Self-dual sector of QCD amplitudes

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We provide an action for self-dual Yang-Mills theory which is a simple truncation of the usual Yang-Mills action. Only vertices that violate helicity conservation maximally are included. One-loop amplitudes in the self-dual theory then follow as a subset of the Yang-Mills ones. In light-cone gauges this action is almost identical to previously proposed actions, but in this formulation the vanishing of all higher-loop amplitudes is obvious; the explicit perturbative S-matrix is known. Similar results apply to gravity. [S0556-2821(96)04122-7]

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I. INTRODUCTION

Certain S-matrix amplitudes in the high-energy, or massless, limit of quantum chromodynamics take a particularly simple form both at the tree and one-loop levels. These amplitudes describe processes where (almost) all external outgoing lines possess the same helicity. Such single-helicity configurations have a natural interpretation in terms of the scattering of a self-dual gauge field.

Specifically, the n-point gluon tree amplitudes with all, or all but one, helicities the same vanish (as implied by a supersymmetric identity [1,2]). Those with two helicities opposite, the Parke-Taylor amplitudes, have the momentum dependence [3]

$$A_n^{\text{tree}}(g_1^-,g_2^-,g_3^+,\ldots,g_n^+)=i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{1}$$

We have written the result in a color-ordered form and used the twistor language [4], also known as spinor helicity [5], to express the helicities and on-shell massless momenta. All quantities are written in terms of two-component SL$(2,C)$ (Weyl) spinors; in both matrix and (van der Waerden) index notation we have, for $p^2=k^2=0$,

$$p=|p\rangle[p|\Leftrightarrow p_{\alpha\dot\beta}=p_\alpha p_{\dot\beta}, \tag{2a}$$

$$\langle pk\rangle=p^\alpha k_\alpha,\quad [pk]=p^{\dot\alpha}k_{\dot\alpha}, \tag{2b}$$

where a four-vector is represented as a 2×2 matrix (whose determinant is the usual Lorentz square). The amplitude with the opposite helicity configuration is found by complex conjugation.

Furthermore, the leading-color component of the n-point one-loop gluon amplitudes with all helicities the same has the simple dependence [6]

$$A_{n;1}^{[1]}(g_1^+,g_2^+,\ldots,g_n^+)=\sum_{ijkl\ \text{cyclic}}-\frac{i}{192\pi^2}\frac{[ij]\langle jk\rangle[kl]\langle li\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{3}$$

where the sum is over cyclic orderings of any four numbers i,j,k,l in the range 1 to n. The nonleading-color component is a sum of permutations of the leading term [7]. (We refer the reader to [2,8] for a detailed discussion on the techniques used in calculating tree and loop amplitudes in gauge theories.) In a supersymmetric theory the corresponding gluon amplitude vanishes to all orders in perturbation theory. The loop amplitudes (3) have only two-particle poles and no cuts, and thus resemble tree graphs. The absence of cuts is due to the vanishing of the on-shell maximally helicity-violating (MHV) tree amplitudes appearing in the Cutkosky rules. No higher-loop amplitudes in pure Yang-Mills theory have these simple features; the cuts of two- or more-loop amplitudes are proportional to phase space integrals of nonvanishing lower-order amplitudes.

Bardeen [9] proposed that the simple form of these amplitudes could be derived from a self-dual Yang-Mills theory. Previously one of us had pointed out [10] that the light-cone [11] superspace action for self-dual $N=4$ super Yang-Mills theory is a truncation of the corresponding non-self-dual action [12] to chiral terms, and had given a Lorentz-covariant component action that generates it. In this paper we show that the self-dual theory based on the chiral truncation gives the subset of the Yang-Mills light-cone vertices that are maximally helicity violating. The S matrices derived in our formulation of self-dual Yang-Mills theory are automatically the subset of those in light-cone Yang-Mills theory consisting of amplitudes of $1-l$ gluons with helicity -1 and all the rest $+1$, where l is the number of loops. Explicitly, they consist of (1) the tree graphs with one helicity -1 and all the rest $+1$, (2) the one-loop graphs with all helicities $+1$, and (3) no graphs at all at two or more loops.

The two physical polarizations of the gauge field in the light-cone action are represented in our formulation by the highest and lowest components of a chiral superfield, as defined in the $N=4$ light-cone supersymmetric action given below or its $N=0,1,2$ truncations. Both fields appear in the theory's truncation to self-dual form. However, the self-dual action given here is not identical to previous self-dual actions [11,13,14], which have the field content of only one of the two physical polarizations. Specifically, as required by Lorentz covariance, the light-cone Yang-Mills field has two transverse components describing the two helicities, which are present in *both* the self-dual and non-self-dual theories.

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In the self-dual theory one of the two components appears only linearly in the classical action, and thus to order $1-l$ in perturbation theory.

In the following section we derive the self-dual action by truncation of the usual Yang-Mills action in the light-cone formalism. We prove this action describes self-duality and that the truncation preserves Lorentz covariance by deriving it from a Lorentz-covariant self-dual form in a way that is exact within perturbation theory. In Sec. III we compare our action with other actions proposed to describe self-dual Yang-Mills theory. The other actions, unlike ours, are not Lorentz covariant, have a dimensionful coupling constant, and at more than one loop generate nonvanishing diagrams that do not relate to Yang-Mills theory. Finally, in Sec. IV we speculate on relations to anomalies and string theory.

II. $N=4$ SUPERSYMMETRY AND SELF-DUAL LAGRANGIANS

We first consider the light-cone action for $N=4$ supersymmetric Yang-Mills theory [12]; the reduction to pure Yang-Mills theory is achieved by simply dropping the lower-spin fields.

We adopt the notation of [15], so that all quantities are written in terms of SL(2,C) two-component spinor indices. Four-vectors are written as $x^{\alpha\dot{\alpha}}$, and the component $x^{-\dot{-}}$ represents the ''time'' coordinate of the light-cone formalism. Spinor indices are raised and lowered according to $\chi^{\pm}=\mp i\chi_{\mp}$, $\chi^{\dot{\pm}}=\pm i\chi_{\dot{\mp}}$, and the Lorentz inner product is $p^2=-\det p_{\alpha\dot{\beta}}$.

In the light-cone formalism the field content of the $N=4$ vector multiplet is described by a complex chiral superfield whose components contain only the physical states. The chiral superfields relevant to $N=4$ light-cone superspace are defined by the chirality condition

$$\overline{D^a}\phi=0\Rightarrow\phi(x,\theta,\overline{\theta})=\exp(\theta^a\overline{\theta_a}i\partial_{+\dot{+}})\hat{\phi}(x,\theta) \tag{4}$$

in terms of the anticommuting derivatives

$$D_a=\frac{\partial}{\partial\theta^a}+\overline{\theta_a}i\partial_{+\dot{+}}, \quad \overline{D^a}=\frac{\partial}{\partial\theta_a}+\theta^a i\partial_{+\dot{+}}. \tag{5}$$

Here a is a four-valued index of the internal SU(4) symmetry of $N=4$ supersymmetry, and we adopt the normalization $\int d^4\theta\ \theta^4=1$. In addition, we impose the ''reality'' condition on ϕ:

$$D^4\phi=(i\partial_{+\dot{+}})^2\overline{\phi}. \tag{6}$$

Expanding ϕ in θ^a gives the various component fields, but only those corresponding to physical polarizations. In $N=4$ light-cone superspace, ϕ and $d^4\theta$ have helicity assignments $1,-2$, respectively (and opposite for the conjugates). The θ expansion of ϕ is an expansion in the component fields of helicity equal to 1 minus half the order in θ; there are 1,4,6,4,1 fields possessing helicity $+1,+1/2,0,-1/2,-1$.

The $N=4$ light-cone action can be written simply in light-cone superspace [12] as

$$S=S_2+S_{3,c}+S_{3,\overline{c}}+S_4, \tag{7a}$$

$$S_2=\frac{1}{g^2}\,\mathrm{Tr}\int d^4x\ d^4\theta\ \tfrac{1}{2}\phi\Box\phi, \tag{7b}$$

$$S_{3,c}=\frac{1}{g^2}\,\mathrm{Tr}\int d^4x\ d^4\theta\ \tfrac{1}{3}i\phi(\partial_{+}^{\dot{\alpha}}\phi)(\partial_{+\dot{\alpha}}\phi), \tag{7c}$$

$$S_4=\frac{1}{g^2}\,\mathrm{Tr}\int d^4x\ d^4\theta\ d^4\overline{\theta}\big(\tfrac{1}{8}[\phi,\overline{\phi}]^2-\tfrac{1}{4}[\phi,\partial_{+\dot{+}}\phi] \times(\partial_{+\dot{+}})^{-2}[\overline{\phi},\partial_{+\dot{+}}\overline{\phi}]\big), \tag{7d}$$

where $S_{3,\overline{c}}$ is the complex conjugate of $S_{3,c}$. (Note that $\partial_{+}{}^{\dot{\alpha}}\phi\partial_{+\dot{\alpha}}\phi=-\partial_{+\dot{\alpha}}\phi\partial_{+}{}^{\dot{\alpha}}\phi$.) Further, S_2 is real because ϕ satisfies the reality condition (6). Using this constraint the action may be written with only $d^4\theta\ d^4\theta$ in a way where reality is manifest. We have further written ϕ in matrix notation with Hermitian group generators.

The usual transverse components $A_{\pm\dot{\mp}}$ of the gauge fields appear in ϕ as

$$\phi=\frac{1}{\partial_{+\dot{+}}}A_{-\dot{+}}+\cdots-\theta^4\partial_{+\dot{+}}A_{+\dot{-}}. \tag{8}$$

The two circular polarizations of the gauge fields then reduce to particle and antiparticle assignments of the complex field $A_{+\dot{-}}$.

The total helicity of the external fields at any vertex in the action (7) follows from counting the powers of ϕ and θ: S_2 and S_4 have total helicity 0, $S_{3,c}$ has $+1$, and $S_{3,\overline{c}}$ has -1. (Since total angular momentum is conserved, the helicity may alternatively be read off from the spacetime derivatives, which give the orbital angular momentum.) The vertex which gives the maximal helicity violation is $S_{3,c}$, while $S_{3,\overline{c}}$ gives the minimal (negative) violation. Consider the truncation to S_2 and $S_{3,c}$:

$$S=\frac{1}{g^2}\,\mathrm{Tr}\int d^4x\ d^4\theta\ \tfrac{1}{2}\phi\Box\phi+\tfrac{1}{3}i\phi(\partial_{+}^{\dot{\alpha}}\phi)(\partial_{+\dot{\alpha}}\phi). \tag{9}$$

The θ expansion generates all of the (three-point) couplings between the $N=4$ matter fields in which the total outgoing helicity is 1; it generates Feynman diagrams, and amplitudes, possessing maximal helicity violation when regarded as a subset of the complete Lagrangian (7). In the supersymmetric form (9), we may replace ϕ by $\hat{\phi}$ since there is no θ dependence.

Upon further reduction to just the nonsupersymmetric Yang-Mills fields, the action (9) becomes

$$S=\frac{1}{g^2}\,\mathrm{Tr}\int d^4x\ \phi_-[\Box\phi_++i(\partial_{+}^{\dot{\alpha}}\phi_+)(\partial_{+\dot{\alpha}}\phi_+)]. \tag{10}$$

We have written the fields as they naturally appear in the θ expansion of ϕ: ϕ_+ is the lowest component and ϕ_- is the highest. This results in a Jacobian factor of 1 in going from $A_{-\dot{+}}$ to ϕ_+ and $A_{+\dot{-}}$ to ϕ_-, where the complex fields are formally treated as independent. Since ϕ_- appears only linearly in both terms in Eq. (10), it can be used to count loops;

the number of external ϕ_- lines is just $1-l$. (It can absorb the factor $1/\hbar$ multiplying the action in the functional integral, just like the dilaton in string theory.) Note that the action (10) does not require a dimensionful coupling constant; ϕ_- and ϕ_+ have mass dimensions 0 and 2.

Thus the action (10) is unable to generate diagrams with external ϕ_- lines except at tree level, in which case only one external ϕ_- state is possible. The one-loop contributions generate the amplitudes (2), as seen upon comparison with the pure Yang-Mills (YM) sector of the non-self-dual light-cone theory (7). The other vertices in the YM action (7) are quadratic in ϕ_-, and thus generate contributions to S matrices with more external lines of helicity -1 (i.e., amplitudes that are not MHV). There are no further loop corrections to the S matrix from the action (10). Furthermore, as we will prove below, this action can be obtained by quantization of an action that describes self-dual Yang-Mills theory in a manifestly Lorentz-covariant way.

[The MHV gluon amplitudes calculated in the supersymmetric theory (9) vanish to all orders in perturbation theory [1,2]. The S matrix of external gauge bosons is trivial in this case, although there are nonvanishing contributions to amplitudes between lower-spin fields.]

Self-dual Yang-Mills theory is defined only in four spacetime dimensions, and because of reality properties, only with an even number of time dimensions. If we include spinors (twistors or physical fermions), then only $2+2$ dimensions are allowed because $4+0$ has no Majorana spinors. (This is also the case relevant to the $N=2$ string [16].) However, we are interested in using the self-dual theory to describe a sector of the physical (non-self-dual) theory, which resides in $3+1$ dimensions. We now briefly clarify the differences between the actions (9) and (10) in spacetimes with these two signatures. In $3+1$ dimensions the fields ϕ and $\bar{\phi}$ are treated asymmetrically—they are complex conjugates, as are θ^a and $\bar{\theta}_a$, while $A_{\alpha\dot{\beta}}$ and $x^{\alpha\dot{\beta}}$ are Hermitian matrices. In this case, the two truncated actions (9) and (10) are then complex.

Alternatively, one can treat our actions in $D=2+2$ dimensions after a Wick rotation. In this case all covering groups for (super-)space-time symmetries become real. In particular, the SL(2,C) Lorentz symmetry becomes SL(2)⊗SL(2), and the internal SU(4) goes into SL(4). [Furthermore, conformal SU(2,2)→SL(4) and super-conformal SU(2,2|4)→SL(4|4).] Thus all the objects ϕ, $\bar{\phi}$, $A^{\alpha\dot{\beta}}$, $x^{\alpha\dot{\beta}}$, θ^a, $\bar{\theta}_a$ become separately real; θ and $\bar{\theta}$ are then independent, while the constraint (6) determines $\bar{\phi}$ in terms of ϕ.

We complete our discussion of the light-cone self-dual actions in (9) and (10) by giving a manifestly Lorentz-covariant theory which reproduces them upon going to the light cone. We start with the $N=4$ supersymmetric self-dual action [10]

$$S=\frac{1}{g^2}\,\mathrm{Tr}\int d^4x\,\tfrac{1}{2}G^{\alpha\beta}F_{\alpha\beta}+\chi^{a\alpha}\nabla_\alpha{}^{\dot{\beta}}\chi_{a\dot{\beta}}+\epsilon^{abcd}\big(\tfrac{1}{8}\phi_{ab}\Box\phi_{cd}+\tfrac{1}{4}\phi_{ab}\chi_c{}^{\dot{\alpha}}\chi_{d\dot{\alpha}}\big). \tag{11}$$

The field $G^{\alpha\beta}$ is an anti-self-dual Lagrange multiplier (which has mass dimension 2); the anti-self-dual part of the Yang-Mills field strength is

$$F_{\alpha\beta}=\partial_{(\alpha}{}^{\dot{\gamma}}A_{\beta)\dot{\gamma}}+i[A_\alpha{}^{\dot{\gamma}},A_{\beta\dot{\gamma}}]. \tag{12}$$

All the components are related by $N=4$ supersymmetry; when truncated to $N\leqslant 2$ super-Yang-Mills theories, the fields form two separate multiplets.

The action can be reduced to the light-cone form at the quantum level. We only examine here the nonsupersymmetric gauge sector, $S=(1/g^2)\mathrm{Tr}\int d^4x\,\frac{1}{2}G^{\alpha\beta}F_{\alpha\beta}$. The equations of motion are

$$F_{\alpha\beta}=0,\quad \nabla^{\alpha\dot{\alpha}}G_{\alpha\beta}=0, \tag{13}$$

which classically choose only the self-dual part $F_{\dot{\alpha}\dot{\beta}}$ of the Yang-Mills field strength to survive, while giving the anti-self-dual field $G_{\alpha\beta}$ the same field equation that would be satisfied by $F_{\alpha\beta}$ in the non-self-dual theory. The various Lorentz components expanded out give

$$L=\tfrac{1}{2}G^{\alpha\beta}F_{\alpha\beta}=-\tfrac{1}{2}G_{--}F_{++}+G_{+-}F_{+-}-\tfrac{1}{2}G_{++}F_{--}, \tag{14}$$

where, explicitly,

$$F_{++}=-2i(\partial_{+\dot{+}}A_{+\dot{-}}-\partial_{+\dot{-}}A_{+\dot{+}})+2[A_{+\dot{+}},A_{+\dot{-}}], \tag{15a}$$

$$F_{+-}=-i(\partial_{+\dot{+}}A_{-\dot{-}}-\partial_{+\dot{-}}A_{-\dot{+}}+\partial_{-\dot{+}}A_{+\dot{-}}-\partial_{-\dot{-}}A_{+\dot{+}})+([A_{+\dot{+}},A_{-\dot{-}}]+[A_{-\dot{+}},A_{+\dot{-}}]), \tag{15b}$$

$$F_{--}=-2i(\partial_{-\dot{+}}A_{-\dot{-}}-\partial_{-\dot{-}}A_{-\dot{+}})+2[A_{-\dot{+}},A_{-\dot{-}}]. \tag{15c}$$

We first choose the light-cone gauge $A_{+\dot{+}}=0$; as usual, the Faddeev-Popov ghosts decouple. In this gauge the G_{--} term has only an Abelian component,

$$L^{\mathrm{LC}}_{--}=iG_{--}\partial_{+\dot{+}}A_{+\dot{-}}, \tag{16}$$

and may also be functionally integrated out; this enforces $A_{+\dot{-}}=0$. (The constant Jacobian det $\partial_{+\dot{+}}$ decouples, as in the Faddeev-Popov determinant of the previous step.) The surviving contribution for G_{+-} is now also Abelian,

$$L^{\mathrm{LC}}_{+-}=-iG_{+-}(\partial_{+\dot{+}}A_{-\dot{-}}-\partial_{+\dot{-}}A_{-\dot{+}}), \tag{17}$$

and can be solved to give the final expression for the gauge potentials:

$$A_{+\dot{\alpha}}=0,\quad A_{-\dot{\alpha}}=\partial_{+\dot{\alpha}}\phi_+. \tag{18}$$

We are left with the L^{LC}_{++} term; upon relabeling $G_{++}=i\phi_-$ we find the action (9).

The manipulations we have just performed are exact within perturbation theory, and prove the equality of the covariant (11) and light-cone (10) forms of the S matrix elements to all orders, in the gauge sector. The complex-conjugate Lagrangian may be derived using an (anti-) self-dual covariant action, i.e., with dotted and undotted indices reversed in Eq. (11). (As usual, we freely invert the ''spatial'' derivative $\partial_{+\dot{+}}$, which is legal with appropriate boundary conditions. Also, since the theory is Lorentz covariant, $\partial^{-1}_{+\dot{+}}$ cannot generate poles by itself in $D>2$. Furthermore, since we neglect only determinants of free derivatives, any modes which might be missed by inverting such derivatives are those that decouple.)

Finally, we make a few remarks about how helicity is defined and its relation in the self-dual and non-self-dual actions. The simplest way to define helicity is in terms of field strengths. This method is not only Lorentz and gauge covariant, but also applies to interacting states. For example, $F_{\dot\alpha\dot\beta}$ describes helicity $+1$, while $F_{\alpha\beta}$ (or $G_{\alpha\beta}$ in the self-dual formulation, where $F_{\alpha\beta}=0$) describes -1. The helicity is simply half the number of dotted minus undotted indices, which follows from the fact that any field strength satisfies a Weyl equation on each spinor index. This translates into counting the twistors that carry these indices: In the free theory, or for asymptotic states,

$$F_{\dot\alpha\dot\beta}=p_{\dot\alpha}p_{\dot\beta}f_+, \quad F_{\alpha\beta} \quad \text{or} \quad G_{\alpha\beta}=p_\alpha p_\beta f_- \tag{19}$$

in terms of some scalar twistor-space functions $f_\pm$. These expressions have close analogs in ordinary coordinate (or momentum) space; in the usual Yang-Mills theory in the light-cone gauge, where $A_{+\dot+}=0$ and $A_{-\dot-}$ is eliminated by its field equation, we have

$$F_{\dot\alpha\dot\beta}=-i\partial_{+\dot\alpha}\partial_{+\dot\beta}\partial_{+\dot+}{}^{-1}A_{-\dot+}+O(A^2),$$

$$F_{\alpha\beta}=-i\partial_{\alpha\dot+}\partial_{\beta\dot+}\partial_{+\dot+}{}^{-1}A_{+\dot-}+O(A^2) \tag{20}$$

on shell. In the LMP-type light-cone gauge for self-dual Yang-Mills theory we have

$$F_{\dot\alpha\dot\beta}=-i\partial_{+\dot\alpha}\partial_{+\dot\beta}\phi_+, \quad F_{\alpha\beta}=0, \tag{21}$$

$$G_{\alpha\beta}=\partial_{+\dot+}{}^{-1}\nabla_{\alpha\dot+}\partial_{+\dot+}{}^{-1}\nabla_{\beta\dot+}\phi_-=\partial_{\alpha\dot+}\partial_{\beta\dot+}\partial_{+\dot+}{}^{-2}\phi_- + O(\phi^2). \tag{22}$$

In 2+2 dimensions, we have the freedom to scale p_α and $p_{\dot\alpha}$ oppositely in $p_{\alpha\dot\beta}=p_\alpha p_{\dot\beta}$. (In 3+1 the invariance is a phase, and we generally have to write $p_{\alpha\dot\beta}=\pm p_\alpha p_{\dot\beta}$ to treat both positive and negative energy. These problems are also avoided by our Wick rotation from 2+2.) This allows us to choose

$$p_+=1 \Rightarrow p_{\dot\alpha}=p_{+\dot\alpha}. \tag{23}$$

This makes Feynman graph calculations in the self-dual theory almost indistinguishable from twistor calculations, since noncovariant vertex factors $p_{+\dot\alpha}$ can be replaced with covariant twistors $p_{\dot\alpha}$ after being expressed in terms of (on-shell) external momenta.

III. RELATIONS TO OTHER PROPOSED SELF-DUAL ACTIONS

Except for the θ integration, the above truncated $N=4$ light-cone action (9) is the one proposed by Leznov and Mukhtarov, and Parkes (LMP) [11] to describe self-dual Yang-Mills theory,

$$S_{\text{LMP}}=\frac{1}{\lambda^2}\,\text{Tr}\int d^4x\,\tfrac{1}{2}\phi\Box\phi+\tfrac{1}{3}i\phi(\partial_+^{\dot\alpha}\phi)(\partial_{+\dot\alpha}\phi). \tag{24}$$

However, the action we use has several important differences. The most important is that, after truncation to the nonsupersymmetric Yang-Mills sector, we have *two* polarizations, as required for Lorentz invariance, not one. In the kinetic term the lowest order in θ component of the superfield ϕ (helicity $+1$) couples to the highest one (helicity -1).

The fact that the LMP action has only one field has two immediate consequences: (1) The LMP action is not Lorentz invariant, not even in a hidden way. (2) The coupling constant in the LMP action has the wrong (engineering) dimension. The above $N=4$ action, and its $\phi_\pm$ truncation, have neither of these problems.

We now compare the S matrices of our action to those of the LMP action in the nonsupersymmetric case. (The supersymmetric forms are almost trivial since all loop amplitudes vanish for both theories.) (1) In our case the propagator has a "+" at one end and a "−" at the other; the vertex has 2+'s and a −. In the LMP case no lines are distinguished. (2) There is no difference at the tree level, since tree S matrices vanish, except for the three-point vertex, which is nonvanishing in 2+2 dimensions. (In 3+1, kinematic constraints force it to vanish.) The three-point contribution is indistinguishable in the two theories because of the symmetry of the vertex, and because the normalization can be absorbed by a redefinition of the coupling or of ϕ_-. (3) At the one-loop level the LMP action gives the same result *except* for an additional factor of 1/2, since there is only one field and not two. As usual for one-loop graphs, this normalization cannot be modified. (4) At higher loops all graphs vanish for our action. There is no such implication for the LMP action, which apparently has higher-loop contributions.

Another action to compare against is that proposed by Donaldson, and Nair and Schiff [14], based on Yang's [13] form of the self-dual equations (YDNS). We find a similar action from the above covariant form (11) by slightly modifying the above steps to the light cone. As before, we choose the gauge $A_{+\dot+}=0$ in Eq. (14) and functionally integrate out G_{--}, so $A_{+\dot-}=0$. Instead of examining the G_{+-} term, however, we Abelianize the G_{++} term by the field redefinitions

$$A_{-\dot-}=-ie^{-i\phi}\partial_{-\dot-}e^{i\phi},$$

$$A_{-\dot+}=-ie^{-i\phi}(\partial_{-\dot+}+iA'_{-\dot+})e^{i\phi},$$

$$G_{++}=e^{-i\phi}G'_{++}e^{i\phi}. \tag{25}$$

The L_{++} term is then

$$L_{++}=-G'_{++}\partial_{-\dot-}A'_{-\dot+}. \tag{26}$$

Integrating out G'_{++} sets $A'_{-\dot+}=0$, after dropping the irrelevant Jacobian factor det $\partial_{-\dot-}$.

Up till now all Jacobians have been constants. Another type of trivial Jacobian is one of a functional determinant involving no derivatives: If such determinants are written in terms of Faddeev-Popov-like ghosts, the ghosts have nonderivative propagators. Such determinants produce $\delta^4(0)$ terms, which can be neglected. (For example, they vanish in dimensional regularization.) The Jacobian from the change of variables (25) reduces to that for the first redefinition, times nonderivative determinants of this type. This remaining contribution to the effective action can be represented by a Faddeev-Popov-like expression

$$S_c = \mathrm{Tr} \int d^4x\ \tilde{C} Q(e^{-i\phi}\partial_{-\dot{-}} e^{i\phi}) \quad \text{with } Q\phi = C, \tag{27}$$

where Q is a Becchi-Rouet-Stora-Tyutin- (BRST-)like operator, which acts in the same way as a derivative or variation. The action may be reorganized as

$$S_c = -\mathrm{Tr} \int d^4x (e^{-i\phi}\tilde{C} e^{i\phi})\partial_{-\dot{-}}(e^{-i\phi} Q e^{i\phi}). \tag{28}$$

We next perform two successive field redefinitions on the ghosts, the Jacobians of which are trivial [$\delta^4(0)$ and constant terms, respectively],

$$\tilde{C} = e^{i\phi}\tilde{C}' e^{-i\phi}, \quad \tilde{C}' = \frac{1}{\partial_{-\dot{-}}} \tilde{C}'', \tag{29}$$

and obtain the contribution

$$S_c = \mathrm{Tr} \int d^4x\ \tilde{C}''(e^{-i\phi} Q e^{i\phi}). \tag{30}$$

This ghost term may be path-integrated out since it is algebraic. The final expression for the potential is

$$A_{+\dot{\alpha}} = 0, \quad A_{-\dot{\alpha}} = -ie^{-i\phi}\partial_{-\dot{\alpha}} e^{i\phi}. \tag{31}$$

The resulting action comes from the L^{LC}_{+-} term, and gives the Yang field equation, but from a two-field action

$$S = -i\ \mathrm{Tr} \int d^4x\ G_{+-}\partial_+{}^{\dot{\alpha}}(e^{-i\phi}\partial_{-\dot{\alpha}} e^{i\phi}). \tag{32}$$

This action thus also gives S matrices equal to those of non-self-dual Yang-Mills theory restricted to certain helicities.

On the other hand, the YDNS action gives S matrices that disagree in the same way as described above for the LMP action. The YDNS action gives the same field equations as Eq. (32), but in terms of one field instead of two:

$$\delta S = \int f(\phi)\delta G_{+-} + h(\phi, G_{+-})\Delta\phi,$$

$$S_{\mathrm{YDNS}} = \int f(\phi)\Delta\phi; \quad \Delta\phi \equiv -ie^{-i\phi}\delta e^{i\phi}; \tag{33}$$

where we have used the covariant variation $\Delta\phi$. (Using the covariant variation instead of the naive one just introduces another trivial determinant.) The one-loop S matrix is expressed in terms of the one-loop effective action, which is the determinant of the second functional derivative of the classical action:

$$S_{\mathrm{eff,YDNS}} = -\tfrac{1}{2} \ln \det\left(\frac{\delta^2 S_{\mathrm{YDNS}}}{\Delta\phi\Delta\phi}\right) = -\tfrac{1}{2} \ln \det\left(\frac{\delta f}{\Delta\phi}\right),$$

$$S_{\mathrm{eff}} = -\tfrac{1}{2} \ln \det \begin{pmatrix} \dfrac{\delta^2 S}{\Delta\phi\Delta\phi} & \dfrac{\delta^2 S}{\Delta\phi\delta G_{+-}} \\ \dfrac{\delta^2 S}{\delta G_{+-}\Delta\phi} & \dfrac{\delta^2 S}{\delta G_{+-}\delta G_{+-}} \end{pmatrix}$$

$$= -\tfrac{1}{2} \ln \det \begin{pmatrix} \dfrac{\delta h}{\Delta\phi} & \dfrac{\delta f}{\Delta\phi} \\ \dfrac{\delta f}{\Delta\phi} & 0 \end{pmatrix} = -\ln \det\left(\frac{\delta f}{\Delta\phi}\right). \tag{34}$$

We have thus proven the equivalence of our modifications of the LMP and YDNS actions, and that the original LMP and YDNS actions give the same one-loop S matrices (both differing from ours by a factor of 1/2).

The YDNS action has also been proposed to describe the $N=2$ (open) string [16]. However, it is also possible to interpret that string in terms of our two-field modification of that action: States in that string in different pictures are usually interpreted as the same state, since their couplings are the same. However, in ordinary QCD we know maximally helicity violating couplings are helicity independent. If we use helicity (i.e., Lorentz transformations) to distinguish otherwise-identical states [17], then (at least) two different states appear in Lorentz-invariant amplitudes.

Similar remarks can be made regarding gravity. The analog of the YDNS action for self-dual gravity, the Plebański action [18], must be modified to contain the fields describing both ± 2 helicities. The light-cone action for gravity [20] can easily be truncated for maximal helicity violation to give the analog of the LMP action [19]; the infinite number of terms reduce to one interaction plus the kinetic term. All the other terms generate amplitudes which contain at least one more negative-helicity external state.

Remarks made in the introduction carry over to the gravitational case. As with Yang-Mills theory, the MHV graviton scattering amplitudes vanish at tree level and must be cut free at one loop. However, the all-plus one-loop scattering amplitudes have not been calculated beyond four-point [21]; complete solutions to the self-dual theory, unlike SDYM, are not known explicitly.

IV. DISCUSSION

Bardeen has conjectured that these amplitudes are related to anomalies. The effective action for our self-dual theory receives contributions only at one loop. A possible candidate for this one-loop contribution is the trace anomaly, which leads to very simple effective actions in two-dimensional theories. For example, in the Schwinger model a fermion loop generates exactly $F\Box^{-1}F$ for the effective action. The four-dimensional analog would be $F\Box^{\epsilon}F/\epsilon = F^2/\epsilon + F(\ln\Box)F$, where the divergent term vanishes upon integration for self-dual F (and $\Box$ is gauge covariant). We have been unable to verify, however, that the latter term is in fact the complete effective action.

Another, more interesting, possibility is that the one-loop contribution might be generated by a *local* term in the effective action through the introduction of extra fields. This also has an analog in the Schwinger model, where the fermion's contribution to $S_{\mathrm{eff}}[A]$ may be reproduced by introducing an extra scalar field (the fermion-antifermion condensate that comes from bosonization), resulting in the Stueckelberg action for a massive vector.

The existence of a local term is suggested not only by the appearance of only poles in the one-loop S matrices, but by string theory: The $N=2$ open string is known to describe self-dual Yang-Mills theory [12] (or its supersymmetric generalizations [15]). One-loop diagrams in open-string theory are equivalent to *tree* graphs in the combined theory of open and closed strings [22]. In the one-loop planar graph, the loop can be pulled out to represent a closed string propagator connecting an open string tree to the vacuum; the one-loop double-twisted graph can be stretched to produce a closed string propagator connecting two open string trees. This suggests the introduction of fields without physical polarizations to represent the closed string. A likely candidate would be a dilaton, namely the Weyl scale mode of the metric, which in ordinary gravity has no physical degrees of freedom (although it has a nontrivial kinetic term). Also, it couples to the trace of the energy-momentum tensor, which relates to the previous conjecture concerning the trace anomaly.

Explicit calculations in string theory [23], however, have indicated the vanishing of all one-loop graphs with more than three external lines in all $N=2$ string theories. These string results are in direct contradiction with field theory. This suggests some subtlety was missed, possibly signaling the presence of an anomaly in the world-sheet theory describing the string.

Note added. After this work was completed, Cangemi [24] showed by explicit calculation that the light-cone action for self-dual Yang-Mills theory gives the one-loop S matrices for ordinary Yang-Mills theory with all external helicities the same.

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