Quasinormal modes of nearly extreme Reissner-Nordström black holes

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(Received 23 July 1996)

We present detailed calculations of the quasinormal modes of Reissner-Nordström black holes. While the first few, slowly damped, modes depend on the charge of the black hole in a relatively simple way, we find that the rapidly damped modes show several peculiar features. The higher modes generally spiral into the value for the extreme black hole as the charge increases. We also discuss the possible existence of a purely imaginary mode for the Schwarzschild black hole: Our data suggest that there is a quadrupole quasinormal mode that limits to $\omega M = -2i$ as $Q \to 0$. [S0556-2821(96)03324-3]

PACS number(s): 04.70.Bw, 04.25.Nx, 97.60.Lf

I. PERTURBING THE REISSNER-NORDSTRÖM BLACK HOLE

Quasinormal modes of black holes have been studied ever since the seminal work of Vishveshwara [1] and Press [2] in the early 1970s. It soon became evident that exponentially damped mode oscillations will dominate most processes involving perturbed black holes (see [3] for references). This means that the quasinormal modes provide a unique opportunity to identify a black hole, a possibility that hopefully will become reality when large-scale laser-interferometric detectors for gravitational waves come into operation in the near future [4]. In order to extract as much information as possible from a gravitational-wave signal it is important that we understand exactly how the quasinormal modes depend on the parameters of the black hole.

The parameters of main astrophysical importance are the black holes mass and angular momentum. That is, the Kerr solution is the most relevant one from an astrophysical point of view. The solution that describes an electrically charged, nonrotating black hole — due to Reissner and Nordström — is of less direct importance because it seems unlikely that black holes with a considerable charge will exist in the Universe. Nevertheless, the Reissner-Nordström solution has several interesting features that warrant a closer inspection. The most intriguing one concerns the possible conversion of electromagnetic energy into gravitational energy and vice versa: In a charged environment an electromagnetic wave will inevitably give rise to gravitational waves. The Reissner-Nordström metric provides the simplest framework for studies of this effect.

For this and several other reasons there has been a number of studies of perturbed Reissner-Nordström black holes. The equations governing a weak (massless) field in the geometry of an electrically charged black hole were first derived by Zerilli [5] and Moncrief [6]. Basically, these equations can be (i) split into axial and polar perturbations (also known as odd and even parity, respectively) and then (ii) reduced to two decoupled wave equations in each case. The final wave equations describe two variables Ψ_1 and Ψ_2, from which all components of the electromagnetic field and the perturbed metric can be reconstructed. In general, both Ψ_1 and Ψ_2 correspond to a combination of electromagnetic waves and gravitational ones, but in the limiting case of an uncharged black hole the two functions reduce to pure electromagnetic and gravitational waves, respectively.

Originally, the perturbation equations were used to study the stability of the metric [5,6]. Later the equations were used to investigate the already mentioned conversion of electromagnetic energy into gravitational energy [7–9]. There have also been several studies of quasinormal modes for Reissner-Nordström black holes [10–12]. But although the methods used in those studies provide accurate numerical results, there is still the need for more information. Specifically, one would like to know what happens to the quasinormal modes as the black hole becomes extremely charged. There are two parts to this problem, which seemingly must be studied separately. In the typical case each of the two horizons, at $r_\pm = M \pm \sqrt{M^2 - Q^2}$ (where M is the mass and $Q \leqslant M$ the charge of the hole), corresponds to a second order singularity in the linearized equations. But in the extreme case these two singularities merge (at $r = M$) into a single one of fourth order. This means that the methods that have been devised to determine the entire spectrum of quasinormal modes break down in the extreme limit [11,12].

Even though the extremely charged case is only of theoretical interest, it is worth considering. One reason (apart from natural curiosity) is the fact that the existence of two coalescing horizons is a prominent feature also of rapidly rotating black holes. Thus one may hope that a study of the mode behavior as $Q \to M$ can lead to some insight also for the rotating case or, at least, that methods that prove reliable for high charge can be adapted to the Kerr case. Recently Leaver's continued fraction method was amended in such a way that it could be applied to the case of extreme charge [13]. Although it seems plausible that other methods, such as the numerical integration of Andersson [12], can be extended in a similar way, there have been no attempts to do this.

Anyway, at present the main question concerns the reliability of the various approaches for nearly extreme black holes. It is clear that the methods in [11,12] will break down for $Q = M$, and also that the method of Onozawa *et al.* [13]

can be used only for the extreme case. Thus, we do not yet know to what extent the available results for high charge (and perhaps also rapid rotation in the case of Kerr [15]) can be trusted. The work presented in this paper was motivated by a desire to obtain a better understanding of this issue. We also wanted to unveil the detailed behavior of the quasinormal modes as the charge of the black hole was increased. Specifically, previous evidence (see Fig. 1 in [11] and Figs. 2 and 3 in [12]) indicate that this behavior is somewhat peculiar for the higher overtones of the black hole. As we will show in the following section, the behavior of the highly damped Reissner-Nordström quasinormal modes is, indeed, very strange.

II. RESULTS

A. Numerical work

Since the equations that describe a perturbed Reissner-Nordström black hole are available in the literature (most notably in Chandrasekhar's exhaustive book [16]), we will not list them here. For the present discussion it is sufficient to know that the equations take the general form

$$\frac{d^2\Psi}{dr_*^2}+[\omega^2-V(r)]\Psi=0, \tag{1}$$

where we have assumed that the time dependence of the perturbation is $e^{-i\omega t}$. The tortoise coordinate that is defined by

$$\frac{d}{dr_*}=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\frac{d}{dr} \tag{2}$$

has the effect that the event horizon of the black hole is "pushed away" to $r_*=-\infty$.

The function Ψ can be either an axial (odd parity) or polar (even parity) perturbation of the black hole. Also, for each case there are two different functions Ψ_1 and Ψ_2, which correspond to pure electromagnetic and gravitational waves in the Schwarzschild limit. In total there are thus four equations, all with slightly different effective potentials. For obvious reasons this difference in the potentials leads to somewhat different quasinormal modes for Ψ_1 and Ψ_2. Analogously, one might expect the quasinormal modes for axial and polar perturbations to be different. But as pointed out by Chandrasekhar [16], the mathematical theory of black holes is an intricately entangled web where many quantities are related (often in a surprising way). Thus it turns out that the quasinormal modes for axial and polar perturbations are identical. The underlying reason is that two effective potentials can, although different, contain much the same physical information. This means that it is sufficient to restrict a study of the Reissner-Nordström problem to either the axial or the polar case. Once Ψ_1 and Ψ_2 are found for one kind of perturbation, the corresponding solutions for the other case are easily generated [16].

The quasinormal modes of a black hole correspond to solutions to Eq. (1) that satisfy the causal condition that no information should leak out through the event horizon of the black hole and at the same time correspond to purely outgoing waves at spatial infinity. This means that a typical mode-solution will behave like

$$\Psi\sim\begin{cases} e^{i\omega r_*} & \text{as } r_*\to+\infty, \\ e^{-i\omega r_*} & \text{as } r_*\to-\infty. \end{cases} \tag{3}$$

Since one would expect the black hole to be stable against a small perturbation, the mode frequencies should be complex. This implies that an identification of quasinormal modes is nontrivial. In order for a mode solution to be damped with time at a fixed r_* the frequency ω must have a negative imaginary part. But then the corresponding solution will diverge both at the event horizon and spatial infinity [cf. Eq. (3)] at a fixed time. Several methods have been devised to deal with this difficulty. Most notable are Leaver's continued fraction approach [15] and Andersson's complex-coordinate integration method [17]. These two methods provide highly accurate numerical results also for rapidly damped modes. Furthermore, both methods have been used to study the first few quasinormal modes for Reissner-Nordström black holes.

As already mentioned in the previous section, it is clear that all existent methods will fail for a nearly extreme black hole. The reason is that the two horizons of the black hole merge as $Q\to M$, and this changes the singularity structure of the problem [this is easy to see if Eq. (1) is expanded in such a way that all derivatives are taken with respect to r rather than r_*]. Thus, the extreme case must be considered separately. This was recently done by Onozawa *et al.* [13]. As a test of the reliability of that study and also to provide a better understanding of the behavior of the quasinormal modes for highly charged black holes, we decided to make an exhaustive study of the problem. We made detailed calculations (using both the continued fraction method [11] and numerical integration [12] to make sure that the results were reliable) for the first nine dipole ($\ell=1$) modes of Ψ_1 and the first nine quadrupole ($\ell=2$) modes of Ψ_2. These are the lowest radiating multipoles for each case. The results of this investigation are shown in Figs. 1 and 2. We now proceed to discuss them in more detail.

B. Slowly damped modes

For the first few modes the behavior of the mode frequencies is readily described. The damping rate typically reaches a maximum for $Q/M\approx0.7$–0.8, and the oscillation frequency generally increases with Q. This is clear from the results for $n=0$–3 in Fig. 1. Moreover, this behavior agrees with the understanding from the WKB approximation [10]. It is relatively straightforward to verify that, when the lowest order of approximation is used, the WKB formulas suggest the approximate behavior (for the slowest damped mode)

$$\mathrm{Re}\,\omega\approx\left(\ell+\frac{1}{2}\right)\left[\frac{M}{r_0^3}-\frac{Q^2}{r_0^4}\right]^{1/2}, \tag{4}$$

$$\mathrm{Im}\,\omega\approx-\frac{1}{2}\left[\frac{M}{r_0^3}-\frac{Q^2}{r_0^4}\right]^{1/2}\left[\frac{3M}{r_0}-\frac{4Q^2}{r_0^2}\right]^{1/2}, \tag{5}$$

in the limit $\ell\gg1$. Here we have defined r_0 as the position where the black-hole potential attains its maximum value.

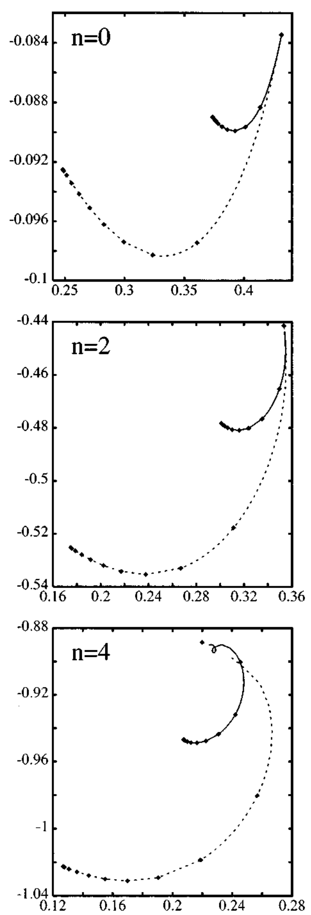

FIG. 1. The behavior of the first six quasinormal mode frequencies (we show $\mathrm{Im}\,\omega M$ as a function of $\mathrm{Re}\,\omega M$) for a Reissner-Nordström black hole as the charge is increased. The solid curves correspond to $\ell=2$ and Ψ_2 (which reduces to pure gravitational waves in the Schwarzschild limit) and the dashed curves to $\ell=1$ and Ψ_1 (which limits us to pure electromagnetic waves). To show that the mode frequencies change dramatically for $Q>0.9M$ we have indicated the charge of the black hole by diamonds (at increments in Q of $0.1M$). The frequencies generally move counterclockwise in the figures as the charge is increased.

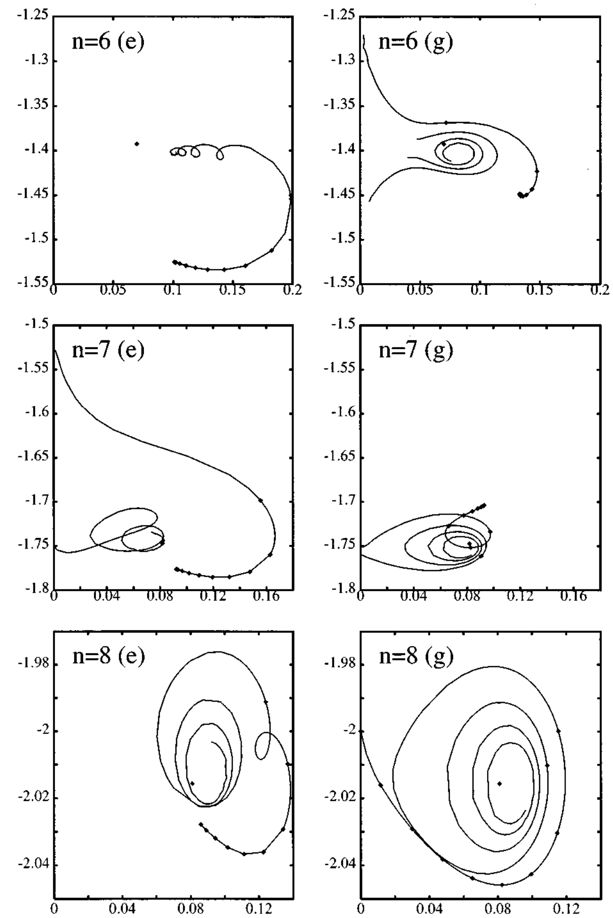

FIG. 2. The behavior of the rapidly damped modes (we show $\mathrm{Im}\,\omega M$ as a function of $\mathrm{Re}\,\omega M$) of a Reissner-Nordström black hole as the charge is increased. The right frames [labeled (g)] correspond to $\ell=2$ and Ψ_2 (which reduces to pure gravitational waves in the Schwarzschild limit) and the left frames [labeled (e)] are for $\ell=1$ and Ψ_1 (which limits us to pure electromagnetic waves). To show that the mode frequencies change dramatically for $Q>0.9M$ we have indicated the charge of the black hole by diamonds (at increments in Q of $0.1M$). The frequencies generally move counterclockwise in the figures as the charge is increased.

This means that $2r_0\approx 3M+\sqrt{9M^2-8Q^2}$. That is, r_0 corresponds to the position of the unstable, circular photon orbit in the Reissner-Nordström spacetime.

To understand better the physics that lead to this behavior we can use an argument that is due to Goebel [18]: Consider a congruence of null rays circling the black hole in the unstable photon orbit. To circle the black hole would require a coordinate time

$$\Delta t=2\pi r_0\left(1-\frac{2M}{r_0}+\frac{Q^2}{r_0^2}\right)^{-1/2}. \tag{6}$$

The fundamental mode frequency then follows (if the beam contains ℓ cycles) from

$$\omega\approx\frac{2\pi\ell}{\Delta t}. \tag{7}$$

In a similar way, one can infer the damping rate of the quasinormal mode from the decay rate of the congruence if the null orbit is slightly perturbed [19]. From this information it is easy to convince oneself that the oscillation frequency of the modes should increase as Q increases. The results for the first few modes in Fig. 1 agree nicely with this description.

A remarkable fact, which is notable in Fig. 1, is that the dipole frequencies for Ψ_1 approach the quadrupole frequencies for Ψ_2 as the black hole becomes extreme. This effect was first noted by Onozawa *et al.* [13] (for a study of the spin-3/2 case, see [14]). They also showed that this surprising phenomenon arises because the corresponding two effective potentials are related. That is, the effective potential for ℓ and Ψ_1 is related to that for $\ell+1$ and Ψ_2. Specifically, $\Psi_1(\ell)$ corresponds to $\Psi_2(-\ell-1)$. As is easily verified this is true also for the polar potential, but in that case the relation does not take the simple form of Eq. (C3) in [13].

C. Highly damped modes

While the behavior for the first few modes has been studied in detail before [10–12] there have been no studies of the highly damped modes for Reissner-Nordström black holes. Thus, the results for $n=5$–8 in Figs. 1 and 2 are new. They are also truly remarkable. While the behavior is rather simple for the first modes ($n=0$–3), the rapidly damped modes show many strange features. Our calculations have discovered a small zoo of seemingly different species. Let us discuss them separately.

The first surprising feature occurs for $n=4$ (see Fig. 1) at a charge $Q\approx 0.938M$: The Ψ_2 mode goes through a tiny loop that closes at $Q\approx 0.962M$. For the following mode ($n=5$) this loop has grown. Several similar loops are obvious in the results for $n=6$ and Ψ_1, and when one continues up the spectrum one finds that these loops are a common feature of many modes. One interesting aspect of this result is that black holes with different charge may share a specific mode frequency.

For a few modes the mode frequencies approach the negative Imω axis as the charge varies. Then our numerical methods [11,12] become unreliable. This is the reason for the gaps in the data for $n=6$ for Ψ_2 and $n=7$ for both Ψ_1 and Ψ_2. Whether or not these modes cross the axis is impossible to say given the available techniques. If they do, it would be very surprising, but given the data in our figures it does not seem impossible. One added peculiarity is that it seems as if some modes are ''multivalued.'' That is, for $n=6$ and Ψ_2 we find a mode both in the upper branch close to the imaginary axis and in the lower branch for a small range of the black-hole charge ($0.8985\leq Q/M\leq 0.9105$). A similar behavior can be observed for $n=7$ and Ψ_1 when $0.9330\leq Q/M\leq 0.9425$. At first this result seems nonsensical and obviously wrong, but it is confirmed by both the continued fraction method and the numerical integration scheme. This could indicate that it should be taken seriously.

Our study also sheds some light on an issue that has been debated for Schwarzschild black holes for some time. Is there a quasinormal mode *on* the imaginary axis at $\omega M=-2i$? Some methods indicate that there should be such a mode and that it corresponds to $n=8$. Other methods say that no such mode exists. To resolve this issue is difficult, basically because none of the proposed methods is reliable close to the axis. An added complication is that the case $\omega M=-2i$ is a very special one. That frequency corresponds to a so-called algebraically special perturbation and, as was shown by Chandrasekhar [23], one can find analytic solutions to the corresponding perturbation equations. Although one can convince oneself that Chandrasekhar's special solutions do not satisfy the quasinormal-mode boundary conditions, the issue is not resolved. To prove the existence of a quasinormal mode one must study the analytic properties close to the mode frequency, and this turns out to be difficult in this specific case. Anyway, once the black hole acquires some small charge it is clear that a mode exists ($n=8$ for Ψ_2 in Fig. 2). This mode approaches the suggested value $\omega M=-2i$ as $Q\to 0$. But still this does not prove the existence of a mode for the Schwarzschild black hole. It is plausible that the mode in Fig. 2 and its symmetric counterpart in the left half of the complex ω plane will coalesce at $Q=0$,

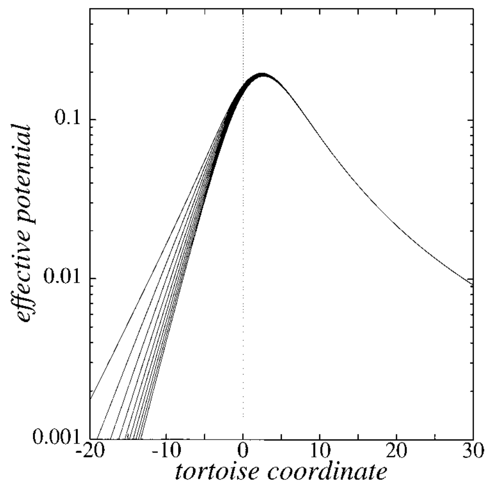

FIG. 3. The effective potential for $\ell=2$ and Ψ_2 (which corresponds to gravitational perturbations in the Schwarzschild limit) is shown for $Q/M=0.9$–1.0. The potential falls off slower towards $r_*=-\infty$ as Q increases. The behavior is similar for other values of ℓ and also for the potential that governs Ψ_1.

and perhaps the two modes that exist for the charged case then cancel each other in some way. It is also worth mentioning here that there are algebraically special solutions also for the Reissner-Nordström black hole. For $\Psi_{1,2}$ these correspond to frequencies [23]

$$\omega^s_{1,2}=-\frac{i}{2}\ell(\ell-1)(\ell+1)(\ell+2) \times\left[3M\mp\sqrt{9M^2+4Q^2(\ell-1)(\ell+2)}\right]^{-1}. \quad (8)$$

In general we cannot see any correlation between these frequencies and the quasinormal modes of the charged black hole.

For higher overtones than those displayed in Fig. 2 the behavior tends to be qualitatively similar to the presented ones. The mode frequency spirals into the value for the extreme black hole. The spirals get tighter for more rapidly damped modes.

How are we to understand these results? Well, at present it is very hard to make much sense out of the data presented in Fig. 2. The obvious answer is that the change in the mode frequencies occurs because the effective potential changes as the charge increases. For the slowly damped modes it is straightforward to show that the oscillation frequency depends on the height of the peak of the potential, while the damping rate is related to the second derivative of V at the peak [10]. But it is much harder to draw similar conclusions for the rapidly damped modes. Present methods can reliably calculate the mode frequencies (and also account for the excitation of the modes in a dynamical process [20–22]), but we have no clear understanding of the relation between the highly damped modes and the details of the effective potential. On the other hand, we have seen that the mode frequencies change dramatically as the charge increases from $Q\approx 0.9M$. If one studies the corresponding change in the effective potential (cf. Fig. 3), one can infer that as one approaches the extreme black hole (i) the potential does not change much for $r\gg r_0$, (ii) the peak of the potential stays almost constant, and (iii) the main change in the potential is the falloff rate as $r_* \to -\infty$. This suggests the possibility that the high overtones of the black hole are in some way related to the effective potential close to the horizon. This issue will have to be studied further in the future.

III. CONCLUDING REMARKS

In this paper we have presented the results of a detailed study of the quasinormal modes of a Reissner-Nordström black hole. Our work complements previous studies in several ways. For the first time we have considered (i) the rapidly damped modes and (ii) how the mode frequencies approach the anticipated value for the extremely charged black hole. We have given special attention to the mode behavior near the extreme limit. The two independent algorithms (from [11] and [12]) that we used for the calculations provide consistent results even for a nearly extreme black hole. These values are also in nice agreement with the results obtained for the extreme black hole [13]. That is, the quasinormal modes converge towards the values for the extreme case as $Q \to M$.

We find that while the slowly damped modes can be understood (for example, by a WKB argument), the behavior of the rapidly damped modes as the charge increases is harder to explain. In general, the modes tend to spiral into the extreme value. Moreover, there are cases where the quasinormal modes seem to be double valued (in cases when the mode frequency approaches the imaginary ω axis). At one level, these are just peculiar results without much relevance, but they lead to several questions. For example, will the modes of a Kerr black hole show similar features? This does not seem unlikely, because the Kerr problem is in many ways similar to the Reissner-Nordström one. There will be two horizons that merge as the black hole becomes extreme. Although the slowest damped modes of the Kerr black hole have been calculated [15], the rapidly damped ones have not yet been considered. Since rotating black holes should be astrophysically relevant, a study of the rapidly damped modes for Kerr could yield physically interesting results.

The present study also leads to questions regarding the quasinormal modes themselves: Which features of the effective potential govern the behavior of the rapidly damped modes? The present evidence (cf. Fig. 3) seems to indicate that the falloff rate towards the event horizon plays an important role. At first sight it may seem that this idea can be tested by calculations for higher values of ℓ (since the falloff of the potential towards $r_* = -\infty$ is different for different values of ℓ). However, after doing a sample of such calculations we find this not to be a useful approach. As should be expected from the Schwarzschild case [3], we find that the mode behavior for the higher multipoles is qualitatively similar to that for the lowest radiating ones (cf. Figs. 1 and 2). But since we still do not have an intuitive understanding of the origin of the rapidly damped black-hole modes, we cannot associate this behavior with the change in a black-hole potential in a meaningful way. Much further work is required to test the assertion that the behavior of the potential close to the event horizon is particularly relevant for the high black-hole overtones. The information needed to resolve this problem is likely to be contained in the quasinormal-mode eigenfunctions. A detailed study of the actual solutions that correspond to the highly damped modes may thus provide the desired answer. But such a study goes far beyond the scope of the present paper and we will have to return to it in the future.

A final question regards the relevance of the algebraically special modes. In general, our results do not indicate any relation between the algebraically special mode (which has a purely imaginary frequency) and the quasinormal modes for a charged black hole. But it is well known that the algebraically special mode coalesces with the ninth gravitational quasinormal mode in the Schwarzschild limit. Is there a profound reason for this or is it just a coincidence? To answer this question is difficult because all present methods for quasinormal modes break down in the region close to the imaginary frequency axis. On the other hand, it is known that the algebraically special modes of the Kerr black hole move away from the imaginary axis. Hence, it seems likely that a study of the Kerr problem may prove illuminating also for this issue.

Several questions concerning quasinormal modes thus remain to be resolved. The present work has contributed a few new pieces to the puzzle, but many other pieces need to be added before the picture becomes completely clear. Still, the present results should help us get a better understanding not only of the Reissner-Nordström black hole, but also of the quasinormal-mode phenomenon in general.

ACKNOWLEDGMENTS

H.O. would like to thank A. Hosoya and H. Ishihara for continuous encouragement. The work of N.A. was supported by NSF (Grant No. PHY 92-2290) and NASA (Grant No. NAGW 3874). The work of H.O. was supported in part by the JSPS Research Program for Young Scientists and the Scientific Research Fund of the Ministry of Education.

[1] C. V. Vishveshwara, Nature (London) **227**, 936 (1970).
[2] W. H. Press, Astrophys. J. **170**, L105 (1971).
[3] N. Andersson, in *Physics of Black Holes*, edited by I. D. Novikov and V. Frolov (Kluwer Academic, Dordrecht, 1996), Chap. 4.
[4] K. S. Thorne, in *Relativistic Cosmology*, Proceedings of the Eighth Nishinomiya-Yukawa Memorial Symposium, edited by M. Sasaki (Universal Academy Press, Kyoto, 1994), pp. 67–82.
[5] F. J. Zerilli, Phys. Rev. D **9**, 860 (1974).
[6] V. Moncrief, Phys. Rev. D **9**, 2707 (1974); **10**, 1057 (1974); **12**, 1526 (1975).
[7] M. Johnston, R. Ruffini, and F. Zerilli, Phys. Rev. Lett. **31**, 1317 (1973).

[8] D. W. Olson and W. G. Unruh, Phys. Rev. Lett. **33**, 1116 (1974).
[9] D. C. Gunter, Philos. Trans. R. Soc. London **A299**, 497 (1980).
[10] K. D. Kokkotas and B. F. Schutz, Phys. Rev. D **37**, 3378 (1988).
[11] E. W. Leaver, Phys. Rev. D **41**, 2986 (1990).
[12] N. Andersson, Proc. R. Soc. London **A442**, 427 (1993).
[13] H. Onozawa, T. Mishima, T. Okamura, and H. Ishihara, Phys. Rev. D **53**, 7033 (1996).
[14] H. Onozawa, T. Okamura, T. Mishima, and H. Ishihara, ''Perturbing Supersymmetric Black Hole,'' Report No. TIT/HEP-336/COSMO-74, 1996 (unpublished).
[15] E. W. Leaver, Proc. R. Soc. London **A402**, 285 (1985).
[16] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, 1983).
[17] N. Andersson, Proc. R. Soc. London **A439**, 47 (1992).
[18] C. J. Goebel, Astrophys. J. **172**, L95 (1972).
[19] B. Mashhoon, Phys. Rev. D **31**, 290 (1985).
[20] E. W. Leaver, Phys. Rev. D **34**, 384 (1986).
[21] N. Andersson, Phys. Rev. D **51**, 353 (1995).
[22] N. Andersson, ''Evolving test-fields in a black hole geometry,'' Phys. Rev. D (to be published).
[23] S. Chandrasekhar, Proc. R. Soc. London **A392**, 1 (1984).