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From vacuum fluctuations to radiation. II. Black holes

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We address the problem of the physical relevance of the ''trans-Planckian'' frequencies which occur in Hawking radiation. We first show that these frequencies characterize the fluctuations of the energy-momentum tensor around its regular mean value. These fluctuations are isolated, and their properties obtained, by considering the energy density correlated to a specific final state of the Hawking radiation. This conditional energy density is expressed in terms of an off-diagonal matrix element and is complex. The dynamical relevance of these conditional fluxes is then proven in the context of perturbation theory in a *S* matrix formulation. In particular, we show how this analysis can be used to study back reaction effects to the production of a single quantum. Furthermore these conditional fluxes offer a historical description of the emergence of Hawking quanta from vacuum fluctuations. It is shown that initially these fluctuations are located around the lightlike geodesic that shall generate the horizon and have exponentially large energy densities. Upon exiting from the star they break up into two pieces. The external one is red shifted and becomes an on mass shell quantum, the other, its ''partner,'' ends up in the singularity. [S0556-2821(96)04524-9]

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I. INTRODUCTION

Pair creation in a strong external and classical field is a well-known aspect of quantum matter field theory. For instance, in a constant electric field, e^+e^- pairs are spontaneously created [1]. In addition, the subsequent emission of photons by accelerated electrons [2] is closely related to the thermalization of the Unruh's detector [3–5] as well as to the Hawking's flux engendered by the time-dependent geometry of an incipient black hole [6].

At present the back reaction of these quanta on the external field which produces them is far from being understood. The semiclassical treatment alone does not give rise to difficulty since the external field remains purely classical and only the mean value of the matter current operator acts on it as a source [7–10]. All the quantum properties of the matter, including its fluctuations and correlations, are completely discarded. It is probable that the semiclassical theory correctly predicts certain properties of the full theory such as the rate of particle production or the large scale structure of the geometry. But it will necessarily fail when considering more detailed questions related to correlations between produced particles.

The difficult task is then to determine in which circumstances the importance of the fluctuations will invalidate the semiclassical treatment. This task involves two steps. First, one needs to identify and describe the relevant fluctuations, and secondly, one should compute the modifications of the physical quantities induced by these fluctuations. Both steps have been carried out in the context of the Unruh's detector, in [11,12]. In that case, the classical accelerated trajectory is replaced by a dynamical wave function and the consequences of the recoils have been evaluated.

In the black hole situation, at present, there is no consensus on either aspect. The main point which has been stressed by 't Hooft [13] and Jacobson [14] is that Hawking's derivation of black hole radiation should no longer be valid as soon as gravitational interactions are taken into account because it makes appeal to the structure of the vacuum configurations on exponentially small scales. Furthermore, 't Hooft [15] and others [16–18] claim that these gravitational interactions will invalidate the semiclassical scenario to the extent that the solution will be completely different, even at macroscopical scales. On the other side, following the early work of Hawking [19], there is the opinion that the quantum fluctuations will not prevent the formation of the event horizon nor the loss of information for asymptotic observers [20]. This argument principally relies on the weakness of the curvature invariants at the horizon until the residual mass approaches the Planck mass.

In order to clarify the debate, one first needs a precise description of the quantum fluctuations relevant for black hole evaporation. To obtain their properties is the principal aim of this paper. The second aspect of the problem, which is concerned by the consequences of these fluctuations, will only be schematically discussed. Thus we consider the first aspect of the problem: namely, (a) how to isolate the fluctuations within black hole radiance, (b) how to describe them, and (c) what are their properties?

We isolate the fluctuations [point (a)] by considering the

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field configurations correlated to the presence (or absence) of a specific asymptotic Hawking quantum. We then reveal the properties of these field configurations [point (b)] through the study of appropriate off diagonal matrix elements of the energy momentum operator $T_{\mu\nu}$. This approach to isolate and study certain field configurations has already been used in [21] to describe the emergence of a specific e^+e^- pair produced in an electric field. We have also applied it in companion article [22] to study the field configurations which are correlated to the transitions of an accelerated detector. Since the matrix elements are off diagonal, they are complex. Therefore, it requires some care to properly interpret them. We refer to [21,23] for a discussion of these aspects. We also refer to the work of Aharonov and collaborators [24] where these off diagonal matrix elements were first introduced and where their dynamical relevance was first discussed.

In this paper we emphasize that these matrix elements arise in any S-matrix calculation of gravitational back reaction effects. Furthermore, being off diagonal, they control back reaction effects which cannot be described by the mean theory wherein only the expectation value of $T_{\mu\nu}$ is used. Indeed, to first order in the perturbation, the modification of the amplitude of finding the specific final state is proportional to such a matrix element of $T_{\mu\nu}$ and both its real and imaginary part intervene. Others authors have also advocated the study of these matrix elements, see [15,25,26].

The result of our calculation [point (c)] is to obtain the pattern of the energy density correlated to the emission of a specific Hawking quantum. When a Hawking quantum is emitted at retarded time u_0 with asymptotic frequency λ, the energy density correlated to this emission forms a dipolar structure located around the lightlike geodesic $u=u_0$. In the remote past, the energy density of this vacuum fluctuation is $O(\omega^2)$ and located on a distance scale of order ω^{-1} where $\omega=\lambda e^{u_0/4M}$ (where M is the mass of the star). Therefore, after a time $u_0=O(4M\ln M)$ for a typical $\lambda=O(M^{-1})$, ω is greater than the Planck frequency. Outside the star, one piece of the vacuum fluctuation is gradually redshifted until when it reaches large radius its frequency has become λ. The other piece, the ''partner,'' is located beyond the horizon and ends up in the singularity. We recall that the presence of this partner ensures that Hawking radiation appears to be in a thermal density matrix for external observers.

All these properties result from the following two assumptions: *free* field in a *given* geometry. In particular, free propagation implies that the energy density experiences the classical redshift along the geodesics. Therefore the fluctuations inevitably reach the trans-Planckian regime. However these matrix elements of $T_{\mu\nu}(x)$ also control the first corrections to both of the assumptions. Thus the exponentially growing energy densities may give rise to unbounded corrections and completely invalidate Hawking's assumptions. This is the trans-Planckian hiatus [15] made explicit.

There have been two attitudes in the literature to confront this hiatus. The first is to try to guess what could be the physics at the Planck scale near the horizon and how the Hawking radiation emerges therefrom see, e.g., [18,27–29]. The second has been to use Einstein equations to investigate how back reaction effects modify the production of Hawking photons [16,30]. The present article places itself in this latter vein and, as a preliminary exercise, we evaluate the modification of the probability to find a specific Hawking quantum induced by a fluctuation of the mass of the infalling star. To first order, this modification is entirely determined by the (imaginary part of the) conditional value of $T_{\mu\nu}$. However, despite its trans-Planckian character, the modification of the probability is finite and related to the additional small fluctuating mass only. This indicates that there is, at least in some cases, a washing out mechanism which prevents the trans-Planckian fluctuations from showing up in physical amplitudes. This mechanism is reminiscent of the recent work of Unruh [31,32] wherein it was found that the properties of emitted particles were unaffected by a *phenomenological* modification of the high frequency spectrum. Hence further analysis is required to establish *dynamically* when and how the trans-Planckian densities invalidate the semiclassical theory.

There is another important issue which must be addressed upon evaluating the fluctuations. It is concerned with the final specification of the field configurations since the conditional value of $T_{\mu\nu}$ it defines is *singular* in certain cases, such as when the final state contains a definite number of quanta. This could have deep implications for black hole physics since it suggests that one should impose a boundary to space time at the horizon as in 't Hooft's brick wall model [13,33] or generalizations thereof. However, when the final state is specified by the transitions of a particle detector, there is no singularity on the horizon. In view of this, it is at present unclear whether this singularity is an artifact of certain specifications of the final state, or whether it has deep physical meaning.

This article is organized as follows: We first review the quantization of a massless scalar field in the collapsing geometry. We then show how the energy density correlated to the emission of a specific Hawking quanta is given by a certain off diagonal matrix elements of $T_{\mu\nu}$. The following three sections are devoted to evaluating these matrix elements. They are first renormalized, then they are evaluated for several specifications of the final state of the radiation and finally they are considered in detail when the final state is specified by a transition of a two level detector. The article concludes with a discussion of how these matrix elements of $T_{\mu\nu}$ enter into some specific transition amplitudes.

II. THE SCATTERED MODES

In this section, we review the main properties of the collapsing geometry and the scattering of a massless field in this geometry. In particular, we insist on the divergent character of the outgoing modes on the future horizon and on the correlations of the field configurations existing on both sides of this horizon. Both of these properties play a determinant role upon computing the fluctuations within Hawking radiance.

Following Hawking [6] and Unruh [3], we work in the background metric of a spherically symmetric collapsing star of mass M. Outside the star the geometry is described by the Schwarzschild metric

$$ds^2=\left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2-r^2d\Omega^2$$
$$=\left(1-\frac{2M}{r}\right)dudv-r^2d\Omega^2,$$

$$v,u=t\pm r^*,\quad r^*=r+2M\ln\frac{r-2M}{2M}. \tag{1}$$

For simplicity, we consider the collapse of a spherically symmetric thin shell of pressureless massless matter. Inside the shell space is then flat and the metric reads

$$ds^2=d\tau^2-dr^2-r^2d\Omega^2=dUdv-r^2d\Omega^2,$$
$$v,U=\tau\pm r, \tag{2}$$

where v is the same coordinate in Eqs. (1) and (2) since on $\mathcal{I}^-$ ($u=-\infty$) space-time is flat on both sides. The collapsing shell follows the geodesic $v=v_S$. The connection between the two metrics is obtained by imposing the continuity of r along the shell's trajectory [3,34]:

$$dU=du\left(1-\frac{2M}{r(u,v_S)}\right)=du\left(1-\frac{4M}{v_S-U}\right). \tag{3}$$

Then by choosing $v_S=4M$ one obtains

$$u(U)=U-4M\ln\left(\frac{-U}{4M}\right). \tag{4}$$

With this choice of v_S, the incoming light ray which shall generate the future horizon at $U=0$, is $v=0$ since $r=0$ reads $v=U$.

In the static space-time outside the star, the massless Klein-Gordon equation for a mode of the form $\varphi_{l,m}=Y_{lm}(\theta,\varphi)\psi_l(t,r)/\sqrt{4\pi r^2}$ reads

$$\left[\partial_t^2-\partial_{r*}^2-\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right)\right]\psi_l(t,r)=0. \tag{5}$$

Near the horizon $r-2M\ll 2M$, it becomes the wave equation for a massless field in $1+1$ dimensions. By considering only the s-wave sector of a massless field and dropping the residual ''quantum potential'' $2M(r-2M)/r^4$, the conformal invariance holds everywhere, inside as well as outside the star. From now on we shall work in this simplified context in which the wave equation becomes

$$\partial_U\partial_v\psi_{l=0}=\partial_u\partial_v\psi_{l=0}=0. \tag{6}$$

The only difference with a massless field in (1+1)-dimensional flat space is that $\psi_{l=0}$ must vanish at the origin: $\psi_{l=0}(r=0)=0$.

In second quantization, the Heisenberg state $|0\rangle$ is chosen to be the initial vacuum, i.e., vacuum with respect to the modes which have positive v frequency on $\mathcal{I}^-$. These modes are

$$\psi_\omega(v,u)=\frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega v}-e^{-i\omega U(u)}). \tag{7}$$

The u part is determined by the condition that ψ_ω vanish at $r=v-U=0$, see [34,23] for more details. Hence, by virtue of Eq. (3), for $u>4M$, or even on both sides of the horizon for $-M<U<M$, the state of the field tends exponentially quickly (in u) to the Unruh vacuum [3], i.e., a vacuum with respect to the modes

$$\frac{1}{\sqrt{4\pi\omega}}\exp(-i\omega v)\quad\text{and}\quad\frac{1}{\sqrt{4\pi\omega}}\exp(i\omega 4Me^{-u/4M}). \tag{8}$$

To the modes ψ_ω are associated the Kruskal creation and destruction operators $a_\omega^\dagger$, a_ω and the Heisenberg vacuum $|0\rangle$ is the state annihilated by all a_ω.

The ''Schwarzschild'' u modes $\chi_{\lambda,R}(u)=\theta(r-2M)e^{-i\lambda u}/\sqrt{4\pi\lambda}$ are needed to analyze the particle content of the scattered modes ψ_ω on $\mathcal{I}^+$ since they correspond to Minkowskian spherical wave at $\mathcal{I}^+$. In terms of U given in Eq. (4) they take the form

$$\chi_{\lambda,R}(U)=\theta(-U)\frac{1}{\sqrt{4\pi\lambda}}\left(\frac{-U}{4M}\right)^{i\lambda 4M}e^{-i\lambda U}. \tag{9}$$

To these modes are associated the destruction operators $a_{\lambda,R}$. It is useful to define an additional set of Schwarzschild modes which live only inside the horizon $\chi_{\lambda,L}(U)=\chi_{\lambda,R}^*(-U)$ and the corresponding destruction operator $a_{\lambda,L}$. These operators define the Schwarzschild vacuum outside and inside the horizon: $a_{\lambda,R}|0_R\rangle=0$ and $a_{\lambda,L}|0_L\rangle=0$. The state containing no Schwarzschild particle is the Boulware vacuum $|B\rangle=|0_R\rangle\otimes|0_L\rangle$.

The exact Bogolyubov coefficient between φ_ω and χ_λ is given by the overlap

$$\alpha_{\omega,\lambda}=\langle\varphi_\omega,\chi_{\lambda,R}\rangle=\frac{1}{4\pi}\sqrt{\frac{\omega}{\lambda}}\Gamma(1+i4M\lambda)$$
$$\times[4M(\omega-\lambda)]^{-i4M\lambda-1}e^{\pm 2\pi M\lambda} \tag{10}$$

where the $\pm$ is to be understood as $+$ if $\omega>\lambda$ and $-$ if $\omega<\lambda$. The expression for $\beta_{\omega,\lambda}=\langle\varphi_\omega,\chi_{\lambda,R}^*\rangle$ is obtained by taking λ into $-\lambda$.

In the limit $\omega\to+\infty$ (which corresponds to the late time limit, $u\to+\infty$, because of the classical Doppler shift relating ω to λ, see [35]), these Bogolyubov coefficients tend to the ones obtained by Hawking:

$$\alpha_{\omega,\lambda}=\frac{1}{4\pi}\sqrt{\frac{\omega}{\lambda}}\Gamma(1+i4M\lambda)(4M\omega)^{-i4M\lambda-1}e^{2\pi M\lambda}$$
$$=\beta_{\omega,\lambda}^*e^{4\pi M\lambda}. \tag{11}$$

In this late time limit, the black hole emits a steady thermal flux at the Hawking temperature $1/8\pi M$ since $|\beta_{\omega,\lambda}/\alpha_{\omega,\lambda}|^2=e^{-8\pi M\lambda}$ for all ω.

In order to display the nature of the singularity of the modes at fixed outgoing frequency λ, we introduce the ''Unruh'' wave functions [3]

$$\psi_{\lambda,K} \underset{\epsilon\to 0^+}{\sim} \frac{1}{\sqrt{4\pi\lambda}} \frac{1}{\sqrt{e^{4M\pi\lambda}-e^{-4M\pi\lambda}}}\{[(\epsilon+iv)/4M]^{-i4M\lambda} - [(\epsilon+iU)/4M]^{-i4M\lambda}\} \tag{12}$$

where $\epsilon>0$ specifies how the functions have to be continued in the complex U plane. ϵ also regulates the modes at the horizon. It is important to take the limit $\epsilon\to 0$ only at the end of the calculation in order that the $\psi_{\lambda,K}$ constitute an alternative basis of in-modes [36]. The essential advantage of the basis $\varphi_{\lambda,K}$ is that, at late times, when the factor $e^{-i\lambda U}$ in χ_λ, Eq. (9), can be neglected, the Bogolyubov transformation between the modes χ_λ and $\varphi_{\lambda,K}$ is diagonal in λ (this is due to the stationary character of Hawking radiation)

$$\psi_{\lambda,K}=\alpha_\lambda \chi_{\lambda,R}+\beta_\lambda \chi^*_{\lambda,L}, \quad \psi_{-\lambda,K}=\alpha_\lambda \chi_{\lambda,L}+\beta_\lambda \chi^*_{\lambda,R}, \tag{13}$$

where $\beta_\lambda/\alpha_\lambda=e^{-4\pi\lambda M}$ and $\alpha_\lambda^2-\beta_\lambda^2=1$. It is then easy to express the Heisenberg vacuum as an entangled state of Schwarzschild quanta living on both sides of the future horizon

$$|0\rangle=\prod_\lambda \frac{1}{\alpha_\lambda} e^{\beta_\lambda/\alpha_\lambda a^\dagger_{\lambda,L} a^\dagger_{\lambda,R}}|0_R\rangle \otimes |0_L\rangle. \tag{14}$$

Since an external observer has no access to the field configurations beyond the horizon, he must trace over them and this leads to a thermal density matrix for the outgoing radiation [37].

III. THE FLUCTUATIONS WITHIN BLACK HOLE RADIANCE

We shall show that the energy density correlated to a specific final state of the Hawking radiation can be expressed as a normalized off diagonal matrix element of $T_{\mu\nu}$. To illustrate the various aspects of its physical significance, we shall present three different ways to obtain it. The first two derivations are dynamical and show how this conditional energy density naturally arises in a perturbative expansion of S matrix elements. The last derivation relates it to usual conditional values in probability theory.

Then we shall generalize these procedures in order to cope with the facts that an external observer has no access to the internal field configurations and that the outgoing modes are singular on the horizon.

To display the dynamical relevance of this matrix element, we introduce the interaction between the quantized scalar field ϕ with the gravitational field. To first order in $h^{\mu\nu}$, it is given by the Hamiltonian

$$\int dt H_{\rm int}=-\frac{1}{2}\int d^4x\sqrt{-g}h^{\mu\nu}(x)T_{\mu\nu}(x) \tag{15}$$

where $h^{\mu\nu}(x)$ is the fluctuating part of the metric, i.e., $ds^2=(g_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu$ with $g_{\mu\nu}(x)$ the background metric. Equation (15) follows from the definition of the energy momentum tensor as the derivative of the matter action with respect to $g^{\mu\nu}$:

$$T_{\mu\nu}=\frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{16}$$

Let us use Eq. (15) to calculate how a classical (c number) fluctuation of the background geometry modifies the amplitude to find a specific final state. To this end, we consider a set of states which completely specify the field configurations on $\mathcal{I}^+$, i.e., particle content in terms of asymptotic Hawking quanta:

$$|\psi_{\{n_\lambda\}}\rangle=\prod_\lambda \frac{(a^\dagger_{\lambda,R})^{n_\lambda}}{\sqrt{n_\lambda!}}|0_R\rangle \prod_{\lambda'} \frac{(a^\dagger_{\lambda',L})^{n_{\lambda'}}}{\sqrt{n_{\lambda'}!}}|0_L\rangle. \tag{17}$$

In the background geometry, when the Heisenberg state is the in-vacuum, Eq. (14), the amplitude of probability for the radiation to be in the ψ state is

$$S^0_{\{n_\lambda\}}=\langle\psi_{\{n_\lambda\}}|0\rangle. \tag{18}$$

In the perturbed geometry, to first order in $h^{\mu\nu}$, this amplitude becomes

$$S^1_{\{n_\lambda\}}=\langle\psi_{\{n_\lambda\}}|1-i\int d^4x\frac{-1}{2}\sqrt{-g}h^{\mu\nu}T_{\mu\nu}|0\rangle = S^0_{\{n_\lambda\}}\left(1-i\int d^4x\frac{-1}{2}\sqrt{-g}h^{\mu\nu}\frac{\langle\psi_{\{n_\lambda\}}|T_{\mu\nu}|0\rangle}{\langle\psi_{\{n_\lambda\}}|0\rangle}\right). \tag{19}$$

The normalized matrix element of $T_{\mu\nu}$,

$$\langle T_{\mu\nu}(x)\rangle_{\{n_\lambda\}}=\frac{\langle\psi_{\{n_\lambda\}}|T_{\mu\nu}(x)|0\rangle}{\langle\psi_{\{n_\lambda\}}|0\rangle} \tag{20}$$

appears therefore to be the *energy density correlated to the final state* $|\psi_{\{n_\lambda\}}\rangle$. Notice that both its real part and its imaginary part modify the complex amplitude $S_{\{n_\lambda\}}$. In Sec. VII, we shall explicitly compute the change in the probability to find $|\psi_{\{n_\lambda\}}\rangle$ and relate this change to the properties of $\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}$.

We now consider another situation in which $\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}$ comes up in order to confirm that its correct interpretation is indeed the one of a conditional energy density. Suppose that the gravitational field has been quantized and that its Heisenberg state is $|0_g\rangle$. Then, the first order modification of the state of the gravitational field due to the presence of the quanta $\{n_\lambda\}$ introduces some entanglement between the matter part and the gravitational part of the wave function. To first order, the final entangled state is

$$\left(1-i\int d^4x\frac{-1}{2}\sqrt{-g}\hat{h}^{\mu\nu}T_{\mu\nu}\right)|0\rangle|0_g\rangle \tag{21}$$

where $\hat{h}^{\mu\nu}(x)$ is now a quantized field operator.

When the final state of the radiation is $|\psi_{\{n_\lambda\}}\rangle$, the state of the gravitational field correlated to this outcome is obtained by projecting the bra $\langle\psi_{\{n_\lambda\}}|$ onto the coupled state equation (21) and one finds

$$|\Phi_{\{n_\lambda\}}\rangle = |\psi_{\{n_\lambda\}}\rangle\langle\psi_{\{n_\lambda\}}|\left(1-i\int d^4x\frac{-1}{2}\times\sqrt{-g}\hat{h}^{\mu\nu}T_{\mu\nu}\right)|0\rangle|0_g\rangle$$
$$= S^0_{\{n_\lambda\}}|\psi_{\{n_\lambda\}}\rangle\left(1-i\int d^4x\frac{-1}{2}\times\sqrt{-g}\hat{h}^{\mu\nu}\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}\right)|0_g\rangle. \tag{22}$$

Then, the value of the gravitational field fluctuation which is conditional to the fact that the final state is $|\psi_{\{n_\lambda\}}\rangle$ reads

$$\langle h_{\alpha\beta}(y)\rangle_{\{n_\lambda\}} = \frac{\langle_{\{n_\lambda\}}\Phi|\hat{h}_{\alpha\beta}(y)|\Phi_{\{n_\lambda\}}\rangle}{\langle_{\{n_\lambda\}}\Phi|\Phi_{\{n_\lambda\}}\rangle}$$
$$= \mathrm{Im}\left[\int d^4x\sqrt{-g}\langle T_{\mu\nu}(x)\rangle_{\{n_\lambda\}}\times\langle 0_g|\hat{h}_{\alpha\beta}(y)\hat{h}^{\mu\nu}(x)|0_g\rangle\right]. \tag{23}$$

It is given in terms of the conditional value of $T_{\mu\nu}(x)$ and the graviton propagator evaluated in the unperturbed gravitational state, see Sec. III A in [22] for a more traditional example of this kind of response function.

In summary, from Eqs. (22) and (23), we see that the Hamiltonian which acts on the gravitational state is given by

$$\int d^4x\frac{-1}{2}\sqrt{-g}\hat{h}^{\mu\nu}\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}.$$

This confirms that $\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}$, Eq. (20), is indeed the conditional energy momentum.

We now present the third derivation for $\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}$ which makes manifest the relation between this formalism and the usual notion of the conditional value in probability theory. This derivation proceeds through the decomposition of the mean value of the energy-momentum tensor in terms of a complete set of final states, Eq. (17):

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \sum_{\{n_\lambda\}}\langle 0|\psi_{\{n_\lambda\}}\rangle\langle\psi_{\{n_\lambda\}}|T_{\mu\nu}(x)|0\rangle$$
$$= \sum_{\{n_\lambda\}}|\langle 0|\psi_{\{n_\lambda\}}\rangle|^2\frac{\langle\psi_{\{n_\lambda\}}|T_{\mu\nu}(x)|0\rangle}{\langle\psi_{\{n_\lambda\}}|0\rangle}$$
$$= \sum_{\{n_\lambda\}}|S^0_{\{n_\lambda\}}|^2\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}. \tag{24}$$

Since $|S^0_{\{n_\lambda\}}|^2$ is the probability to obtain the final state $|\psi_{\{n_\lambda\}}\rangle$, this expression confirms that $\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}$ has the interpretation of a conditional value, *exactly as in probability theory*. Indeed, we have rewritten the mean energy density as a sum over all possible outcomes of the product of the probability of each outcome times the conditional value of the energy density if that outcome is realized. Note that since the left-hand side of Eq. (24) is real, this implies the imaginary parts of the terms on the right-hand side sum to zero.

In the remainder of this section we shall generalize Eq. (20) for $\langle T_{\mu\nu}\rangle_{\{n_\lambda\}}$ in two different ways and for two different reasons. Namely, we want to take into account the inaccessible character of the field configurations on the other side of the horizon and the divergent properties of the outgoing modes on that horizon.

The set of states $|\psi_{\{n_\lambda\}}\rangle$ specify the state of the radiation on *both* sides of the future horizon. But since only the region outside the horizon is accessible to the asymptotic observer, we must consider *partial* specifications of the final state. These can be introduced by using a complete set of projectors which act as the identity operator when applied to the field configurations located beyond the horizon:

$$\Pi_{\{n_\lambda\}} = I_L\otimes\prod_{\{n_\lambda\}}\frac{(a^\dagger_{\lambda R})^{n_\lambda}}{\sqrt{n_\lambda!}}|0_R\rangle\langle 0_R|\frac{(a_{\lambda R})^{n_\lambda}}{\sqrt{n_\lambda!}}. \tag{25}$$

We can now decompose the mean energy momentum density using these projectors as

$$\langle T_{\mu\nu}\rangle = \sum_{\{n_\lambda\}}\langle 0|\Pi_{\{n_\lambda\}}T_{\mu\nu}|0\rangle = \sum_{\{n_\lambda\}}P_{\{n_\lambda\}}\frac{\langle 0|\Pi_{\{n_\lambda\}}T_{\mu\nu}|0\rangle}{\langle 0|\Pi_{\{n_\lambda\}}|0\rangle} \tag{26}$$

where $P_{\{n_\lambda\}} = \langle 0|\Pi_{\{n_\lambda\}}|0\rangle$ is the probability to obtain the final configurations specified by the projector $\Pi_{\{n_\lambda\}}$. This decomposition is once more of the type used in usual conditional probabilities and

$$\langle T_{\mu\nu}\rangle_{\Pi_{\{n_\lambda\}}} = \frac{\langle 0|\Pi_{\{n_\lambda\}}T_{\mu\nu}|0\rangle}{\langle 0|\Pi_{\{n_\lambda\}}|0\rangle} \tag{27}$$

is the energy correlated to these final configurations. Note that one recovers Eq. (20) if $\Pi_{\{n_\lambda\}} = |\psi_{\{n_\lambda\}}\rangle\langle\psi_{\{n_\lambda\}}|$, i.e., if $\Pi_{\{n_\lambda\}}$ is the projector onto the pure state $|\psi_{\{n_\lambda\}}\rangle$.

We now address the second problem, namely the difficulties engendered by the singular behavior of the Schwarzschild modes. Indeed, as displayed in Eq. (12), the Schwarzschild modes specified on $\mathcal{I}^+$ are singular on the horizon (for the same reasons that the Rindler modes are singular in flat space-time, see [36]). In the next section, we shall prove that this leads to singular conditional energy densities in most cases except when the final specification is carried out by particle detectors. Thus, we turn to these specifications when the particle detector sits still at a large radius $r_p \gg 2M$. The model of the two level particle detector we shall use is the one studied in detail in [22]. Here, we shall only repeat the salient aspects which will intervene upon computing the quantum response to the fluctuations among Hawking radiation.

The two level detector with ground state $|-\rangle$ and excited state $|+\rangle$ is coupled to the field ϕ by the interaction Hamiltonian:

$$\int dtd^3x\mathcal{H}^D_{\text{int}}(t,x)=gm\int dt[f(t)e^{-imt}|-\rangle \times\langle+|+f^*(t)e^{imt}|+\rangle\langle-|]\phi(t,r_p) \tag{28}$$

where g is a dimensionless coupling constant that shall be taken for simplicity small enough that second order perturbation theory be valid, m is the difference of energy between the ground and the excited state of the atom, and $f(t)$ is the dimensionless function that governs when and how the interaction is turned on and off.

If the detector is initially in its ground state and the radiation described by the in-vacuum, the state of the coupled system, at late times, is

$$|\Psi\rangle=|0\rangle|-\rangle-igm\int dtf^*(t)e^{imt}\phi(t,r_p)|0\rangle|+\rangle +O(g^2). \tag{29}$$

To second order in g, the probability to find the detector in its excited state is

$$P_e=\langle\Psi|+\rangle\langle+|\Psi\rangle =g^2m^2\int dt'f(t')e^{-imt'}\int dtf^*(t)e^{imt}\langle 0|\phi(t',r_p) \times\phi(t,r_p)|0\rangle. \tag{30}$$

Since $r_p \gg M$, the s wave solution of the Klein-Gordon equation separates into a left and a right moving part $\phi=f(u)+g(v)$. It is therefore consistent to take the detector to be coupled to the u part only. From now on we shall make this assumption which isolates the effects of the Hawking radiation. (The v modes would in any case give rise only to exponentially small effects since they are in the vacuum state relative to $e^{-i\omega t}$, see [22].)

If the function $f(t)$ is equal to 1 for a long time T and tends to 0 outside this interval, the concept of a rate of transition emerges. More precisely, in the ''golden rule'' limit, $T\gg M$ with g^2T finite, one finds

$$P_e=\frac{1}{2}g^2mT\frac{1}{e^{8\pi Mm}-1}. \tag{31}$$

The factor $(e^{8\pi Mm}-1)^{-1}=\beta^2_{\lambda=m}$ is the Planckian distribution of outgoing Hawking quanta, see Eq. (13).

If the detector makes a transition there necessarily was a particle emitted by the black hole. Thus we can use the final state of the detector to isolate the field configurations containing specific outgoing quanta. Then, the energy density correlated to the transitions of the detector is obtained in strict analogy with Eq. (22) by considering how the state of the gravitational field is correlated to the final state of the detector. The specification that the detector is in its excited or ground state is carried out by the projectors $\Pi_+=|+\rangle\langle+|$ and $\Pi_-=|-\rangle\langle-|$. Because we are only specifying partially the final state it is necessary to work in a density matrix formulation. The density matrix of the correlated field ϕ, gravitational field, and detector is

$$\rho=\left[\exp\left(-i\int dtH^D_{\text{int}}\right)\left(1-i\int d^4x\frac{-1}{2}\sqrt{-g}\hat{h}^{\mu\nu}T_{\mu\nu}\right)\right] \times|0\rangle|0_g\rangle|-\rangle\langle-|\langle 0_g|\langle 0|[\text{H.c.}] \tag{32}$$

where the interaction with the detector acts after the interaction with the gravitational field since the detector is located at arbitrarily large distance from the black hole. We now project onto the excited state of the detector and trace over the field ϕ to obtain the reduced density matrix of the gravitational field correlated to the excitation of the detector. To second order in g and to first order in $h^{\mu\nu}$, the reduced density matrix is

$$\begin{aligned}\text{Tr}_{\phi,\text{detector}}[\Pi_+\rho]=&\text{Tr}_\phi\bigg[-g^2m^2\int dt'f(t')e^{-imt'}\phi(t',x)\\ &\times\int dtf^*(t)e^{imt}\phi(t,x)\\ &\times\left(1-i\int d^4x\frac{-1}{2}\sqrt{-g}\hat{h}^{\mu\nu}T_{\mu\nu}\right)\\ &\times|0\rangle|0_g\rangle\langle 0_g|\langle 0|(\text{H.c.})\bigg]\\ =&P_e\bigg[\left(1-i\int d^4x\frac{-1}{2}\sqrt{-g}\hat{h}^{\mu\nu}\langle T_{\mu\nu}\rangle_+\right)\\ &\times|0_g\rangle\langle 0_g|(\text{H.c.})\bigg]\end{aligned} \tag{33}$$

where

$$\langle T_{\mu\nu}\rangle_+=\frac{\langle 0|\mathcal{O}T_{\mu\nu}|0\rangle}{\langle 0|\mathcal{O}|0\rangle}, \tag{34}$$

where $\mathcal{O}=\int dt'f(t')e^{-imt'}\phi(t',r_p)\int dtf(t)^*e^{imt}\phi(t,r_p)$.

$\langle T_{\mu\nu}\rangle_+$ is thus the energy density correlated to the excitation of the detector. When compared with Eq. (27), one sees that the role of the projector $\Pi_{\{n_\lambda\}}$ is now played by the field operator $\mathcal{O}$.

IV. MATRIX ELEMENTS IN CURVED SPACE-TIME

The off diagonal matrix elements of $T_{\mu\nu}$ which describe the energy density correlated to a specific final state obtained in the previous section are formally infinite and have to be renormalized. We now address this point since in the textbooks only the renormalization of the diagonal part $T_{\mu\nu}$ in curved space-time is described.

Wald has proposed a set of eminently reasonable conditions that a renormalized energy momentum operator should satisfy [38]. By Wald's argumentation one deduces that the renormalized energy momentum operator $T^{(\text{ren})}_{\mu\nu}(x)$ can be written in the following way

$$T^{(\text{ren})}_{\mu\nu}(x)=T_{\mu\nu}(x)-t_{\mu\nu(\text{S})}(x)I \tag{35}$$

where $T_{\mu\nu}(x)$ is the bare energy momentum operator and I the identity operator. The subtraction term $t_{\mu\nu(\text{S})}(x)$ is an (infinite) conserved c-number function only of the geometry at x [34]. It can be understood [39,40] as the (infinite) ground state energy of the ''local inertial vacuum:'' that state which most resembles Minkowski vacuum at x. Numerous techniques have been developed to calculate $t_{\mu\nu(\mathrm{S})}$ and we refer the reader to Ref. [34] for a review.

In the Heisenberg vacuum $|0\rangle$, the renormalized expectation value of $T_{\mu\nu}$ takes the form

$$\langle 0|T^{(\mathrm{ren})}_{\mu\nu}(x)\,|0\rangle = \langle 0|T_{\mu\nu}(x)\,|0\rangle - t_{\mu\nu(\mathrm{S})}(x) \qquad (36)$$

where both terms on the right-hand side are infinite but their difference is finite. We remind the reader that the mean value of the energy density in the Heisenberg states is regular on the future horizon $U=0$ and that Hawking radiation can be conceived as the matter response that gives regular mean energy densities since $T_{\mu\nu}(x)$ computed in Boulware vacuum diverges on the horizon. We refer to Refs. [34,23] for further discussion of the mean flux.

Inserting Eq. (35) into the expression for the conditional value of $T_{\mu\nu}$, Eq. (27) yields

$$\langle T^{\mathrm{ren}}_{\mu\nu}(x)\rangle_{\Pi} = \frac{\langle 0|\Pi T_{\mu\nu}(x)|0\rangle}{\langle 0|\Pi|0\rangle} - t_{\mu\nu(\mathrm{S})}(x). \qquad (37)$$

Then by expressing the operator $T_{\mu\nu}(x)$ in terms of the operators which annihilate the in-vacuum one obtains

$$\langle T^{\mathrm{ren}}_{\mu\nu}(x)\rangle_{\Pi} = \int_0^\infty d\omega \int_0^\infty d\omega' \, \frac{\langle 0|\Pi a^\dagger_\omega a^\dagger_{\omega'}\,|0\rangle}{\langle 0|\Pi|0\rangle} \hat{T}_{\mu\nu}(x) \times [\varphi^*_\omega \varphi^*_{\omega'}] + \langle 0|T^{(\mathrm{ren})}_{\mu\nu}(x)|0\rangle \qquad (38)$$

where $\hat{T}_{\mu\nu}(x)$ is the classical differential operator which acting on the in-waves φ^*_ω gives their energy density. The renormalized energy density correlated to transitions of a detector takes a similar form, with the operator Π replaced by the field operator $\mathcal{O}$ [see Eq. (34)].

The important point is that the renormalized conditional value contains two contributions which have different origins and play different roles in dynamical processes. The first term is the fluctuating part which depends on the particle content of the state specified by Π. It is complex. The second term is the (real) mean energy density, Eq. (36), obtained when no specification on the final state is added.

Equation (37) warrants a few additional comments. First, notice that there are parts of $\langle T_{\mu\nu}\rangle_{\Pi}$ that are entirely contained in the subtraction. Most notably there is the trace anomaly and those components of the energy momentum tensor which are related to it by energy conservation. For instance, under the neglect of the potential term in the wave equation for s waves, the classical differential operator $\hat{T}_{\mu\nu}$ acting on s waves is

$$\hat{T}_{vv}(x)[\varphi\varphi'] = \frac{1}{r^2}\partial_v[r\varphi(x)]\partial_v[r\varphi'(x)] = \frac{1}{4\pi r^2}\partial_v\psi(x)\partial_v\psi'(x) = \frac{1}{4\pi r^2}T_{vv}(x)[\psi\psi'],$$

$$\hat{T}_{uu}(x)[\varphi\varphi'] = \frac{1}{r^2}\partial_u[r\varphi(x)]\partial_u[r\varphi'(x)] = \frac{1}{4\pi r^2}\partial_u\psi(x)\partial_u\psi'(x) = \frac{1}{4\pi r^2}T_{uu}(x)[\psi\psi'],$$

$$\hat{T}_{uv}(x)[\varphi\varphi'] = 0. \qquad (39)$$

Thus $(4\pi r^2\langle T_{uu}\rangle_{\Pi})_{,v}$, $(4\pi r^2\langle T_{vv}\rangle_{\Pi})_{,u}$, and $\langle T_{uv}\rangle_{\Pi}$ are independent of Π or, expressed differently, do not fluctuate. This implies that the specification of an outgoing particle on $\mathcal{I}^+$ affects the ingoing flux $\langle T_{vv}\rangle_{\Pi}$ only because of the reflection condition at $r=0$. Thus on the future horizon $\langle T_{vv}(r=2M)\rangle_{\Pi}$ is unaffected by such a specification on $\mathcal{I}^+$. This last effect disappears partially when considering the potential barrier in the wave equation (5).

From Eq. (39) we see that $4\pi r^2 T_{\mu\nu}[\psi\psi']$ takes an extremely simple form. Therefore, when dealing with s waves only, it is convenient to multiply $T_{\mu\nu}$ by $4\pi r^2$. We shall do so in Secs. V and VI which are purely kinematical in character. However in Sec. VII, upon considering dynamical backreaction effects, the four-dimensional character of the problem can no longer be neglected.

V. THE CONDITIONAL VALUE OF $T_{\mu\nu}$

The purpose of this section is to obtain explicit expressions for the energy density correlated to specific final states of the Hawking radiation. This will be done by using the formal expressions of the renormalized energy momentum, Eqs. (20), (27), (34), and the properties of the modes of the field ϕ. In addition, we will identify which class of final states gives rise to regular conditional values of $T_{\mu\nu}(x)$ on the horizon. We shall see that the exponentially growing Doppler factor relating U to u, see Eq. (4), imposes severe restrictions on the acceptable final states.

We consider three cases, namely when the final state contains no particles, when the final state contains one particle, and when the final state is specified by the transition of a detector at large distance from the black hole.

The conditional value in Boulware vacuum. The projector which specifies that the final state contains no (s-wave) particles is

$$\Pi_B = I_L \otimes |0_R\rangle\langle 0_R|. \qquad (40)$$

Due to the correlations between left and right quanta in Eq. (14), one finds that

$$\Pi_B\,|0\rangle = |0_L\rangle \otimes |0_R\rangle = |B\rangle, \qquad (41)$$

i.e., if no external particles are emitted, their necessarily are no Schwarzschild quanta beyond the horizon. The energy density (multiplied by $4\pi r^2$) correlated to this absence of radiation is

$$\langle T^{\rm ren}_{\mu\nu}\rangle_{\Pi_B}=\frac{\langle 0|\Pi_B T^{\rm ren}_{\mu\nu}|0\rangle}{\langle 0|\Pi_B\,|0\rangle}=\frac{\langle B|T_{\mu\nu}|0\rangle}{\langle B|0\rangle}-\langle 0|T_{\mu\nu}\,|0\rangle+\langle 0|T^{\rm ren}_{\mu\nu}\,|0\rangle$$
$$=\int_0^\infty d\omega\int_0^\infty d\omega'\ \frac{\langle B|a^\dagger_\omega a^\dagger_{\omega'}|0\rangle}{\langle B|0\rangle}\times T_{\mu\nu}[\psi^*_\omega,\psi^*_{\omega'}]+\langle 0|T^{\rm ren}_{\mu\nu}\,|0\rangle \qquad (42)$$

where, as in Eq. (38), the second term is the renormalized energy density in the Heisenberg state $|0\rangle$. The component T_{UU} is

$$\left(\frac{\langle B|T_{UU}|0\rangle}{\langle B|0\rangle}-\langle 0|T_{UU}\,|0\rangle\right)=-2\int_0^\infty d\lambda\,\frac{\beta_\lambda}{\alpha_\lambda}\partial_U\psi^*_{\lambda,K}\partial_U\psi^*_{-\lambda,K}=-\frac{\pi}{12}\,\frac{1}{(8\pi M)^2}\,\frac{(4M)^2}{(U+i\epsilon)^2}. \qquad (43)$$

The first equality results from the Wick contractions between the Kruskal operators in T_{UU} and the Kruskal operators in the expression of Boulware vacuum in terms of Kruskal quanta $|B\rangle=\Pi_\lambda(1/\alpha_\lambda)e-(\beta_\lambda/\alpha_\lambda)a^\dagger_{\lambda,K}a^\dagger_{-\lambda,K}\,|0\rangle$. Thus the difference between the conditional density and the mean density as measured in the coordinate system of an in-falling observer is negative. But on the horizon it is positive and infinite in the limit $\epsilon\to 0$ [36].

Similarly, before reflection at $r=v-U=0$, the conditional value of the in-falling flux, T_{vv}, is

$$\left(\frac{\langle B|T_{vv}|0\rangle}{\langle B|0\rangle}-\langle 0|T_{vv}\,|0\rangle\right)=-\frac{\pi}{12}\,\frac{1}{(8\pi M)^2}\,\frac{(4M)^2}{(v+i\epsilon)^2}. \qquad (44)$$

It is singular on the light ray $v=0$ which shall generate the future horizon. The component T_{Uv} vanishes since classically the trace $\hat{T}_{Uv}$ vanishes (see the remark at the end of the previous section).

Using the Jacobian $du/dU=1-4M/U\simeq-4M/U$ [see Eq. (4)], one obtains from Eq. (43) the Schwarzschild energy density correlated to the absence of emitted particles:

$$\left(\frac{\langle B|T_{uu}|0\rangle}{\langle B|0\rangle}-\langle 0|T_{uu}\,|0\rangle\right)=-\frac{\pi}{12}\,\frac{1}{(8\pi M)^2} \qquad (45)$$

which is minus the mean flux of Hawking quanta. Thus, the conditional flux $\langle T^{\rm ren}_{uu}(r=\infty)\rangle_{\Pi_B}$ vanishes as expected since one has specified that no quanta are emitted to infinity.

The conditional value when one quantum is present. The projector which imposes that only *one* quantum of energy λ defined at $\mathcal{I}^+$ is emitted is given by

$$\Pi_\lambda=I_L\otimes a^\dagger_{\lambda,R}\,|0_R\rangle\,\langle 0_R|a_{\lambda,R} \qquad (46)$$

and one has, see Eq. (14),

$$\Pi_\lambda\,|0\rangle=a^\dagger_{\lambda,R}a^\dagger_{\lambda,L}\,|B\rangle. \qquad (47)$$

Thus the specification of one asymptotic particle automatically implies that there is a partner beyond the horizon. The energy density correlated to this final state is

$$\frac{\langle 0|\Pi_\lambda T^{\rm ren}_{\mu\nu}\,|0\rangle}{\langle 0|\Pi_\lambda\,|0\rangle}=\frac{2}{\alpha_\lambda\beta_\lambda}T_{\mu\nu}[\psi^*_{\lambda,K}\psi^*_{-\lambda,K}]+\frac{\langle B|T^{\rm ren}_{\mu\nu}\,|0\rangle}{\langle B|0\rangle} \qquad (48)$$

which is easily obtained using the identity

$$a^\dagger_{\lambda,R}a^\dagger_{\lambda',L}\,|B\rangle=\frac{1}{\alpha_\lambda\alpha'_\lambda}a^\dagger_{\lambda,K}a^\dagger_{-\lambda',K}\,|B\rangle+\frac{\beta_\lambda}{\alpha_\lambda}\delta(\lambda-\lambda')\,|B\rangle. \qquad (49)$$

Thus the conditional energy density decomposes into two terms. The second is the energy density when no quanta are emitted, see Eq. (42). The first term describes the energy density correlated to the quantum λ. Its UU component is given by

$$\langle T_{UU}\rangle_\lambda=\frac{2}{\alpha_\lambda\beta_\lambda}\partial_U\psi^*_{\lambda,K}\partial_U\psi^*_{-\lambda,K}=\frac{\lambda}{2\pi}\,\frac{(4M)^2}{(U+i\epsilon)^2}. \qquad (50)$$

In Schwarzschild coordinate u, it becomes

$$\langle T_{uu}\rangle_\lambda=\frac{\lambda}{2\pi} \qquad (51)$$

which corresponds to a constant flux of energy at infinity whose total energy is infinite. Therefore it is appropriate (especially since the aim is to consider backreaction effects) to consider a quantum described by a normalized wave packet $\int_0^\infty d\lambda g_\lambda e^{-i\lambda u}/\sqrt{4\pi\lambda}$ with $\int_0^\infty d\lambda|g_\lambda|^2=1$. The corresponding projector is

$$\Pi_{g_\lambda}=I_L\otimes\int_0^\infty d\lambda g_\lambda a^\dagger_{\lambda R}\,|0_R\rangle\,\langle 0_R|\int_0^\infty d\lambda' g^*_{\lambda'}a_{\lambda' R} \qquad (52)$$

and the correlations between left and right quanta lead to

$$\Pi_{g_\lambda}\,|0\rangle=\left(\int_0^\infty d\lambda g_\lambda a^\dagger_{\lambda R}\right)\left(\int_0^\infty d\lambda'\,\frac{\beta_{\lambda'}}{\alpha_{\lambda'}}g^*_{\lambda'}a^\dagger_{\lambda' R}\right)|B\rangle. \qquad (53)$$

Note that the partner beyond the horizon is not described by the same wave packet as the specified particle. (This will have important consequences for the conditional energy.)

As in Eq. (48), the conditional energy density in this state contains two terms. We consider the ''first'' one which depends on g_λ:

$$\langle T_{\mu\nu}\rangle_{g_\lambda}=\frac{\langle 0|\Pi_{g_\lambda}T^{\rm ren}_{\mu\nu}|0\rangle}{\langle 0|\Pi_{g_\lambda}|0\rangle}-\frac{\langle B|T^{\rm ren}_{\mu\nu}|0\rangle}{\langle B|0\rangle}$$
$$=2\left[\int_0^\infty d\lambda\int_0^\infty d\lambda'\,\frac{\beta_{\lambda'}}{\alpha_\lambda\alpha^2_{\lambda'}}g_\lambda g_{\lambda'}\times T_{\mu\nu}[\psi^*_{\lambda K}\psi_{\lambda K}]\right]\Bigg/\left[\int_0^\infty d\lambda\,\frac{\beta^2_\lambda}{\alpha^2_\lambda}\left|g_\lambda\right|^2\right]. \quad (54)$$

This conditional energy density turns out to be singular on the horizon, for *all* g_λ in the limit $\epsilon\to 0$, as in Eq. (50). In order to prove this, we shall consider the energy correlated to a transition of a two level detector.

The conditional energy correlated to a transition of the detector. This energy density is given by Eqs. (34) and (38). The Wick contractions which arise upon evaluating the fluctuating part of these expressions can be expressed in terms of the two functions (see [22] for more details):

$$\mathcal{C}_+(U)=\int dte^{-imt}f(t)\,\langle 0|\phi(t,r_p)\phi(U)\,|0\rangle$$

$$\mathcal{C}_-(U)=\int dte^{+imt}f^*(t)\,\langle 0|\phi(t,r_p)\phi(U)\,|0\rangle \quad (55)$$

where we recall that $\phi(t,r_p)$ is the u part of the field operator evaluated on the detector trajectory $r=r_p$. Thus one obtains

$$\langle T_{UU}\rangle_e=\langle T^{ren}_{UU}\rangle_+-\langle 0|T^{ren}_{UU}|0\rangle=\frac{2g^2m^2}{P_e}\partial_U\mathcal{C}_+(U)\partial_U\mathcal{C}_-(U). \quad (56)$$

To study the possible singularities that can arise in $\langle T_{UU}\rangle_e$ we explicitize the Wightman propagators in Eq. (55) as $\langle 0|\phi(t,r_p)\phi(U)\,|0\rangle=(1/4\pi)\ln[U(t,r_p)-U-i\epsilon]\simeq(1/4\pi)\ln(-Ce^{-t/4M}-U-i\epsilon)$ where C is a constant. This shows that $\partial_U\,\mathcal{C}_+(U)$ can be singular only at the horizon $U=0$ where it takes the form

$$\partial_U\,\mathcal{C}_+(U=0)=-\frac{1}{4\pi}\int dt\,\frac{1}{-Ce^{-t/4M}-i\epsilon}f(t)e^{-imt}\simeq\frac{1}{4\pi C}\int dte^{t/4M}f(t)e^{-imt}. \quad (57)$$

The last integral is finite if and only if $f(t)$ decreases for $t\to-\infty$ quicker than $e^{-t/4M}$. This is the necessary condition which ensures finite energy densities on the horizon.

To understand why wave packets specified by Π_{g_λ} lead to singular densities, it is appropriate to express $f(t)e^{-imt}$ in Fourier transform

$$f(t)e^{-imt}=\int_{-\infty}^{+\infty}d\lambda\,\frac{c_\lambda}{2\pi}e^{-i\lambda t}. \quad (58)$$

The exponential decrease of $f(t)$ required to have finite energy densities is equivalent to imposing that its Fourier transform c_λ be analytic in the strip $0\leqslant{\rm Im}(\lambda)<1/4M$. However $g_\lambda=0$ for $\lambda<0$, thus g_λ is not an analytic function of λ. Therefore, the energy density correlated to the presence of one asymptotic quantum, Eq. (54), is singular on the horizon.

Equation (56) reexpressed in terms of the Fourier components c_λ reads

$$\langle T_{UU}\rangle_e=\left[\int_{-\infty}^{+\infty}d\lambda\int_{-\infty}^{+\infty}d\lambda'\,c^*_\lambda c_{\lambda'}\,\frac{1}{4\pi\sqrt{\lambda\lambda'}}\beta_\lambda\alpha_\lambda\partial_U\psi^*_{\lambda,K}\times\partial_U\psi^*_{-\lambda',K}\right]\Bigg/\left[\int_{-\infty}^{+\infty}d\lambda\,\frac{|c_\lambda|^2}{4\pi\lambda}\beta^2_\lambda\right]. \quad (59)$$

This expression is obtained by expressing the field operators in Eq. (55) in terms of the Kruskal modes $\psi_{\lambda,K}$ and $f(t)$ in terms of c_λ, then carrying out the integral over t and inserting the resulting expression for $\partial_U\mathcal{C}_\pm$ into Eq. (56). Up to a factor m^2g^2 the denominator in Eq. (59) is the probability to get excited, see Eq. (31),

$$P_e=g^2m^2\int_{-\infty}^{+\infty}d\lambda\,\frac{|c_\lambda|^2}{4\pi\lambda}\beta^2_\lambda. \quad (60)$$

Equations (59) and (54) are extremely similar. The only (but important) differences are factors of α_λ, which come from Bose statistic, and the domains of integration over λ, which allows regular energy density when extended from $-\infty\to+\infty$.

VI. FROM VACUUM FLUCTUATIONS TO BLACK HOLE RADIATION

Our aim is to describe the properties of the energy density of the field configurations subject to the double specification that the initial state is vacuum and that the detector gets excited. In the next section, we shall see how these properties control the gravitational corrections to some transition amplitudes. We take the coupling to be such that the detector, of resonance frequency $\lambda=m$, will be excited around the retarded time $u=u_0$. In [22] a particular form of the Fourier components c_λ of the switch off function $f(t)$ was given for which the energy density can be computed exactly. They are given by

$$c_\lambda=D\,\frac{\lambda}{m}e^{i\lambda u_0}e^{-(\lambda-m)^2T^2/2}(1-e^{-2\pi\lambda/a}) \quad (61)$$

where D is a normalization constant, see [22]. T is the interval of time during which the atom is coupled to the field and u_0 is taken well inside the region $u>0$, see Eq. (4), where the isomorphism of the scattered waves, Eq. (7), and the Unruh modes, Eq. (12), is achieved. Thus $u_0\gg T$.

We first consider the fluctuating part of the energy density, given in Eq. (59) and evaluated on $\mathcal{I}^-$. In the golden rule limit, $T\gg m^{-1}$ and $T\gg 4M$, using Eqs. (59) and (61) and the reflection at $r=0$, one finds

$$\langle T_{vv}(\mathcal{I}^-)\rangle_e \simeq \left(\frac{4M}{v}\right)^2 \frac{m\alpha_m^2}{\sqrt{4\pi}T} \times \exp\left(-\left\{\frac{1}{T}\left[4M\ln\left(\frac{-v-i\epsilon}{4M}\right)+u_0\right]\right\}^2\right). \tag{62}$$

The reader interested by the exact expressions will consult [22]. Due to the vanishing of the modes at $r=0$, see Eq. (7), and the lightlike character of the propagation, the selected final state eschews from a spherically symmetric vacuum fluctuation which extends from $\mathcal{I}^-$ and which is centered around the light ray $v=0$ that shall become the future horizon $U=0$. From the v dependence of Eq. (62) we see that this fluctuation is located in an *exponentially small* region

$$|\Delta v| \simeq 4Me^{-u_0/4M} \tag{63}$$

since $u_0 \gg T$ is the condition to be in the stationary Hawking regime. Similarly, the energy density is enhanced by the exponential Doppler shift between light rays on $\mathcal{I}^+$ and $\mathcal{I}^-$. This shift appears here as the Jacobian $(4M/v)^2$. Thus, for a typical quantum $m=O(1/M)$ and $T=O(M)$, $\langle T_{vv}\rangle_e$ becomes ''trans-Planckian'' and located within a distance Δv smaller than the Planck length as soon as $u_0 > 4M\ln M$.

This is also true for the energy density seen by an infalling inertial observer crossing the horizon $U=0$. Indeed, as mentioned above, see Eq. (7), the reflection condition at $r=v-U=0$ implies

$$\langle T_{UU}\rangle_e = \langle T_{vv}\rangle_e|_{v=U}. \tag{64}$$

A few remarks about Eqs. (62) and (64) are warranted. First note how the $i\epsilon$ in Eq. (62) defines the logarithm $\ln(-v/4M-i\epsilon)$ as $\ln|v/4M|$ for $v<0$ and as $\ln|v/4M|-i\pi$ for $v>0$. As expected from the analysis of the previous section, upon taking the limit $\epsilon\to 0$ no singularity occurs on the light ray which generates the horizon. In fact $\langle T_{UU}\rangle_e$ vanishes for $U=0$. This is due to the particular form of c_λ chosen in Eq. (61). It has zeros for $\lambda=ina$, $n=\dots,-1,0,1,\dots$ which imply that upon evaluating Eq. (59) at $U=0$ by contour integration the poles of $\beta_\lambda\alpha_\lambda$ have zero residue. However, from the expression for $\partial_U \mathcal{C}_\pm(U=0)$ given in Eq. (57), it results that the generic behavior of T_{UU} is to stay finite as $U\to 0$. In more physical terms this corresponds to saying that the vacuum fluctuation correlated to the transition of the detector straddles the horizon with no clear cut separation between the pieces in the left and right quadrants.

Secondly, ϵ, which specified the analytical properties of the modes $\psi_{\lambda,K}$, see Eq. (12), leads to the vanishing of the integrals

$$\int_{-\infty}^{+\infty} dv\, \langle T_{vv}\rangle_e = \int_{-\infty}^{+\infty} dU\, \langle T_{UU}\rangle_e = 0 \tag{65}$$

by contour integration. This follows from the fact that $|0\rangle$ is the ground state on $\mathcal{I}^-$, hence $\int dv\, T_{vv}\,|0\rangle = 0$: vacuum fluctuations carry no energy.

We now consider the Schwarzschild energy for $r>2M$. It is given by (see [22])

$$\langle T_{uu}(r>2M)\rangle_e = \frac{g^2m^2}{P_e}\int d\lambda \int d\lambda'\, c_\lambda c^*_{\lambda'} \frac{1}{(4\pi)^2}\beta_\lambda^2\alpha_\lambda^2 e^{-i(\lambda-\lambda')u} \simeq \frac{m\alpha_m}{2\sqrt{\pi}T}e^{-(u-u_0-i4\pi M)^2/T^2}. \tag{66}$$

By getting excited, the two level atom imposes that the Hawking radiation contains at least one particle along the geodesic $u=u_0$. Furthermore since energy flows along the lines $u=cst$, $\langle T_{uu}(r>2M)\rangle_e$ is centered around $u=u_0$ with at spread $\Delta u=T$. It carries a Schwarzschild energy obtained by integrating Eq. (66)

$$\int du\, \langle T_{uu}(r>2M)\rangle_e = \frac{\int d\lambda |c_\lambda|^2\beta_\lambda^2\alpha_\lambda^2}{\int d\lambda |c_\lambda|^2 \frac{1}{\lambda}\beta_\lambda^2} \simeq m\alpha_m^2. \tag{67}$$

Thus the mean energy emitted is m times the factor $\alpha_m^2 = 1/(1-e^{-8\pi Mm})$ which takes correctly into account the Bose statistics of the field since Eq. (67) corresponds to evaluating $\langle n^2\rangle/\langle n\rangle$ in a thermal distribution.

Because of the strict correlations between left and right quanta, there is a corresponding energy density on the other side of the horizon, for $r<2M$. It is given by

$$\langle T_{uu}(r<2M)\rangle_e = \frac{g^2m^2}{P_e}\left|\int d\lambda\, c_\lambda \frac{1}{4\pi}\beta_\lambda^2 e^{4\pi M\lambda}e^{-i\lambda u}\right|^2 \simeq \frac{m\alpha_m^2}{2\sqrt{\pi}T}e^{-(u-u_0)^2/T^2}. \tag{68}$$

The Schwarzschild energies on either side of the horizon are opposite:[1]

$$\int du\, \langle T_{uu}(r>2M)\rangle_e = \int du\, \langle T_{uu}(r<2M)\rangle_e \tag{69}$$

as can be seen explicitly from Eqs. (66) and (68). The equality of Schwarzschild energies on either side of the horizon results from the invariance under Schwarzschild time translations of the Unruh vacuum. Indeed for u modes, the Schwarzschild time translation operator can be written as $i\partial_t = \int du T_{uu}(r>2M) - \int du T_{uu}(r<2M)$, in complete analogy with the boost operator in Minkowski vacuum, see [22]. The invariance under time translations implies $i\partial_t|U\rangle = 0$ where $|U\rangle$ is the Unruh vacuum. Since the Heisenberg state coincides with $|U\rangle$ at late times, all physical processes which occur at late times are time translation invariant. Finally we note that although the integrals of T_{uu} on either side of the horizon are equal, the energy densities are not. This stems from the asymmetry between the particle and partner when they are specified in a wave packet [see Eq. (53)]. In particular $\langle T_{uu}(r<2M)\rangle_e$ is real, see Eq. (68). This results from causality. Since the final state of the radiation is specified only outside the horizon by the transitions of the two

[1] Behind the horizon ∂_u is directed towards the past so the Schwarzschild energy for $r<2M$ is $-\int du T_{uu}(r<2M)$.

level atom, the field operator $\mathcal{O}$ which appears in Eq. (34) depends only on the field operator ϕ for $r>2M$. Therefore $\mathcal{O}$ and $T_{uu}(r<2M)$ commute. This implies that $\langle T_{uu}(r<2M)\rangle_e$ is real.

Conclusions. We have a complete picture of the energy density correlated to the transition of the detector. On $\mathcal{I}^-$, the total energy carried by these field configurations vanishes because one is in the vacuum state [see Eq. (65)]. However their energy density is enhanced by the Jacobian $du/dU=e^{u/4M}$ centered around $u=u_0$. Hence when u_0 is larger than $O(4M\ln M)$, the energy densities T_{UU} and T_{vv} (rescaled by $4\pi r^2$) become ''trans-Planckian'' and located within a distance Δv much smaller than the Planck length. After issuing from $\mathcal{I}^-$, the vacuum fluctuation contracts until it reaches $r=0$ and then reexpands along $U=$const lines. Upon crossing the surface of the star in a region $\Delta U=\Delta v\simeq 4Me^{(T-u_0)/4M}$ centered on the horizon, it separates into two pieces. The first one, the partner, falls into the singularity and carries a negative Schwarzschild energy equal to $-m\alpha_m^2$. The second piece keeps expanding, escapes to $\mathcal{I}^+$, and constitutes the quantum that induces the transition of the atom. It carries a positive Schwarzschild energy equal to $m\alpha_m^2$.

It is interesting to obtain a description of the energy density in the intermediate regions in order to interpolate between the descriptions on $\mathcal{I}^-$ and $\mathcal{I}^+$. One possible interpolation consists in using a set of static observers at constant r. Then the ''Rindler'' description would be used everywhere outside the star. However a difficulty arises in this scheme if one really considers a set of material ''fiducial'' [27] detectors at constant r. For upon interacting with the field and thermalizing at the local temperature $(8\pi M\sqrt{1-2M/r})^{-1}$ the detectors will emit large amounts of ultraviolet Kruskal quanta (see Ref. [22], Sec. III B). The back reaction of these on mass shell quanta cannot be neglected and cannot be evaluated perturbatively owing to the trans-Planckian energy they carry.

An alternative interpolation consists in giving the value of $T_{\mu\nu}$ in the local inertial coordinate system (Riemann normal coordinates). This stems from the idea that local physics should be described locally in such a coordinate system. This approach has been used in defining the subtraction necessary to renormalize the energy momentum tensor [34,39,40]. For spherically symmetric situations, the local inertial radial coordinates are easy to construct since $\tilde{u}=r(u,v)$ is an affine parameter along the geodesics $v=$constant. The outgoing flux in these coordinates is

$$T_{\tilde{u}\tilde{u}}(\tilde{u})=\left(\frac{du(r,v)}{dr}\right)^2 T_{uu}(u(r,v)). \tag{70}$$

This is represented in a Penrose diagram in Fig. 1 and Eddington-Finkelstein coordinates in Fig. 2. The inertial coordinate $\tilde{u}$ will come up spontaneously in the next section upon investigating back reaction effects, thereby justifying dynamically this local description.

Up to now, we have considered a two-level atom coupled to s wave only. If more realistically, we take a two-level atom coupled locally to the field (i.e., coupled to all the modes $l>0$), it will select particles coming out of the black hole in its direction (θ_0,φ_0). Then the picture that emerges is essentially the same as for an s wave except that on $\mathcal{I}^-$ the vacuum fluctuation is localized on the antipodal point of the detector, i.e., $(\pi-\theta_0,\varphi_0+\pi)$.

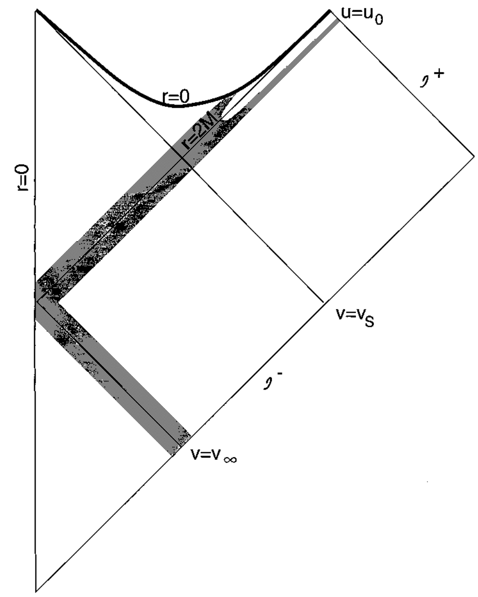

FIG. 1. The local description of a vacuum fluctuation giving rise to a Hawking photon emitted around $u=u_0$ is represented in a Penrose diagram. The shaded areas correspond to the regions where $T_{\tilde{u}\tilde{u}}(\tilde{u})$ is nonvanishing. $v=v_S$ is the trajectory of the collapsing spherically symmetric shell of massless matter.

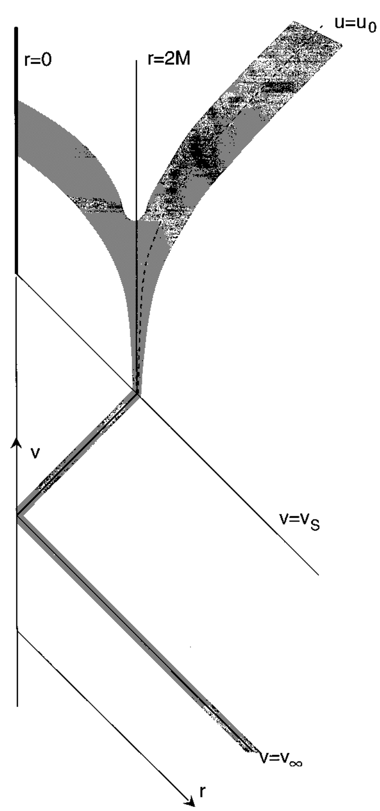

FIG. 2. The same as in Fig. 1 drawn in Eddington-Finkelstein coordinates.

We take the opportunity of mentioning $l>0$ modes to point out that upon considering *all* the modes and specifying that we are in Boulware vacuum, the resulting conditional value of $T_{\mu\nu}$ is much more singular than Eq. (43). Indeed near the horizon it behaves like[2]

$$\left(\frac{\langle B|T_{UU}|0\rangle}{\langle B|0\rangle}-\langle 0|T_{UU}|0\rangle\right)$$
$$\simeq -\frac{1}{(8\pi M)^2}\frac{(4M)^2}{(U+i\epsilon)^2}\frac{2M}{r-2M} \qquad (71)$$

where the factor $r/(r-2M)$ takes into account the number of excited $l>0$ modes which can propagate up to r. We refer to [41] for a discussion of the fluctuations of the metric induced by these thermally distributed modes.

VII. THE GRAVITATIONAL BACK REACTION

The aim of this section is to illustrate how the various properties of the (complex) conditional value of $T_{\mu\nu}$ correlated to the transition of a detector modifies the transition amplitudes of certain processes. We recall that $T_{\mu\nu}$ controls perturbative corrections through the minimal coupling of the field to gravity, Eq. (15). We first illustrate the role of $T_{\mu\nu}$ by explicitly calculating the change in the probability to find the atom excited engendered by a fluctuation of the geometry. We then propose a perturbative scheme for taking into account self interactions among Hawking quanta.

As in Sec. III, we consider a change in the background geometry $g_{\mu\nu}\rightarrow g_{\mu\nu}+h_{\mu\nu}$. This perturbation modifies both the mean values of the flux at later times as well as more detailed properties such as the probability to find a specific photon on $\mathcal{I}^+$. We focus on this latter type of changes. In order to have finite energy densities on the horizon, we consider how $h_{\mu\nu}$ modifies the probability for a two level detector to absorb a Hawking photon and get excited.

In the unperturbed background geometry, this probability is given by Eq. (31). In the modified geometry, the probability to find the same photon is, in the interaction representation with respect to the perturbation equation (15), cf. Eq. (33):

$$P_e^{g+h}=\langle\Psi|e^{i\int dt H_{\rm int}}\Pi_+ e^{-i\int dt H_{\rm int}}|\Psi\rangle. \qquad (72)$$

To first order in $h_{\mu\nu}$ the relative change in probability is

$$\frac{P_e^{g+h}-P_e^g}{P_e^g}=\frac{\langle\Psi|\Pi_+(-i\int dt H_{\rm int})|\Psi\rangle}{\langle\Psi|\Pi_+|\Psi\rangle}+\text{ H.c.}$$
$$=\int d^4x\sqrt{-g}h^{\mu\nu}\,\mathrm{Im}\langle T_{\mu\nu}\rangle_+ . \qquad (73)$$

Only the imaginary part of $\langle T_{\mu\nu}\rangle_+$ controls the change in probability induced by $h_{\mu\nu}$. And since the mean flux $\langle 0|T^{\rm ren}_{\mu\nu}|0\rangle$ is real, *only* the fluctuating term $\langle T_{\mu\nu}\rangle_e$, see Eq. (56), which depends explicitly on the selected quantum contributes to $P_e^{g+h}-P_e^g=\delta P$.

We now illustrate by a specific example how the properties of the fluctuating part $\langle T_{\mu\nu}\rangle_e$ intervene into δP. We take the simple case wherein $h_{\mu\nu}$ is due to the in-fall of an additional lightlike shell of mass δM at time $v=v'$ with $v'\geqslant v_S$ where v_S is the trajectory of the star's shell. Then for $v_S<v<v'$ the metric, Eq. (1) and Eq. (4) is unchanged,

$$ds^2=\left(1-\frac{2M}{r}\right)dv^2-2dv\,dr-r^2d^2\Omega, \qquad (74)$$

whereas for $v>v'$ it is

$$ds^2=\left(1-\frac{2M+2\delta M}{r}\right)dv^2-2dv\,dr-r^2d^2\Omega \qquad (75)$$

where we have used for obvious convenience the Eddington-Finkelstein coordinates v and r.

The change in the matter action $S=\int d^4x\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is, for s waves,

$$\delta S=-\int dv H_{\rm int}=\int_{v'}^{+\infty}dv\int_0^\infty dr 4\pi r^2(\delta M/r)\partial_r\phi\partial_r\phi$$
$$=\delta M\int_{v'}^{+\infty}dv\int_0^\infty dr 4\pi r\left[\frac{T_{\tilde u\tilde u}(r,v)}{4\pi r^2}\right] \qquad (76)$$

where $T_{\tilde u\tilde u}$ is the rescaled energy density given by Eq. (70). As emphasized at the end of Sec. VI, it is the energy momentum in Riemann normal coordinates which appears automatically in such problems since the matter response to a local change in the geometry is local as well.

In the case when $v'=v_S$, one simply has a single shell of mass $M+\delta M$. Thus the probability to find the atom excited is simply, see Eq. (31),

$$P_e^{M+\delta M}=\frac{1}{2}g^2mT\frac{1}{e^{8\pi(M+\delta M)m}-1}=\frac{1}{2}g^2mT|\beta_m^{(M+\delta M)}|^2 \qquad (77)$$

hence

$$\frac{\delta P}{P}=-\delta M(8\pi m)|\alpha_m^{(M)}|^2. \qquad (78)$$

Thus, in this case, the trans-Planckian character of the energy density is washed out by the integration in Eq. (76).

It is interesting to analyze more closely how this washing out mechanism follows from the behavior of $\langle T_{uu}\rangle_e$. We first recall that the imaginary part of $\langle T_{uu}\rangle_e$ vanishes on the other side of the horizon, for $r<2M$ [see discussion after Eq. (69)]. From Eq. (76) we understand that this is dictated by causality: a change in $g_{\mu\nu}$ in the region $r<2M$ cannot affect the probability to find a specific Hawking photon on this side. Because of this vanishing one rewrites the contribution to $\delta P/P$ of the integral over r at fixed v as

[2]Equation (71) differs from the s-wave result equation (43) by the factor $2M/(r-2M)$. This factor is estimated by counting the number of high angular momentum modes that can propagate up to $r-2M$ and multiplying the s-wave result by this number. From Eq. (5) the high angular modes with typical energy $\lambda\simeq 1/M$ that can propagate up to r must obey the condition $l(l+1)<2M/(r-2M)$.

$$\int_0^\infty dr 4\pi r \,\mathrm{Im}\left[\frac{\langle T_{rr}\rangle_e}{4\pi r^2}\right] = \int_{2M}^\infty dr \frac{1}{r}\mathrm{Im}\langle T_{rr}\rangle_e$$

$$= \int_{-\infty}^{+\infty} du \frac{2}{r-2M}\mathrm{Im}\langle T_{uu}\rangle_e ; \quad (79)$$

see Eqs. (73) and (76). We have also used the Jacobian $dr/du|_v$, see Eq. (1).

Since the trans-Planckian frequencies appear close to the horizon, we first analyze the behavior of $\langle T_{UU}\rangle_e$ for $r-2M \ll 2M$. There, $r-2M$ is

$$r-2M = 2Me^{(v-u-4M)/4M}(1-e^{(v-u-4M)/4M} + (3/2)e^{(v-u-4M)/2M} + \cdots) \quad (80)$$

and Eq. (79) becomes

$$\int_{-\infty}^{+\infty} du \frac{2\mathrm{Im}\langle T_{uu}\rangle_e}{r-2M} = \int_{-\infty}^{+\infty} \frac{du}{M}\mathrm{Im}\langle T_{uu}\rangle_e (e^{(-v+u+4M)/4M} + 1 - (1/2)e^{(v-u-4M)/4M} + O(e^{(v-u-4M)/2M})). \quad (81)$$

The first term is proportional to $\int du \mathrm{Im}\langle T_{uu}\rangle_e e^{u/4M}$. Hence it vanishes because it is equal to $\int dU \mathrm{Im}\langle T_{UU}\rangle_e$, see Eq. (65). The second term, being proportional to $\int du \mathrm{Im}\langle T_{uu}\rangle_e$, vanishes as well, see Eq. (69). The third term vanishes also. Indeed, under the change $u \to -u$, this term behaves like the first term which vanishes for all selected wave packets. Hence one is left with the fourth term which gives a contribution proportional to

$$-\frac{1}{2M}e^{(v-u_0-4M/2M)}\int_{-\infty}^{+\infty} d\Delta u e^{-\Delta u/2M}\mathrm{Im}\langle T_{uu}(u_0+\Delta u)\rangle_e$$

$$= -\frac{1}{2M}\left(\frac{r_0(v)-2M}{2M}\right)^2 C \quad (82)$$

where we have changed variables to $u = u_0 + \Delta u$ and $r = r_0(v)$ is the expression for the geodesic $u = u_0$ in r,v. C is a constant with respect with u_0. Therefore, the contribution to the change in probability near the horizon is bounded. As announced all the trans-Planckian oscillations have been washed out.

In addition, at large r_0 (i.e., $r_0 \gg 2M$), where $\Delta r|_v = -(1/2)\Delta u$, Eq. (79) decreases rapidly. Indeed, one finds

$$\int_{-\infty}^{+\infty} du \frac{\mathrm{Im}\langle T_{uu}\rangle_e}{r(u,v)-2M} = \int_{-\infty}^{+\infty} d\Delta u \mathrm{Im}\langle T_{uu}(u_0+\Delta u)\rangle_e \times\left(\frac{1}{r_0-2M} + \frac{\Delta u}{2(r_0-2M)^2}\right). \quad (83)$$

The first term vanishes since it is proportional to $\int d\Delta u \mathrm{Im}\langle T_{uu}\rangle_e$. Hence the contribution at large r_0 decreases as $1/r_0^2$.

Thus we have obtained a local description of the quantum matter response to a modification of the classical background geometry. We first recall that the *mean* value of $T_{\mu\nu}$, being real, cannot contribute to this response. We then insist on the localized character of this response. Even though $\langle T_{\mu\nu}\rangle_e$ extends from $\mathcal{I}^-$ to $\mathcal{I}^+$, it is only the imaginary part of $\langle T_{uu}\rangle_e$ which contributes in a well localized region to $\delta P/P$. This region lies along the classical trajectory $u = u_0$ between $r = 2M$ and $r \leq O(6M)$. This local response furnishes a precise answer to the longstanding question: where is a Hawking photon ‘‘born’’? [42].

We now present a perturbative scheme to take into account backreaction effects among Hawking quanta. The first step in this procedure is relatively simple. It consists in quantizing $h_{\mu\nu}$ as in Eq. (21) *et seq.* Then the mean value of $h_{\mu\nu}$ is obtained by integrating Einstein's equations with the mean energy momentum tensor as a source, see Sec. III of [22] for a more conventional example. This corresponds to the linear approximation to the semiclassical solution [10]. But one can also evaluate the *conditional value* of $h_{\mu\nu}$ correlated to a specific final state. This has been carried out formally in Eq. (23). We recall that the change in $h_{\mu\nu}$ is given by

$$\langle h^{\alpha\beta}(x)\rangle_e = \mathrm{Im}\left[\int d^4x' \sqrt{-g}G^{\mu\nu\alpha\beta}(x,x')\langle T_{\mu\nu}(x')\rangle_e\right] \quad (84)$$

where $G^{\mu\nu\alpha\beta}$ is the graviton propagator and we have subtracted the contribution of the mean energy $\langle 0|T^{\mathrm{ren}}_{\mu\nu}|0\rangle$. Since the total energy carried by the conditional value of $T_{\mu\nu}$ vanishes from $\mathcal{I}^-$ till the emergence of the fluctuation from the star after reflection on $r=0$, $\langle h_{\mu\nu}\rangle_e$ will vanish outside the interval Δv centered around $v=0$ and given in Eq. (63). Within that (exponentially small) interval the precise shape of $\langle h_{\mu\nu}\rangle_e$ will depend on the particular choice of wave packet to which the two level atom responds. On the contrary, outside the star, for $r>4M$ and $u>u_0$, i.e., in the middle of the two members of the pair, $\langle h_{\mu\nu}\rangle_e$ will encode the mass loss $\lambda = m$ and in fact describes a new classical real valued Schwarzschild space where the mass is $M-m$.

The next step consists in taking into account the effect of $h_{\mu\nu}$ on the production of Hawking photons themselves. To first order this gravitational self-interaction is encoded in a interaction Hamiltonian for the quantum field ϕ of the form

$$\int dt\tilde{H}_{\mathrm{int}} = \int d^4x\sqrt{-g}\int d^4x'\sqrt{-g}T_{\mu\nu}(x)G^{\mu\nu\alpha\beta} \times(x,x')T_{\alpha\beta}(x'). \quad (85)$$

This is in strict analogy with the interaction among charged pairs created in an electric field. Indeed, in this case, upon functionally integrating over the electro magnetic field, one obtains an effective current-current interaction for the charged field of the form $\int d^4x \int d^4x' J_\mu(x)D^{\mu\alpha}(x,x')J_\alpha(x')$ where $D^{\mu\alpha}$ is the photon propagator.

We now obtain an expression for $G^{\mu\nu\alpha\beta}$ valid for the spherically symmetric case from the form of the Berger-Chitre-Moncrief-Nutku (BCMN) Hamiltonian for a self interacting spherically symmetric field considered in [43], corrected in [3], and generalized to the case of a background black hole in [44]. To first order in the gravitational interaction the BCMN Hamiltonian can be written in the form

$$H=M+\int_0^\infty dr h(r)\left[1-\frac{2M}{r}-\frac{2}{r}\int_0^r dr' h(r')\left(1-\frac{2M}{r'}\right)\right],$$

$$h(r)=\frac{1}{2}\left(\frac{\Pi_\phi^2}{r^2}+r^2(\partial_r\phi)^2\right), \tag{86}$$

where Π_ϕ is the momentum conjugate to ϕ. The metric associated to the solution of Eq. (86) is

$$ds^2=(1+\alpha)L^{-2}dt^2-L^2dr^2-r^2d\Omega^2 \tag{87}$$

where

$$L^{-2}=1-\frac{2M}{r}-\frac{2}{r}\int_0^r dr' h(r')\left(1-\frac{2M}{r'}\right) \tag{88}$$

and α is a slowly varying function of h. This shows that the gravitational interactions arise from the existence of a self-consistent r dependent mass in the Schwarzschild solution. Furthermore one can show that the energy-momentum tensor is given by

$$T_{rr}=\frac{1}{4\pi r^2}h, \quad T^{tt}=\frac{1}{4\pi r^2}\frac{h}{(1+\alpha)}. \tag{89}$$

So that the interaction Hamiltonian can be written as

$$\tilde{H}_{\rm int}=-\int_0^\infty dr 4\pi r^2 T_{rr}\frac{2}{r}\int_0^r dr' 4\pi r'^2 T_{r'r'}\left(1-\frac{2M}{r'}\right) \tag{90}$$

which is the sought for expression for the propagator $G^{\mu\nu\alpha\beta}$. Equation (90) should be compared with Eq. (76). It is now $2\int_0^r dr' 4\pi r'^2 T_{r'r'}(1-2M/r')/r$ which plays the role of change of the geometry $2\delta M/r$. Note also the similarity between these expressions and the instantaneous Coulomb-Coulomb interaction in electrodynamics.

Upon calculating $\delta P/P$ to first order due to this self-interaction, it is the conditional value of $\tilde{H}_{\rm int}$ which will come up. However, $\tilde{H}_{\rm int}$ being quartic in the field, the calculation of its conditional value is much more complicated than the evaluation of the conditional value of $T_{\mu\nu}$ carried out in the preceding section. Among other difficulties one must renormalize the infinities which arise in loops.

However in both electroproduction and Hawking radiation, this interaction Hamiltonian not only modifies the probability of creating a specific particle but also induces correlations among the produced particles. In particular the probability to find two particles will no longer factorize into the products of the probabilities to find them independently. An important problem is to evaluate these correlations. To first order in $\tilde{H}_{\rm int}$ they are finite, i.e., it is a zero loop correction. They are given by $C_{12}=P_{12}-P_1P_2$ where P_{12} is the probability to find both particles 1 and 2, while P_1 (P_2) is the probability to find particle 1 (2) independently of particle 2 (1). To first order in $\tilde{H}_{\rm int}$ one obtains

$$C_{12}=-i\langle 0|\Pi_1\Pi_2\int dt{:}\tilde{H}_{\rm int}{:}|0\rangle+\text{H.c.} \tag{91}$$

where Π_1 (Π_2) is the projector which specifies that particle 1 (2) is present and $:\tilde{H}_{\rm int}:$ is the interaction Hamiltonian normal ordered with respect to the Heisenberg vacuum.

Because of the normal ordering, upon evaluating C_{12} one will obtain both a direct term and a cross term but no self-interaction term. The direct term is given entirely in terms of the conditional values of $T_{\mu\nu}$:

$$\begin{aligned}\frac{C_{12}|_{\rm direct}}{P_1P_2}&=-i\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g}\frac{\langle 0|\Pi_1 T_{\mu\nu}(x)|0\rangle}{\langle 0|\Pi_1|0\rangle}\\&\quad\times G^{\mu\nu\alpha\beta}(x,x')\frac{\langle 0|\Pi_2 T_{\alpha\beta}(x')|0\rangle}{\langle 0|\Pi_2|0\rangle}+\text{H.c.}\\&=2\,\text{Im}\bigg[\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g}\langle T_{\mu\nu}(x)\rangle_{\Pi_1}\\&\quad\times G^{\mu\nu\alpha\beta}(x,x')\langle T_{\alpha\beta}(x')\rangle_{\Pi_2}\bigg].\end{aligned} \tag{92}$$

Probably the dominant (long lasting) effect is encoded in Eq. (92) and corresponds to the fact one particle lives in a geometry where the mass is reduced by the energy of the other particle. In view of the fact that the effect of a classical change in mass on the probability to produce one particle was insensitive to the trans-Planckian frequencies, we are inclined to believe that the correlations C_{12} will also be finite and small even though in the integrand of Eq. (92) (and of the corresponding cross term) there appear trans-Planckian energy densities. However this remains to be proven by a detailed calculation.

VIII. CONCLUSION

In resumé, we have described the field configurations correlated to the presence of a specific Hawking quantum by using the conditional value of $T_{\mu\nu}$. This conditional value extends from $\mathcal{I}^-$ to $\mathcal{I}^+$. On $\mathcal{I}^-$ the energy density grows exponentially with the time u_0 at which the photon reaches $\mathcal{I}^+$. This results from the two hypothesis of Hawking, namely free propagation in a given classical background. It is important to realize that the conditional value of $T_{\mu\nu}$ becomes trans-Planckian on $\mathcal{I}^-$ in the unique inertial coordinate system at rest with respect to the star. If one introduces a new coordinate system such that $T_{\mu\nu}$ on $\mathcal{I}^-$ becomes smoother by rescaling v around $v=0$, then any regular field configuration on $\mathcal{I}^-$ would become trans-Planckian in the new system.

We then showed how a modification of the geometry induces, through the minimal coupling $H_{\rm int}$, Eq. (15), a change in the probability to find a specific Hawking quantum. It is only the imaginary part of the fluctuating term of $\langle T_{\mu\nu}\rangle_e$ which determines the change in probability. Furthermore, when the modification of the geometry is due to an in-falling shell, the contribution comes from a region comprised between $2M<r<O(6M)$. Thus, one can interpret this region as the locus where Hawking radiation is produced. Finally, we showed that the trans-Planckian oscillations of $\langle T_{\mu\nu}\rangle_e$

average out since only a small change, proportional to the mass increase, is obtained. We also suggested how to compute the correlations between successively emitted quanta and argued that these correlations may essentially be due to the *sole* decrease of mass induced by the emitted quanta.

Nevertheless the trans-Planckian energies might induce important consequences in the full treatment of the back reaction. In Ref. [26], together with Englert, we argued that the nonlinearity of general relativity cannot accommodate these densities and that there must be a dynamical taming mechanism if Hawking radiation does exist. To illustrate the possible effects of a taming, in Refs. [31,32] the dispersion relation of the free field theory in a Schwarzschild background was modified so that energies above the Planck scale can no longer occur. The remarkable result is that the thermal properties of Hawking radiation are completely unaffected by the modification of the theory. This suggests that the gravitational interactions may induce a taming mechanism which does not modify the properties of Hawking radiation. However this still needs to be proven.

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