q}{[(m+sp+tq)^2]^{D/2-1}}. \tag{B20}$$

In the following we are interested only in the divergent parts of these integrals. The integrals I_1 and I_2 are calculated in Appendix D. The results are

$$I_1 = -\frac{1}{4-D} + \text{finite} \tag{B21}$$

and

$$I_2(p,q) = \frac{1}{4-D}\gamma\cot\gamma + \text{finite}, \tag{B22}$$

where γ is the angle between p and q. The integral I_3 is finite as long as $m\neq 0$.

To specify the renormalization factors Z we choose the minimal subtraction scheme K_γ^{MS} as described in [18]. In dimensional regularization all the divergences are given by sums of pole terms. We define every Z factor to be given just by the respective sum. The important property of this renormalization scheme is that the Z factors depend on the angles only and not on the length of the loop or of any part of the loop. Using Eqs. (B21) and (B22) the Z factors can be read off from Eq. (B17) (cf. [21]):

[4]Only a factor μ^{D-4} comes in since g_R as given in Eq. (B2) is dimensionless in contrast with g.

$$\delta Z(\overline{C_\alpha})=\frac{N^2-1}{2N}\frac{1}{4\pi^2}\frac{1}{4-D}(\alpha\cot\alpha-1), \quad \text{(B23)}$$

$$\delta Z(\overline{C_{\alpha'}})=\frac{N^2-1}{2N}\frac{1}{4\pi^2}\frac{1}{4-D}(\alpha'\cot\alpha'-1), \quad \text{(B24)}$$

$$\delta Z(\overline{C_\beta})=\frac{N^2-1}{2N}\frac{1}{4\pi^2}\frac{1}{4-D}(\beta\cot\beta-1), \quad \text{(B25)}$$

$$\delta Z(\overline{C_{\beta'}})=\frac{N^2-1}{2N}\frac{1}{4\pi^2}\frac{1}{4-D}(\beta'\cot\beta'-1), \quad \text{(B26)}$$

$$\delta Z_{11}=\frac{N^2-1}{2N}\frac{1}{4\pi^2}\frac{1}{4-D}[(\gamma_{x'y}\cot\gamma_{x'y}-1)+(\gamma_{xy'}\cot\gamma_{xy'}-1)], \quad \text{(B27)}$$

$$\delta Z_{12}=\frac{N^2-1}{2N}\frac{1}{4\pi^2}\frac{1}{4-D}[\gamma_{xx'}\cot\gamma_{xx'}+\gamma_{yy'}\cot\gamma_{yy'}-(\pi-\gamma_{x'y'})\cot(\pi-\gamma_{x'y'})-(\pi-\gamma_{xy})\cot(\pi-\gamma_{xy})]. \quad \text{(B28)}$$

Now, we exploit the fact that only one of the angles, namely γ_{xy} depends on x *and* y. All the other ones depend only on x or on y separately, or on none of them. This simplifies Eq. (B15) drastically:

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0=\lim_{D\to4}\delta^{ab}\frac{2N}{N^2-1}\mu^{D-4}\lim_{\substack{x'\to x\\ y'\to y}}\partial_\mu^x\partial_\nu^y\times(\delta Z_{12}+\delta\widetilde{W}_1)$$
$$=\lim_{D\to4}\left[\delta^{ab}\mu^{D-4}\partial_\mu^x\partial_\nu^y\left(\frac{1}{4\pi^2}\frac{1}{4-D}\times(\pi-\gamma_{xy})\cot\gamma_{xy}\right)+\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\rm rad}^0\right], \quad \text{(B29)}$$

where we have used Eq. (B16) to get the last expression.

Before discussing some properties of the renormalized expression for the free propagator we shall show that the counterterm

$$C_{\mu\nu}^{ab}(x,y):=\delta^{ab}\mu^{D-4}\partial_\mu^x\partial_\nu^y\left(\frac{1}{4\pi^2}\frac{1}{4-D}(\pi-\gamma_{xy})\cot\gamma_{xy}\right) \quad \text{(B30)}$$

exactly cancels the divergence of the propagator (3.7), i.e., $\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0$ really is finite. To this end we use some technical results derived in Appendix D. The divergent part of the propagator (3.7) is given by

$$U_{\mu\nu}^{ab}(x,y):=-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\delta^{ab}\times\partial_\mu^x\partial_\nu^y\int_0^1 ds\int_0^1 dt\,x\cdot y[(sx-ty)^2]^{1-D/2}. \quad \text{(B31)}$$

Using Eqs. (D4) and (D14) below, we find

$$U_{\mu\nu}^{ab}(x,y)=\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\delta^{ab}\partial_\mu^x\partial_\nu^y I_2(x,-y)$$
$$=\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\delta^{ab}\partial_\mu^x\partial_\nu^y\left(\frac{1}{4-D}(\pi-\gamma_{xy})\times\cot(\pi-\gamma_{xy})+\text{finite}\right)$$
$$=-\frac{1}{4\pi^2}\delta^{ab}\partial_\mu^x\partial_\nu^y\times\left(\frac{1}{4-D}(\pi-\gamma_{xy})\cot\gamma_{xy}\right)+\text{finite} \quad \text{(B32)}$$

and thus

$$U_{\mu\nu}^{ab}(x,y)+C_{\mu\nu}^{ab}(x,y)=\text{finite}. \quad \text{(B33)}$$

Note that if one tries to guess a finite expression such as $\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0$ one would have to introduce a scale μ by hand without interpretation. In our derivation this scale appears naturally as the typical renormalization scale of the $\mathcal{R}$ operation.

The counter term $C_{\mu\nu}^{ab}(x,y)$ has some interesting properties. It is symmetric with respect to an exchange of all variables and it obeys the gauge condition

$$x^\mu C_{\mu\nu}^{ab}(x,y)=0=C_{\mu\nu}^{ab}(x,y)y^\nu. \quad \text{(B34)}$$

Thus $\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0$ is finite in the limit $D\to4$ but still can be interpreted as a gluonic two-point function which satisfies the radial gauge condition

$$x^\mu\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0=0. \quad \text{(B35)}$$

However, $C_{\mu\nu}^{ab}(x,y)$ and thus also $\langle A_\mu^a(x)A_\nu^b(y)\rangle_R^0$ is ill defined at the origin and hence conflicts with the field strength formula (1.2). Note that the regularized propagator in contrast with the renormalized ''propagator'' is well defined and vanishes if one of its arguments approaches zero, as pointed out after Eq. (3.7). We, therefore, conclude that we may use the regularized propagator in perturbative calculations and can ensure to preserve relations such as the field strength formula or Eqs. (2.11) or (3.1) throughout the calculation.

APPENDIX C: WILSON LOOP CALCULATION WITH AN ALTERNATIVE RADIAL PROPAGATOR

In Sec. IV we have calculated the g^2 contribution of the Wilson loop along the path (4.1) in Feynman gauge as well as in the radial gauge using the propagator (3.7). We have shown that the two results I_f and I_r agree with each other. However, using the propagator (5.5) for the same Wilson loop we end up with a different result as we shall show now.

We have to compare $I_f=I_r$ as given in Eq. (4.9) with

$$I_w=-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\Bigg[\int_0^1 d\sigma\int_0^1 d\tau\dot{w}_\mu(\sigma)\dot{w}^\mu(\tau)[(w(\sigma)-w(\tau))^2]^{1-D/2}+\frac{1}{2}\left(\int_\infty^1 ds\int_0^1 dt+\int_0^1 ds\int_\infty^1 dt\right)(w_\mu(1)w^\mu(1)$$
$$\times\{[sw(1)-tw(1)]^2\}^{1-D/2}+w_\mu(0)w^\mu(0)\{[sw(0)-tw(0)]^2\}^{1-D/2}-w_\mu(1)w^\mu(0)\{[sw(1)-tw(0)]^2\}^{1-D/2}$$
$$-w_\mu(0)w^\mu(1)\{[sw(0)-tw(1)]^2\}^{1-D/2})-\frac{1}{2}\left(\int_0^1 ds+\int_\infty^1 ds\right)\int_0^1 d\tau\dot{w}_\mu(\tau)(w^\mu(1)\{[sw(1)-w(\tau)]^2\}^{1-D/2}-w^\mu(0)$$
$$\times\{[sw(0)-w(\tau)]^2\}^{1-D/2})-\frac{1}{2}\left(\int_0^1 dt+\int_\infty^1 dt\right)\int_0^1 d\sigma\dot{w}_\mu(\sigma)(w^\mu(1)\{[w(\sigma)-tw(1)]^2\}^{1-D/2}$$
$$-w^\mu(0)\{[w(\sigma)-tw(0)]^2\}^{1-D/2})\Bigg]. \tag{C1}$$

Obviously, it is very tedious and on the other hand not very illuminating to calculate all the integrals in Eqs. (4.9) and (C1). Therefore, we restrict ourselves to the most interesting part, the cusp divergence at the origin, i.e., we calculate the contributions $f_r(\delta)$ and $f_w(\delta)$ with

$$I_{r,w}=\frac{1}{4-D}f_{r,w}(\delta)+\text{contr. indep. of}\,\delta+\text{finite contr.} \tag{C2}$$

and

$$\cos\delta:=\frac{w(0)\cdot w(1)}{|w(0)||w(1)|}. \tag{C3}$$

To prove that $I_r\neq I_w$ holds, it is sufficient to show that $f_r\neq f_w$.

f_r is determined by the contributions (3,1) and (1,3) in Eq. (4.9): i.e.,

$$\begin{aligned}\frac{1}{4-D}f_r(\delta)=&-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\int_0^1 ds\int_0^1 dt(-w_\mu(1)w^\mu(0)\\&\times\{[sw(1)-tw(0)]^2\}^{1-D/2}-w_\mu(0)w^\mu(1)\\&\times\{[sw(0)-tw(1)]^2\}^{1-D/2})\\=&-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\,2I_2(w(0),-w(1))\end{aligned} \tag{C4}$$

with I_2 given in Eq. (D4). Strictly speaking, the first equality sign in Eq. (C4) holds up to finite contributions. Using Eq. (D14) we get

$$f_r(\delta)=-\frac{1}{2\pi^2}(\pi-\delta)\cot(\pi-\delta). \tag{C5}$$

Similarly, f_w is given by

$$\begin{aligned}\frac{1}{4-D}f_w(\delta)=&-\frac{1}{2}\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\left(\int_\infty^1 ds\int_0^1 dt+\int_0^1 ds\int_\infty^1 dt\right)\\&\times(-w_\mu(1)w^\mu(0)\{[sw(1)-tw(0)]^2\}^{1-D/2}\\&-w_\mu(0)w^\mu(1)\{[sw(0)-tw(1)]^2\}^{1-D/2})\\=&-\frac{\Gamma(D/2-1)}{4\pi^{D/2}}[I_4(w(0),-w(1))\\&+I_4(w(1),-w(0))]\end{aligned} \tag{C6}$$

with

$$I_4(p,q):=\int_\infty^1 ds\int_0^1 dt\frac{p\cdot q}{[(sp+tq)^2]^{D/2-1}}. \tag{C7}$$

As shown in Appendix D the integral I_4 is finite for $D\rightarrow 4$; thus

$$f_w=0 \tag{C8}$$

which obviously differs from Eq. (C5). This proves that $I_r\neq I_w$.

APPENDIX D: SOME IMPORTANT INTEGRALS

The integrals I_1 and I_2 play an important part in the renormalization procedure of Appendix B and determine the divergences of the naive free radial propagator introduced in Sec. III. The integral I_4 is important for the comparison of the radial propagator obtained here with the one presented in [15]. All these integrals are discussed in detail below.

To calculate

$$I_1:=\int_0^1 ds\int_0^1 dt\Theta(s-t)\frac{1}{[(s-t)^2]^{D/2-1}}, \tag{D1}$$

we introduce the substitution

$$g=s-t,\quad h=s+t \tag{D2}$$

to get

$$\begin{aligned}I_1&=\frac{1}{2}\int_0^1 dg\int_g^{2-g}dh g^{2-D}=\int_0^1 dg(1-g)g^{2-D}\\&=\frac{\Gamma(2)\Gamma(3-D)}{\Gamma(5-D)}=\frac{1}{(4-D)(3-D)}.\end{aligned} \tag{D3}$$

For the calculation of

$$I_2(p,q):=\int_0^1 ds\int_0^1 dt\frac{p\cdot q}{[(sp+tq)^2]^{D/2-1}} \tag{D4}$$

we have to distinguish the two cases $p\neq\alpha q$ where the only divergence that appears is for $s=t=0$ and $p=\alpha q$ with an additional divergence at $s=t\alpha$. Here, we will only need the former [cf. the discussion after Eq. (3.7)].

As a first step it is useful to separate off the divergence at the origin by the substitution

$$\lambda=s+t,\quad x=s/\lambda. \tag{D5}$$

This yields

$$\begin{aligned}I_2(p,q)&=\left(\int_0^{1/2}dx\int_0^{1/(1-x)}d\lambda\right.\\&\left.+\int_{1/2}^{1}dx\int_0^{1/x}d\lambda\right)\lambda^{3-D}\frac{p\cdot q}{\{[xp+(1-x)q]^2\}^{D/2-1}}\\&=\int_0^{1/2}dx\frac{(1-x)^{D-4}}{4-D}\frac{p\cdot q}{\{[xp+(1-x)q]^2\}^{D/2-1}}\\&+\int_{1/2}^{1}dx\frac{x^{D-4}}{4-D}\frac{p\cdot q}{\{[xp+(1-x)q]^2\}^{D/2-1}}.\end{aligned} \tag{D6}$$

As long as $p\neq\alpha q$ holds there are no divergences in the x integration since

$$u(x):=xp+(1-x)q \tag{D7}$$

never vanishes. We introduce the angle between p and q,

$$\cos\gamma:=\frac{p\cdot q}{|p||q|}, \tag{D8}$$

and the substitution [18]

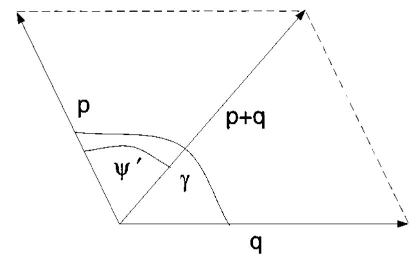

FIG. 4. The geometry of the variables appearing in the calculation of I_2 (C4).

$$e^{2i\psi}=\frac{x|p|+(1-x)|q|e^{i\gamma}}{x|p|+(1-x)|q|e^{-i\gamma}}. \tag{D9}$$

Note that ψ is nothing but the angle between p and $u(x)$. To perform this substitution in Eq. (D6) we need

$$x=|q|\sin(\gamma-\psi)/N(\psi),1-x=|p|\sin\psi/N(\psi), \tag{D10}$$

$$[u(x)]^2=p^2q^2\sin^2\gamma/[N(\psi)]^2,\text{and}\frac{d\psi}{dx}=-\frac{[N(\psi)]^2}{|p||q|\sin\gamma} \tag{D11}$$

with

$$N(\psi):=|p|\sin\psi+|q|\sin(\gamma-\psi). \tag{D12}$$

In addition, it is useful to introduce

$$\psi':=\psi(x=1/2) \tag{D13}$$

which is the angle between p and $p+q$ (cf. Fig. 4).

Using all that, we end up with

$$\begin{aligned}I_2(p,q)&=\int_\gamma^{\psi'}d\psi\frac{-|p||q|\sin\gamma}{N^2}\left(\frac{|p|\sin\psi}{N}\right)^{D-4}\left(\frac{N^2}{p^2q^2\sin^2\gamma}\right)^{D/2-1}\frac{p\cdot q}{4-D}\\&+\int_{\psi'}^{0}d\psi\frac{-|p||q|\sin\gamma}{N^2}\left(\frac{|q|\sin(\gamma-\psi)}{N}\right)^{D-4}\left(\frac{N^2}{p^2q^2\sin^2\gamma}\right)^{D/2-1}\frac{p\cdot q}{4-D}=\frac{-\cos\gamma\sin^{3-D}\gamma}{4-D}\left(\left|q\right|^{4-D}\int_\gamma^{\psi'}d\psi\sin^{D-4}\psi\right.\\&\left.+\left|p\right|^{4-D}\int_{\psi'}^{0}d\psi\sin^{D-4}(\gamma-\psi)\right)=\frac{-\cos\gamma\sin^{3-D}\gamma}{4-D}\left(\left|q\right|^{4-D}\int_\gamma^{\psi'}d\psi\sin^{D-4}\psi+\left|p\right|^{4-D}\int_\gamma^{\gamma-\psi'}d\psi\sin^{D-4}\psi\right)\\&=\frac{1}{4-D}\gamma\cot\gamma+\text{finite}.\end{aligned} \tag{D14}$$

The integral

$$I_4(p,q):=\int_\infty^1 ds\int_0^1 dt\frac{p\cdot q}{[(sp+tq)^2]^{D/2-1}} \tag{D15}$$

can be calculated in the same way as I_2. Again, we use the substitution (D5) to derive

$$I_4(p,q)=-\int_{1/2}^{1}dx\int_{1/x}^{1/(1-x)}d\lambda\lambda^{3-D}\frac{p\cdot q}{\{[xp+(1-x)q]^2\}^{D/2-1}}. \tag{D16}$$

Obviously, I_4 is finite for $D\to4$ as long as $p\neq\alpha q$ holds.

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