Quantization of anomalous gauge field theory and BRST-invariant models of two-dimensional quantum gravity

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We analyze the problems with the so-called gauge-invariant quantization of the anomalous gauge field theories originally due to Faddeev and Shatashvili (FS). Our analysis is a generalization of the FS method, which allows us to construct a series of classically equivalent theories that are nonequivalent at the quantum level. We prove that these classical theories are all consistent with the BRST invariance of the original gauge symmetry with a suitably augmented field content. As an example of such a scenario, we discuss the class of physically distinct models of two-dimensional induced gravity, which are a generalization of the David-Distler-Kawai model. [S0556-2821(96)01024-7]

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I. INTRODUCTION

The consistent quantization of (classical) gauge-invariant field theory requires the complete cancellation of anomalies [1,2]. Here, ''consistent'' means that we want not only to require renormalizability (perturbative finiteness), but also unitarity of the *S* matrix, non-violation of Lorentz invariance, etc. Moreover, in the physical four-dimensional 4D world, the anomaly cancellation condition itself often leads to physical predictions. A well-known example is the equality of numbers of quarks and leptons in the standard model of Weinberg and Salam.

On lower-dimensional (e.g., $d=2$) field theory, the cancellation of anomalies is still the crucial ingredient for model building. The critical string dimension $d=26$ is often quoted [3] as a consequence of the anomaly-free condition for a bosonic string (although in this example the cancellation of the anomaly does not guarantee full consistency of the model in the above sense, due to the presence of tachyons).

In the case of lower-dimensional field theory ($d<4$), one often tries to quantize a gauge field theory when there is no way of canceling its anomaly. The classical example of this situation is the attempt to quantize the chiral Schwinger model by Jackiw and Rajaraman [4,5]. They have shown that the model can be consistently quantized (free field theory) even when gauge invariance is broken through the anomaly.

In general there seem to be two ways for attempting the quantization of the anomalous gauge field theory.

(1) Gauge-nonvariant method. One ignores the breaking of gauge symmetry and tries to show that the theory can be quantized even without gauge invariance. The example of this approach is the above Jackiw-Rajaraman quantization of the chiral Schwinger model. The problem here is that it is not easy to develop the general techniques covering a wide class of physically relevant models with an anomaly.

(2) Gauge-invariant method. In this case, one first tries to recover gauge invariance by introducing new degrees of freedom. The theory is anomalous when one cannot find a local counterterm to cancel the gauge noninvariance due to the one loop ''matter'' integrals in the presence of gauge fields, by making use exclusively of the degrees of freedom (fields) already present in the classical action.

In Ref. [6], Faddeev and Shatashvili (FS) have tried to justify the introduction of new degrees of freedom that are necessary to construct the anomaly-canceling counterterm. Their argument is based on the idea of a projective representation of the gauge group. They observe that the appearance of an anomaly does not mean the simple breakdown of (classical) gauge symmetry, but rather it signals that the symmetry is realized projectively (this is related to the appearance of anomalous commutators of relevant currents). Such a realization, through projective representations, necessitates the enlargement of physical Hilbert space. Thus they argued that the introduction of new fields in the model is not an *ad hoc* (and largely arbitrary) construction.

Independently of their ''philosophy,'' the FS method gives the gauge-invariant action at the price of introducing the extra degrees of freedom (generally physical). The serious problem of this method is, however, that the gauge invariance thus ''forced'' upon the theory, does not automatically guarantee the consistency of the theory. This is in

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contrast with our experience with some 4D models such as the standard model.

For example, one may apply the FS method to the celebrated case of chiral Schwinger model [4,5]. In this case, we have the classical action

$$S_0=\int \frac{dz\wedge d\bar{z}}{2i}\left[\bar{\psi}_R\gamma_{\bar{z}}(\bar{\partial}+R)\psi_R+\bar{\psi}_L\gamma_z\partial\psi_L+\tfrac{1}{4}\ \mathrm{Tr}F^2\right],$$

where

$$\psi_{R/L}=\frac{1\pm\gamma_5}{2}\ \psi,$$

$$R/L=A_1\pm iA_2,\quad F=\bar{\partial}L-\partial R+[R,L]$$

(we are using the Euclidian notation).

This is invariant under the gauge transformation

$$\psi_R\to\psi_R^g=S(g)\psi_R,$$

$$\psi_L\to\psi_L,$$

$$A_\mu=gA_\mu g^{-1}+g\partial_\mu g^{-1}$$

for any $g(z,\bar{z})\in G$.

The theory is anomalous because the one loop integral

$$e^{-W_R(R)}=\int \mathcal{D}\psi_R\mathcal{D}\bar{\psi}_R\exp\left(-\int \bar{\psi}_R\gamma_{\bar{z}}(\bar{\partial}+R)\psi_R\right)$$

is not gauge invariant under

$$R\to gRg^{-1}+g\bar{\partial}g^{-1}$$

(for any choice of the regularization).

Following the FS technique (see the next section), however, one can introduce the local counterterm $\Lambda(R,L;g)$, $(g(z,\bar{z})\in G)$ so that the gauge variation of Λ cancels the noninvariance of $W_R(R)$.

There is certain arbitrariness in the choice of Λ but the convenient one is

$$\Lambda(R,L;g)=-\left(\alpha_L(L,g)+\frac{1}{4\pi}\int\ \mathrm{Tr}(RL)\right),$$

where

$$\alpha_L(L,g)=\frac{1}{4\pi}\Bigg[-\int \frac{dz\wedge d\bar{z}}{2i}\mathrm{Tr}(g^{-1}\bar{\partial}g,L)+\frac{1}{2}\int \frac{dz\wedge d\bar{z}}{2i}\mathrm{Tr}(g\partial g^{-1},g\bar{\partial}g^{-1})-\frac{1}{2}\int_0^1 dt\int \frac{dz\wedge d\bar{z}}{2i}\times\mathrm{Tr}(g'\partial_t g'^{-1},[g'\partial g'^{-1},g'\bar{\partial}g'^{-1}])\Bigg],$$

$$g'(0,z,\bar{z})=1,\quad g'(1,z,\bar{z})=g(z,\bar{z})$$

is the Wess-Zumino-Novikov-Witten action corresponding to the anomaly of left fermion ψ_L, $\bar{\psi}_L$ (α_L is not globally a local action but it is so far ''small'' $g\simeq 1+i\xi$). That is, one can write

$$\alpha_L(L,g)=W_L(L^g)-W_L(L),$$

where

$$e^{-W_L(L)}=\int \mathcal{D}\psi_L'\mathcal{D}\bar{\psi}_L'\exp\left(-\int \bar{\psi}_L'\gamma_z(\partial+L)\psi_L'\right)$$

(note that ψ_L', $\bar{\psi}_L'$ have nothing to do with $\psi_L,\bar{\psi}_L$ in S_0).

With this choice of counterterm, one can show that the theory is equivalent to (a) free decoupled fermion $\psi_L,\bar{\psi}_L$ and (b) the vector Schwinger model. In fact, the added bosonic degree of freedom $g(z,\bar{z})\in G$ can be ''fermionized'' to act as missing ψ_L', $\bar{\psi}_L'$ with the right coupling to the left component L of gauge field.

However, there is still a point missing in this story. In fact, after introducing the new degree of freedom g, there is no reason to exclude the other type of invariant local counterterm such as

$$\frac{a}{4\pi}\int\ \mathrm{Tr}(L^gR^g)=\frac{a}{4\pi}\int\ \mathrm{Tr}[(gLg^{-1}+g\partial g^{-1}),(gRg^{-1}+g\bar{\partial}g^{-1})]$$

[one can also attribute it to the indefinite—regularization dependent—part of the fermionic integral, i.e., $W_R(R)+W_L(L)+(a/4\pi)\int\mathrm{Tr}(RL)$].

It is well known [4] that the arbitrary constant a enters the physical spectrum. For the Abelian case, $G=U(1)$, the mass square of the massive boson is given by

$$m^2=\frac{e^2a^2}{a-1},$$

thus, for $a<1$, the theory is not consistent although the requirement of gauge invariance is satisfied.

In the fermionized version of the theory [5(b)], a enters the charges of the left and right fermions as

$$e_{R/L}=\frac{e}{2}\left(\sqrt{a-1}\pm\frac{1}{\sqrt{a-1}}\right).$$

This means that the condition $a>1$ is necessary also for the real coupling constant, or the Hermitian Hamiltonian.

In general, the consistency of the theory can be proved if one can set up the Beechi-Rouet-Stora-Tyutin (BRST) scheme with certain physical conditions at the start, such as Hermiticity of the Hamiltonian [11].

In what follows, we discuss the possibility of recasting the FS method into BRST formalism, thus facilitating the analysis of the consistency of the theory.

II. FADDEEV-SHATASHVILI METHOD

A. Path integral formalism

We shall briefly describe the FS method of quantizing anomalous gauge field theory in the path integral formalism, following the work of Harada and Tsutsui [7] and Babelon, Shaposnik, and Vialet [8].

Let us take a generic gauge field theory described by the classical action

$$S_0(A,X)=S_G(A)+S_M(X;A), \tag{1}$$

where $\{A(x)\}$ and $\{X(x)\}$ represent, respectively, gauge fields and ''matter fields,'' gauge invariantly coupled to the former.

The total action S_0 as well as the pure gauge part S_G and the matter part S_M are invariant under the local gauge transformation

$$A\to A'=A^g,\quad X\to X'=X^g,\quad g(x)\in G. \tag{2}$$

Being anomalous generally means that the one loop matter integral [assuming that $S_M(X,A)$ is quadratic in X]

$$\int \mathcal{D}X\ e^{-S_M(X,A)}\equiv e^{-W(A)} \tag{3}$$

cannot be regularized in such a way as to preserve the gauge invariance of the functional $W(A)$:

$$W(A^g)-W(A)=\alpha(A;g)\neq 0. \tag{4}$$

Naturally, $\alpha(A;g)$ depends on the regularization used, but there is no way of canceling it completely by adding some local counter term $\Lambda(A,X)$ to the action.

One can understand Eq. (4) as the noninvariance of the path integral measure, $\mathcal{D}X$:

$$\mathcal{D}X^g\neq\mathcal{D}X. \tag{5}$$

In fact, as shown by Fujikawa [9], one can write the ''anomaly equation''

$$W(A^g)-W(A)=\alpha(A;g),$$

$$\det\left(\frac{\mathcal{D}X^g}{\mathcal{D}X}\right)=e^{-\alpha(A;g)}=e^{\alpha(A;g^{-1})}. \tag{6}$$

In this situation, clearly one cannot hope that the usual Faddeev-Popov (FP) ansatz to quantize the theory may go through.

If one inserts the δ function identity,

$$1=\Delta(A)\int\mathcal{D}g\ \delta(F(A^g)), \tag{7}$$

where $F(A)$ is a gauge-fixing function, into the path integral expression for the partition function

$$Z=\int\mathcal{D}A\int\mathcal{D}X\ e^{-[S_G(A)+S_M(X;A)]}$$

then one obtains

$$Z=\int\mathcal{D}A\int\mathcal{D}X\ \Delta(A)e^{-[S_G(A)+S_M(X;A)]}\int\mathcal{D}g\,\delta(F(A^g))$$
$$=\int\mathcal{D}A\int\mathcal{D}g\ \Delta(A)e^{-S_G(A^g)}\int\mathcal{D}X\ e^{-S_M(X^g;A^g)}\delta(F(A^g)). \tag{8}$$

The second equality follows from the gauge invariance of the classical action: $S_0(A^g;X^g)=S_0(A;X)$.

In the case of usual gauge field theory, such as the chiral Schwinger model, we can make a series of assumptions on the remaining functional measures $\mathcal{D}A$ and $\mathcal{D}g$.

First, we assume

$$(1)\quad \mathcal{D}A=\mathcal{D}A^g. \tag{9}$$

Then with the change of variable $A^g\to A$ and $X^g\to X$ in Eq. (7), we get

$$Z=\int\mathcal{D}g\int\mathcal{D}A\ \Delta\,\delta(F(A))(A^{g^{-1}})e^{-S_G(A)}$$
$$\times\int\mathcal{D}X\ e^{-[S_M(X;A)+\alpha(A;g^{-1})]}, \tag{10}$$

where we have used Eq. (5), i.e., $\mathcal{D}X=\mathcal{D}X^{gg^{-1}}=\mathcal{D}X^g e^{-\alpha(A;g^{-1})}$.

Further, one can assume, for the usual gauge group, the invariance of Haar measure $\mathcal{D}g$, i.e., for any h in G,

$$(2)\quad \mathcal{D}(gh)=\mathcal{D}(hg)=\mathcal{D}g, \tag{11}$$

which results, as is well known, in the invariance of the FP factor $\Delta(A)$:

$$\Delta(A^{g^{-1}})=\Delta(A). \tag{12}$$

Thus, we get the expression for Z proposed in Refs. [6] and [7]:

$$Z=\int\mathcal{D}g\int\mathcal{D}A\ \Delta(A)\delta(F(A))\int\mathcal{D}X\ e^{-S_{\rm eff}(X,A;g)} \tag{13}$$

with

$$S_{\rm eff}(X,A;g)=S_0(X,A;g)+\alpha(A;g^{-1}). \tag{14}$$

As one can see from Eq. (4) the effect of the counterterm $\alpha(A;g^{-1})$ is to transform the one loop path integral $W(A)$, Eq. (3), to $W(A^{g^{-1}})$, which is trivially gauge invariant under the extended gauge transformation

$$A\to A^h,\quad X\to X^h,$$
$$g\to hg \tag{15}$$

and thus the model is invariant up to the one loop level.

We have repeated here the above well known manipulations [7] to emphasize the relevance of the invariance conditions (1) and (2) [Eqs. (9) and (11)]. In many familiar examples, such as the chiral Schwinger model, these conditions are trivially satisfied.

One well-known case where these conditions become problematic is the two-dimensional (2D) induced gravity or off-critical string. In this case, if one fixes the path integral measures $\mathcal{D}\phi$ for the Weyl factor of the metric and $\mathcal{D}\sigma$ for the Weyl group element by the invariance under the diffeomorphisms of the 2D manifold, then they are not invariant under the translations, e.g., $\sigma \rightarrow \sigma + \alpha$ (i.e., the Weyl transformation). Thus, the path integral measure (i.e., $\mathcal{D}A\mathcal{D}g$) can never be invariant under the whole gauge group

$$G = \text{diffeo} \otimes \text{Weyl}$$

B. BRST quantization [10]

A more rigorous strategy to have a consistent formulation of a gauge field theory is to recast it in the BRST formalism. In this way, one may discuss the physically important questions such as the unitarity of the S matrix [11].

In a simpler example such as the chiral gauge field theory where the invariance of the measure $\mathcal{D}g\mathcal{D}A$ [Eqs. (9) and (11)] under the gauge transformations is respected, there is no difficulty in setting up the BRST procedure once the anomaly has been removed.

One replaces the ''heuristic'' FP factor

$$\Delta(A)\,\delta(F(A)) = \det\left(\frac{\delta F(A^h)}{\delta h}\bigg|_{h=1} \right) \delta(F(A))$$

with BRST gauge-fixing term

$$\exp\left(-\int \hat{s}[\overline{c}F(A)] \right) = \exp\left(-\int \left[BF(A) - \overline{c}\, \frac{\delta F(A^h)}{\delta h}\bigg|_{h=1} c \right] \right),$$

where $c, \overline{c}$ are the BRST ghosts corresponding to the gauge group G while B (''Lagrange multiplier'') is the Nakanishi-Lautrup field [12]. Under the BRST operator $\hat{s}$, one has, in particular,

$$\hat{s}\overline{c} = B,$$

$$\hat{s}B = 0$$

$$(\hat{s}^2 - 0).$$

With the counterterm $\alpha(A;g)$ canceling the one loop anomaly, one can show easily the validity of the Slavnov-Taylor identity,

$$\frac{\delta\tilde{\Gamma}}{\delta A}\frac{\delta\tilde{\Gamma}}{\delta K} + \frac{\delta\tilde{\Gamma}}{\delta \Phi_i}\frac{\delta\tilde{\Gamma}}{\delta K_i} + \frac{\delta\tilde{\Gamma}}{\delta c}\frac{\delta\tilde{\Gamma}}{\delta L} \quad (\tilde{\Gamma} \times \tilde{\Gamma} = 0), \tag{16}$$

up to one loop.

$\tilde{\Gamma}$ is the generating functional of the one particle irreducible part Γ (with an added external source for composite operators) minus the ''gauge fixing term'' [in Eq. (16), A and c are the classical counterparts of the gauge fields A and ghost c, while $\{\Phi_i\}$ are the classical fields for the matter X and newly introduced field g; K, K_i, and L are the usual external sources for the gauge variations $\hat{\delta}A$, $\hat{\delta}\Phi_i$, and δc, respectively]. One then hopes that it is possible to choose the higher-order local counterterm in such a way that Eq. (16) is satisfied to all orders.

Let us now imagine, however, that the invariance conditions (1) and (2) for the measure $\mathcal{D}A\mathcal{D}g$ [Eqs. (9) and (11)] are not satisfied [11(b)]. This means that one should take account of one or both of the following situations.

(1′) Condition (1) is not satisfied, i.e., $\mathcal{D}A \neq \mathcal{D}A^g = \mathcal{D}Ae^{-\alpha'}(A;g)$, where $\alpha'(A;g)$ is the ''Fujikawa determinant'' associated with the non-gauge-invariance of the measure over the gauge field itself. (2′) Condition (2) is not satisfied, i.e., $\Delta(A^g) \neq \Delta(A)$.

First of all, the noninvariance property (2′) means that the factor $\Delta(A)\,\delta(\mathrm{F(A)})$ in Eq. (11) must be replaced by $\Delta(A^{g^{-1}})\,\delta(F(A))$.

Thus, instead of a BRST gauge fixing term (14) one ends up with

$$\int \hat{s}[\overline{c}F(A)] + \ln\left(\frac{\Delta(A^{g^{-1}})}{\Delta(A)} \right). \tag{17}$$

The trouble is that one cannot transform $-\ln \Delta(A)$ into a BRST-invariant local term in the action. In fact, the BRST gauge fixed action would appear something like

$$S_{\text{eff}} = S_0 + \alpha(A;g^{-1}) + \alpha'(A;g^{-1}) + \ln\left(\frac{\Delta(A^{g^{-1}})}{\Delta(A)} \right) + \int \hat{s}[\overline{c}F(A)]. \tag{18}$$

The extra one loop term $\alpha'(A;g)$ does not cause any trouble for the BRST scheme to work at least in the example we are interested in. One way to push through the BRST scheme may be to replace Eq. (18) with

$$S'_{\text{eff}} = S_0 + \alpha(A;g^{-1}) + \alpha'(A;g^{-1}) + \int \hat{s}[\overline{c}F(A)]. \tag{19}$$

It is likely that the effective action (19) leads to a consistent BRST quantization. One may only add that it does not correspond to the path integral method of Refs. [7] and [8] when $\Delta(A^g) \neq \Delta(A)$. To reconcile the ''path integral'' formulation of the FS method with the BRST scheme, we propose another possibility.

It must be realized that once the new degree of freedom g is admitted in the theory then there is no reason to exclude new local counterterms of the right dimension, which are BRST invariant and which may also depend on g. Naturally this will change the model and its ''physics,'' but nevertheless it can remain consistent, insofar as the BRST invariance is maintained.

Let us then introduce the following counterterm in our theory:

$$\tilde{\Lambda}_G(A,g;c,\overline{c},c',\overline{c'},B) = \left[BG(A^{g^{-1}}) - \overline{c'}\, \frac{\delta G((A^{g^{-1}})^h)}{\delta h}\bigg|_{h=1} c' \right] - \left[BG(A) - \overline{c}\, \frac{\delta G(A^h)}{\delta h}\bigg|_{h=1} c \right], \tag{20}$$

where the second pair of ''ghosts'' $c',\overline{c'}$ are defined as the BRST singlet

$$\hat{\delta}\overline{c'} = 0, \quad \hat{s}c' = 0 \tag{21}$$

and $G(A)$ is the ''pseudo-gauge-fixing,'' which is generally different from $F(A)$.

The first term in $\tilde{\Lambda}_G$ is trivially BRST invariant since all the fields involved are either gauge invariant by themselves or appear as invariant combinations. The second term, on the other hand, can be written as

$$\hat{s}[\overline{c}G(A)],$$

so it is invariant too.

The effective action now takes the form

$$S_{\text{eff}} = S_0 + \alpha(A;g^{-1}) + \alpha'(A;g^{-1}) + \int \tilde{\Lambda}_G(A,g;c,\overline{c},c',\overline{c'},B) + \int \hat{s}[\overline{c}F(A)]. \tag{22}$$

Note that the gauge freedom of the BRST-invariant theory (22) is represented by the (arbitrary) gauge-fixing function $F(A)$ while each different choice of ''pseudo-gauge-function'' $G(A)$ defines a new model.

Each choice of $G(A)$ then results in a gauge-invariant model, which must then be gauge fixed by choosing a particular form for $F(A)$. In the limit of singular gauge

$$F(A) \to G(A) \tag{23}$$

the effective action (22) gives the series of models depending on $G(A)$ alone. The corresponding effective action can be formally written as

$$S_{\text{eff}} = S_0 + \alpha(A;g^{-1}) + \alpha'(A;g^{-1}) + \int \left[BG(A^{g^{-1}}) - \overline{c'}\, \frac{\delta G((A^{g^{-1}})^h)}{\delta h}\bigg|_{h=1} c' \right]. \tag{24}$$

Note that in Eq. (24) the gauge is already fixed (with a singular gauge). To see the gauge-invariance property of the model (24), one must go back to Eq. (22) with Eq. (20): i.e.,

$$S_{\text{eff}}^{\text{inv}} = S_0 + \alpha(A;g^{-1}) + \alpha'(A;g^{-1}) + \int \left[BG(A^{g^{-1}}) - \overline{c'}\, \frac{\delta G((A^{g^{-1}})^h)}{\delta h}\bigg|_{h=1} c' \right] - \int \left[B[F(A)-G(A)] - \overline{c}\, \frac{\delta}{\delta h}[F(A^h) - G(A^h)]\bigg|_{h=1} c \right]. \tag{25}$$

We have seen in this way that the FS method of formulating an anomalous theory within the path integral formalism apparently generates a series of physically distinct and BRST-invariant gauge field theories. We will discuss the possible candidate for such a scenario in the next section.

Before leaving this section, however, one has to consider the following question: i.e., in what sense can the effective action of Eqs. (24) or (25) be considered as the quantum version of the classical action Eq. (1)?

Apart from the inevitable [6] $g(x)$, one has new ghosts $c'(x)$ and $\overline{c'}(x)$. They are BRST singlets as stated above [Eq. (21)]. Thus, it is not apparent that these extra new ghosts decouple from the theory even in the classical limit. To show that such a decoupling actually takes place—albeit only in the classical limit—we start from the gauge-fixed action, Eq. (24). Since we are interested in the classical limit, we may further simplify the discussion by considering instead

$$S'_{\text{eff}} = S_0(A,X) + \int \left[BG(A^{g^{-1}}) - \overline{c'}\frac{\delta G((A^{g^{-1}})^h)}{\delta h}\bigg|_{h=1} c' \right], \tag{26}$$

forgetting temporarily the one loop counterterms $\alpha(A;g^{-1})$ and $\alpha'(A;g^{-1})$ in Eq. (24).

The second term in Eq. (26), i.e., the ''pseudo-gauge-fixing term,'' certainly cannot be interpreted as the gauge fixing term for the original gauge symmetry (which has been fixed already). On the other hand, one may still wonder if there can be any accidental gauge symmetry (which may be anomalous at one loop) realized in the action Eq. (24).

Let us consider the altered gauge transformations

$$(G')\begin{cases} A \to A' = A^{\gamma}, \\ X \to X' = X^{\gamma}, \\ g \to g' = \gamma g \gamma^{-1} \end{cases}. \tag{27}$$

These transformations differ from the original gauge transformations of Eq. (15), only by the transformation of the Faddev-Shatashvili field $g(x)$. In Eq. (15) we have

$$(G)\begin{cases} A \to A' = A^{\gamma} \\ X \to X' = X^{\gamma} \\ g \to g' = \gamma g \end{cases}$$

so that $A^{g^{-1}}$ is invariant.

Starting from Eq. (27), we define the new BRST transformations corresponding to (G') with FP ghosts c' and $\overline{c'}$

replacing c and $\overline{c}$. We must therefore define, apart from the BRST variation of A, X, and g,

$$s'c'=\tfrac{1}{2}c'\wedge c',$$

$$s'\overline{c'}=B(\text{not } B'),$$

$$s'B=0.$$

Note here that the new (G') transformation of $g(x)$ field causes the linear transformation of the ''mean field'' $\pi(x)$ defined by

$$g(x)=e^{\pi(x)},$$

while the original $g\rightarrow g'=\gamma g$ induces the nonlinear transformation of the π field.

With the new BRST operator s' defined above, one can see immediately that the pseudo-gauge-fixing terms in Eq. (24) or Eq. (26) can be written as

$$\int \left[BG(A^{g^{-1}})-\overline{c'}\frac{\delta G((A^{g^{-1}})^{h})}{\delta h}\bigg|_{h=1}c'\right]=s'[\overline{c'}G(A^{g^{-1}})],$$

i.e., the effective action (26) is invariant under s' and the pseudo-gauge-fixing becomes a true one with respect to the gauge symmetry G'. The gauge-fixing function $G(A^{g^{-1}})$ depends explicitly on the extra ''scalar'' field $g(x)$ in the same way as the 't Hooft gauge fixing for the ''spontaneously broken'' gauge field theory (Higgs). The G' gauge symmetry and the corresponding BRST operation s' have nothing to do with the original gauge symmetry of the theory [Eq. (25)]. But they have all the necessary characteristics for defining a BRST invariant system.

Formally (i.e., without worrying about loop corrections) one can define the new BRST operator Q' such that

$$s'\Phi\equiv[Q',\Phi]$$

with the nilpotency $Q'^{2}=0$. Thus, the theory defined (forgetting the anomaly problem for a moment) by the effective action (26) can be interpreted as an ordinary gauge field theory with gauge fixing:

$$\int s'[\overline{c'}G(A^{g^{-1}})],$$

where the FP ghosts are c' and $\overline{c'}$ and with original Nakanishi-Lautrup field B. If the accidental G' gauge symmetry was good to all orders, then one could have further restricted the physical S-matrix elements so that the new ghosts c' and $\overline{c'}$ would not have entered the physical spectrum. One could have repeated the whole Kugo-Ojima argument [11] starting from the definition of the physical state

$$Q'|\text{phys}\rangle=0$$

(together with $Q|\text{phys}\rangle=0$). As it is, G' is broken by the one loop effects, which are (i) α and α' in Eq. (24), which are not G' invariant, and (ii) the anomaly caused by the matter-gauge coupling in $S_0(A,X)$, Eq. (1). Point (ii) could be, in principle, dealt with exactly as before, i.e., by introducing the counterterms $\alpha(A,g'^{-1})$ and $\alpha'(A,g'^{-1})$, which are, however, not G invariant (g' would be the new FS field). Thus, at one loop, one loses the new ghosts $(c',\overline{c'})$ free definition of the physical S-matrix elements and the G-BRST invariants c' and $\overline{c'}$ start to contribute to the physical particle spectrum. In other words, beyond zero loops, the symmetry G' (and corresponding s') cannot discriminate against gauge-noninvariant c' and $\overline{c'}$. However, as we have shown, the new ghosts c' and $\overline{c'}$ should decouple in the classical limit by virtue of behaving like true FP ghosts of the accidental (exact at zero loops) gauge symmetry G' of Eq. (27).

III. TWO-DIMENSIONAL INDUCED GRAVITY

In this section we would like to apply the FS method of Sec. I to analyze the quantization problem of 2D gravity [13] (off critical string) in conformal gauge [14]. The theory at the classical level is defined in terms of the Polyakov action

$$S_0=\sum_{\mu=1}^{d}\int d^2x\sqrt{g}g^{ab}\partial_a X_\mu \partial_b X^\mu, \tag{28}$$

where $\{X^\mu(x)\}_{\mu=1,d}$ are the bosonic matter fields coupled to the 2D metric g_{ab} (in the string language, the string is immersed in a d-dimensional target space).

We use a Euclidian metric and introduce the complex coordinates

$$z=x_1+ix_2,$$

$$\overline{z}=x_1-ix_2.$$

The invariant line element can be written as

$$ds^2=g_{ab}dx^a dx^b=e^{\phi}|dz+\mu d\overline{z}|^2. \tag{29}$$

Thus, one can conveniently parametrize the metric as

$$g_{zz}=\overline{\mu}e^{\phi},\quad g_{\overline{z}\,\overline{z}}=\mu e^{\phi},$$

$$g_{z\overline{z}}=g_{\overline{z}z}=\frac{1+\overline{\mu}\mu}{2}\,e^{\phi}.$$

In terms of the parameters μ, $\overline{\mu}$, and ϕ the classical action (28) takes the form [15]

$$S_0=\sum_{\mu=1}^{d}\int \frac{dz\wedge d\overline{z}}{2i}\frac{(\overline{\partial}-\mu\partial)X_\mu(\partial-\overline{\mu}\overline{\partial})X^\mu}{1-\mu\overline{\mu}}.$$

It is understood that μ and $\overline{\mu}$ are constrained by

$$|\mu|^2<1.$$

The classical action S_0 is invariant under the gauge group G, which is the semidirect product of diffeomorphisms (general coordinate transformations) and Weyl transformations. These symmetry groups imply, respectively, (1) the symmetry under the general coordinate transformation

$$z\rightarrow z'=f(z,\overline{z}),$$
$$\overline{z}\rightarrow\overline{z'}=\overline{f}(z,\overline{z}), \tag{30}$$

where the relevant fields transform as

$$X^{\mu}(z,\overline{z})\rightarrow X'^{\mu}(z',\overline{z'})=X^{\mu}(z,\overline{z}) \quad \text{(scalar)},$$

$$\mu(z,\overline{z})\rightarrow\mu'(\overline{z'},\overline{z'})=-\frac{\overline{\partial f}-\mu\partial f}{\overline{\partial f-\mu\partial f}}(z,\overline{z}), \tag{31}$$

$$\phi(z,\overline{z})\rightarrow\phi'(z',\overline{z'})=\phi(z,\overline{z})+\ln\frac{(\overline{\partial f-\mu\partial f})(\partial f-\overline{\mu}\overline{\partial f})}{D_f^2},$$

where

$$D_f=\det\begin{pmatrix}\frac{\partial f}{\partial f} & \frac{\overline{\partial f}}{\overline{\partial f}}\end{pmatrix}.$$

(2) The symmetry under the local rescaling of the 2D metric

$$g_{ab}\rightarrow e^{\sigma}g_{ab}$$

or in terms of the μ, $\overline{\mu}$, and ϕ variables,

$$\mu\rightarrow\mu, \quad \overline{\mu}\rightarrow\overline{\mu}, \quad \phi\rightarrow\phi+\sigma. \tag{32}$$

It is well known that the theory is anomalous; i.e., one cannot regularize the path integral in a way that conserves the whole G=diffeo×Weyl group. One can see this easily, examining the matter integral measure $\mathcal{D}X^{\mu}$. With the simplest (translationally invariant or ''flat'') regularization $\mathcal{D}_0X^{\mu}$, one has

$$\prod_{\mu=1}^{d}\int\mathcal{D}_0X^{\mu}e^{S_0(X,\mu,\overline{\mu})}=\exp\left(-\frac{d}{24\pi}[W(\mu)+\overline{W(\mu)}]\right), \tag{33}$$

where $W(\mu)$ is Polyakov's ''light cone gauge'' action [13]. This is naturally Weyl invariant (S_0 does not contain the variable ϕ). On the other hand, it is equally clear that one has lost diffeomorphism's invariance, since the invariance under general coordinate transformations means

$$\delta W(\mu)=0 \tag{34}$$

under $\delta\mu=(\overline{\partial}-\mu\partial+\partial\mu)(\epsilon+\mu\overline{\epsilon})$, which corresponds to the infinitesimal version of Eq. (31) with $f(z,\overline{z})=\epsilon(z,\overline{z})$, $\overline{f(z,\overline{z})}=\overline{\epsilon(z,\overline{z})}$.

Equation (34) is equivalent to the functional differential equation

$$(\overline{\partial}-\mu\partial-2\partial\mu)\frac{\delta W}{\delta\mu(z,\overline{z})}=0.$$

A well-known computation [16] gives, instead,

$$(\overline{\partial}-\mu\partial-2\partial\mu)\frac{\delta W}{\delta\mu(z,\overline{z})}=\partial^3\mu\neq 0. \tag{35}$$

Thus, $\mathcal{D}_0X^{\mu}$ cannot be invariant under diffeomorphisms. One can define the diffeomorphism-invariant measure $\mathcal{D}_{\text{diffeo}}X^{\mu}$ by introducing the local counterterm

$$\Lambda(\mu,\overline{\mu},\phi)=-\frac{1}{2}\int\frac{dz\wedge d\overline{z}}{2\pi}\Bigg[\frac{1}{1-\mu\overline{\mu}}\{(\partial-\overline{\mu}\overline{\partial})\phi(\overline{\partial}-\mu\partial)\phi-2[\overline{\partial\mu}(\overline{\partial}-\mu\partial)+\partial\mu(\partial-\overline{\mu}\overline{\partial})]\phi\}+F(\mu,\overline{\mu})\Bigg], \tag{36}$$

where $F(\mu,\overline{\mu})$ is a local function of μ and $\overline{\mu}$ only. We do not need the explicit form of F [17].

The new effective action

$$W_{\text{cov}}(\mu,\overline{\mu},\phi)=W(\mu)+\overline{W(\mu)}+\Lambda(\mu,\overline{\mu},\phi)$$

is invariant under diffeomorphisms.

One can write $W_{\text{cov}}(\mu,\overline{\mu},\phi)$ compactly in the form

$$W_{\text{cov}}(\mu,\overline{\mu},\phi)=\int\frac{dz\wedge d\overline{z}}{2\pi}\frac{(\partial-\overline{\mu}\overline{\partial})\Phi(\overline{\partial}-\mu\partial)\Phi}{1-\mu\overline{\mu}}$$
$$=\int d^2x\sqrt{g}g^{ab}\partial_a\Phi\partial_b\Phi, \tag{37}$$

where $\Phi=\phi-\ln\partial\zeta\overline{\partial\zeta}$ and $\mu=\overline{\partial}\zeta/\partial\zeta$ (Beltrami differentials). The nonlocal (with respect to μ and $\overline{\mu}$) parameter $\zeta(z,\overline{z})$ is a Polyakov meson field (13) in 2D gravity.

One characterizes the diffeomorphism-invariant measure $\mathcal{D}_{\text{diffeo}}X^{\mu}$ by

$$\prod_{\mu=1}^{d}\int\mathcal{D}_{\text{diffeo}}X^{\mu}e^{S_0(X,\mu,\overline{\mu})}=\exp\left(-\frac{d}{24\pi}W_{\text{cov}}(\mu,\overline{\mu},\phi)\right). \tag{38}$$

(One can understand the appearance of the ϕ field, which is absent in the classical action, as due to the introduction of a covariant regularization: Λ_{cov}, $ds^2\sim e^{\phi}|dz|^2>\Lambda_{\text{cov}}^2$.)

Following, for instance, Distler, David, and Kawai (DDK) [14], in what follows we consistently make use of the diffeomorphism-invariant measure. Thus, except when indicated explicitly otherwise,

$$\mathcal{D}X^{\mu}\equiv\mathcal{D}_{\text{diffeo}}X^{\mu}, \tag{39}$$

and more generally $\mathcal{D}\varphi\equiv\mathcal{D}_{\text{diffeo}}\varphi$ for any other filed φ. Evidently, the diffeomorphism-invariant measure $\mathcal{D}X^{\mu}$ cannot be invariant under the Weyl transformation

$$\phi\rightarrow\phi+\sigma.$$

Thus, one establishes that the theory is G anomalous. Having seen that our model for 2D gravity is anomalous, one would like to apply to it the Faddeev-Shatashvili method of ''gauge-invariant'' quantization of Sec. I. As in Sec. I, we ''preestablish'' the gauge choice for the full group G=diffeo ×Weyl:

$$\mu=\mu_0,$$
$$\overline{\mu}=\overline{\mu_0} \quad \text{diffeomorphisms},$$
$$F(\phi)=0 \quad \text{Weyl}. \tag{40}$$

Since our regularization preserves the diffeomorphisms we assume that the gauge-fixing problem (with relevant ''b,c'' ghosts) for diffeomorphisms has been already taken care of.

To deal with anomalous Weyl symmetry, we have to introduce an extra degree of freedom, a scalar field $\sigma(z,\bar{z})$, corresponding to the element of Weyl symmetry group $g=e^{\sigma(z,\bar{z})}$. The anomaly canceling counterterm suggested by FS is then given by

$$\alpha(\mu,\bar{\mu},\phi;-\sigma)=W_{\rm cov}(\mu,\bar{\mu},\phi-\sigma)-W_{\rm cov}(\mu,\bar{\mu},\phi)=-\frac{1}{2}\int\frac{dz\wedge d\bar{z}}{2i}\frac{1}{1-\mu\bar{\mu}}\{(\partial-\bar{\mu}\bar{\partial})\sigma(\bar{\partial}-\mu\partial)\sigma+2(\partial-\bar{\mu}\bar{\partial})\sigma \times(\bar{\partial}-\mu\partial)\phi-2[\bar{\partial}\bar{\mu}(\bar{\partial}-\mu\partial)+\partial\mu(\partial-\bar{\mu}\bar{\partial})]\phi\}. \tag{41}$$

Note that the nonlocal part of $W_{\rm cov}$ is canceled and $\alpha(\mu,\bar{\mu},\phi;-\sigma)$ is perfectly local. Naturally, one needs the counterterm α for each covariant one loop integral corresponding not only to the matter field $\{X_\mu\}_{\mu=1}^{d}$, but also to the diffeomorphism ghosts, b,c and $\bar{b},\bar{c}$, as well as to the ϕ field contained in $W_{\rm cov}(\mu,\bar{\mu},\phi)$.

Thus, the effective action in sense of Sec. II. is given by

$$S_{\rm eff}=S_0(X,\mu,\bar{\mu})+S_{\rm GF}^{(d)}(b,c,\bar{b},\bar{c},B,\bar{B},\mu,\bar{\mu}) +\gamma'\alpha(\mu,\bar{\mu},\phi;-\sigma), \tag{42}$$

where $S_{\rm GF}^{(d)}$ is the gauge fixing term with respect to the nonanomalous diffeomorphism symmetry.

As explained above, the coefficient γ' is contributed by all the relevant fields, that is $\{X^\mu\}_{\mu=1}^{d}\Rightarrow d,(b,c,\bar{b},\bar{c})\Rightarrow-26,\phi\Rightarrow1$, which gives $\gamma'=(d-26+1)/24\pi=(d-25)/24\pi$. Note that the contribution of the ϕ field is due to the fact that $\mathcal{D}_{\rm diffeo}\phi\neq\mathcal{D}_0\phi$, or in the terminology Sec. II; that one needs the ''second'' FS counterterm $\alpha'(\phi;\sigma)$. One can now write down the partition function Z with the FS prescription [within the path integral formalism of Ref. [7], see Eq. (8) of Sec. I]. Integrating out the ''matter fields'' $(X^\mu,b,c,\bar{b},\bar{c},)$, one has

$$Z\sim\int\mathcal{D}\sigma\mathcal{D}\phi\Bigg[\exp\left(-\gamma'\int\frac{dz\wedge d\bar{z}}{2i}\frac{1}{1-\mu_0\bar{\mu}_0}\right) \times\{(\partial-\bar{\mu}_0\bar{\partial})(\phi-\sigma)(\bar{\partial}-\mu_0\partial)(\phi-\sigma) -2[\bar{\partial}\bar{\mu}_0(\bar{\partial}-\mu_0\partial)+\partial\mu_0(\partial-\bar{\mu}_0\bar{\partial})](\phi-\sigma)\}\Bigg] \times\Delta(\phi-\sigma)\delta(F(\phi)), \tag{43}$$

where the local action in the exponential is essentially a Liouville action $S_L'(\phi')$ $(\phi'=\phi-\sigma)$. The last two factors come from the δ-function insertion

$$\Delta(\phi)\int\mathcal{D}\sigma\ \delta(F(\phi+\sigma))=1. \tag{44}$$

Note that, since $\mathcal{D}\sigma\equiv\mathcal{D}_{\rm diffeo}\sigma\neq\mathcal{D}_0\sigma$ ($\mathcal{D}_0\sigma$ ''flat'' measure)

$$\Delta(\phi-\sigma)\neq\Delta(\phi). \tag{45}$$

Formally, one can write the $\Delta(\phi-\sigma)$ factor as a local action with the help of the ''Weyl ghosts'' ψ and $\bar{\psi}$

$$\Delta(\phi-\sigma)=\int\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\exp\left(-\int\bar{\psi}\frac{\delta F(\phi-\sigma)}{\delta\phi}\psi\right). \tag{46}$$

The path integral argument of Sec. I is at best heuristic. It may suggest the possible models but one cannot prove in this way their consistency. As argued in Sec. I, one may start a more precise discussion after setting up the BRST quantization procedure.

The BRST properties of the type of models we are dealing with here have been studied in detail for the critical case, i.e., for $d=26$, where the theory is not anomalous. In Ref. [15], the BRST transformation properties of the fields are given. They may be used to study our (off critical) model.

One has [see Eq. (31)]

$$\begin{aligned}\hat{\delta}X^\mu&=(\xi\cdot\partial)X^\mu,\\ \hat{\delta}_\mu&=(\bar{\partial}-\mu\partial+\partial\mu)c,\\ \hat{\delta}\phi&=\psi+(\xi\partial)\phi+(\partial\xi)+\mu\partial\bar{\xi}+\bar{\mu}\bar{\partial}\bar{\xi},\\ \hat{\delta}\xi&=(\xi\cdot\partial)\xi,\\ \hat{\delta}c&=c\partial c,\\ \hat{\delta}\psi&=(\xi\cdot\partial)\psi,\end{aligned} \tag{47}$$

where $\xi\cdot\partial$ means $\xi\partial+\bar{\xi}\bar{\partial}$. Here $\hat{\delta}$ stands for the both Weyl and diffeomorphism symmetries. The diffeomorphism ghosts $c,\bar{c}$ are related to the original $(\xi,\bar{\xi})$ [corresponding to $\delta z=\epsilon(z,\bar{z})$, $\delta\bar{z}=\bar{e}(z,\bar{z})$] by

$$\begin{aligned}c&=\xi+\mu\bar{\xi},\\ \bar{c}&=\bar{\xi}+\bar{\mu}\xi.\end{aligned} \tag{48}$$

To Eq. (47), we must add the transformation of the auxiliary field $\sigma(z,\bar{z})$. Since σ must be a scalar with respect to diffeomorphisms one has

$$\hat{\delta}\sigma=\psi+(\xi\cdot\partial)\sigma. \tag{49}$$

Together with the formulas in Eqs. (47)–(49), one consistently finds

$$\hat{\delta}^2=0. \tag{50}$$

One should add also the diffeomorphism antighost $(b,\overline{b})$ and Weyl antighost ψ with the corresponding Nakanishi-Lantrup fields B and D. Their transformation properties are

$$\hat{s}b=B, \quad \hat{s}\overline{b}=\overline{B}, \quad \hat{s}\,\overline{\psi}=D, \qquad \hat{s}B=\hat{s}\overline{B}=\hat{s}D=0. \tag{51}$$

We have seen, however, that the Faddeev-Popov factor $\Delta(\phi)$ is not Weyl invariant [Eq. (45)]. Thus, according to the result of Sec. I, one needs to correct the effective action $S_{\rm eff}$ by modifying the factor $\Delta(\phi-\sigma)\delta(F(\phi))$ into a BRST gauge-fixing term. As we have seen in Sec. I, such a prescription is not unique. Formally, any action of the form

$$\text{BRST(invariant)}+\hat{s}[\psi F(\phi)](\text{BRST exact})$$

will do the job.

Now the factor $\Delta(\phi-\sigma)\delta(F(\phi))$ can be rewritten in the form

$$\exp\left[-\int\left(\mathcal{D}F(\phi)+\overline{w'}\,\frac{\delta F}{\delta\phi}\,(\phi-\sigma)\psi'\right)\right].$$

Thus, in order to follow this expression as close as possible, we suggest to add a counterterm of the form of Eq. (20) in Sec. I,

$$\begin{aligned}\tilde{\Lambda}_G(\phi,\sigma;\psi,\overline{\psi},\psi',\overline{\psi'},D)&=\left[DG(\phi-\sigma)+\overline{\psi'}\,\frac{\delta G}{\delta\phi}\,(\phi-\sigma)\psi'\right]\\&\quad-\left[DG(\phi)+\overline{w}\,\frac{\delta G}{\delta\phi}\,(\phi)\psi\right],\end{aligned} \tag{52}$$

where we have introduced the function $G(\phi)$ to distinguish it from the true gauge-fixing term $s[\psi F(\phi)]$. The new fields ψ' and $\overline{\psi'}$ in Eq. (52) [c' and $\overline{c'}$ in Eq. (20)] are Weyl singlet and transform as

$$\hat{\delta}\overline{\psi'}=0, \qquad \hat{\delta}\psi'=(\xi\cdot\partial)\psi'. \tag{53}$$

With the addition of the counterterm $\tilde{\Lambda}_G$, the effective action now reads

$$\begin{aligned}\tilde{S}_{\rm eff}&=S_L''(\phi-\sigma)+\int\tilde{\Lambda}_G(\phi,\sigma;\psi,\overline{\psi},\psi',\overline{\psi'},D)\\&\quad+\int\hat{s}[\psi F(\phi)]\\&=S_L''(\phi-\sigma)+\int\left[DG(\phi-\sigma)+\overline{\psi'}\,\frac{\delta G}{\delta\phi}\,(\phi-\sigma)\psi'\right]\\&\quad+\int\hat{s}[\overline{\psi}(F-G)(\phi)].\end{aligned} \tag{54}$$

The expression for $\tilde{S}_{\rm eff}$ contains two arbitrary functions $F(\phi)$ and $G(\phi)$. Their roles are completely different. While $F(\phi)$ is a genuine gauge-fixing function, each choice of $G(\phi)$ actually defines a new model.

Naturally, the "series" of models (at arbitrary gauge) includes the familiar cases. For example, if one fixes the model by choosing

$$G=0,$$

one reproduces the physically equivalent formulations of the DDK model. Alternatively, for any given G, one may consider the singular gauge limit

$$F\to G.$$

In this limit the model formally corresponds to the action

$$\tilde{S}_{\rm eff}=S_L''(\phi-\sigma)+\int\left[DG(\phi-\sigma)+\overline{\psi'}\,\frac{\delta G}{\delta\phi}\,(\phi-\sigma)\psi'\right]. \tag{55}$$

This is the type of model treated in Ref. [18]. One may further add the BRST invariant term $(\lambda/2)\int D^2$ and transform $\tilde{S}_{\rm eff}$ into

$$\tilde{S}'_{\rm eff}=S_L''(\phi-\sigma)+\int\left[\frac{1}{2\lambda}\,G^2(\phi-\sigma)+\overline{\psi'}\,\frac{\delta G}{\delta\phi}\,(\phi-\sigma)\psi'\right]. \tag{56}$$

Note that the Weyl-invariant "new ghosts" ψ' and $\overline{\psi'}$ should decouple from the theory in the classical limit, as we have explained at the end of Sec. II. Under the "accidental" gauge symmetry G', the BRST transformations of ψ' and $\overline{\psi'}$ are

$$\delta'\,\overline{\psi'}=D,$$

$$\delta'\,\psi'=(\xi\cdot\partial)\psi'.$$

Because the Weyl transformation is Abelian, the transformation (27) for the FS field σ is

$$\sigma\to\sigma,$$

while under the original G, one has of course

$$\sigma\to\sigma+\alpha.$$

Equation (55) [or Eq. (56)] seems to be the closest BRST quantization scheme corresponding to the FS prescription given by the insertion

$$1=\Delta(\phi)\int\mathcal{D}\sigma\;\delta(G(\phi+\sigma)). \tag{57}$$

In Ref. [18], and in some later works, the choice

$$G(\phi)=R(\phi)-R_0 \tag{58}$$

with R the scalar curvature, has been made. Using Eq. (58), the effective action (56) becomes

$$\tilde{S}'_{\rm eff}[(\phi'=\phi-\sigma),\psi',\overline{\psi},]$$
$$=S''_L(\phi')+\int\left[\frac{1}{2\lambda}[R(\phi')-R_0]^2\right.$$
$$\left.\times(\phi-\sigma)+\overline{\psi'}\frac{\delta R}{\delta\phi}(\phi-\sigma)\psi'\right]. \quad (59)$$

Note that the model defined by Eq. (59) is fully interacting. In particular (a) the presence of propagating ψ' and ψ' fields and (b), more importantly, the presence of ψ', ψ', and ϕ' (Yukawa) interaction in Eq. (59), change the parameters in the Liouville-type action $S''_L(\phi')$. Such a change, which affects the low-energy dynamics of Eq. (59), cannot be calculated exactly. It is not easy to develop a systematic perturbation expansion [20]. We believe [18,19] that the modification represented by Eq. (59) may result in deviations from the classical DDK result, when one uses Eq. (59) to calculate physical quantities such as the string tension and the anomalous dimensions.

Lastly, it must be mentioned that the BRST-invariant term

$$\int\overline{\psi'}\frac{\delta G}{\delta\phi}(\phi-\sigma)\psi' \quad (60)$$

in Eq. (55) could also be obtained from the alternative gauge fixing

$$S_{\rm GF}=\int\hat{s}\left[\overline{\psi}\frac{\delta G}{\delta\phi}(\phi-\sigma)\sigma\right]. \quad (61)$$

In this case, one can dispense with the extra BRST-invariant (for Weyl transformation) ψ' and ψ' degrees of freedom. The gauge-fixing function is

$$F(\phi,\sigma)=\frac{\delta G(\phi-\sigma)}{\delta\phi}(\phi-\sigma)\sigma. \quad (62)$$

It looks as if this model is gauge equivalent to the DDK model, since the gauge choice $G(\phi)=\phi$ gives the effective action

$$S_{\rm eff}=S'_L(\phi-\sigma)+\int(\psi\overline{\psi}+D\sigma)\sim S'_L(\phi)\quad(\sigma=0). \quad (63)$$

The Liouville action S'_L here is identical to Eq. (44) without further renormalization [Eq. (59) is a free-field action].

In the next section, we apply the DDK-type [14] consistency arguments to analyze the consequences of the model [Eq. (59)], paying attention to the influence of the Yukawa term $(\psi'\psi'\phi)$ in Eq. (59).

IV. PHYSICAL CONSEQUENCES (MODIFIED KPZ-DDK MODEL)

After reading the last section, one may wonder if the counterterm such as

$$DG(\phi-\sigma)+\overline{\psi'}\frac{\delta G}{\delta\phi}(\phi-\sigma)\psi' \quad (64)$$

[Eq. (52) of Sec. III] may indeed influence the physics in any way. In fact, it is very probable that such an influence is washed away for a large class of ‘‘pseudo-gauge-functions’’ $G(\phi)$ by the renormalization-group argument.

However, for the specific choice of Ref. [18], i.e., [Eq. (58) of Sec. III],

$$G(\phi)=R(\phi)-R_0,$$

it gives actually the possibility to modify the classical Knizhnik-Polydkov-Zamolodchikov (KPZ) [28] results on the string tension and anomalous conformal dimensions.

The effective action that corresponds to the above choice of $G(\phi)$ is given by Eq. (59). As remarked previously, this action is equivalent to the well-known Kawai-Nakayama R^2 model [21], if one omits precisely the ‘‘fake’’ FP term

$$\int\overline{\psi'}\frac{\delta R}{\delta\phi}(\phi-\sigma)\psi' \quad (65)$$

in Eq. (59).

Now, in Ref. [21], it has been shown that the Kawai-Nakayama model with the R^2 [or $(R-R_0)^2$] term gives the same scaling behavior for large distances as the original DDK model. That is, for the fixed area partition function

$$Z(A)=\int\mathcal{D}(\text{fields})(\text{Jacobians})$$
$$\times\exp\left[-S_{\rm eff}\times\delta\left(\int dx^2\sqrt{g}-A\right)\right]$$
$$\sim\text{const }A^{-\Gamma(h)-3} \quad (66)$$

as $A\to\infty$, except for the $1/A$ correction in the exponent. The string tension is the same as the KPZ result (h is the genus)

$$\Gamma(h)=(1-h)\frac{25-d+\sqrt{(1-d)(25-d)}}{12}. \quad (67)$$

It is not a simple matter to calculate the possible change with respect to this result in the presence of the pseudo-FP term Eq. (65). The difficulty is due to the fact that we have here the genuine interacting theory instead of an effective Gaussian model such as the original DDK case [14]. Here we present the approximate analysis, which is, at best, valid for the low-energy (large distance) regime.

Writing down the pseudo-FP term Eq. (65) in detail, we have

$$S''=\int\frac{dz\wedge d\overline{z}}{2i}\sqrt{g}\overline{\psi'}[-\partial\overline{\partial}+\partial\overline{\partial}(\phi-\sigma)+R]\psi' \quad (68)$$

(in the conformal gauge where $\mu=\mu_0=0$, $\overline{\mu}=\overline{\mu_0}=0$).

In S'', the free part for ψ' and ψ' has the structure of the so-called bc ghost system

$$S''(\text{free})=\int b\overline{\partial}c\frac{dz\wedge d\overline{z}}{2i} \quad (69)$$

if one identifies

$$b\equiv\partial\overline{\psi'},$$
$$c\equiv\psi', \quad (70)$$

with the stress energy tensor and ghost number current given by

$$T = -b\partial c,$$
$$J = bc \tag{71}$$

(note that the conformal dimensions of b and c here is, respectively, 1 and 0).

Then one can give an equivalent bosonic system with

$$T' = -\tfrac{1}{2}[(\partial\varphi)^2 + Q'\partial^2\varphi],$$
$$J' = i\partial\varphi, \tag{72}$$

which reproduces the same algebraic structure as the system Eq. (71) for the suitable value of $Q'(=i)$, if the new scalar field φ satisfies

$$\varphi(z)\varphi(w) \sim -\ln|z-w|^2.$$

One should still take account of the interaction (Yukawa) term in Eq. (68). To do so, we write

$$\partial\overline{\psi'}\psi' = iA\partial\varphi + \cdots , \tag{73}$$

where the ellipsis represents the higher-order corrections.

Thus, the low energy equivalent of Eq. (68) is

$$S'' \sim \int \sqrt{\hat{g}}[-\varphi\partial\overline{\partial\varphi} - i(1+A)\varphi\hat{R} + i\alpha A\varphi\partial\overline{\partial\phi'}] \tag{74}$$

($\phi' = \phi - \sigma$, see Sec. III).

The undetermined constant A represents the first-order correction due to the interaction. The constant α is the usual gravitational correction ($g_{ab} = e^{\phi}\hat{g}_{ab} \rightarrow e^{\alpha\phi}\hat{g}_{ab}$). To avoid the imaginary coupling constant in Eq. (74) (problem of unitarity in the BRST approach), we ''Wick rotate'' φ, $\varphi \rightarrow i\varphi$. Then, with the redefinition of constants in Eq. (74), one can write a low-energy approximation as

$$S'' = \frac{1}{8\pi}\int \sqrt{\hat{g}}(\varphi\partial\overline{\partial\varphi} - 2B\varphi\partial\overline{\partial\phi'} + \tilde{Q}\varphi\hat{R}). \tag{75}$$

Putting this together with the rest of the effective action in Eq. (59), our low-energy approximation consists of taking a Gaussian model with two scalars:

$$\tilde{S}'_{\text{eff}}(\phi',\varphi) = \frac{1}{8\pi}\int \frac{dz\wedge d\overline{z}}{2i}\sqrt{\hat{g}}(-M_{ij}\Phi^i\Delta_{\hat{g}}\Phi^j - Q_i\hat{R}\Phi^i), \tag{76}$$

where

$$\Phi^i \equiv (\Phi^1,\Phi^2) = (\phi',\varphi),$$
$$Q_i \equiv (Q_1,Q_2) = (Q,-\tilde{Q}),$$

and

$$M_{ij} = \begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}.$$

Such a two-boson system with a ''Lorentzian'' metric as a kind of improvement over the standard Liouville type 2D gravity (DDK model) has been suggested in the past [18,23]. More recently, Cangemi, Jackiw, and Zwiebach have given the thorough field theoretical analysis of such a system for $B=0$, treating it as the ''dilatation gravity'' [24]. Here, however, the presence of the φ-ϕ' coupling term ($B\neq 0$) is crucial for the possible modification of the KPZ-DDK result. The origin of such a term is, of course, the Yukawa coupling in the original counterterm (64).

At this point, one can in principle apply the techniques of Ref. [20] to get the perturbative estimate of the constant A (i.e., B). We leave such an analysis for further publication and content ourselves with repeating the original DDK consistency arguments to indicate that indeed one has the possibility of changing the KPZ-DDK result.

Thus, we would like to apply the effective action (76) to estimate (a) string tension $\Gamma(h)$, and (b) renormalization $\Delta_0 \rightarrow \Delta$ of the conformal dimension of a primary operator O.

From the effective action (76), one can derive the expression for the gravitational stress energy tensor

$$T_{\text{grav}} = -\tfrac{1}{2}(M_{ij}\partial\Phi^i\partial\Phi^j + BQ_i\partial^2\Phi^i), \tag{77}$$

which contributes to the central charge by the amount

$$c_{\text{grav}} = 2 + 3M^{ij}Q_iQ_j, \tag{78}$$

where $M^{ik}M_{kj} = \delta^i_j$.

A. String tension

The details of how to generalize the DDK argument to get the string tension $\Gamma(h)$ in our model are given in Ref. [18]. We limit ourselves, therefore, to the more relevant results.

The consistency conditions lead to the determination of Q_i's as

$$Q_1 = -\frac{1}{\sqrt{3}}\left[B\sqrt{1+Bd} - \sqrt{1+B^2+(B-1)d}\right],$$
$$Q_2 = \left(\frac{1+BD}{3}\right)^{1/2}. \tag{79}$$

Then the string tension $\Gamma(h)$ is given, just as in Ref. [14], by

$$\Gamma(h) = \chi(h)\,\frac{Q_1}{\alpha} + 2$$

[$\chi(h) = 2(1-h)$ is the Euler index]. α can be calculated again as in Ref. [14] from

$$\dim(e^{\alpha\phi'}\sqrt{\hat{g}}) = 1,$$

which gives

$$\alpha = -\frac{\sqrt{1+B^2}}{2\sqrt{3}}\left[\sqrt{25+(B-1)d} - \sqrt{1+(B-1)d}\right]. \tag{80}$$

Thus, one obtains the string tension in our model as the function of B,

$$\Gamma(h)=\frac{2(1-h)}{\sqrt{1+B^2}}\frac{B\sqrt{1+Bd}-\sqrt{(1+B^2)25+(B-1)d}}{\sqrt{25+(b-1)d}-\sqrt{1+(B-1)d}}+2, \tag{81}$$

which reduces to the KPZ expression if $B=0$ (i.e., $d\leqslant 1$). Note that $\Gamma(h)$ is real for $B\geqslant 1-1/d$ with an arbitrary positive d. In view of excellent agreement between the KPZ formula and ''experiments'' for $d<1$, we might expect some sort of phase transition behavior

$$B\propto\theta(d-1)$$

[where $\theta(x)$ is the step function] but it would be very hard to show such behavior by the limited techniques available [20]. For the various ''improvements'' and applications to statistical mechanic of Eq. (81) we refer to [19].

B. Anomalous dimension

The calculation of the renormalization of conformal dimension is more straightforward. Let the base conformal dimensions of an operator O be (Δ_0,Δ_0). We would like to construct the globally defined operators $\int e^{\alpha\phi}\sqrt{\hat g}$ and $\int e^{\beta\phi}\sqrt{\hat g}O$. This requirement implies

$$\dim(e^{\alpha\phi}\sqrt{\hat g})=(1,1),$$

$$\dim(e^{\beta\phi}\sqrt{\hat g}O)=(1,1).$$

These conditions give rise to

$$\alpha^2+(Q_1+BQ_2)\alpha+2(1+B^2)=0,$$
$$\beta^2+(Q_1+BQ_2)\beta+2(1+B^2)(1-\Delta_0)=0. \tag{82}$$

The renormalized dimension Δ of the operator O can be read off from the asymptotic formula

$$F_O(A)=\int\mathcal{D}\phi'\mathcal{D}\varphi e^{-\tilde S'_{\rm eff}}\delta\left(\int e^{\alpha\phi'}\sqrt{\hat g}-A\right)\times\int Oe^{\beta\phi'}\sqrt{\hat g}\Big/Z(A)\sim K_O A^{1-\Delta}.$$

This gives, just as in Ref. [14],

$$1-\Delta=\frac{\beta}{\alpha}. \tag{83}$$

From Eqs. (82) and (83), one gets the equation determining Δ:

$$\Delta-\Delta_0=-\frac{1}{2}\frac{1}{1+B^2}\alpha^2\Delta(\Delta-1)$$

Needless to say, this too reduces to the KPZ result when $B=0$.

V. CONCLUSION

In this paper, we have tried to analyze further consequences of the Faddeev-Shatashvili method of quantizing anomalous gauge-field theories. In contrast with other authors [5(b)], we did not try to show the equivalence with the gauge-noninvariant method of which the Jackiw-Rajaraman treatment of the chiral Schwinger model is a distinguished example. On the contrary, we have argued that, in certain cases of physical interest, the FS method can be used to generate models. The series of 2D gravity models proposed here includes the models in Refs. [18] and [19] as well as the Kawai-Nakayama type $(R-R_0)^2$ (or R^2) models [21,22]. The analysis presented in Sec. IV with respect to the model Eq. (59) is at best heuristic and we certainly cannot (and do not) claim to have ''solved'' the famous $d=1$ barrier problem in 2D gravity. We merely indicate possible ways to modify the original DDK model.

To see if the possibility of enlarging in this way the 2D (induced) gravity models really throws some light on the problem of the $d=1$ barrier in 2D gravity, we need a more thorough analysis of the consistency of these models as well as a better understanding of their physical consequences. We would like to end by mentioning a further peculiarity about the anomalous diffeomorphism-Weyl gauge symmetry of 2D gravity. It is natural to ask whether, instead of somehow trying to conserve the entire gauge symmetry of the anomalous classic model, one may still have a physical consistent quantum model by keeping only the ''maximal'' anomalous free part of the classical symmetry (up to local counterterms).

Recently precisely such a suggestion has been made by Jackiw and others [25]. They counter the conventional argument favouring the diffeomorphism symmetry (over Weyl) by pointing out the even greater difficulty of conserving the whole diffeomorphism symmetry in the quantum canonical Hamiltonian approach [26].

Thus, in Ref. [25], it has been suggested to conserve Weyl symmetry plus area (volume) preserving diffeomorphism (i.e., diffeomorphism $x^\mu\to x'^\mu=f^\mu(x)$ with the constrain $\det[\partial f^\mu/\partial x^\nu]=1$).

Jackiw's formalism can be generalized to the series of models that are symmetric under the modified diffeomorphisms $D^{(k)}$:

$$x^\mu\to x'^\mu=f^\mu_{(k)}(x),$$
$$g_{\mu\nu}(x)\to g'_{\mu\nu}(x')=g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\left[\det\left(\frac{\partial x^\eta}{\partial x'^\lambda}\right)\right]^{(k-1)/k}, \tag{84}$$

where $0<k\leqslant 1$. While $k=1$ corresponds to the usual diffeomorphism-invariant DDK-like model, the limit $k\to 0$ can be shown to give the improved Weyl invariant model of Jackiw *et al.* Superficially, these models parametrized by k correspond to different gauge symmetries and, in particular, one might expect a drastic change of the physics between the two limits $k=1$ (diffeomorphisms) and $k\to 0$ (Weyl and area preserving diffeomorphisms). However, there are reasons to believe that they actually correspond to the same physics. (1) One can move formally from the ''standard'' $D^{(1)}$ invariant model to the $D^{(k\neq 1)}$ defined through Eq. (84) by a simple changing of variables. In terms of Beltrami parametrization of the 2D metric in Sec. III; this changing of variables is given by

$$\mu \to \mu^{(k)} = \mu,$$

$$\overline{\mu} \to \overline{\mu^{(k)}} = \overline{\mu},$$

$$\phi \to \phi^{(k)} = k\phi + (1-k)\ln\left(\frac{1}{1-\mu\overline{\mu}}\right) = \phi + (k-1)\ln\sqrt{-g}. \tag{85}$$

At the quantum level, Eq. (85) amounts to a different choice of the local counterterm. (2) In Ref. [27], the two-dimensional Hawking radiation has been calculated using Jackiw's Weyl-invariant model as well as the general $D^{(k)}$-invariant model. In either case, the result is identical with the standard ($D^{(1)}$-invariant) model. This fact means that at least the black-hole thermodynamics is independent from the parameter k.

If one conjectures from these facts that the choice of invariant gauge group is in some sense irrelevant (at least for 2D gravity), the implication for the anomalous gauge field theory is not clear.

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[1] C. Bouchia, J. Iliopoulos, and Ph. Mayer, Phys. Lett. **38B**, 519 (1972).

[2] D. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[3] M. Kato and K. Ogawa, Nucl. Phys. **B212**, 443 (1983).

[4] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. **54**, 1219 (1985).

[5] (a) R. Rajaraman, Phys. Lett. **154B**, 305 (1985); (b) There is a large number of works accumulated on this subject. We list below a few of them that seem to be relevant for the present discussion. The list is by no means complete however: H. O. Girotti, H. J. Rothe, and K. D. Rothe, Phys. Rev. D **34**, 592 (1986); I. G. Halliday, E. Rabinovici, A. Schwimmer, and M. Chanowitz, Nucl. Phys. **B268**, 413 (1986); D. Boyanovsky, *ibid.* **B294**, 223 (1987); L. Caneschi and V. Montalbano, Pisa Report No. IFUP Th-20/86 (unpublished); M. Mosha and Y. Oz, Phys. Lett. B **224**, 145 (1989); C. Pittori and M. Testa, Z. Phys. C **47**, 487 (1990).

[6] L. D. Faddeev and S. L. Shatashvili, Phys. Lett. **167B**, 225 (1986); L. D. Faddeev (unpublished).

[7] K. Harada and I. Tsutsui, Phys. Lett. B **183**, 311 (1987).

[8] O. Babelon, F. A. Shaposnik, and C. M. Vialet, Phys. Lett. B **177**, 385 (1986).

[9] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).

[10] C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N.Y.) **98**, 287 (1976); (unpublished).

[11] (a) T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66**, 1 (1979); (b) Jordi Paris, Phys. Lett. B **300**, 104 (1993).

[12] N. Nakanishi, Prog. Theor. Phys. **35**, 1111 (1966); B. Lautrup, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **35**, 1 (1967).

[13] (a) A. M. Polyakov, Mod. Phys. Lett. A **2**, 893 (1987); V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, *ibid.* **3**, 819 (1988); (b) The consistent BRST quantization of 2D induced gravity, while taking note of its anomalous nature, has been attempted by various authors from different points of view. See, for instance, T. Fujiwara, Y. Igarashi, J. Kubo, and T. Tabei, Phys. Rev. D **48**, 1836 (1993); F. De Jonghe, R. Siebhink, and W. Troost, Phys. Lett. B **306**, 295 (1993); F. Brandt, W. Troost, and A. Von Proeyen, in Geometry of Constrained Dynamical Systems, Proceedings of the Isaac Newton Institute Conference, Cambridge, England, 1994 (unpublished), Report No. hep-th 940706 (unpublished); J. Gomis and J. Paris, Nucl. Phys. **B431**, 378 (1994).

[14] J. Distler and H. Kawai, Nucl. Phys. **B321**, 509 (1989); F. David, Mod. Phys. Lett. A **3**, 1651 (1988).

[15] L. Baulieu, C. Becchi, and R. Stora, Phys. Lett. B **180**, 55 (1986); L. Baulieu and M. Bellon, *ibid.* **196**, 142 (1987); C. Becchi, Nucl. Phys. **B304**, 513 (1988); see also K. Fujikawa, *ibid.* **B291**, 583 (1987); K. Fujikawa, T. Inagaki, and H. Suzuki, Phys. Lett. B **213**, 279 (1988).

[16] L. Alvarez-Gaumè and E. Witten, Nucl. Phys. **B234**, 269 (1984).

[17] M. N. Sanielvici, G. W. Senenoff, and Y. Shi-Wu, Phys. Rev. Lett. **60**, 2571 (1988).

[18] M. Martellini, M. Spreafico, and K. Yoshida, Phys. Lett. A **7**, 1281 (1992); Report No. 21/26-9-1992 (unpublished).

[19] M. Martellini, M. Spreafico, and K. Yoshida, Mod. Phys. Lett. A **9**, 2009 (1994); Rome Report No. 5/1994 (unpublished).

[20] H. Kawai, Y. Kitazawa, and M. Ninomiya, Nucl. Phys. **B393**, 280 (1993); **B404**, 684 (1993).

[21] H. Kawai and R. Nakayama, Phys. Lett. B **306**, 224 (1993).

[22] T. Burwick, Nucl. Phys. **B418**, 257 (1994).

[23] J. Cohn and V. Periwal, Phys. Lett. B **270**, 18 (1991).

[24] D. Cangemi, R. Jackiw, and B. Zwiebach, Ann. Phys. (N.Y.) **245**, 408 (1996).

[25] R. Jackiw, ''Another View on Massless Matter-Gravity Fields in Two Dimensions,'' Report No. hep/th 9501016 (unpublished); D. R. Karakhanyan, R. P. Manvelyan, and R. L. Mkrtchyan, Phys. Lett. B **329**, 185 (1994); M. Martellini, Ann. Phys. (N.Y.) **167**, 437 (1986).

[26] See, e.g., R. Jackiw, ''Quantal Modifications to the Wheeler De Witt Equation,'' Report No. hep/th 9506037 (unpublished).

[27] J. Navarro-Salas, M. Navarro, and C. F. Talavera, Phys. Lett. B **356**, 217 (1995); G. Amelino-Camelia and D. Seminara, Class. Quantum Grav. **13**, 881 (1996).

[28] V. G. Kwizhnik, A. M. Polyakov, and A. B. Zdmolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).