

## Black hole area in Brans-Dicke theory

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We show that the dynamics of the scalar field  $\phi(x) = "G^{-1}(x)"$  in Brans-Dicke theories of gravity makes the surface area of the black hole horizon *oscillatory* during its dynamical evolution. It explicitly explains why the area theorem does not hold in Brans-Dicke theory. However, we show that there exists a certain nondecreasing quantity defined on the event horizon which is proportional to the black hole entropy for the case of stationary solutions in Brans-Dicke theory. Some numerical simulations are demonstrated for Oppenheimer-Snyder collapse in Brans-Dicke theory. [S0556-2821(96)05022-9]

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### I. INTRODUCTION

Scheel, Shapiro, and Teukolsky [1] have numerically shown that Oppenheimer-Snyder collapse in Brans-Dicke theories of gravity produces black holes that are identical to those of general relativity in the final stationary stage, but behave quite differently during dynamical evolution. For example, in general relativity the apparent horizon of a black hole is always inside the event horizon and the total surface area of the event horizon never decreases in *any* classical process provided that null energy condition for matter fields (i.e.,  $T_{ab}k^ak^b \geq 0$  for all null vectors  $k^a$ ) and cosmic censorship conjecture are satisfied [2]. In Brans-Dicke (BD) theories of gravity, however, they found that there are some initial epochs in Oppenheimer-Snyder collapse during which not only the apparent horizon passes outside the event horizon, but also the surface area of the event horizon decreases in time. Thus the area increase theorem for black holes in general relativity does not hold in Brans-Dicke theory. As mentioned briefly in Ref. [1], these different behaviors are possible because the null convergence condition, i.e.,  $R_{ab}k^ak^b \geq 0$  for all null vectors  $k^a$ , is violated for a dynamical black hole in Brans-Dicke theory. At the present paper, we investigate how the behavior of the auxiliary scalar field  $\phi(x) = "G^{-1}(x)"$  in Brans-Dicke theory causes this violation and the oscillatory behavior of the surface area in detail.

One may think that the violation of the area theorem for black holes in Brans-Dicke theory causes some problem to the black hole thermodynamics in BD gravity since the black hole entropy in Einstein gravity is proportional to the surface area of the horizon and so the area theorem automatically serves as the classical second law in black hole mechanics. We show that there exists a certain quantity defined on the event horizon which never decreases during any classical process in BD gravity. Moreover, this quantity coincides with the entropy in the case of stationary black holes in BD theory. Thus the black hole thermodynamic second law in BD theory is established. Since BD theories of gravity are dynamically related to other theories of gravity such as higher curvature theories of gravity and dilaton gravity, the results for black holes in BD theory also give some insights

for those in other theories of gravity.

In Sec. II, we explain why black holes in BD theory behave differently from those in Einstein gravity by investigating the behavior of the auxiliary scalar field  $\phi(x)$ . In Sec. III, a quantity is constructed on the event horizon from various points of view and proved that it is always nondecreasing for arbitrary dynamical processes. Finally, the validity of assumptions on the scalar field  $\phi(x)$  used in the proof is discussed. General features of the proof and possible application to other theorems are also mentioned briefly.

### II. BLACK HOLE AREA IN BRANS-DICKE THEORY

Let us consider the change of the total surface area of the horizon of any black hole along the null congruence of the horizon generators orthogonal to the spacelike cross section  $\mathcal{H}$ :

$$\frac{dA(\lambda)}{d\lambda} = \frac{d}{d\lambda} \oint_{\mathcal{H}} d^2x \sqrt{h} = \oint_{\mathcal{H}} d^2x \sqrt{h} \theta(\lambda), \quad (1)$$

where  $\lambda$  is the affine parameter of null geodesics whose tangents are  $k^a$ , i.e.,  $k^a \nabla_a = d/d\lambda$ , and  $\theta = d(\ln \sqrt{h})/d\lambda = \nabla_a k^a$  is the expansion of the horizon generators. One can easily see that, if  $\theta(\lambda) \geq 0$  everywhere on the horizon and at any point in  $\lambda$ ,  $dA/d\lambda \geq 0$  and so the surface area is nondecreasing always. Otherwise, it either still increases or decreases depending on the value of integration of  $\theta$  over the whole event horizon [3].

Now let us consider the evolution of expansion along a hypersurface orthogonal null congruence of geodesics in general. It is determined by Raychaudhuri equation

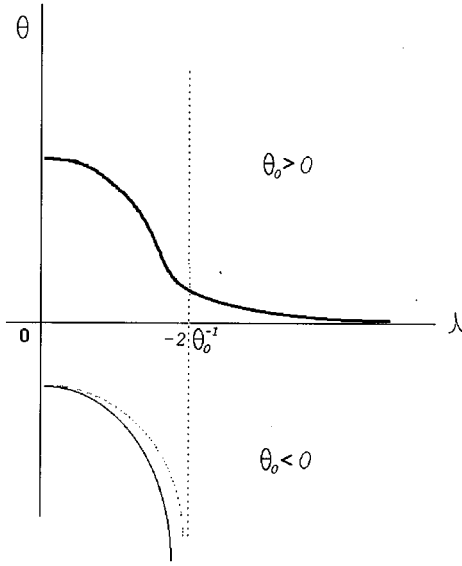
$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}k^ak^b, \quad (2)$$

where  $\sigma^2$  is the square of the shear. In the case that  $R_{ab}k^ak^b \geq 0$  for any null vector  $k^a$ ,  $d\theta/d\lambda$  is negative-definite and

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \quad (3)$$

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or

FIG. 1. Behavior of the expansion  $\theta$  in general relativity.

$$\frac{d\theta^{-1}}{d\lambda} \geq \frac{1}{2}. \quad (4)$$

Suppose one has  $\theta < 0$  at some point on a null geodesic. Then the above inequality equation shows that  $\theta^{-1}$  should meet zero within some finite affine parameter. That is to say,  $\theta$  reaches  $-\infty$  at some finite affine parameter, giving a conjugate point as can be seen in Fig. 1.

When applied to black holes in a strongly asymptotically predictable spacetime (thus, no naked singularity exists outside the black hole region), this behavior of expansion leads that  $\theta$  cannot be negative, i.e.,  $\theta \geq 0$  everywhere on the horizon and at any point in  $\lambda$  [2]; a similar derivation will be explained more detailly in Sec. III. It, therefore, shows  $dA/d\lambda \geq 0$  proving the area theorem: *the total surface area never decreases in any classical process.*

For black holes in Einstein gravity, the null convergence condition is equivalent to the null energy condition since field equations are

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi GT_{ab}. \quad (5)$$

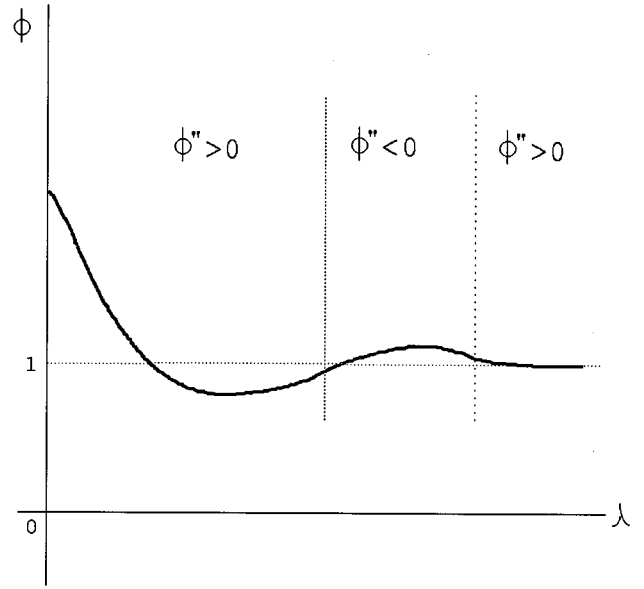
For black holes in BD gravity, however,  $R_{ab}k^ak^b$  becomes *indefinite* as follows:

$$\mathcal{L}_{\text{BD}} = \frac{1}{16\pi G} \left( R\phi - \frac{\omega}{\phi} \nabla_a \phi \nabla^a \phi \right) + \mathcal{L}_{\text{matter}}. \quad (6)$$

Field equations are

$$\begin{aligned} \left( R_{ab} - \frac{1}{2}Rg_{ab} \right) \phi = 8\pi GT_{ab} + \frac{\omega}{\phi} \left( \nabla_a \phi \nabla_b \phi \right. \\ \left. - \frac{1}{2}g_{ab} \nabla_c \phi \nabla^c \phi \right) + \nabla_a \nabla_b \phi - g_{ab} \nabla_c \nabla^c \phi. \end{aligned} \quad (7)$$

And so

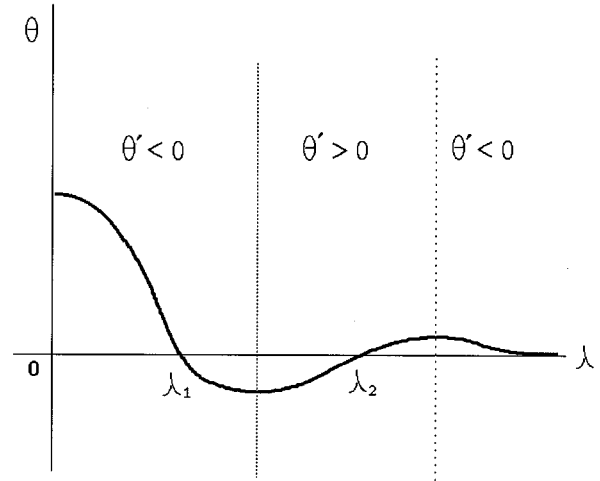
FIG. 2. Oscillatory behavior of the scalar field  $\phi(x)$  in Brans-Dicke theory.

$$\begin{aligned} R_{ab}k^ak^b = 8\pi G\phi^{-1}T_{ab}k^ak^b + \omega\phi^{-2}(k^a\nabla_a\phi)^2 \\ + \phi^{-1}k^ak^b\nabla_a\nabla_b\phi. \end{aligned} \quad (8)$$

One sees that, even if null energy condition and positivity of  $\phi$  and  $\omega$  are satisfied,  $R_{ab}k^ak^b$  is *indefinite* since the last term,  $\phi^{-1}k^ak^b\nabla_a\nabla_b\phi = \phi^{-1}k^a\nabla_a(k^b\nabla_b\phi) = \phi^{-1}d^2\phi/d\lambda^2 = \phi^{-1}\phi''$ , could be strongly negative. Now Eq. (2) becomes

$$\begin{aligned} \theta' = - \left[ \frac{1}{2}\theta^2 + \sigma^2 + 8\pi G\phi^{-1}T_{ab}k^ak^b + \omega(\phi'/\phi)^2 \right] \\ - \phi^{-1}\phi''. \end{aligned} \quad (9)$$

Therefore, if the auxiliary scalar field  $\phi(x)$  behaves as in Fig. 2 during the dynamical evolution of black holes, the expansion  $\theta(\lambda)$  possibly behaves as in Fig. 3 along the horizon generators. In other words, before  $\theta$  hits the negative

FIG. 3. Possible behavior of the expansion  $\theta$  in Brans-Dicke theory corresponding to that of  $\phi$  in Fig. 2.

infinity after  $\lambda_1$  in Fig. 3, the negativeness of  $\phi''$  starts to increase  $\theta$  up over zero. Consequently, the total surface area of the horizon is increasing until  $\lambda = \lambda_1$ , decreasing between  $\lambda_1$  and  $\lambda_2$ , increasing again after  $\lambda = \lambda_2$ , and finally approaches constant.

For Oppenheimer-Snyder collapse in BD theory where an initially uniform particle distribution within a sphere is allowed to collapse to a final state, the numerical results obtained in Ref. [1] demonstrate this behavior described above. That is, Fig. 10 in Ref. [1] shows the indefiniteness of  $R_{ab}k^ak^b$  for different values of  $\omega$ , and Fig. 9 in Ref. [1] shows that the surface area is strongly oscillating for small  $\omega$  and becomes monotonically increasing for large  $\omega$ . A monotonic behavior for large  $\omega$  appears because, even though BD theory does not completely reduce to Einstein gravity in general as  $\omega \rightarrow \infty$ , it does for physical cases as pointed out in Ref. [1]. Figure 9 in Ref. [1] also shows that apparent horizons pass over the event horizons during epochs of  $\theta \leq 0$ . It is simply because the expansion of null geodesics is negative on the event horizon for those periods. Then trapped surfaces can exist outside the event horizon in contrary to black hole solutions in general relativity. The outer boundary of this trapped region where expansion vanishes will be the apparent horizon. Interestingly, ‘‘outgoing’’ null geodesics from a trapped surface outside the event horizon do not hit the singularity. Instead they again expand after while perhaps due to the last term in Eq. (9) and escape to the null infinity finally.

### III. A NONDECREASING QUANTITY

In the thermodynamic analogy of classical black hole mechanics in Einstein gravity, the total surface area of the event horizon plays a somewhat special role. It is proportional to the black hole entropy, which has been later justified by considering quantum effect on matter fields around the black

hole. Hawking’s area increase theorem then establishes the classical second law of black hole mechanics in Einstein gravity. In the previous section, however, we have seen that the area theorem does not hold for black holes in Brans-Dicke theory. Does this mean that the second law does not hold for black holes in BD theory? This question turns out to be irrelevant since in general ‘‘area-entropy’’ relation no longer holds in other theories of gravity [4]. Indeed this modification of ‘‘area-entropy’’ relation happens for black holes in BD theory.

The entropy for stationary black holes in Einstein gravity is given by

$$S_E = A/4G = \frac{1}{4G} \oint_{\mathcal{H}} d^2x \sqrt{h}. \quad (10)$$

Since Newton’s constant  $G$  becomes dynamical in BD theory, a natural candidate for the entropy of stationary black holes in BD theory will be

$$S_{BD} = \frac{1}{4G} \oint_{\mathcal{H}} d^2x \sqrt{h} \phi(x). \quad (11)$$

Note that we defined  $\phi(x)$  to be dimensionless. This quantity is indeed the black hole entropy for stationary solutions in BD theory evaluated by many other methods such as Noether charge method, field redefinition, and Euclidean method [5].

The formula above can also be obtained by considering a ‘‘physical process’’ derivation of the first law in black hole thermodynamics introduced by Wald [6] and extended to a certain class of higher curvature theories in Ref. [7]. Consider a black hole stationary initially, being perturbed by small amount of matter falling, and finally settling down to a stationary state again. Small amount of matter falling into a stationary black hole,  $\Delta M = \int_O T_{ab} \xi^a d\Sigma^b$  and  $\Delta J = - \int_O T_{ab} \varphi^a d\Sigma^b$  in the asymptotically flat region, produces some change at the event horizon as follows:

$$\Delta M - \Omega \Delta J = \int_O T_{ab} \chi^a d\Sigma^b = - \int_H T_{ab} \chi^a d\Sigma^b = \kappa \int_H T_{ab} k^a k^b \lambda \sqrt{h} d\lambda d^2x, \quad (12)$$

where  $\chi^a = \xi^a + \Omega \varphi^a$  is the Killing generator of the horizon in leading order of the perturbation and  $H = \mathcal{H} \otimes \mathbf{R}$  the black hole horizon; see details in Ref. [7]. From field equations in Eq. (7) one obtains, in leading order,

$$\begin{aligned} \int_H d^2x d\lambda \sqrt{h} \lambda T_{ab} k^a k^b &= \frac{1}{8\pi G} \int_{\mathcal{H}} d^2x \int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{h} \lambda \left( \phi R_{ab} k^a k^b - \frac{\omega}{\phi} \phi'^2 - \phi'' \right) \\ &\simeq \frac{1}{8\pi G} \int_{\mathcal{H}} d^2x \int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{h} \lambda (-\phi \theta' - \phi'') \\ &\simeq \frac{1}{8\pi G} \int_{\mathcal{H}} d^2x \left[ (\sqrt{h} \phi) \Big|_{\lambda_i}^{\lambda_f} - \sqrt{h} \lambda (\phi \theta + \phi') \Big|_{\lambda_i}^{\lambda_f} \right]. \end{aligned} \quad (13)$$

Since  $\theta = \phi' = 0$  on both initial and final stationary stages, one finds

$$\Delta M - \Omega \Delta J \simeq \frac{\kappa}{8\pi G} \int_{\mathcal{H}} d^2x (\sqrt{h} \phi) \Big|_{\lambda_i}^{\lambda_f} = \frac{\kappa}{2\pi} \Delta S_{BD}, \quad (14)$$

giving the formula in Eq. (11) for black hole entropy in BD theory.

Now let us assume that the quantity in Eq. (11) is the black hole entropy even at any moment of dynamical evolution of the horizon and see how it behaves. To be more general in the proof, let us consider the quantity

$$S = \frac{1}{4G} \oint_{\mathcal{H}} d^2x \sqrt{h} e^\rho, \quad (15)$$

where  $e^\rho$  is a scalar function locally defined on the horizon. As before, the change of  $S(\lambda)$  along the null congruence generating the event horizon is

$$\frac{dS}{d\lambda} = \frac{1}{4G} \oint_{\mathcal{H}} d^2x \sqrt{h} e^\rho \tilde{\theta} \quad (16)$$

with

$$\tilde{\theta} = \theta + \partial_\lambda \rho. \quad (17)$$

Now let us apply the method developed in Ref. [7] which basically follows Hawking's proof of the area theorem, with  $\tilde{\theta}$  in place of  $\theta$ . Again the question is whether or not there can exist a point along the null geodesics at which  $\tilde{\theta}$  becomes negative. The Raychaudhuri equation shows

$$\partial_\lambda \tilde{\theta} = -\frac{1}{2} \theta^2 - \sigma^2 - \omega^2 - (R_{ab} - \nabla_a \nabla_b \rho) k^a k^b. \quad (18)$$

If the last term is positive-definite, then one has

$$\partial_\lambda \tilde{\theta} \leq -\frac{1}{2} \theta^2, \quad (19)$$

or

$$\partial_\lambda \tilde{\theta}^{-1} \geq \frac{1}{2} (\theta/\tilde{\theta})^2. \quad (20)$$

Now suppose  $\tilde{\theta} < 0$  at some point on the horizon. Then in a neighborhood of that point one can deform a spacelike slice of the horizon slightly outward to obtain a compact spacelike surface  $\Sigma$  which enters  $J^-(\mathcal{I}^+)$  and has  $\tilde{\theta} < 0$  everywhere on  $\Sigma$ ,  $\tilde{\theta}$  being defined along the outgoing null geodesic congruence orthogonal to  $\Sigma$ . If cosmic censorship is assumed, then there is necessarily some null geodesic orthogonal to  $\Sigma$  that remains on the boundary of the future of  $\Sigma$  all the way out to  $\mathcal{I}^+$  [2]. In other words, this geodesic has no conjugate point between  $\Sigma$  and  $\mathcal{I}^+$  and is future complete. However, this is impossible for the following reason. Asymptotic flatness implies that  $\rho \rightarrow 0$  like  $\lambda^{-1}$  at infinity for the case of Eq. (11) and so does  $\theta$ , where  $\lambda$  is the affine parameter along an outgoing null geodesic. Therefore  $\theta/\tilde{\theta} \rightarrow 1 + O(\lambda^{-1})$ , so the inequality (20) implies that, as one follows the geodesic outwards from  $\Sigma$ ,  $\tilde{\theta}$  reaches  $-\infty$  within some finite affine parameter. Since  $\tilde{\theta} = \theta + \partial_\lambda \rho$ , this means that either  $\theta$  or  $\partial_\lambda \rho$  goes to  $-\infty$ . In the former case we have a contradiction, as in the area theorem, since it implies there is a conjugate point on the geodesic. In the latter case it leads to a naked singularity outside the horizon, contradicting to cosmic censorship, as shall be shown for BD theory below.

Therefore, one finally proves the entropy increase theorem for black holes in BD theory by checking whether or not the positivity of  $(R_{ab} - \nabla_a \nabla_b \rho) k^a k^b$  is satisfied. Since  $e^\rho = \phi(x)$ , one has

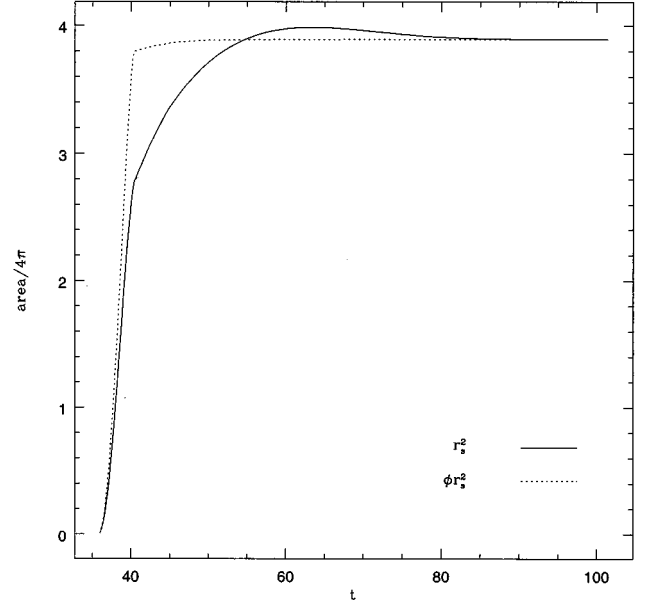


FIG. 4. Behavior of the surface area of the horizon (solid line) and black hole entropy (dotted line) in numerical simulation for Oppenheimer-Snyder collapse in Brans-Dicke theory with  $\omega=0$ . Figure courtesy of Mark A. Scheel and Richard O'Shaughnessy. The figure was produced using the numerical code described in Ref. [12].

$$\begin{aligned} (R_{ab} - \nabla_a \nabla_b \rho) k^a k^b &= R_{ab} k^a k^b - k^a k^b \phi^{-1} \nabla_a \nabla_b \phi \\ &\quad + (k^a \nabla_a \phi / \phi)^2 \\ &= 8\pi G \phi^{-1} T_{ab} k^a k^b + (1 + \omega) \\ &\quad \times (k^a \nabla_a \phi / \phi)^2, \end{aligned} \quad (21)$$

which is manifestly non-negative provided that null energy condition for matter fields and positivity of  $\phi$  and  $\omega$  are satisfied. Since  $\partial_\lambda e^\rho = k^a \nabla_a \phi = e^\rho \partial_\lambda \rho = \phi \partial_\lambda \rho$  and  $\phi$  is assumed to be positive ( $\neq 0$ ), the divergence of  $\partial_\lambda \rho$  implies a curvature singularity from the equation (8) unless it is cancelled by other terms on the right-hand side of the equation. Note finally that the proof described above does not require the existence of regular event horizon as in Hawking's area theorem. A numerical simulation for Oppenheimer-Snyder collapse in BD theory with  $\omega=0$  in Fig. 4 demonstrates that the quantity defined in Eq. (11) is always increasing even if the surface area of the event horizon is oscillatory. From the behavior of the surface area of the horizon one sees that  $\theta$  is positive initially, zero near  $t=60$ , becomes negative, and increases to zero finally. In order for  $\theta$  to increase from some negative value near the final stationary stage, the only possible way is that  $\phi''$  is negative for that period and dominant in the equation (9). In fact  $\phi''$  turns to be negative near  $t=75$  in Fig. 5. For other values of  $\omega$  one can refer to Figure 12 in Ref. [1]. The black hole area in conformally related Einstein frame is indeed the same as black hole entropy in BD theory as shall be shown in Eq. (26) below.

#### IV. DISCUSSION

In summary we have shown that the decrease of black hole area during the dynamical epoch of Oppenheimer-

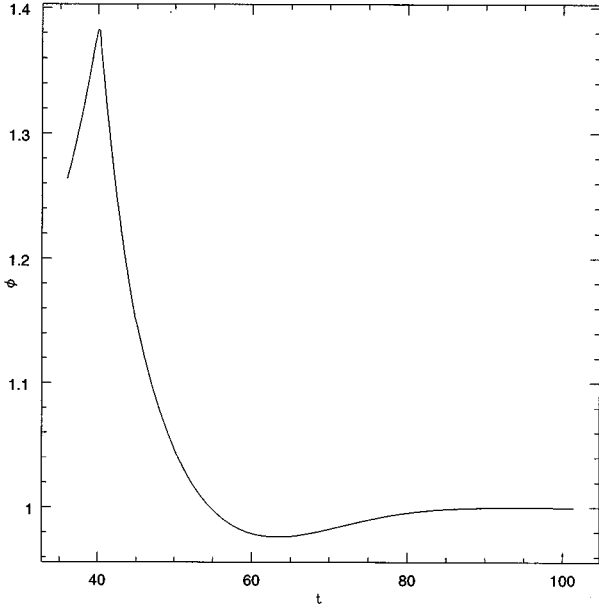


FIG. 5. Positivity and oscillatory behavior of the scalar field  $\phi(x)$  corresponding to Fig. 4. Figure courtesy of Mark A. Scheel and Richard O’Shaughnessy. The figure was produced using the numerical code described in Ref. [12].

Snyder collapse in BD theory is due to the oscillatory behavior of the auxiliary scalar field  $\phi(x) = “G^{-1}(x)”$  in strong gravitational field region. We also have proved that the quantity defined in Eq. (11), which is indeed the entropy of stationary black holes in BD theory, never decreases under arbitrary dynamical evolutions. Thus it establishes a second law of black hole thermodynamics in BD theory.

Results for dynamical black holes in BD theory also give some hints on the behavior of black holes in other theories of gravity. For example, let us consider curvature scalar squared theories of gravity

$$I_0 = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R + \alpha R^2) + \mathcal{L}_m(\psi, g) \right]. \quad (22)$$

This higher curvature theory is dynamically equivalent to the following theory with no higher curvature term but with one auxiliary field  $\Phi(x)$  [8]:

$$I_1 = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (1 + 2\alpha\Phi)R - \frac{\alpha}{16\pi G} \Phi^2 + \mathcal{L}_m(\psi, g) \right]. \quad (23)$$

By redefining the scalar field  $1 + 2\alpha\Phi = \phi^q$  with  $q^2 = 1 + 2\omega/3$  and conformally scaling the metric  $g_{ab} = \phi^{1-q} \bar{g}_{ab}$ , one finds that  $I_1$  has the form of BD theory with a potential term and unconventional couplings between the scalar field  $\phi$  and matter fields as follows:

$$I_2 = \int d^4x \sqrt{-\bar{g}} \left[ \frac{1}{16\pi G} \left( \phi \bar{R} - \frac{\omega}{\phi} \bar{\nabla}_a \phi \bar{\nabla}^a \phi \right) - V(\phi) + \phi^{2(1-q)} \mathcal{L}_m(\psi, \phi^{1-q} \bar{g}) \right], \quad (24)$$

where  $V(\phi) = \phi^2(1 - \phi^{-q})^2/16\pi G\alpha$ . Further conformally transforming  $\bar{g}_{ab} = \phi^{-1} \tilde{g}_{ab}$  and redefining the field  $\varphi = \beta^{-1} \ln \phi$  with  $\beta = \sqrt{16\pi G/(3+2\omega)}$ , one reaches the form of ordinary Einstein gravity [9,8]

$$I_3 = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{16\pi G} \tilde{R} - \frac{1}{2} \tilde{\nabla}_a \varphi \tilde{\nabla}^a \varphi - e^{-2\beta\varphi} V(\varphi) + e^{-2\beta q \varphi} \mathcal{L}_m(\psi, e^{-\beta q \varphi} \tilde{g}) \right]. \quad (25)$$

Now let us ask how the black hole area in curvature squared theories behaves. First of all, notice that the null convergence condition is not satisfied in this theory as well. Thus one expects that the black hole area will be oscillatory during dynamical epochs as in the case of BD theory. The relationship between  $I_0$  and  $I_2$  shows it more explicitly. Assume an initial Cauchy surface such as that of Oppenheimer-Snyder collapse in  $I_0$  where spacetime is almost flat, e.g.,  $R \approx 0$ . Since  $\Phi = R$  in  $I_1$ ,  $\phi \approx 1$  and  $V(\phi) \approx 0$  on the corresponding Cauchy surface in  $I_2$ . From the results for BD theory above, one then expects that the behavior of  $\phi$  is oscillatory during dynamical epoch of the collapse. The “extended” null convergence condition in Eq. (21) to prove the increment of  $\sqrt{\bar{h}}\phi$  is still satisfied in  $I_2$  even in the presence of the potential  $V(\phi)$  and the unconventional couplings between  $\phi(x)$  and matter fields as long as  $\phi$  and  $\omega$  are positive. Thus the quantity  $\sqrt{\bar{h}}\phi = \sqrt{\bar{h}}\phi^q$  on the event horizon never decreases. Then the “area” element  $\sqrt{\bar{h}} = (\sqrt{\bar{h}}\phi)/\phi^q$  will be in general oscillatory for black holes in  $I_0$  unless the radial increase of  $\sqrt{\bar{h}}\phi$  is bigger than that of  $\phi^q$  for the increasing period. However, the positivity of  $\phi(x)$  is guaranteed only for  $\alpha > 0$ . It follows because the form of  $V(\phi)$  confines the field  $\phi$  in the positive region only if  $\alpha > 0$  since the potential barrier increases exponentially as  $\varphi \rightarrow -\infty$  in  $I_3$  (i.e.,  $\phi \rightarrow 0$  in  $I_2$ ). When  $\alpha < 0$ , the potential falls down exponentially,  $\phi$  easily becomes negative, and then the conformal factors become singular at  $\phi = 0$ . It also has been shown that the theory  $I_0$  has the well-posed initial value formulation only if  $\alpha > 0$  [10].

As by-products of the relationships above one finds

$$\begin{aligned} \tilde{\Sigma}_{\text{BH}} &= \frac{\tilde{A}}{4G} = \frac{1}{4G} \oint_{\mathcal{H}_3} d^2x \sqrt{\bar{h}} = \frac{1}{4G} \oint_{\mathcal{H}_2} d^2x \sqrt{\bar{h}} \phi \\ &= \frac{1}{4G} \oint_{\mathcal{H}_1} d^2x \sqrt{\bar{h}} (1 + 2\alpha\Phi) \\ &= \frac{1}{4G} \oint_{\mathcal{H}_0} d^2x \sqrt{\bar{h}} (1 + 2\alpha R). \end{aligned} \quad (26)$$

Therefore, one sees that Hawking’s area theorem is transferred to the increase of black hole entropy in BD theory as well as in higher curvature theories through conformal transformations and an introduction of a scalar field [11]. The analysis above can also be extended to more general class of higher curvature theories such as actions polynomial in  $R$  [7].

In the proof given in Sec. III, we assumed the positivity of the scalar field  $\phi(x)$  and the coupling constant  $\omega$  in BD

theory. The positivity of  $\omega$  is natural since, otherwise, it gives unphysical negative energy matter in the theory. However, the positivity of  $\phi(x)$  at any moment is highly non-trivial in BD theory and should be guaranteed in order for the quantity in Eq. (11) to be interpreted as ‘‘entropy,’’ which is positive by definition in statistical mechanics. From field equations in Eq. (7) we see that any constant  $\phi$  can be a solution. Thus each set of solutions for a given  $\phi = \text{const}$  (which could be negative as well) in BD theory with large  $\omega$  gives Einstein-like gravity. For Einstein gravity this constant is determined by considering Newtonian limit. It gives a positive constant  $\phi = 1$  in the unit of  $G^{-1}$  and so the black hole entropy (10) in Einstein gravity is always positive. In BD theory, however,  $\phi(x)$  is highly dynamical and so the quantity in Eq. (11) could be negative at some stage, which is problematic to be an entropy of dynamical black holes in BD theory. We are in fact interested in a physical system in which the matter is initially distributed in an almost flat space. In other words,  $\phi(x)$  is initially positive near the unit. Now the question is whether or not there is a Cauchy surface in the future evolution where the field  $\phi(x)$  passes zero and becomes negative. As can be seen in field equations (7), it implies a curvature singularity unless the total energy-momentum tensor on the right-hand side cancels this singular behavior. Therefore,  $\phi \rightarrow 0$  corresponds to a naked singularity on the event horizon or outside the horizon, violating the assumption of cosmic censorship in BD theory which we used in the proof. To see whether the dynamics of the field  $\phi$  can prevent this singular behavior, let us examine the relationship between BD theory  $I_2$  and Einstein gravity  $I_3$ . In the theory  $I_3$ ,  $\phi = 0$  corresponds to  $\varphi \rightarrow -\infty$ . In general, the dynamics of the field  $\varphi$  is determined by the potentials coming from couplings with matter Lagrangian as well as  $V(\phi)$ . If the net effect of potentials gives rising up as  $\varphi \rightarrow -\infty$ , the field  $\varphi$  cannot reach to  $-\infty$ . One example is the higher curvature squared theory with  $\alpha > 0$  through conformal rescalings as explained above. For BD theory without potential terms, that is,  $V = 0$  and  $q = 1$  in  $I_2$ , the rising up of potential barrier as  $\varphi \rightarrow -\infty$  is not guaranteed in general. For example, suppose a typical matter field,  $\mathcal{L}_m(\psi, \bar{g}) \sim -\frac{1}{2} \bar{\nabla}_a \psi \bar{\nabla}^a \psi - V(\psi)$ . Then,

$$e^{-2\beta\varphi} \mathcal{L}_m(\psi, e^{-\beta\varphi} \bar{g}) \sim -\left[ \frac{1}{2} (\bar{\nabla}_a \psi \bar{\nabla}^a \psi) e^{-\beta\varphi} + e^{-2\beta\varphi} V(\psi) \right]. \quad (27)$$

Since  $\bar{\nabla}_a \psi \bar{\nabla}^a \psi$  could be negative in general, the potential could fall exponentially, leading to  $\varphi \rightarrow -\infty$  and so  $\phi \rightarrow 0$ . As far as we know, however, we have not found any physical system in the literature which shows vanishing of  $\phi(x)$ . Numerical simulation for the case of Oppenheimer-Snyder collapse in Fig. 5 also shows that the scalar field  $\phi(x)$  is always positive during its evolution.

Finally, let us extract some general feature of the proof shown in Sec. III. One of the most important behaviors of expansion  $\theta$  used in proving many global properties of spacetime such as singularity theorem and area theorem is that  $\theta$  reaches  $-\infty$  within some finite affine parameter if  $\theta$  is negative at some point and convergence condition (i.e.,  $R_{ab} v^a v^b \geq 0$  for all nonspacelike vectors  $v^a$ ) is satisfied as explained in Sec. II briefly. In general relativity the convergence condition is satisfied if some suitable energy condition for matter fields is assumed. In other theories of gravity, however, this is not true any more as seen, for instance, in Sec. II. Thus it is not clear whether or not some global theorems in general relativity or modified forms of them where the convergent behavior of  $\theta$  is used for proof still hold in other theories of gravity. In this paper we have shown that a modified ‘‘expansion’’  $\tilde{\theta}$  plays the same role as  $\theta$  under same energy conditions for matter fields. For example, one may define a modified ‘‘trapped’’ surface as a compact, two-dimensional, smooth spacelike surface where all ‘‘ingoing’’ and ‘‘outgoing’’ null geodesics orthogonal to it have negative  $\tilde{\theta}$ , instead of having negative  $\theta$ . In fact, for BD theory, this  $\tilde{\theta}$  is nothing but the expansion in the conformally related Einstein frame with an overall multification factor. However, the general feature of the proof in this paper may still be applicable to other class of gravity theories which are not conformally related to Einstein gravity. In addition, it also gives some hints on investigating how robust many global properties of spacetime in general relativity are under the change of dynamics of gravitational fields. In other words, by using the convergent behavior of  $\tilde{\theta}$ , we may extend the validity of many global results in general relativity to other theories of gravity as done for the area theorem in this paper.

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