PHYSICAL REVIEW D VOLUME 54, NUMBER 12 15 DECEMBER 1996

Thermal partition function of photons and gravitons in a Rindler wedge

Devis Iellici* and Valter Moretti†

Dipartimento di Fisica, Università di Trento, and Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Trento, I-38050 Povo (TN), Italy

(Received 8 July 1996)

The thermal partition function of photons in any covariant gauge and gravitons in the harmonic gauge, propagating in a Rindler wedge, are computed using a local ζ-function regularization approach. The correct Planckian leading order temperature dependence T^4 is obtained in both cases. For the photons, the existence of a surface term giving a negative contribution to the entropy is confirmed, as earlier obtained by Kabat, but this term is shown to be gauge dependent in the four-dimensional case and, therefore, is discarded. It is argued that similar terms could appear dealing with any integer spin $s \geqslant 1$ in the massless case and in more general manifolds. Our conjecture is checked in the case of a graviton in the harmonic gauge, where different surface terms also appear, and physically consistent results arise dropping these terms. The results are discussed in relation to the quantum corrections to the black hole entropy. [S0556-2821(96)06222-4]

PACS number(s): 04.62.+v, 04.70.Dy

I. INTRODUCTION

In recent years, many papers have been concerned with the first quantum correction to the Bekenstein-Hawking black hole entropy. According to 't Hooft [1], the main contribution to these corrections comes from quantum fields propagating in the region outside the horizon. An important tool used to compute these corrections is the approximation of the metric of a large mass Schwarzschild black hole given by the simpler Rindler metric. In this approximation the quantum corrections are identified with the entropy of thermal states of quantum fields in the Rindler space-time. Many different methods have been employed to compute this entropy and, among them, the method of the conical singularity is one of the most used: one follows the usual prescription to compute the thermal partition function of a quantum field, that is, to evaluate the Euclidean path integral over all the field configurations that are periodic in the imaginary time and identify the period β with the inverse of the temperature. In doing this, the Rindler manifold acquires a conical singularity with angular deficit $2\pi-\beta$, and so one sees that, in order to avoid the singularity, there is only one possible temperature for the system, i.e., the Unruh-Hawking temperature $\beta=2\pi$. However, if one wants to compute thermodynamical quantities such as the entropy and the internal energy using standard thermodynamical relations such as $S_\beta=\beta^2\partial_\beta F_\beta$, then one needs to go "off shell," i.e., consider $\beta\neq 2\pi$ and so manifolds with a conical singularity. Therefore, many techniques have been developed to compute the one-loop quantum corrections on manifolds with conical singularities. In this respect, it is important to note that the standard use of heat kernel plus proper time regularization yields the wrong temperature dependence of the free energy and the other thermodynamical quantities, at least when the dimension of the space-time is not two [2]. In four dimensions, in particular, the leading term in the high temperature limit of the free energy should be Planckian, namely, proportional to β^{-4} [3–5], while the heat kernel gives β^{-2} independently of the dimension.

In this context, Zerbini, Cognola, and Vanzo [6], starting from a previous work of Cheeger [7], have recently introduced a new method to compute the effective action of a scalar field on manifolds with conical singularities using the ζ-function regularization. This method, in addition to giving the correct temperature dependence and allowing one to work directly with massless fields, has the advantage that it does not require the regularization of the conical singularity or transforming the cone in a compact manifold, procedures which do not have a clear physical meaning if one is interested in the (Euclidean) Rindler space. The drawbacks are that this method is technically difficult to apply in the case of massive fields and especially that it yields for the part of the free energy proportional to β^{-2} a numerical coefficient different from that obtained with the point splitting and the optical metric methods [3,8–10]. This latter problem is shared with the heat-kernel approach and the reason for this discrepancy is not yet understood.

Most of the work on the quantum corrections to the black hole entropy is carried on using the scalar field. Results for higher spins have been obtained translating earlier results obtained for the closely related cosmic string background [4]. Last year, in an interesting paper [11] Kabat investigated the corrections to the black hole entropy coming from scalar, spinor, and vector fields by explicitly writing the field modes in the Euclidean Rindler space and then using the heat-kernel and the proper-time regularization. In the vector field case he has obtained an unexpected "surface" term, which corresponds to particle paths beginning and ending at the horizon. This term gives a negative contribution to the entropy of the system and, in fact, is large enough to make the total entropy negative at the equilibrium temperature. Kabat argues that this term corresponds to the low-energy limit of string processes which couple open strings with both ends attached to the horizon and closed strings propagating outside the horizon diagrams and discussed by Susskind and Uglum [5] as responsible for black hole entropy within string theory.

*Electronic address: iellici@science.unitn.it
†Electronic address: moretti@science.unitn.it

In this paper, we apply the method of [6] to the case of the Maxwell field and the graviton field. As a result, in the case of the photon field we confirm that there is a ''surface term'' which would give a negative contribution to the entropy, as obtained by Kabat in [11]. However, besides getting a different temperature dependence, we show that it depends on the gauge-fixing parameter and so we discuss how it is possible to discard it. In this way we also avoid embarrassing negative entropies. In the case of the graviton we get similar surface terms and show that one can get consistent physical results by discarding them. We also discuss the appearance of similar terms in more general manifolds. After discarding the surface terms we get the reasonable result that the effective action and all the thermodynamical quantities are just twice those of the minimally coupled scalar field: this is in agreement with the results of the point-splitting method [9,10], the heat kernel method [12–14], and, apart from the surface terms, also with Kabat [11].

We remind that the Rindler wedge is a globally hyperbolic manifold defined by the inequality $x>|t|$, in the usual set of rectangular coordinates (t,x,y,z) of Minkowski spacetime. In this wedge we can define a new set of static coordinates by setting $t=r\sinh\tau$ and $x=r\cosh\tau$, with $0<r<\infty$ and $-\infty<\tau<\infty$. Then the Minkowski metric takes the form of the Rindler metric:

$$ds^2=-r^2d\tau^2+dr^2+dy^2+d^2z. \tag{1}$$

One can see that lines of constants r,y, and z are trajectories of uniformly accelerated particles, with proper acceleration $a=r^{-1}$.

As we said above, the importance of the Rindler metric is mainly due to the fact that it can be seen as an approximation of the metric of a large mass Schwarzschild black hole outside the event horizon. Indeed, consider the Schwarzschild metric, which describes an uncharged, nonrotating black hole of mass M:

$$ds^2=-\left(1-\frac{2GM}{R}\right)dT^2+\left(1-\frac{2GM}{R}\right)^{-1}dR^2+R^2d\Omega_2,$$

$$d\Omega_2=d\theta^2+\sin\theta d\varphi^2,$$

where M is the mass of the black hole. In the region outside the event horizon, namely, $2GM<R<\infty$, we can define new coordinates τ and r by

$$\tau=\frac{T}{4GM}, \tag{2}$$

$$r=\sqrt{8GM(R-2GM)}, \tag{3}$$

and so the metric takes the form

$$ds^2=-r^2\left(1+\frac{r^2}{16G^2M^2}\right)^{-1}d\tau^2+\left(1+\frac{r^2}{16G^2M^2}\right)dr^2 \tag{4}$$

$$+4G^2M^2\left(1+\frac{r^2}{16G^2M^2}\right)^2d\Omega_2. \tag{5}$$

If we take the large mass limit, the last term becomes the metric of a spherical surface with very large radius that can be approximated by a flat metric dy^2+dz^2. Then, in this limit, the metric becomes the Rindler one, Eq. (1). Actually, even if we do not consider the large mass limit, the approximation should become better and better as we approach the event horizon, $r=0$.

The Rindler metric is also related with the study of the cosmic string background: the metric around an infinitely long, static, straight and with zero thickness cosmic string can be written as

$$ds^2=-dt^2+dz^2+dr^2+r^2d\varphi,\quad 0\leqslant\varphi\leqslant\alpha,$$

where the polar angle deficit $2\pi-\alpha$ is related to the mass per unit length of string μ by $2\pi-\alpha=8\pi G\mu$. Since the metric is ultrastatic, we can perform a Wick rotation, $t\to it$, and the metric becomes equal to the Euclidean Rindler metric. Therefore, we can identify the thermal partition function of a field at temperature α^{-1} in the Rindler wedge with the zero-temperature, Euclidean-generating functional of the same field in a cosmic string background.

The rest of this paper is organized as follows. In Sec. II we compute the one-loop effective action for the electromagnetic field on the manifold $C_\beta\times R^2$ using the ζ-function regularization. We use this result to compute the quantum correction to the black hole entropy in the framework of conical singularity method. In Sec. III we formulate a general conjecture on the appearance of Kabat-like surface terms in the case of integer spin and general manifolds. In Sec. IV the conjecture is checked in the case of the graviton. Section V is devoted to the discussion of the results.

II. EFFECTIVE ACTION FOR THE PHOTON FIELD

In a curved space-time with Lorentz signature the action of the electromagnetic field is $S=\int\mathcal{L}(x)\sqrt{-g}d^4x$, where the Lagrangian scalar density[1] is [15]

$$\mathcal{L}_{\rm em}(x)=-\frac{1}{4}F_{ab}F^{ab},$$

$$F_{ab}=\nabla_aA_b-\nabla_bA_a=\partial_aA_b-\partial_aA_b. \tag{6}$$

We need also the gauge-fixing term and the contribution of the ghosts:

$$\mathcal{L}_G=-\frac{1}{2\alpha}(\nabla^aA_a)^2, \tag{7}$$

$$\mathcal{L}_{\rm ghost}=\frac{1}{\sqrt{\alpha}}g^{ab}\partial_ac\partial_bc^*, \tag{8}$$

[1]We adopt the convention that the indices $a,b,\ldots=\tau,r,y,z$ are for the whole manifold, the greek indices are for the pure cone, $a,b,\ldots=\tau,r$, and the indices $i,j,\ldots=y,z$ are for the transverse flat directions.

where c and c^* are anticommuting scalar fields. The dependence on the gauge-fixing parameter α of the ghost action is relevant only in presence of a scale anomaly. It is not the case here, and, therefore, we shall ignore it.

We are interested in the finite temperature theory and so we change $\tau \to i\tau$ and identify τ and $\tau+\beta$. The metric of the Rindler space-time turns to Euclidean signature, $ds^2=r^2d\tau^2+dr^2+dy^2+dz^2$, and the vector D'Alembertian operator $\Box$ becomes the vector Laplace-Beltrami operator Δ. In the following this operator will be simply called Laplacian. The one-loop effective action for this theory will then be given by the determinants

$$\ln Z_\beta = -\frac{1}{2}\ln\det\mu^{-2}\left[g^{ab}(-\Delta)-R^{ab}+\left(1-\frac{1}{\alpha}\right)\nabla^a\nabla^b\right] + \ln Z_{\beta,\text{ghosts}}, \tag{9}$$

where μ^2 is the renormalization scale and the effective action of the ghosts is minus twice the effective action of a scalar massless field, which is well known [7,6]. It is important to note that the determinant has to be evaluated on the whole set of eigenfunctions, not only on the physical ones [16].

We work on the manifold $C_\beta \times R^2$, where C_β is the cone with angular deficit equal to $2\pi-\beta$. This manifold is flat everywhere but on the tip of the cone, where the curvature has a δ-function singularity. Nevertheless, the modes we use vanish on the tip, and so we can consider $R_{ab}=0$. Note also that, due to the flatness, the covariant derivatives commute. Hence, we are left with the problem of computing the determinant of the operator $\{g^{ab}(-\Delta)+[1-(1/\alpha)]\nabla^a\nabla^b\}$ acting on vectors. In order to define this determinant we use the ζ-function regularization: first, suppose we have a complete set of eigenfunctions of the operator, indicated as $A_a^{(i,n\lambda\mathbf{k})}(x)$, with eigenvalue $\nu_i^2(n\lambda\mathbf{k})$. Here, $\mathbf{k}=(k_y,k_z)$, $a=\tau,r,y,z$, and $i=1,\ldots,4$ is the polarization index. In this notation we have taken into account the triviality of the transverse dimension and the fact that we have a discrete index n since the τ coordinate is compact and we impose periodic boundary conditions. Then we can define the local, diagonal heat kernel as

$$K^{(i)}(t;x)=\sum_n \int d\mu(\lambda)d^2\mathbf{k}e^{-t\nu_i^2}g^{ab}A_a^{(i)}(x)A_b^{(i)*}(x), \tag{10}$$

where $d\mu(\lambda)$ is an appropriate integration measure. The corresponding local spin-traced ζ function can be obtained through a Mellin transform:

$$\zeta(s;x)=\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\sum_i K^{(i)}(t;x). \tag{11}$$

Alternatively, we can define the local ζ function as the inverse power of the kernel of the above differential operator: the spectral representation gives directly

$$\zeta(s;x)=\sum_i\sum_n\int d\mu(\lambda)d^2\mathbf{k}[\nu_i^2(n\lambda\mathbf{k})]^{-s}g^{ab}A_a^{(i)}(x) \times A_b^{(i)*}(x). \tag{12}$$

In general, both the Mellin transform and the inverse power of the operator require analytic continuation arguments to be defined at the physical values of s.

We can also define a global ζ function by tracing over the space indices:

$$\zeta(s)=\int d^4x\sqrt{g}\zeta(s;x). \tag{13}$$

This last step is delicate: in general, the operation of tracing over the space indices requires the introduction of a smearing function, since the manifold is noncompact and there can be nonintegrable singularities in the local ζ function, and a particular choice of the smearing function could sweep away important information. This is one of the reasons why we prefer to work with a local formalism as long as possible. Once we have computed and analytically continued the ζ function, we can write the effective Lagrangian density and the effective action as

$$\mathcal{L}_\beta(x)=\frac{1}{2}\zeta'(s=0;x)+\frac{1}{2}\zeta(s=0;x)\ln\mu^2,$$

$$\ln Z_\beta=\int d^4x\sqrt{g}\mathcal{L}_\beta(x). \tag{14}$$

Of course, to the above expression we have to add the contribution of the ghosts, which is minus two times the effective Lagrangian density of a scalar field.

A suitable set of normalized eigenfunctions of the operator $\{g^{ab}(-\Delta)+[1-(1/\alpha)]\nabla^a\nabla^b\}$ (equivalent to Kabat's set [11] if $\alpha=1$) is the following: setting $k=|\mathbf{k}|$,

$$A_a^{(\mathrm{I},n\lambda\mathbf{k})}=\frac{1}{k}\epsilon_{ij}\partial^j\phi=\frac{1}{k}(0,0,ik_z\phi,-ik_y\phi),$$

$$A_a^{(\mathrm{II},n\lambda\mathbf{k})}=\frac{\sqrt{g}}{\lambda}\epsilon_{\mu\nu}\nabla^\nu\phi=\frac{1}{\lambda}\left(r\partial_r\phi,-\frac{1}{r}\partial_\tau\phi,0,0\right),$$

$$A_a^{(\mathrm{III},n\lambda\mathbf{k})}=\frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}\left(\frac{k}{\lambda}\nabla_\mu-\frac{\lambda}{k}\partial_i\right)\phi =\frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}\left(\frac{k}{\lambda}\partial_\tau\phi,\frac{k}{\lambda}\partial_r\phi,-\frac{\lambda}{k}\partial_y\phi,-\frac{\lambda}{k}\partial_z\phi\right),$$

$$A_a^{(\mathrm{IV},n\lambda\mathbf{k})}=\frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}\nabla_a\phi=\frac{1}{\sqrt{\lambda^2+\mathbf{k}^2}}(\partial_\tau\phi,\partial_r\phi,\partial_y\phi,\partial_z\phi), \tag{15}$$

where $\sqrt{g}\epsilon_{\mu\nu}$ is the Levi-Civita pseudotensor on the cone, ϵ_{ij} is the Levi-Civita pseudotensor on R^2 in Cartesian coordinates, and $\phi=\phi_{n\lambda\mathbf{k}}(x)$ is the complete set of normalized eigenfunctions of the Friedrichs self-adjoint extension of the scalar Laplacian on $C_\beta \times R^2$ [17]:

$$\phi_{n\lambda\mathbf{k}}(x)=\frac{1}{2\pi\sqrt{\beta}}e^{ik_y y+ik_z z}\exp\left(i\frac{2\pi n}{\beta}\tau\right)J_{\nu_n}(\lambda r),$$

$$n=0,\pm 1,\ldots,\quad \lambda\in R^+,\quad k_y,k_z\in R,$$

$$\Delta\phi_{n\lambda\mathbf{k}}(x)=-(\lambda^2+\mathbf{k}^2)\phi_{n\lambda\mathbf{k}}(x). \tag{16}$$

Here, J_{ν_n} is the Bessel function of first kind and $\nu_n=(2\pi|n|/\beta)$. Using the relation

$$\int_0^\infty dr\, r J_\nu(\lambda' r)J_\nu(\lambda r)=\frac{1}{\lambda}\delta(\lambda-\lambda'),$$

one can check that the modes (15) are normalized according to

$$(A^{(i',n'\lambda'\mathbf{k}')},A^{(i,n\lambda\mathbf{k})})\equiv\int d^4x\sqrt{g}g^{ab}A_a^{(i',n'\lambda'\mathbf{k}')*}A_b^{(i,n\lambda\mathbf{k})}$$

$$=\delta_{i'i}\delta_{n'n}\delta^{(2)}(\mathbf{k}-\mathbf{k}')\frac{1}{\lambda}\delta(\lambda-\lambda').$$

The first three eigenfunctions (15) satisfy $\nabla^a A_a=0$ and have eigenvalue $\lambda^2+\mathbf{k}^2$, while $A_a^{(\mathrm{IV})}$ is a pure gauge and has eigenvalue $(1/\alpha)(\lambda^2+\mathbf{k}^2)$.

Using these eigenfunctions, we can compute the diagonal ζ function using the spectral representation Equation (12): after the integration over $d\mathbf{k}$, the contributions of the modes to the diagonal ζ function are

$$\zeta^{(\mathrm{I})}(s;x)=\zeta^{\mathrm{scalar}}(s;x),$$

$$\zeta^{(\mathrm{II})}(s;x)=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\,\lambda^{1-2s}\times\left[\frac{\nu_n^2}{r^2}J_{\nu_n}^2(\lambda r)+[\partial_r J_{\nu_n}(\lambda r)]^2\right],$$

$$\zeta^{(\mathrm{III})}(s;x)=\frac{s-1}{s}\zeta^{\mathrm{scalar}}(s;x)+\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s+1)}\times\sum_n\int_0^\infty d\lambda\,\lambda^{1-2s}\left[\frac{\nu_n^2}{r^2}J_{\nu_n}^2(\lambda r)+[\partial_r J_{\nu_n}(\lambda r)]^2\right],$$

$$\zeta^{(\mathrm{IV})}(s;x)=\frac{\alpha^s}{s}\zeta^{\mathrm{scalar}}(s;x)+\frac{\alpha^s\Gamma(s)}{4\pi\beta\Gamma(s+1)}\times\sum_n\int_0^\infty d\lambda\,\lambda^{1-2s}\left[\frac{\nu_n^2}{r^2}J_{\nu_n}^2(\lambda r)+[\partial_r J_{\nu_n}(\lambda r)]^2\right],$$

where the spectral representation of the local ζ function of a minimally coupled scalar field on $C_\beta\times R^2$ is

$$\zeta^{\mathrm{scalar}}(s;x)=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_{n=-\infty}^{\infty}\int_0^\infty d\lambda\,\lambda^{3-2s}J_{\nu_n}^2(\lambda r).$$

Now, looking for a way close to that followed by Kabat [11], we use the following identity, which can be proved using some recursion formulas for the Bessel functions [18]:

$$2\left[\frac{\nu_n^2}{r^2}J_{\nu_n}^2(\lambda r)+[\partial_r J_{\nu_n}(\lambda r)]^2\right]=2\lambda^2J_{\nu_n}^2(\lambda r)+\frac{1}{r}\partial_r r\partial_r J_{\nu_n}^2(\lambda r), \tag{17}$$

and so the spin-traced local ζ function becomes

$$\zeta(s;x)=\left(1+\frac{s-1}{s}+\frac{\alpha^s}{s}\right)\zeta^{\mathrm{scalar}}(s;x)+\frac{s+1+\alpha^s(s-1)}{2s}\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty\lambda^{1-2s}\times\left[2\lambda^2J_{\nu_n}^2(\lambda r)+\frac{1}{r}\partial_r r\partial_r J_{\nu_n}^2(\lambda r)\right],$$

namely,

$$\zeta(s;x)=(3+\alpha^s)\zeta^{\mathrm{scalar}}(s;x)+\frac{s+1+\alpha^s(s-1)}{2s}\zeta^{\mathrm{V}}(s;x), \tag{18}$$

where we have set

$$\zeta^{\mathrm{V}}(s;x)=\frac{1}{r}\partial_r r\partial_r\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_{n=-\infty}^{\infty}\int_0^\infty d\lambda\,\lambda^{1-2s}J_{\nu_n}(\lambda r)^2. \tag{19}$$

Notice that the term $\zeta^{\mathrm{V}}(s;x)$ arises from the "conical" components of the field, i.e., A_τ and A_r. In particular, its source is the second term in the right-hand side of Eq. (17) only. This term will produce the Kabat "surface term" as we will see shortly.

We have taken $(1/r)\partial_r r\partial_r$, which is in fact the Laplacian Δ, outside the integral and the series, but this is a safe short-cut: indeed, one could first let Δ act on the Bessel function using $\partial_r J_\nu(\lambda r)=\lambda J_{\nu-1}(\lambda r)-(\nu/r)J_\nu(\lambda r)$, go through some tedious calculations and get the same result as Eq. (21).

So far, the expressions for ζ^{scalar} and ζ^{V} are just formal, since one can easily see that there is no value of s for which they converge. The correct way to compute ζ^{scalar} in this background has been recently given by Zerbini, Cognola, and Vanzo [6], following an earlier work of Cheeger [7], and the result is

$$\zeta^{\mathrm{scalar}}(s;x)=\frac{r^{2s-4}}{4\pi\beta\Gamma(s)}I_\beta(s-1),$$

where

$$I_\beta(s)=\frac{\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{\pi}}[G_\beta(s)-G_{2\pi}(s)], \tag{20}$$

$$G_\beta(s)=\sum_{n=1}^{\infty}\frac{\Gamma(\nu_n-s+1)}{\Gamma(\nu_n+s)},\quad G_{2\pi}(s)=-\frac{\Gamma(1-s)}{2\Gamma(s)},$$

$$I_\beta(0)=\frac{1}{6}\left(\frac{2\pi}{\beta}-\frac{\beta}{2\pi}\right),$$

$$I_\beta(-1)=\frac{1}{90}\left(\frac{2\pi}{\beta}-\frac{\beta}{2\pi}\right)\left[\left(\frac{2\pi}{\beta}\right)^2+11\right].$$

The function $I_\beta(s)$ is analytic in the whole complex plane but in $s=1$, where it has a simple pole with residue $\frac{1}{2}[(\beta/2\pi)-1]$. Following the same procedure used in [6] to obtain the above result, we can compute the contribution to the ζ function coming from $\zeta^{\rm V}(s;x)$. The essential step to give a sense to Eq. (19) is the separation of the small eigenvalue ν_0 from the others [7]: define

$$\zeta^{\rm V}_{<}(s;x)=\Delta\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\int_0^\infty d\lambda\lambda^{1-2s}J_0^2(\lambda r),$$

$$\zeta^{\rm V}_{>}(s;x)=2\Delta\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_{n=1}^{\infty}\int_0^\infty d\lambda\lambda^{1-2s}J_{\nu_n}^2(\lambda r).$$

The integrals over λ can be computed [18]: for $\frac{1}{2}<{\rm Re}s<1+\nu$

$$\int_0^\infty d\lambda\lambda^{1-2s}J_\nu^2(\lambda r)=r^{2s-2}\frac{\Gamma(s-\frac{1}{2})\Gamma(\nu-s+1)}{2\sqrt{\pi}\Gamma(s)\Gamma(\nu+s)}.$$

Therefore, in the strip $\frac{1}{2}<{\rm Re}s<1$ we get

$$\zeta^{\rm V}_{<}(s;x)=-\Delta\frac{r^{2s-2}\Gamma(s-1)}{4\pi\Gamma(s)^2}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}}G_{2\pi}(s),$$

while

$$\zeta^{\rm V}_{>}(s;x)=\Delta\frac{r^{2s-2}\Gamma(s-1)}{4\pi\Gamma(s)^2}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}}G_\beta(s),$$

which is valid in the strip $1<{\rm Re}s<1+\nu_1$, since the series defining $G_\beta(s)$ converges for $s>1$. Both expressions can now be analytically continued the whole complex plane and then summed, so we can write

$$\zeta^{\rm V}(s;x)=\Delta\frac{r^{2s-2}\Gamma(s-1)}{4\pi\Gamma(s)^2}I_\beta(s)=\frac{(s-1)r^{2s-4}}{\pi\beta\Gamma(s)}I_\beta(s).\tag{21}$$

This result could be obtained directly from Eq. (19), noting that

$$\zeta^{\rm V}(s;x)=\Delta\left[\frac{s}{s-1}\zeta^{\rm scalar}(s+1;x)\right].$$

Note also that $\zeta^{\rm V}(s;x)|_{\beta=2\pi}=0$ and $\zeta^{\rm V}(s=0;x)=0$.

Now we can write the final result for the local ζ function of the electromagnetic field: after adding the contribution of the ghosts, which is just $-2\zeta_\beta^{\rm scalar}(s;x)$, we get

$$\begin{aligned}\zeta^{\rm em}(s;x)&=(1+\alpha^s)\zeta^{\rm scalar}(s;x)+\frac{s+1+\alpha^s(s-1)}{2s}\zeta^{\rm V}(s;x)\\&=(1+\alpha^s)\frac{r^{2s-4}}{4\pi\beta\Gamma(s)}I_\beta(s-1)\\&\quad+\frac{s+1+\alpha^s(s-1)}{2s}\frac{(s-1)r^{2s-4}}{\pi\beta\Gamma(s)}I_\beta(s).\end{aligned}\tag{22}$$

From this expression we can easily see that $\zeta^{\rm em}(s;x)|_{s=0}=0$ and

$$\zeta^{\rm em\prime}(s;x)|_{s=0}=\frac{1}{2\pi\beta r^4}I_\beta(-1)-(1-\tfrac{1}{2}\ln\alpha)\frac{1}{\pi\beta r^4}I_\beta(0).\tag{23}$$

Therefore, the one-loop effective Lagrangian density for the electromagnetic field on $C_\beta\times R^2$ is

$$\begin{aligned}\mathcal{L}^{\rm em}_\beta(x)&=2\mathcal{L}^{\rm scalar}_\beta(x)-\frac{(1-\frac{1}{2}\ln\alpha)}{2\pi\beta r^4}I_\beta(0)\\&=\frac{1}{4\pi\beta r^4}I_\beta(-1)-\frac{(1-\frac{1}{2}\ln\alpha)}{2\pi\beta r^4}I_\beta(0).\end{aligned}\tag{24}$$

Since $I_{2\pi}(s)=0$, we can notice that both terms of the effective Lagrangian density vanish when the conical singularity disappears, $\beta=2\pi$.

A few remarks on this result. First, no surprise that in the effective Lagrangian density we get a contribution which is twice that of a scalar field. More surprising is the second term: after the integration over the spatial variables, it gives rise to what Kabat [11] calls ''surface'' term and interprets as a low-energy relic of stringy effects foreseen by Susskind and Uglum [5]. This term would give a negative contribution to the entropy of the system, at least for $\alpha<e^2$, and actually also the total correction to the entropy at the black hole temperature $\beta=2\pi$ would be negative for $\alpha<e^{6/5}$, which is clearly nonsense if we want to give a state-counting interpretation to the entropy. However, in the four-dimensional case we get that it is not gauge invariant, in contrast with Kabat's result.

With this regard, it is interesting to note that in two dimensions, i.e., on C_β, the result is indeed independent on the gauge-fixing parameter: using the modes of the em field on C_β given by Kabat [11] and following the same procedure as above, before adding the contribution of ghosts we get

$$\zeta^{\rm em}_{d=2}(s;x)=(1+\alpha^s)[\zeta^{\rm scalar}_{d=2}(s;x)+\zeta^{\rm V}_{d=2}(s;x)],$$

where

$$\zeta^{\rm scalar}_{d=2}(s;x)=\frac{r^{2s-2}}{\beta\Gamma(s)}I_\beta(s),$$

$$\zeta^{\rm V}_{d=2}(s;x)=\Delta\frac{r^{2s}}{2\beta\Gamma(s+1)}I_\beta(s+1),$$

and so, adding the contribution of the ghosts we have

$$\mathcal{L}^{\rm em}(x)=\frac{1}{2\pi\beta r^2}(2\pi-\beta),$$

which is gauge independent and, after the integration over the manifold, gives exactly the result of Kabat.

Coming back to the four-dimensional case, we argue that a natural (albeit not the only possible, see the final discussion) procedure to restore the gauge invariance is simply to drop the Kabat term, namely, the last term in Eq. (24), obtaining the reasonable result $\mathcal{L}^{\rm em}(x)=2\mathcal{L}^{\rm scalar}(x)$.

First of all, notice that the gauge invariance must hold for the integrated quantities as the effective action, namely, the logarithm of the integrated effective Lagrangian. In fact, the ghost procedure, which takes into account the gauge invariance, works on integrated quantities. However, in our case, the integration of the Kabat term produces a divergent *gauge-dependent* result, and thus it seems reasonable to discard such a local term. With this regard, it is important to note that Kabat obtains a gauge-independent result because, within his regularization procedure, he has the freedom to choose an independent cutoff parameter for each mode. Instead, in our procedure we have only one cutoff parameter ϵ, to which we give a precise physical meaning, namely, the minimal distance from the horizon.

A more general discussion might be the following. It is worth one's while stressing that, dealing with *smooth compact* manifold, local quantities as local heat kernel and local ζ functions are intrinsically ill defined due to the possibility of adding to them a total covariant derivative with vanishing integral. In such a case, the previous global quantities are well defined, and one can satisfactorily employ these latter instead of local quantities in order to avoid the ill-definiteness problem. Notice also that the gauge-dependent Kabat surface term formally looks such as a Laplacian and thus it should disappear after a global integration, provided regularity conditions on the manifold are satisfied, producing gauge-independent integrated quantities. However, this is not the case for the present situation, where the background is a noncompact manifold with a conical singularity, and the integrated quantities diverge requiring a regularization procedure. We stress that the use of local quantities is preferred on the physical ground, because they lead us to the correct temperature dependency as we will see shortly.

Therefore, in our case the local quantities remain ill defined and require a further regularization procedure in order to fix the possible added total derivative term before we integrate. Furthermore, the integrated quantities are divergent, so we expect we to have to take into account also total derivative terms with a divergent integral. In our case this further regularization procedure consists just in discarding the Kabat term. Notice that this procedure produces gauge-independent local quantities.

Once we have dropped the Kabat's term, we can compute thermodynamical quantities such as internal energy and entropy: we need the effective action and so we have to introduce a smearing function $\varphi(x)$ in order to define the trace: $\ln Z_\beta=\int d^4x\sqrt{g}\mathcal{L}_\beta(x)\varphi(x)$. Actually, since $\mathcal{L}_\beta$ does not depend on the transverse coordinates y and z, the integration on these coordinates simply yields the infinite area of the Rindler horizon, that we indicate as $A_\perp$. This divergence has clear physical meaning. The integration over τ has no problem, while a convenient smearing function for the integration over r is $\varphi(r)=\theta(r-\epsilon)$, and so the effective action becomes

$$\ln Z_\beta(\epsilon)=\frac{A_\perp}{8\pi\epsilon^2}I_\beta(-1). \tag{25}$$

For $\epsilon\to 0$ we have a divergence that can be seen as a ''horizon'' divergence [1], since as $r\to 0$ we approach the horizon of the Rindler wedge.

From Eq. (25) we can compute the free energy, $F_\beta=-(1/\beta)\ln Z_\beta$, which at high temperature, $\beta\to 0$, has a leading behavior $-2(\pi^2 A_\perp/180\epsilon^2\beta^4)$, in perfect agreement with the statistical mechanics result of Susskind and Uglum [5]. Instead, Kabat [11] obtains a leading behavior $-2(A_\perp/8\epsilon^2\beta^2)$, where the behavior β^{-2}, independent of the dimension of the space-time, is typical of the integrated heat-kernel approach. Of particular interest for the black hole physics is the entropy of the system:

$$S_\beta=\beta^2\partial_\beta F_\beta=\frac{A_\perp}{90\beta\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^2+5\right]. \tag{26}$$

This equation gives, in Rindler space approximation, the one-loop quantum correction to the black hole entropy coming form the electromagnetic field propagating in the region outside the horizon. It shows the well-known horizon divergence [1] (see also [19] for a recent review on this topic): unless we suppose the existence of a natural effective cutoff at the Planck scale due to an (unknown) quantum gravity theory or back-reaction horizon fluctuations etc.,[2] we get a divergent entropy which is physically unsatisfactory and contrasts with the finite thermodynamical Bekenstein-Hawking entropy. However, this problem is not peculiar to the photon field, as it occurs for scalar and spinorial fields as well.

We can note that, if we took into account the surface term which we have previously dropped, we would obtain the unphysical, because being gauge dependent, expression

$$S_\beta(\alpha)=\beta^2\partial_\beta F_\beta=\frac{A_\perp}{90\beta\epsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^2+5\right]-(1-\tfrac{1}{2}\ln\alpha)\frac{A_\perp}{6\beta\epsilon^2}.$$

As anticipated above, this expression for the entropy is negative when the singularity is absent, $\beta=2\pi$, and $\ln\alpha<\frac{6}{5}$. Moreover, for $\ln\alpha<\frac{4}{3}$, $S_\beta(\alpha)$ shows a further zero of the entropy corresponding to an inconsistent (gauge-depending) *finite* temperature *pure* quantum state of the field.

Another thermodynamical quantity that we can compute from the effective action (25) is the internal energy. Since it is well known [15] that the usual Minkowski vacuum state, restricted to the Rindler wedge, may be viewed as a Rindler thermal state at temperature $T=1/2\pi$, it is natural to require that the internal energy vanishes when $\beta=2\pi$, namely, when the conical singularity is absent. Hence, we define a renormalized free energy as $F_\beta^{\rm sub}=F_\beta-U_{2\pi}$ which, by means of the relation $U_\beta=(1/\beta)S_\beta+F_\beta$, automatically gives $U_\beta^{\rm sub}=U_\beta-U_{2\pi}$, that trivially vanish at $\beta=2\pi$, while $S_\beta^{\rm sub}=S_\beta$. Explicitly,

[2]However, such a cutoff should depend on the field spin value to produce the correct entropy factor in front of the horizon area. See [20].

$$U_\beta^{\rm sub}=\frac{\pi^2 A_\perp}{30\beta^4\epsilon^2}+\frac{A_\perp}{36\beta^2\epsilon^2}-\frac{13A_\perp}{1440\pi^2\epsilon^2}. \tag{27}$$

From this expression we can also compute the thermal energy-momentum tensor: using the relation $U_\beta = -\int\langle T_0^0\rangle r dr dy dz$, supposing that $\langle T_0^0\rangle$ depends on r only[3] and that it vanishes at $\beta=2\pi$, we get

$$\langle T_0^0\rangle^{\rm sub}=-\frac{\pi^2}{15\beta^4 r^4}-\frac{1}{18\beta^2 r^4}+\frac{13}{720\pi^2 r^4},$$

$$\langle T_{ab}\rangle^{\rm sub}=\frac{1}{3}\langle T_0^0\rangle^{\rm sub}\left[4\frac{\mathcal{K}_a\mathcal{K}_b}{\mathcal{K}^2}-g_{ab}\right], \tag{28}$$

where in the last equation we have supposed a perfect fluid form, $\mathcal{K}_a=(\partial_t)_a$ is the timelike Killing vector associated with the time coordinate of the Rindler space, and $\mathcal{K}^2=\mathcal{K}_a\mathcal{K}^a$. This result for $\langle T_0^0\rangle^{\rm sub}$ is in agreement with twice the local heat-kernel result [21].

As we have already said in the introduction, our results for the thermodynamical quantities differ from those obtained with the point-splitting and the optical metric methods [22,8,4,9,10]. In fact, for $\langle T_0^0\rangle^{\rm sub}$ they give

$$-\frac{\pi^2}{15\beta^4 r^4}-\frac{1}{6\beta^2 r^4}+\frac{11}{240\pi^2 r^4}, \tag{29}$$

for spin 1 and one-half of this quantity for spin 0. Our result for the coefficient of the term proportional to β^{-2} is one-third of that in Eq. (29), while the difference in the numerical coefficient of the term independent of β is unimportant, since it is determined by the other two by requiring the vanishing of the energy-momentum tensor for $\beta=2\pi$. The reason of this discrepancy, which appears also in the heat-kernel approach [21,11–14] is not clear to us and requires further investigations.

III. A GENERAL CONJECTURE

Let us focus our attention back on Kabat's surface term in the effective Lagrangian, Eq. (24): is it an accident which appears in our manifold and in the vector case only, or conversely, is it a more general phenomenon?

We can grasp some insight by studying either the local ζ function, as it appears in Eq. (12), or the local heat kernel of Eq. (10) and passing to the local ζ functions through Eq. (11). In fact, the Kabat term already comes out in the heat kernel and then it remains substantially unchanged passing to the local ζ function through Eq. (11). The components of the modes II, III, and IV contain (covariant) derivatives in both the conical and R^2 indices. Using trivial (covariant) derivative rules and reminding that $\nabla_\mu\nabla^\mu\phi=-\lambda^2\phi$ and $\partial_i\partial^i\phi=-\mathbf{k}^2\phi$ we may transform scalar products of (covariant) derivatives appearing in the integrand of Eq. (10) into a covariant divergence of a vector plus a simple scalar term. Summing over the modes, these parts produce, respectively, the Kabat surface term and the ‘‘twice scalar’’ part of the effective Lagrangian in Eq. (24) (the mode I gives a contribution to this latter part only). This is the general mechanism which produces Kabat's term. Let us illustrate this in more detail. Dealing with the modes IV, we find

$$\begin{aligned} g^{ab}A_a^{(\rm IV)*}A_b^{(\rm IV)}&=\frac{1}{\lambda^2+\mathbf{k}^2}\nabla_a\phi^*\nabla^a\phi\\ &=\frac{1}{\lambda^2+\mathbf{k}^2}[\nabla_a(\phi^*\nabla^a\phi)-\phi^*\nabla_a\nabla^a\phi]\\ &=\frac{1}{\lambda^2+\mathbf{k}^2}[\nabla_a(\phi^*\nabla^a\phi)+(\lambda^2+\mathbf{k}^2)\phi^*\phi]. \end{aligned} \tag{30}$$

Thus, using the particular form of our modes we get

$$g^{ab}A_a^{(\rm IV)*}A_b^{(\rm IV)}=\frac{1}{2(\lambda^2+\mathbf{k}^2)}\Delta J_{\nu_n}^2+J_{\nu_n}^2.$$

The modes III contribute to the local heat kernel and to the effective Lagrangian in the same way. The modes II require a little different care: we have

$$\begin{aligned} g^{ab}A_a^{(\rm II)*}A_b^{(\rm II)}&=\frac{1}{\lambda^2}g^{\mu\nu}\epsilon_{\mu\sigma}\epsilon_{\nu\rho}\nabla^\sigma\phi^*\nabla^\rho\phi\\ &=\frac{1}{\lambda^2}[\nabla^\sigma(g^{\mu\nu}\epsilon_{\mu\sigma}\epsilon_{\nu\rho}\phi^*\nabla^\rho\phi)\\ &\quad -g^{\mu\nu}\epsilon_{\nu\rho}\epsilon_{\mu\sigma}\phi^*\nabla^\sigma\nabla^\rho\phi]\\ &=\frac{1}{\lambda^2}[\nabla^\sigma(g_{\sigma\rho}\phi^*\nabla^\rho\phi)-\phi^*g_{\rho\sigma}\nabla^\rho\nabla^\sigma\phi]\\ &=\frac{1}{\lambda^2}[\nabla_\mu(\phi^*\nabla^\mu\phi)+\lambda^2\phi^*\phi]\\ &=\frac{1}{\lambda^2}[\nabla_a(\phi^*\nabla^a\phi)+\lambda^2\phi^*\phi]. \end{aligned} \tag{31}$$

And thus, reminding the particular form of our modes

$$g^{ab}A_a^{(\rm II)*}A_b^{(\rm II)}=\frac{1}{2\lambda^2}\Delta J_{\nu_n}^2+J_{\nu_n}^2.$$

The contribution to the effective Lagrangian is similar to the previous ones. In both the examined cases, using the specific form of scalar eigenfunctions, we have obtained the right-hand side of Eq. (17) except for some factors which will be arranged summing over all the modes in the final result. The term $\nabla_a(\phi^*\nabla^a\phi)$ $(=\frac{1}{2}\Delta J_{\nu_n}^2)$ contributes only to the second term of the right-hand side of Eq. (18), namely, it contributes only to the Kabat surface term in the effective Lagrangian in Eq. (24). Moreover, the term $\lambda^2\phi^*\phi$ $(=\lambda^2 J_{\nu_n}^2)$ contributes only to the remaining term in the right-hand side of Eq. (18) and thus to the ‘‘twice scalar’’ part of the same effective Lagrangian only.

We further remark that the previously employed covariant derivative identities are exactly the same which one has to use in order to check the correct normalization of the

[3] The remaining coordinates define Killing vectors.

modes.[4] However, in that case the surface terms are dropped after the formal integration in the spatial variables, because they do not contribute, in a distributional sense, to the overall normalization. Conversely, following the local ζ-function method they produce Kabat-like terms.

More generally speaking, following the previous outline, one can avoid specifying the form of the scalar eigenfunction and the use of Eq. (17), remaining on a more general ground.[5] This means that we can consider a more general manifold which is topologically $\mathcal{M}\times R^2$ with the natural product metric, where $\mathcal{M}$ is any, maybe curved, two-dimensional manifold. The photon effective action can be written as

$$\ln Z=-\frac{1}{2}\ln\det\mu^{-2}\left[+\Delta_1-\left(1-\frac{1}{\alpha}\right)d_0\delta_0\right]+\ln Z_{\text{ghost}}, \tag{32}$$

where $\Delta_1=d_0\delta_0+\delta_1 d_1$ is the Hodge Laplacian for one-forms ($\delta_n\equiv d_n^\dagger$ with respect to the Hodge scalar product). The eigenfunctions of the operator appearing in the above equation can still be written as in Eq. (15). Now, $\phi=(1/2\pi)e^{ik_y y+ik_z z}\mathbf{J}_{n,\lambda}(x^\mu)$ where $\mathbf{J}_{n,\lambda}(x^\mu)$ is an eigenfunction of (the Friedrichs extension of) the 0-forms Hodge Laplacian[6] $\Delta_0^{\mathcal{M}}$ on $\mathcal{M}$, with eigenvalue $+\lambda^2$. Employing a bit of n-forms algebra, one can obtain in our manifold the same eigenvalues found in the manifold $\mathcal{C}_\beta\times R^2$. Furthermore, once again $\delta_0 A^{(y)}=0$, namely, $\nabla^a A_a^{(y)}=0$, in case $y=$I, II, III. Then, using Eqs. (30) and (31) and the definition in Eq. (12), we get, before we take into account the ghosts contribution,

$$\zeta_{\mathcal{M}\times R^2}(s;x)=(3+\alpha^s)\zeta^{\text{scalar}}_{\mathcal{M}\times R^2}(s;x)+\frac{s+1+\alpha^s(s-1)}{2s}\zeta^V_{\mathcal{M}\times R^2}(s;x),$$

where the surface term reads

$$\zeta^{\text{V}}_{\mathcal{M}\times R^2}(s;x)=\frac{\Gamma(s-1)}{4\pi\Gamma(s)}\nabla_a\sum_n\int d\lambda\lambda\mathbf{J}^*\nabla^a\mathbf{J}.$$

Notice that, if the manifold is regular and compact, this surface term automatically disappears after we integrate over the spatial variables. Instead, if the manifold $\mathcal{M}$ has conical singularities or boundaries, then this term could survive the integration. We can further suppose that $\mathcal{M}$ contains a Killing vector ∂_τ with compact orbits in such a manner that we can define a temperature $1/\beta$ and interpret the effective action as the logarithm of the photon partition function. Employing coordinates r,τ on $\mathcal{M}$, we can decompose $\mathbf{J}_{n,\lambda}(r,\tau)$ as $\mathbf{J}_{n,\lambda}(r,\tau)=\beta^{-1/2}e^{-2\pi ni\tau/\beta}\mathcal{J}_{n,\lambda}(r)$, $\mathcal{J}_{n,\lambda}(r)$ being real. The surface term reads, in this case,

$$\zeta^{\text{V}}_{\mathcal{M}\times R^2}(s;x)=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\Delta_0\sum_n\int d\lambda\lambda\mathcal{J}_{n,\lambda}(r)^2.$$

Equation (32) holds in very general manifolds, also dropping the requirement of a metric which is Cartesian product of the flat R^2 metric and any other metric.

One can simply prove that, if ϕ is an eigenfunction of Δ_0 with eigenvalue $+\nu^2$ on such a general manifold, $A=d_0\phi$ will be an eigenfunction of the vector operator $\Delta_1+[1-(1/\alpha)]d_0\delta_0$ with gauge-dependent eigenvalue $+\nu^2/\alpha$. Employing the rule in Eq. (30) with ν^2 in place of $\lambda^2+\mathbf{k}^2$, we expect that this latter eigenfunction should produce a (gauge dependent) surface term into the local ζ function.

Dealing with spin $s\geqslant 1$ and massless fields, because of the simple equation of motion form (in Feynman-like gauges at least), we expect to find out some normal modes obtained as covariant derivatives of the scalar field modes opportunely rearranged. Hence, barring miraculous cancellations, the corresponding local heat kernel, local ζ function, and effective Lagrangian, should contain Kabat-like surface terms, due to the previous mechanism. We will check this for the graviton in the next section.[7]

IV. THE GRAVITON ζ FUNCTION IN THE HARMONIC GAUGE

In this section we shall compute the local ζ function in the case of a linearized graviton propagating in the Rindler wedge. We will see that Kabat-like surface terms indeed appear, as we suggested in the previous section. Moreover, we will find out that consistent results arise by discarding all those terms.

Following the same procedure used in [23,24], which employs the harmonic gauge, we decompose the linearized field of a graviton into its symmetric traceless part h_{ab} and its trace part h. Choosing an opportune normalization factor of the fields and dropping boundary terms, the Euclidean action (containing also the gauge-fixing part) looks like

$$S_E[h_{ab},h]=\frac{1}{32\pi G}\int dx^4\sqrt{g}\left\{\frac{1}{2}g^{aa'}g^{bb'}h_{ab}\nabla_c\nabla^c h_{a'b'}+\frac{1}{4}h\nabla_d\nabla^d h\right\}, \tag{33}$$

where g, g^{ab}, and covariant derivatives are referred to the background metric, namely, the Euclidean Rindler metric.

[4]In this case the indices $(n\lambda\mathbf{k})$ which appear in the modes A_a and A_a^* are generally different.

[5]It is clear from our discussion that the Kabat term gets contributions from each mode II,III,IV, not depending on the corresponding eigenvalue. This term does not coincide with the surface term recently suggested by Fursaev and Miele [14] dealing with compact manifolds, because this latter involves zero modes only.

[6]Remind that the Hodge Laplacian coincides with minus the Laplace-Beltrami operator for 0-forms. This generally does not happen for n-forms when $n>0$ in curved manifolds.

[7]We also tried to study the photon case employing a so-called ''physical gauge'' as $A_z=0$. The use of the ζ-function regularization in this case is problematic due to a remaining gauge ambiguity arising whenever one tries to deal with a path integral, nonformal approach in axial gauges. Nevertheless, through the same mechanism, the Kabat term seems to survive in this case as well.

That metric is also used to raise and lower indices. Notice that curvature tensor terms (see [24]) do not appear in the above action and this is due to the flatness of the manifold. It is necessary to point out that we changed the sign of the trace field Lagrangian as this appeared after we performed a ''simple'' Wick rotation toward the imaginary time on the Lorentzian Lagrangian. In fact, in order to obtain an Euclidean Lagrangian producing a formally finite functional integral,[8] it is also necessary to rotate the scalar field h into imaginary values during the Wick rotation. This adjusts the sign in front of the corresponding Lagrangian [23,24]. We can write, as far as the effective action is concerned:

$$\ln Z_{\text{gravitons}} = -\frac{1}{2}\ln\det\mu^{-2}[-g^{aa'}g^{bb'}\nabla_c\nabla^c] - \frac{1}{2}\ln\det\mu^{-2} \times[-\nabla_d\nabla^d] + \ln Z_{\text{grv. ghosts}}. \tag{34}$$

The first determinant has to be evaluated in the L^2 space of traceless symmetric tensorial field. Unessential factors in front of the operators can be dropped into an overall added constant and thus omitted. Furthermore, the ghost contribution has been taken into account through the last term of the previous equation. A usual procedure[9] leads us to [23,24]

$$\ln Z_{\text{grv. ghosts}} = -2\ln Z_{\text{vector}}.$$

The partition function in $\ln Z_{\text{vector}}$ is the partition function obtained quantizing the massless Klein-Gordon vector field. Hence, this also coincides with the photon partition function evaluated in the Feynman gauge, namely, $\alpha=1$ in Eq. (9), *without* taking into account the photon ghost contribution. Thus, from the effective graviton ghost action, two vector $\alpha=1$ Kabat's surface terms (with the sign changed) arise. In order to compute the above functional determinants, we have to look for normalized modes of a self-adjoint extension of the tensorial Laplace-Beltrami operator $\Delta_T = g^{aa'}g^{bb'}\nabla_c\nabla^c$ in the space of symmetric traceless tensors and the scalar Laplace-Beltrami operator $\Delta_S = \nabla_d\nabla^d$. Obviously, the eigenfunctions of Δ_S can be chosen as $h_{n\lambda\mathbf{k}} = \phi_{n\lambda\mathbf{k}}(x)$, where, as before, $\phi = \phi_{n\lambda k}(x)$ indicates the generic eigenfunction of the scalar Laplacian, Eq. (16).

In the tensorial case, we find the following nine classes of symmetric traceless eigenfunctions[10]:

$$h^{(1)}_{n\lambda\mathbf{k}}: \quad \frac{\sqrt{2}}{\lambda^2}\nabla_\mu\nabla_\nu\phi + \frac{1}{\sqrt{2}}g_{\mu\nu}\phi = h^{(1)}_{\mu\nu} = h^{(1)}_{\nu\mu},$$

$$h^{(2)}_{n\lambda\mathbf{k}}: \quad \frac{\sqrt{g}}{\sqrt{2}\lambda^2}\{\epsilon_{\mu\sigma}\nabla^\sigma\nabla_\nu\phi + \epsilon_{\nu\sigma}\nabla^\sigma\nabla_\mu\phi\} = h^{(2)}_{\mu\nu} = h^{(2)}_{\nu\mu},$$

$$h^{(3)}_{n\lambda\mathbf{k}}: \quad \frac{1}{\sqrt{2}k\lambda}\partial_i\nabla_\mu\phi = h^{(3)}_{i\mu} = h^{(3)}_{\mu i},$$

$$h^{(4)}_{n\lambda\mathbf{k}}: \quad \frac{\sqrt{g}}{\sqrt{2}k\lambda}\epsilon_{\mu\nu}\partial_i\nabla^\nu\phi = h^{(4)}_{i\mu} = h^{(4)}_{\mu i},$$

$$h^{(5)}_{n\lambda\mathbf{k}}: \quad \frac{\sqrt{g}}{\sqrt{2}k\lambda}\epsilon_{\mu\nu}\epsilon_{ij}\partial^j\nabla^\nu\phi = h^{(5)}_{i\mu} = h^{(5)}_{\mu i},$$

$$h^{(6)}_{n\lambda\mathbf{k}}: \quad \frac{1}{\sqrt{2}k\lambda}\epsilon_{ij}\partial^j\nabla_\mu\phi = h^{(6)}_{i\mu} = h^{(6)}_{\mu i},$$

$$h^{(7)}_{n\lambda\mathbf{k}}: \quad \frac{\sqrt{2}}{\mathbf{k}^2}\partial_i\partial_j\phi + \frac{1}{\sqrt{2}}\delta_{ij}\phi = h^{(7)}_{ij} = h^{(7)}_{ji},$$

$$h^{(8)}_{n\lambda\mathbf{k}}: \quad \frac{1}{\sqrt{2}\mathbf{k}^2}\{\epsilon_{ik}\partial^k\partial_j\phi + \epsilon_{jk}\partial^k\partial_i\phi\} = h^{(8)}_{ij} = h^{(8)}_{ji},$$

$$h^{(9)}_{n\lambda\mathbf{k}}: \quad \frac{1}{2}g_{\mu\nu}\phi - \frac{1}{2}\delta_{ij}\phi = h^{(9)}_{ab}.$$

Here, $\sqrt{g}\epsilon_{\mu\nu}$ indicates the antisymmetric Levi-Civita pseudotensor on the cone and ϵ_{ij} the antisymmetric Levi-Civita pseudotensor on R^2 in Cartesian coordinates. The previous modes satisfy

$$\Delta_T h^{(y)}_{n\lambda\mathbf{k}} = -(\lambda^2+\mathbf{k}^2)h^{(y)}_{n\lambda\mathbf{k}}, \quad y=1,2,\ldots,9, \tag{35}$$

and

$$\Delta_S h_{n\lambda\mathbf{k}} = -(\lambda^2+\mathbf{k}^2)h_{n\lambda\mathbf{k}}. \tag{36}$$

Finally, the normalization relations are $(y,y'=1,2,\ldots,9)$

$$\int d^4x\sqrt{g}g^{aa'}g^{bb'}h^{(y)*}_{n\lambda\mathbf{k}}(x)_{ab}h^{(y')}_{n'\lambda'k'_t}(x)_{a'b'} = \delta^{yy'}\delta_{nn'}\delta^{(2)}(\mathbf{k}-\mathbf{k}')\frac{\delta(\lambda-\lambda')}{\lambda}$$

and

$$\int d^4x\sqrt{g}h^*_{n\lambda\mathbf{k}}(x)h_{n'\lambda'k'}(x) = \delta_{nn'}\delta^{(2)}(\mathbf{k}-\mathbf{k}')\frac{1}{\lambda}\delta(\lambda-\lambda').$$

Using Eq. (12), we can write the local ζ function as

$$\begin{aligned}\zeta^{\text{gravitons}}(s;x) &= \sum_{y=1}^{9}\zeta^{(y)}(s;x) + \zeta^{\text{scalar}}(s;x)\\ &= \sum_{y=1}^{9}\sum_{n=-\infty}^{\infty}\int_0^\infty d\lambda\lambda\int_{R^2}d^2\mathbf{k}\nu_n^{-2s}g^{aa'}(x)\\ &\quad\times g^{bb'}(x)h^{*(y)}(x)_{ab}h^{(y)}(x)_{a'b'}\\ &\quad+\sum_{n=-\infty}^{\infty}\int_0^\infty d\lambda\lambda\int_{R^2}d^2\mathbf{k}\nu_n^{-2s}h^*(x)h(x).\end{aligned} \tag{37}$$

The latter term takes into account the graviton trace part contribution to local ζ function. Obviously, this is exactly the scalar local ζ function. Let us rather consider the former

[8]Remind that this functional integral contains the exponential $\exp(-S_E)$.

[9]This result holds also for local quantities.

[10]All the components of each eigenfunction class which do not appear in the following list are understood to vanish.

term and, in particular, the contribution due to $h^{(1)}$. Following the sketch of the previous section, we can rearrange this term transforming the product of the covariant derivatives into a scalar term added to several covariant divergences of vector and tensor fields:

$$\zeta^{(1)}(t;x)=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{3-2s}\phi^*\phi$$
$$+4\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{1-2s}\nabla_a(\phi^*\nabla^a\phi)$$
$$+2\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{-2s}\nabla^a\nabla_b[\nabla_a\phi^*\nabla^b\phi].$$

Using different notation, we finally find

$$\zeta^{(1)}(s;x)=\zeta^{\text{scalar}}(s;x)+2\zeta^V(s;x)+2\zeta^W(s;x),\qquad(38)$$

where we defined

$$\zeta^W(s;x)=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{-2s}\nabla^a\nabla_b(\nabla_a\phi^*\nabla^b\phi)$$
$$=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{-2s}\left[\frac{1}{r}\partial_r r\partial_r(\partial_r J_{\nu_n})^2+\frac{1}{r}\partial_r(\partial_r J_{\nu_n})^2-\frac{\nu_n^2}{r}\partial_r\frac{J_{\nu_n}(\lambda r)^2}{r^2}\right].\qquad(39)$$

Thus, we see that in the local ζ function the ($\alpha=1$)-Kabat surface term $\zeta^{\text{V}}(s;x)$ reappears, together with a new surface term $\zeta'^W(s;x)$. The contribution of $h^{(2)}$ is similar to the previous one and it reads

$$\zeta^2(s;x)=\zeta^{\text{scalar}}(s;x)+\zeta^V(s;x)+\zeta^W(s;x)+\zeta^U(s;x),$$

where, provided $\theta^{ab}=\epsilon^{ab}$ when $a,b=\mu,\nu$ and $\theta^{ab}=0$ otherwise,

$$\zeta^U(s;x)=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{-2s}\nabla_a\nabla_b\times[g\theta^{ac}\theta^{bd}\nabla_c\phi^*\nabla_d\phi]$$
$$=\frac{\Gamma(s-1)}{4\pi\beta\Gamma(s)}\sum_n\int_0^\infty d\lambda\lambda^{-2s}\partial_r\left[\frac{J_{\nu_n}^2}{r}-(\partial_r J_{\nu_n})^2\right].$$

The contributions of the remaining terms are much more trivial. In fact, a little algebra leads us to

$$\zeta^{(3)}(s;x)=\zeta^{(4)}(s;x)=\zeta^{(5)}(s;x)=\zeta^{(6)}(s;x)=\zeta^{\text{scalar}}(s;x)+\frac{1}{2}\zeta^V(s;x),$$

and

$$\zeta^{(7)}(s;x)=\zeta^{(8)}(s;x)=\zeta^{(9)}(s;x)=\zeta^{\text{scalar}}(s;x).$$

Finally, we have already noted above that the contribution of the trace terms h is exactly $\zeta^{\text{scalar}}(s;x)$. Then, taking into account the contribution of the ghost Lagrangian, which amounts to $-8\zeta^{\text{scalar}}(s;x)-2\zeta^V(s;x)$, we get the final expression of spin-traced graviton ζ function:

$$\zeta^{\text{gravitons}}(s;x)=2\zeta^{\text{scalar}}(s;x)+3\zeta^{\text{V}}(s;x)+3\zeta^W(s;x)+\zeta^U(s;x).\qquad(40)$$

Dropping the last three surface terms we obtain the reasonable result which agrees with the counting of the true graviton degrees of freedom: $\mathcal{L}^{\text{graviton}}(x)=2\mathcal{L}^{\text{scalar}}(x)$. Hence, all the thermodynamical quantities coincides with those of the previously computed photon fields.

V. DISCUSSION

In this paper we have computed the effective action of the photon and graviton fields in the conical background $C_\beta\times R^2$, and our main result is that it is just what one expects from counting the number of degrees of freedom, i.e., twice that of the massless scalar effective action. Moreover, we have got the correct Planckian temperature dependence of the thermodynamical quantities.

To get this apparently trivial result, we had to deal with unwanted terms arising from the presence of the conical singularity. We discussed how the appearance of those surface terms is quite a general phenomenon dealing with general manifolds in the case of fields with integer nonzero spin. The presence of conical singularities needs some further regularization procedure. In particular, this is necessary while studying the photon field in order to restore the gauge invariance of the integrated quantities. It could be interesting to develop an analogue research in the case of gravitons in any covariant gauge.

In the general case our proposal is the simplest one, namely, to discard all the surface terms. However, we think that, away from our ζ-function local approach, this should not be the only possible treatment of surface terms. In fact, comparing our results with Kabat's, it arises that, except for the two-dimensional case, the necessary treatment of surface terms strongly depends on the general approach used to define and calculate the effective Lagrangian. Moreover, it also depends on the regularization procedure used to define the integrated quantities.

In our local ζ-function approach, the meaning of the only cutoff as the minimal distance from the horizon leads ourselves towards the simple procedure of discarding the surface terms in order to restore the gauge invariance. In Kabat's treatment, the meaning of the employed cutoff is not so strict and permits one to make safe the gauge invariance and take on the surface terms as well. This is due to fine-tuning of mode-depending cutoffs which contain a further gauge-fixing parameter dependence.

In our approach, when integrating the surface term it is not possible to use an α-dependent cutoff different from that used for the rest and such that it cancels the α dependence in the integrated quantity: in fact, no real function $\epsilon(\alpha)$ can absorb the factor $(1-\frac{1}{2}\ln\alpha)$, appearing in the integrated surface term, for all the values of α.

In any case, we think that any procedure which does not discard the surface terms must be able to explain why the consequent result is not in agreement with what one expects from counting the number of degrees of freedom and to deal with the apparently unphysical corrections to the thermodynamical quantities arising from those terms. Maybe this is possible in an effective low-energy string theory which does not coincide with the ordinary quantum field theory.

ACKNOWLEDGMENTS

We are grateful to Guido Cognola, Giuseppe Nardelli, and, in particular, to Luciano Vanzo and Sergio Zerbini for valuable discussions and useful suggestions.

[1] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).
[2] R. Emparan, Phys. Rev. D **51**, 5716 (1995).
[3] J.S. Dowker, Phys. Rev. D **18**, 1856 (1978).
[4] J.S. Dowker, Class. Quantum Grav. **11**, L55 (1994).
[5] L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994).
[6] S. Zerbini, G. Cognola, and L. Vanzo, Phys. Rev. D **54**, 2699 (1996).
[7] J. Cheeger, J. Diff. Geom. **18**, 575 (1983).
[8] M.R. Brown and A.C. Ottewill, Phys. Rev. D **31**, 2514 (1985).
[9] V.P. Frolov and E.M. Serebriany, Phys. Rev. D **35**, 3779 (1987).
[10] B. Allen, J.G. Mc Laughlin, and A.C. Ottewill, Phys. Rev. D **45**, 4486 (1992).
[11] D. Kabat, Nucl. Phys. **B453**, 281 (1995).
[12] F. Larsen and F. Wilczek, Nucl. Phys. **B458**, 249 (1996).
[13] A.O. Barvinsky and S.N. Solodukhin, ''Non-minimal coupling, boundary terms and renormalization of the Einstein-Hilbert action and the black hole entropy,'' Report No. gr-qc/9512047 (unpublished).
[14] D.V. Fursaev and G. Miele, ''Cones, spins and heat kernels,'' Report No. hep-th/9605153 (unpublished).
[15] N.D. Birrel and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[16] D.V. Vassilevich, Phys. Rev. D **52**, 999 (1995).
[17] B.S. Kay and U.M. Studer, Commun. Math. Phys. **136**, 103 (1991).
[18] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[19] F. Belgiorno and M. Martellini, Phys. Rev. D **53**, 7073 (1996).
[20] J.W. York, Jr., Phys. Rev. D **28**, 2929 (1983).
[21] G. Cognola, K. Kirsten, and L. Vanzo, Phys. Rev. D **49**, 1029 (1994).
[22] D. Deutsch and P. Candelas, Phys. Rev. D **20**, 3063 (1979).
[23] S.W. Hawking, in *Euclidean Quantum Gravity*, edited by G.S. Gibbons and S.W. Hawking (World Scientific, Singapore, 1993).
[24] G.W. Gibbons, S.W. Hawking, and M.J. Perry, Nucl. Phys. **B138**, 141 (1978).