Tunneling in a time-dependent setting

Esko Keski-Vakkuri* and Per Kraus†
California Institute of Technology, Pasadena, California 91125

(Received 10 June 1996)

A standard approach to analyzing tunneling processes in various physical contexts is to use instanton or imaginary time path techniques. For systems in which the tunneling takes place in a time-dependent setting, the standard methods are often applicable only in special cases, e.g., due to some additional symmetries. We consider a collection of time-dependent tunneling problems to which the standard methods cannot be applied directly, and present an algorithm, based on the WKB approximation combined with complex time path methods, which can be used to calculate the relevant tunneling probabilities. This collection of problems contains, among others, the spontaneous nucleation of topological defects in an expanding universe, the production of charged particle-antiparticle pairs in a time-dependent electric field, and false vacuum decay in field theory from a coherently oscillating initial state. To demonstrate the method, we present detailed calculations of the time-dependent decay rates for the last two examples. [S0556-2821(96)04124-0]

PACS number(s): 03.65.Sq, 04.62.+v

I. INTRODUCTION

Problems involving quantum-mechanical tunneling in a time-dependent setting can arise in a wide variety of contexts, such as the ionization of atoms by strong laser fields [1], pair creation of charged particles in time-dependent background electromagnetic fields [2–4], spontaneous nucleation of topological defects in expanding universes [5], and false vacuum decay with time-dependent initial states or time-dependent potentials [6]. In some special cases, these systems can be treated by standard instanton or imaginary time path methods; however, these techniques have limited applicability, and confusion often arises when one tries to extend the analysis to more general time-dependent situations. For a discussion of various difficulties, see, e.g., [6].

In this paper we will investigate a collection of generalized time-dependent versions of ''standard'' tunneling problems, where the textbook instanton and imaginary time path methods are inapplicable due to the additional time dependence. The models typically have Lagrangians with an explicit time dependence arising from external backgrounds or involve more complicated nonstatic initial states. Their unifying aspect is that they all can be analyzed via a method that combines the use of the WKB approximation with solutions of the classical equations of motion along complex time paths. We will present a straightforward algorithm which can be used to compute the relevant tunneling or nucleation rates for such systems.

To give a concrete example of this method, let us consider pair creation by a spatially constant electric field. In order to identify the specific features associated with a time-dependent field, it is useful to first review the simple case of a static field. This problem, first solved by Schwinger [7], is most elegantly treated by an instanton approach. Calling the state with no particles present the false vacuum, the decay rate is determined by the imaginary part of the false vacuum energy, which can be extracted from an imaginary time path integral over fields which approach the false vacuum at $\tau = -it = \pm\infty$. In the semiclassical approximation one saturates the path integral by a solution to the (Euclideanized) equations of motion; the corresponding configuration is the instanton. The action of the instanton determines the decay rate: $\Gamma \propto e^{-S_{\rm instanton}}$.

The instanton solution describes the nucleation of a particle-antiparticle pair in the background electric field. This can be seen directly by cutting the instanton in half. Half of the instanton solution corresponds to interpolating between the false vacuum at $\tau=-\infty$ and a turning point, which we can take to occur at $\tau=0$. The turning point configuration is that of a pair of particles momentarily at rest. If we were to continue evolving in imaginary time towards $\tau=\infty$, then the particles would converge and disappear, leaving the system in the false vacuum again. Instead, however, we can continue the solution to real time at the turning point, in which case the particles accelerate away from one another. Thus the full production process can conveniently be described by a combination of real and imaginary time evolution.

For our purposes it is actually more convenient to consider the preceding discussion in the reverse order. We can start by considering the real time expanding solution and then consider evolving it back in time. Eventually, we will reach the turning point, at which point we continue the evolution to imaginary time. If the particle separation proceeds to smoothly shrink to zero size in imaginary time, then the trajectory considered corresponds to a pair production process, and its action determines the decay rate. So to summarize in a way that is most useful for the proceeding discussion, we look for expanding solutions, which can be smoothly shrunk to zero size when evolved back along some complex time contour.

Phrased in this way, it is apparent how to adapt the procedure to the more general problem of a time-dependent electric field. We can again look for solutions describing expanding pairs, but this time the continuation to complex time is more involved. Because of the time dependence in

*Electronic address: esko@theory.caltech.edu
†Electronic address: perkraus@theory.caltech.edu

the problem, we no longer expect that the time contour along which the pair shrinks to zero size is one involving periods of purely real or purely imaginary time evolution; instead, the contour will be a more general curve in the complex time plane. Given that we can find such a contour, we can proceed to evaluate the action to determine the decay rate. The result will be a decay rate with nontrivial time dependence.

By itself, the pair creation problem in a time-dependent electric field has a long history. It was studied rigorously by Brezin and Itzykson [2], using Schwinger's proper time approach. Marinov and Popov [3] treated it as a barrier penetration problem and employed WKB methods. Further, as we discuss in the Appendix A, Audretsch [4] noticed that the problem is isomorphic to overbarrier scattering in quantum mechanics. However, an advantage of the approach outlined above is that it is easily adapted to other problems involving the decay of metastable states via the production of extended objects. Further, it yields an instantaneous pair production rate with an explicit time dependence, so that one sees a time modulation in the flux of produced particles. The result for the pair production rate in [3] applies only at specific times.

As an example of adapting our approach to other physical processes, one can consider generalizing the computation of false vacuum decay in field theory, which proceeds through the nucleation of bubbles of true vacuum, to include field potentials with explicit time dependence. A particularly interesting source of time dependence arises from expanding universes, where it is expected that the expansion gives rise to the spontaneous nucleation of monopoles, strings, and domain walls. The nucleation rate has been computed for a very specific case, namely, de Sitter space, but this is not in fact a time-dependent problem, as the de Sitter geometry is static. For nonstatic geometries the more general approach discussed above is required.

In the next section we write down an action which is general enough to treat the various processes we have referred to and then give an algorithm by which one can compute the time-dependent nucleation rate of the corresponding objects. In most cases, several steps in the procedure must be performed numerically. The simplest case, in which almost everything can be done analytically, is pair production in a time-dependent electric field. We perform these steps in Sec. 3, showing that the production rate takes a compact integral form. A special case of this formula was derived before in [3]. We review the connection of the problem to the problem of above barrier scattering in quantum mechanics. This discussed in detail in Appendix A. We then study the particular example of a sinusoidally varying field and analyze the instantaneous pair production rates. In Sec. IV we turn to the other source of time dependence mentioned above, arising from the initial state rather than from external sources. Specifically, we consider a field theory with a local, but not global, minimum and take the initial state to be one in which the field is undergoing coherent oscillations about the local minimum. We are able to calculate analytically for small oscillations and to obtain the leading correction to the decay rate. Some computational details are relegated to Appendix B. Finally, in Sec. V we summarize our conclusions and discuss directions for further study.

II. TIME-DEPENDENT TUNNELING

In this section we discuss our general approach to tunneling in models with explicit time dependence. Our aim is to show, using complex time contours, how such systems can be treated by a natural extension of the standard instanton method. To start with, we assume that there is an underlying field theory description of the model under consideration and that the system is initially in some metastable state. We further assume that the state can decay via quantum tunneling and that the decay occurs through the production of objects which can be described by a first quantized action. A wide variety of such objects can be described by the action

$$S=\int dt[-a(x,t)\sqrt{1-\dot{x}^2}+b(x,t)]. \tag{1}$$

Some specific cases of interest are

$$a(x,t)=m, \quad b(x,t)=qE(t)x.$$

This is the relativistic action of a particle of mass m and charge q moving in a time-dependent electric field $E(t)$. As will be discussed in detail, this is the appropriate action for considering pair production due to the electric field.

$$a(x,t)=4\pi\sigma(t)x^2, \quad b(x,t)=\tfrac{4}{3}\pi\rho(t)x^3.$$

This is the action of a spherical ''bubble'' of radius x, with time-dependent surface tension $\sigma(t)$ and bulk energy density $\rho(t)$. It describes false vacuum decay in a field theory from a time-dependent initial state or in a time-dependent potential, in instances where the thin wall approximation is valid.

$$a(x,t)=mc(t), \quad b(x,t)=0.$$

This is the action of a massive particle moving in the metric

$$ds^2=c^2(t)[dt^2-dx^2-s^2(x)(d\theta^2+\sin\,\theta^2 d\phi^2)]$$

at fixed θ and ϕ. Here the choices $s(x)=\sin x$, x, and $\sinh x$ give closed, flat, and open Robertson-Walker universes.

$$a(x,t)=2\pi\mu c^2(t)s(x), \quad b(x,t)=0.$$

We then have the action of a circular cosmic string moving in the above metric. The string is located at $\theta=\pi/2$ and is centered at $x=0$.

$$a(x,t)=4\pi\sigma c^3(t)s^2(x), \quad b(x,t)=0.$$

This, similarly, describes a spherical domain wall in the above metric, again centered at $x=0$.

One might question the applicability of first quantized action to describe these systems, which are fundamentally field theories. In the electric field example, it is possible to make the connection rigorous and explicit, as is discussed in Appendix A. We can see no reason why the connection should not be valid in the other examples as well.

We now explain how the action (1) can be used to compute the spontaneous creation rate of the objects which it describes. To begin, we should find the classical, real time, trajectories. The equations of motion are

$$\frac{d}{dt}\left[\frac{a(x,t)\dot{x}}{\sqrt{1-\dot{x}^2}}\right]=-\sqrt{1-\dot{x}^2}a'(x,t)+b'(x,t), \qquad (2)$$

where the prime denotes $\partial/\partial x$. For the purposes of tunneling, the relevant trajectories are those which emanate from a turning point. For a trajectory $x(t)$, the existence of a turning point at time t_f means that the canonical momentum vanishes there:

$$p(t_f)=\left.\frac{a(x,t)\dot{x}}{\sqrt{1-\dot{x}^2}}\right|_{t=t_f}=0.$$

In most cases, this condition will be simply $\dot{x}=0$. A nucleation process corresponds to a trajectory which smoothly shrinks the object down to zero size when evolved back in time along a complex time contour. The continuation from real to complex time occurs at the turning point. Shrinking to zero size[1] means that $x=0$ in the flat space examples, whereas in curved space it is the physical size $c(t)s(x)$ which is required to go to zero. The condition that the shrinking to zero size be smooth is most easily seen in the flat space examples. Then we require that $\dot{x}\rightarrow 0$ when $x\rightarrow 0$; otherwise, in the electric field case, for instance, the joining of the particle trajectory $x(t)$ and the antiparticle trajectory $-x(t)$ will be singular. In curved space a slightly more detailed analysis is necessary, depending on the specific form of $c(t)$. To summarize, the trajectories of interest satisfy $p(t_f)=0$, $x(t_0)=0$, $\dot{x}(t_0)\rightarrow\infty$, t_f=real, t_0=complex, $x(t)$ =real. The problem is then, given some time t_f, find an initial size $x(t_f)$ and complex time t_0 such that the above conditions are satisfied.

Since x is required to be real while t is complex, the easiest way to proceed is to rewrite the equation of motion (2) as an equation for $t(x)$:

$$\frac{d}{dx}\left[\frac{a(x,t)}{\sqrt{t'^2-1}}\right]=-a'(x,t)\sqrt{t'^2-1}+b'(x,t)t'. \qquad (3)$$

The conditions on $t(x)$ become

$$p(t_f)=\left.\frac{a(x,t)}{\sqrt{t'^2-1}}\right|_{t_f}=0, \quad t'(0)=0.$$

The advantage of this form is that it is straightforward to solve Eq. (3) numerically, even if it is not possible to do so analytically. Then we can search for an initial coordinate value $x_f=x(t_f)$, which leads to $t'(0)=0$. Having found an appropriate tunneling trajectory $t(x)$, we can proceed to evaluate its action. Again, it is easiest to change variables in Eq. (1), yielding

$$S=\int_0^{x_f}dx[-a(x,t(x))\sqrt{t'^2(x)-1}+b(x,t(x))t'(x)].$$

[1]In the electric field case, x denotes the position of a particle whose antiparticle is located at $-x$. $x=0$ thus corresponds to zero separation.

The limits of integration run from 0 to x_f simply because that gives the entire contribution to the imaginary part of the action. The imaginary part of the action determines the decay rate:

$$\Gamma(t_f)=e^{-2\,\mathrm{Im}[S(t_f)]}.$$

Notice that the decay rate depends on the time t_f, which is the end point of the complex time path.

Before turning to some concrete examples of our formalism, it is appropriate at this point to discuss the precise meaning of a time-dependent tunneling rate. One might think that the concept is necessarily vague, since tunneling is an intrinsically wavelike phenomenon, implying that a time cannot be assigned to the tunneling process with an accuracy greater than the inverse frequency of the wave. We would now like to show that the above statement is misleading and that, when properly interpreted, the time-dependent tunneling rate has a precise, unambiguous, physical meaning. For the purposes of this discussion, it is sufficient to consider the simple case of a particle impinging on a time-dependent potential in one-dimensional quantum mechanics. There are two relevant cases to consider: The incoming state can be a particlelike localized wave packet or can be a plane wave. In either case one proceeds to solve the Schrödinger equation to find a time-dependent transmitted wave.

In the first case, since the incoming state corresponds in the classical limit to a particle with a well-defined position, one is tempted to inquire as to the exact time at which the tunneling occurred. However, there does not seem to be any useful definition of such a time and certainly no way of measuring one to arbitrary precision since the state is spread over a distance equal to the size of the wave packet. Thus, in this case, since one cannot associate a precise time with the process of tunneling, a time-dependent tunneling rate is not expected to be physically meaningful unless suitably averaged over some duration.

Contrast this now with the case of an incoming plane wave. From the transmitted wave one can compute a time-dependent probability current which can, in principle, be measured to arbitrary accuracy. The current, measured in a region far from the potential, represents the average number of transmitted particles passing through one's detector at a time t_f of the measurement. Although one is not able to specify precisely when a detected particle actually tunneled, the time-dependent tunneling rate is nonetheless meaningful—and measurable—even over short time scales. The tunneling rates which we compute in this paper correspond to the latter situation, and our time-dependent tunneling rates are then to be interpreted as the increase per unit time of the average number of produced particles at time t_f. It should in particular be clear from the above discussion that the time t_f is not at all related to any concept of a ''tunneling time,'' but simply characterizes the time dependence of the probability current of the transmitted wave. In this way we can see that there is no ambiguity in the interpretation of the quantities which we calculate.

III. PARTICLE CREATION IN A TIME-DEPENDENT ELECTRIC FIELD

As a concrete illustration of the method discussed in the previous section, we will compute the probability for production of charged particles in a time-dependent electric field. As noted previously, a rigorous approach to this problem involves starting from the second quantized field theory. For instance, [2] analyzed the pair production in an alternating electric field through a Schwinger proper time approach. An alternative calculation could proceed through a Bogoliubov transformation relating the ''in'' and ''out'' asymptotic states; this approach is outlined in Appendix A. However, as we also discuss in Appendix A, the same results can be obtained from the more intuitive first quantized approach which we consider in this section.

We begin by considering the simple case of a static field. That a (spatially and temporally) constant electric field should be capable of creating charged pairs is evident on energetic grounds, since the requisite energy $2mc^2$ needed to create the pair is supplied by the electric field, if we separate the particles by a distance $2mc^2/qE$. The creation rate is proportional to the probability of the particles, initially located at the same point, to tunnel to this separation. Our starting point is the spin-0 point particle action in the presence of a constant electric field:

$$S=-\int dt[m\sqrt{1-\dot{x}^2}-qEx]. \tag{4}$$

For simplicity, we have taken space to be one dimensional. For ease of comparison with the discussion in Appendix A, it is actually simpler to integrate by parts and use the action

$$S=-\int dt[m\sqrt{1-\dot{x}^2}+qEt\dot{x}]. \tag{5}$$

(This amounts to a different gauge choice for the gauge potential A_μ.) The equation of motion is

$$m\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right]=qE.$$

The solution is

$$x(t)=x(t_0)+\frac{m}{qE}\left\{\sqrt{1+\left[\frac{qE}{m}(t-t_0)+\frac{\dot{x}(t_0)}{\sqrt{1-\dot{x}^2(t_0)}}\right]^2}-\frac{1}{\sqrt{1-\dot{x}^2(t_0)}}\right\}.$$

Since we are interested in a tunneling process, we continue these trajectories to imaginary time $t\mapsto -i\tau$. Let us consider the particular trajectory for which $\tau_0=-m/qE$, $x(\tau_0)=0$, and $\dot{x}(\tau_0)=i\infty$. This yields the circle

$$x^2(\tau)+\tau^2=\left(\frac{m}{qE}\right)^2,$$

a quarter of which is shown in Fig. 1.

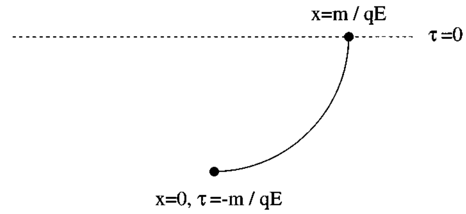

FIG. 1. A trajectory for half of the tunneling process.

The particle comes to rest at $\tau=0$ after having traveled a distance $\Delta x=m/qE$. This portion of the path (one-quarter of the circle) describes half of the tunneling process. The action for this portion is found by substituting the trajectory $x(\tau)$ into

$$S=i\int_{-m/qE}^{0}d\tau\left[m\sqrt{1+\left(\frac{dx}{d\tau}\right)^2}+qE\tau\dot{x}(\tau)\right],$$

where an overdot now means $d/d\tau$. This yields

$$S=\frac{i\pi m^2}{4qE}.$$

The full tunneling process is described by a semicircle (see Fig. 2).

Whereas the part of the trajectory between $x=0$ and $x=m/qE$ depicts the creation of a particle with charge q, the part between $x=0$ and $x=-m/qE$ depicts the creation of the corresponding antiparticle of charge $-q$. We see that the antiparticle trajectory is obtained from the particle trajectory by $q\mapsto -q$, $x(\tau)\mapsto -x(\tau)$. It is easy to see that this path yields a solution for the equation of motion and has the same action as the particle trajectory. The total tunneling action is thus

$$S_{\text{total}}=\frac{i\pi m^2}{2qE}.$$

The rate of pair creation is found by squaring the tunneling amplitude:

$$\Gamma=|e^{-\text{Im}[S_{\text{total}}]}|^2=e^{-\pi m^2/qE}.$$

The exact result obtained by Schwinger [7] is

$$\Gamma=\frac{(qE)^2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-n\pi m^2/qE}.$$

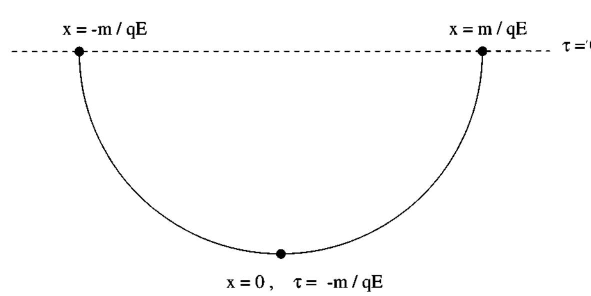

FIG. 2. A trajectory for the full tunneling process.

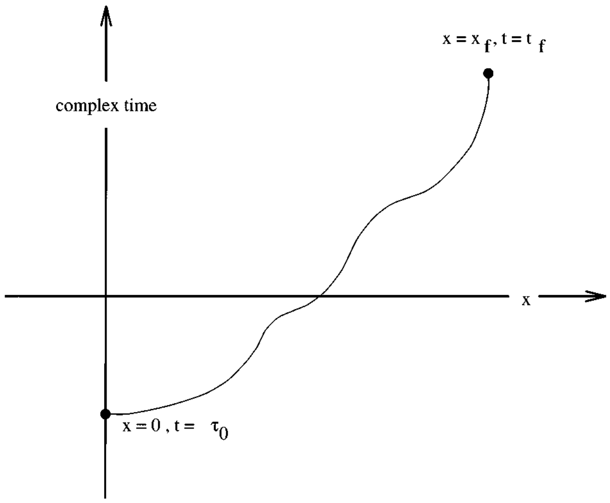

FIG. 3. A trajectory for half of the generic tunneling process.

The WKB result gives a good approximation when $\pi m^2/qE \gg 1$. Finally, we note that after being created, the particles move along hyperbolas, as is seen by continuing the trajectories back to real time.

We now generalize the analysis to treat a time-dependent electric field. As discussed in the previous section, we expect that this can be accomplished by finding a complex time path connecting the initial and final positions.

In particular, we will look for a path as shown in Fig. 3, which describes the creation of a particle at time $t=t_f$. The vertical axis no longer represents purely real or imaginary time, but rather some more general complex time direction (such that t_f is real). At $t=t_f$ the particle is at rest, $\dot{x}(t_f)=0$. At $t=\tau_0$ the velocity should be singular, $\dot{x}(\tau_0)\to\infty$, so that the trajectory for the antiparticle can be smoothly joined onto the particle trajectory. Both of these conditions were, of course, satisfied in the static field case. The remarkable aspect of this problem, as we shall see, is that these conditions allow us to determine the tunneling action without having to find the complex time path explicitly.

The action, after integrating by parts, has the form

$$S=-\int dt[m\sqrt{1-\dot{x}^2}-qA(t)\dot{x}], \tag{6}$$

where we have defined

$$A(t)=-\int_{t_f}^{t} dt' E(t'). \tag{7}$$

The equations of motion then yield

$$\dot{x}(t)=-\frac{qA(t)/m}{\sqrt{1+q^2A^2(t)/m^2}}.$$

We have set the conserved momentum $p_x=\partial L/\partial\dot{x}$ to zero, so that $\dot{x}$ respects the condition $\dot{x}(t_f)=0$. Integrating,

$$x(t)=-\frac{q}{m}\int_{t_f}^{t} dt' \frac{A(t')}{\sqrt{1+q^2A^2(t')/m^2}}+x(t_f).$$

Now t_0 is determined by requiring $\dot{x}(t_0)\to\infty$. For a nonsingular $E(t)$, this implies $A(t_0)=\pm im/q$. We choose to set $A(t_0)=-im/q$.

We will now write the action in terms of $A(t)$ and see that it takes a simple form. Substituting the expression for $\dot{x}$ into Eq. (6) gives

$$S=-m\int_{t_0}^{t_f} dt\sqrt{1+q^2A^2(t)/m^2}.$$

Now we change the integration variable from t to $v=-qA/m$ using

$$dt=\frac{m\,dv}{qE(t(v))},$$

where $t(v)$ is found by inverting Eq. (7). Then,

$$S=-\frac{m^2}{q}\int_i^0 dv\,\frac{\sqrt{1+v^2}}{E(t(v),t_f)}=\frac{im^2}{q}\int_0^{\pi/2} d\theta\,\frac{\cos^2\theta}{E(t(i\sin\theta),t_f)}.$$

This is the action corresponding to the creation of the charge-q particle. As before, the action for the antiparticle is obtained by replacing $q\mapsto -q$, $x(t)\mapsto -x(t)$, which yields the same result as for the particle. Therefore, the pair creation rate is given by

$$\Gamma(t_f)=\exp\{-4\,\mathrm{Im}[S(t_f)]\}.$$

To recapitulate, the essential trick that was used was to map the potentially complicated complex time contour into the complex v plane, where it always takes the simple form of a line from 0 to i. This simplifies the problem considerably, since it is no longer necessary to try to find the complex time contour. We only need to know its image in the v plane, and we do.

A. Example of time-dependent pair creation

Having obtained the general result[2]

$$S=\frac{im^2c^3}{q}\int_0^{\pi/2} d\theta\,\frac{\cos^2\theta}{E(t(ic\sin\theta),t_f)} \tag{8}$$

for the action along the complex time path, we shall now evaluate the pair creation rate

$$\Gamma(t_f)=\exp\left\{-\frac{4}{\hbar}\,\mathrm{Im}[S(t_f)]\right\}$$

in an example case of a time-dependent electric field. But first, we check the formula with a constant electric field $E=E_0$. Now the denominator in Eq. (8) is a constant, and what remains is an elementary integral; so we easily obtain the standard WKB result for the pair creation rate

$$\Gamma=\exp\left\{-\frac{\pi m^2c^3}{\hbar qE_0}\right\}. \tag{9}$$

[2]In this subsection, we restore $\hbar$ and c.

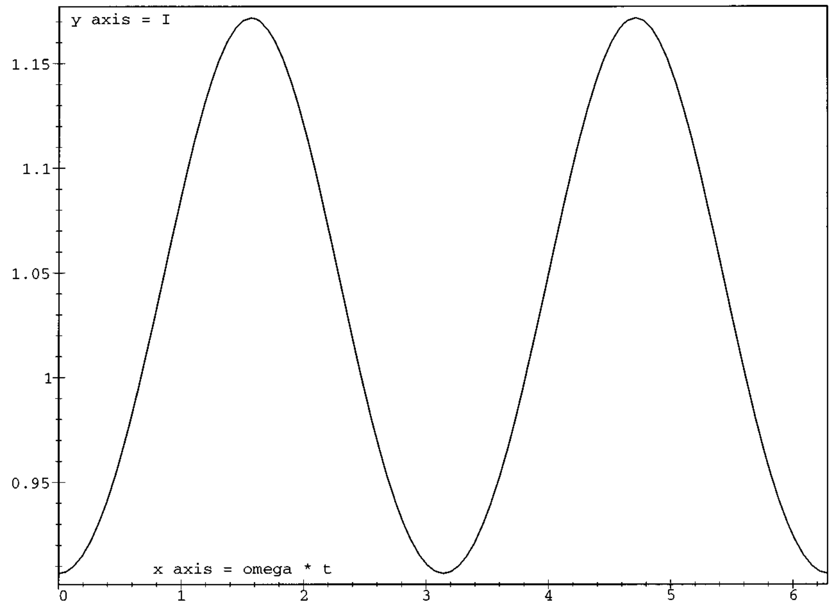

FIG. 4. An example plot of the time modulation function $I(t_f)$ in the action.

This pair creation rate is generally truly small. Even for the strongest electric fields obtained in a laboratory, $E_0 \sim 10^{11}$ N/C [8], the exponent is still enormous: Using $q=e\sim 10^{-19}$ C, $mc^2 \sim m_e c^2 \sim 10^6$ eV$\sim 10^{-13}$ N m, $c\sim 10^8$ m/s, $\hbar \sim 10^{-34}$ J s, we get a vanishingly small production rate

$$\Gamma \sim \exp\{-10^8\}.$$

Now let us consider an example of a time-dependent electric field, an oscillating strong electric field

$$E(t_f)=E_0 \cos\omega t_f .$$

Note that no magnetic field is required to satisfy the Maxwell's equations, only an alternating uniform current density. This case is different from pair creation in the background of an electromagnetic plane wave, which was considered originally by Schwinger [7] in the sense that the intensity of the plane wave is time independent.

We first find v and $E(t(v),t_f)$:

$$v(t,t_f)=\frac{q}{m}\int_{t_f}^{t} dt' E(t')=\frac{qE_0}{m\omega}(\sin\omega t-\sin\omega t_f)$$

and

$$E(t(ic\ \sin\theta),t_f)=E_0\sqrt{1-\left(\frac{im\omega c\ \sin\theta}{qE_0}+\sin\omega t_f\right)^2}.$$

After changing the integration variable to $x=\sin\ \theta$, the action can be written as

$$S=\frac{im^2c^3}{qE_0}\int_0^1 dx\ \frac{\sqrt{1-x^2}}{\sqrt{1-[i(m\omega c/qE_0)x+\sin\omega t_f]^2}}.$$

Finally, isolating the imaginary part of the action, we find the instantaneous pair creation rate to be of the form

$$\Gamma=\exp\left\{-\frac{\pi m^2c^3}{\hbar qE_0}I(t_f)\right\},$$

where $I(t_f)$ is a modulation factor which characterizes the time dependence [compare with Eq. (9)]. We find that

$$I(t_f)=\frac{2\sqrt{2}}{\pi}\int_0^1 dx\ \frac{\sqrt{1-x^2}\sqrt{r(x,t_f)+u(x,t_f)}}{r(x,t_f)},$$

$$u(x,t_f)=1+a^2x^2-\sin^2\omega t_f ,$$

$$r(x,t_f)=\sqrt{u^2(x,t_f)+4a^2x^2\sin^2\omega t_f},$$

$$a\equiv\frac{m\omega c}{qE_0}.$$

This integral can be evaluated numerically, and we show in Fig. 4 a plot of $I(t_f)$ for $a=1(\omega=qE_0/mc)$.

We can see that $I(t_f)$ is a periodic function of time. It reaches a minimum value at times $\omega t_f=0+n\pi$ and a maximum at $\omega t_f=\pi/2+n\pi$. The dependence of the minimum and maximum values of I on the frequency ω turns out to be interesting:

At zero frequency, the minimum value of the time modulation function is equal to 1. The maximum value becomes infinite: At small frequencies the decay rate is completely dominated by the minimum value of the action. Thus at zero frequency the decay rate reduces to that of the static case, as it should. As the oscillation frequency increases, the maximum and minimum values decrease monotonically and both seem to approach the asymptotic value zero. This means that the (average) pair creation rate *increases* as the oscillation frequency of the field increases. Finally, the action becomes so small (and the rate so high) that the WKB approximation is no longer valid.

Let us try to check the asymptotic behavior of the maximum and minimum (see Fig. 5). For the minimum, this can be done rigorously. Setting $\sin\ \omega t_f=0$ simplifies the integral and we can identify it in terms of complete elliptic integrals. We find

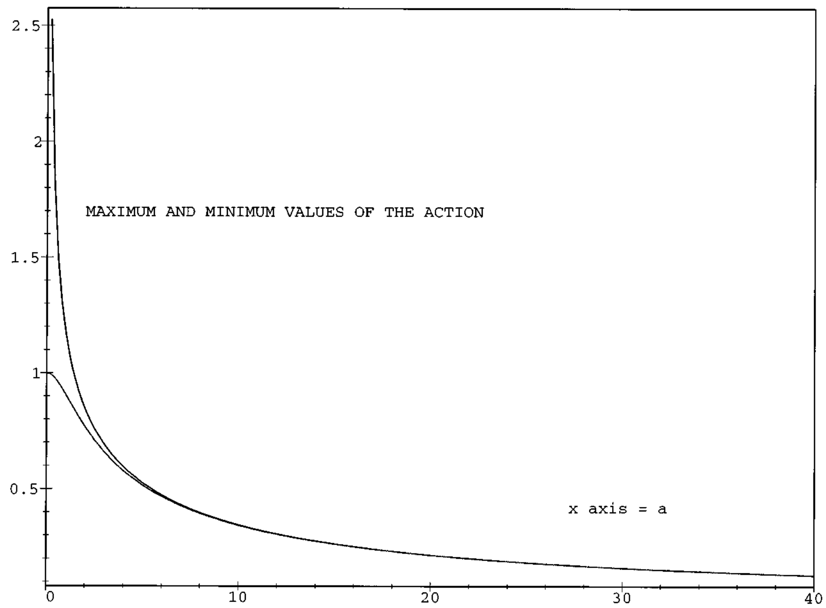

FIG. 5. A plot of the minimum and maximum values of the action as a function of frequency.

$$I_{\min}=\frac{4}{\pi}\frac{1}{a}\sqrt{\frac{1}{a^2}+1}\left\{K\left(\frac{a}{\sqrt{a^2+1}}\right)-E\left(\frac{a}{\sqrt{a^2+1}}\right)\right\}.$$

This result was also found in [3]. As $\omega\to\infty$ $(a\to\infty)$, we get

$$I_{\min}\sim\frac{\ln a}{a}\to 0$$

for the leading asymptotic behavior. For the maximum value, the analysis is a bit more complicated, but we find the same leading asymptotic behavior

$$I_{\max}\sim\frac{\ln a}{a}\to 0.$$

IV. DECAY OF A FALSE VACUUM

We now turn to another problem involving tunneling with nontrivial time dependence, whose treatment requires methods slightly different from those we have discussed to this point. Let us consider a scalar field

$$S=\int d^4x\left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right] \tag{10}$$

and take the potential to be of the form shown in Fig. 6.

We take the initial state to be one in which the field is concentrated in the well centered at $\phi=\phi_f$. The field will be taken to be constant in space, but can have a nontrivial time dependence. In particular, we have in mind semiclassical-looking states in which the field oscillates coherently in the well. Such a configuration, though stable classically provided the amplitude of oscillation is below the barrier, is expected to be unstable quantum mechanically. The presence of the well at $\phi=\phi_t$ signals a decay process whereby the field can tunnel through the barrier. This process is described by bubble nucleation, meaning that regions of field concentrated at ϕ_t spontaneously form within the initial configuration and rapidly expand. We would like to know the rate at which bubble nucleation occurs and describe the resulting bubble trajectory. It is also important to determine the state of the field inside the bubble—we might imagine that the oscillations about the false vacuum outside the bubble feed into the interior of the bubble, causing the field there to oscillate about the true vacuum. By solving the field equations, we will see that such oscillations are actually confined to a region near the bubble wall, so that the interior field in the bulk of the bubble is frozen at the true vacuum.

We begin by considering the simplest case, where the field is initially located at the bottom of the leftmost well, $\phi(t)=\phi_f$. This, of course, is the case considered by Coleman [9]. Our strategy will be to look for an expanding bubble solution which can be shrunk to zero size when evolved back along a complex time contour. This problem is most efficiently solved by utilizing the SO(3,1) symmetry of the theory. However, we will later be considering initial states which oscillate coherently and break the SO(3,1) symmetry to SO(3). Therefore we will discuss the solution in a language which explicitly uses only the latter symmetry.

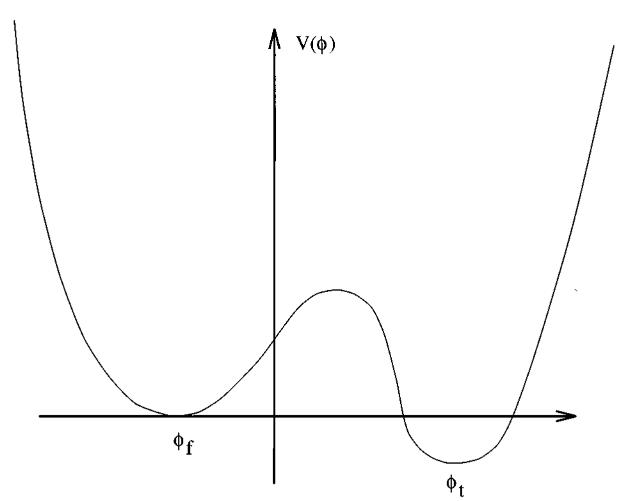

FIG. 6. Potential.

For simplicity, we will work in the thin wall approximation, which is valid provided the difference in energies between the true and false vacua, $\rho \equiv V(\phi_f) - V(\phi_t)$, is sufficiently small. The bubble solution then has the form of a spherical region of true vacuum separated by a thin wall from the outside region of false vacuum:

$$\phi(r,t) \approx \begin{cases} \phi_t & \text{for } r < R(t), \\ \phi_f & \text{for } r > R(t). \end{cases} \tag{11}$$

The trajectory of the bubble wall, $R(t)$, can be determined from energy conservation. Consider the energy in the region $r \leqslant R$. There are two contributions to the energy. The interior of the bubble contributes $E(\text{inside}) = \frac{4}{3}\pi V(\phi_t) R^3$. There is also an energy proportional to the area of the bubble wall associated with the field gradient in passing from true vacuum to false vacuum: $E(\text{wall}) = 4\pi\sigma_0 R^2/\sqrt{1-\dot{R}^2}$. Here σ_0 is the energy density of the wall:

$$\sigma_0 = \int_{\text{wall}} dr \left[\tfrac{1}{2}(\phi')^2 + V(\phi)\right].$$

In the next section we will compute its value from the field equations. The energy $E(\text{inside}) + E(\text{wall})$ must be equal to the energy present in the region before the nucleation of the bubble: $E_{\text{total}} = \frac{4}{3}\pi V(\phi_f) R^3$. So

$$\frac{4\pi\sigma_0 R^2}{\sqrt{1-\dot{R}^2}} - \tfrac{4}{3}\pi\rho_0 R^3 = 0, \tag{12}$$

with

$$\rho_0 = V(\phi_f) - V(\phi_t).$$

The trajectory is then

$$R(t) = \sqrt{R_0^2 + t^2},$$

with

$$R_0 = \frac{3\sigma_0}{\rho_0}. \tag{13}$$

For $t > 0$ this describes an expanding bubble solution. To consider tunneling, we evolve the solution back to the turning point at $t = 0$ and then try to shrink the bubble to zero size along a complex time contour. In the present case this step is trivial—the contour displayed in Fig. 7 does the job.

It remains to determine the amplitude for the tunneling process, and for this we require the classical action. The bubble has a Lagrangian

$$L_{\text{bubble}} = -4\pi\sigma_0 R^2 \sqrt{1-\dot{R}^2} - \tfrac{4}{3}\pi V(\phi_t) R^3.$$

Since we wish to compute the relative probability of bubble nucleation versus remaining in the false vacuum, what we actually want is the difference in action between the bubble solution and the false vacuum state. This is given by

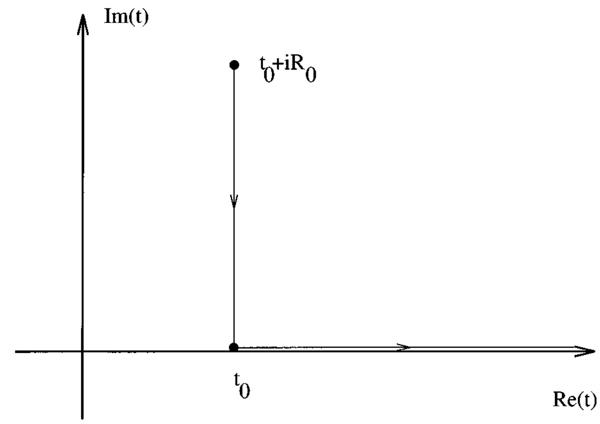

FIG. 7. Time path for bubble nucleation.

$$S = \int dt \left[L_{\text{bubble}} + \tfrac{4}{3}\pi V(\phi_f) R^3\right]$$

$$= -\int dt \left[4\pi\sigma_0 R^2 \sqrt{1-\dot{R}^2} - \tfrac{4}{3}\pi\rho_0 R^3\right].$$

The action can be put in a useful form by inserting the equation of motion (12) and changing variables to R:

$$S = \int dR\, R^2 \sqrt{\left(\frac{4\pi}{3}\rho_0\right)^2 R^2 - (4\pi\sigma_0)^2}$$

$$= \int dR\, \frac{4\pi\rho_0 R^3}{3} \sqrt{1 - \left(\frac{R_0}{R}\right)^2}.$$

The action has an imaginary part coming from the part of the trajectory $0 < R < R_0$, when the bubble is tunneling:

$$\text{Im } S = \frac{4\pi\rho_0}{3} \int_0^{R_0} dR\, R^3 \sqrt{\left(\frac{R_0}{R}\right)^2 - 1} = \frac{27\pi^2\sigma_0^4}{4\rho_0^3}.$$

The nucleation rate is then

$$\Gamma \approx e^{-2\,\text{Im}[S]} = \exp\left\{-\frac{\pi^2}{6}\rho_0 R_0^4\right\},$$

which is Coleman's result.

Now let us generalize to the case where the field is initially oscillating around the false vacuum: $\phi = \phi_f(t)$. In Sec. IV A we will study the bubble solutions with this initial condition, and we summarize the results here. For small oscillations $\phi_f(t) = \phi_f + \alpha(t)$, the bubble looks like

$$\phi_{\text{bubble}}(r,t) \approx \begin{cases} \phi_t & \text{for } r < R - \Delta, \\ \phi_t + \dfrac{R}{r}\exp[(r-R)/\Delta]\alpha(t) & \text{for } R - \Delta < r < R, \\ \phi_f(t) & \text{for } r > R, \end{cases} \tag{14}$$

where Δ is small compared to the typical size of the bubble R. In other words, the field oscillations only penetrate a relatively small distance into the bubble; the bulk of the bubble's interior simply sits at the true vacuum as before. As we have already mentioned, although one might have expected the bubble to leave a state oscillating about the true vacuum, we see that this is not the case.

Now we invoke the same energy considerations as before. Since the field oscillations inside the bubble are localized near the wall, the energy inside the bubble is essentially given by the true vacuum configuration ϕ_t. Thus

$$E_{\rm bubble}({\rm inside})=\tfrac{4}{3}\pi V(\phi_t)R^3.$$

The bubble wall has the energy

$$E_{\rm bubble}({\rm wall})=\frac{4\pi\sigma_E^{\rm bubble}R^2}{\sqrt{1-\dot{R}^2}},$$

where

$$\sigma_E=\int_{\rm wall}dr\{\tfrac{1}{2}(\dot{\phi}_{\rm bubble})^2+\tfrac{1}{2}(\phi'_{\rm bubble})^2+V(\phi_{\rm bubble})\}.$$

σ_E is time independent. The initial energy is also divided into two contributions

$$E_{\rm initial}({\rm inside})=\tfrac{4}{3}\pi R^3[\tfrac{1}{2}(\dot{\phi}_f(t))^2+V(\phi_f(t))]\equiv\tfrac{4}{3}\pi\rho_E^{\rm FV}R^3$$

and

$$E_{\rm initial}({\rm wall})=\frac{4\pi\sigma_E^{\rm FV}R^2}{\sqrt{1-\dot{R}^2}},$$

where

$$\sigma_E^{\rm FV}=\int_{\rm wall}dr[\tfrac{1}{2}(\dot{\phi}_f(t))^2+V(\phi_f(t))].$$

Conservation of energy then requires

$$\frac{4\pi\sigma_E R^2}{\sqrt{1-\dot{R}^2}}-\tfrac{4}{3}\pi\rho_E R^3=0,$$

with

$$\sigma_E=\sigma_E^{\rm bubble}-\sigma_E^{\rm FV}, \qquad (15)$$

$$\rho_E=\rho_E^{\rm FV}-V(\phi_t). \qquad (16)$$

We emphasize that ρ_E and σ_E are constants. This fact means that the complex time contour relevant for tunneling runs in the purely imaginary direction, just as in the static case. However, we expect that this behavior is an accident of the analysis in the limit of small oscillations; more generally, ρ_E and σ_E will acquire time dependence and the time contour will be a more complicated curve in the complex time plane. Now we define

$$R_0=\frac{3\sigma_E}{\rho_E},$$

so that the trajectory is

$$R(t)=\sqrt{R_0^2+(t-t_0)^2}. \qquad (17)$$

Now that we have obtained the bubble trajectory, we turn to the evaluation of the decay rate. As before, the decay rate is found by integrating the action over an imaginary time contour running from some initial time t_0 to t_0+iR_0, where the bubble shrinks to zero size. The action which is to be integrated is the difference between the bubble action and the false vacuum action. The bubble action is

$$S_{\rm bubble}=-\int dt\left[4\pi\sigma_L^{\rm bubble}(t)R^2(t)\sqrt{1-\dot{R}^2}+\frac{4\pi}{3}V(\phi_t)R^3\right],$$

where

$$\sigma_L^{\rm bubble}=-\int_{\rm wall}dr\{\tfrac{1}{2}(\dot{\phi}_{\rm bubble})^2-\tfrac{1}{2}(\phi'_{\rm bubble})^2-V(\phi_{\rm bubble})\}$$
$$=\sigma_E^{\rm bubble}-\int_{\rm wall}dr\ \dot{\phi}^2_{\rm bubble}.$$

The false vacuum action is

$$S_{\rm FV}=-\int dt\left[4\pi\sigma_L^{\rm FV}R^2\sqrt{1-\dot{R}^2}+\frac{4\pi}{3}\rho_L^{\rm FV}R^3\right],$$

where

$$\sigma_L^{\rm FV}=\sigma_E^{\rm FV}-\int_{\rm wall}dr\ \dot{\phi}_f^2,$$

$$\rho_L^{\rm FV}=\rho_E^{\rm FV}-\dot{\phi}_f^2.$$

The action to be integrated is thus

$$S=-\int dt\left[4\pi\sigma_L(t)R^2(t)\sqrt{1-\dot{R}^2}-\frac{4\pi}{3}\rho_L(t)R^3\right], \qquad (18)$$

where

$$\sigma_L(t)=\sigma_E-\int_{\rm wall}dr[\dot{\phi}^2_{\rm bubble}-\dot{\phi}_f^2], \qquad (19)$$

$$\rho_L(t)=\rho_E-\dot{\phi}_f^2. \qquad (20)$$

A crucial point is that although σ_E and ρ_E are constants, σ_L and ρ_L are time dependent—their time dependence determines the time dependence of the decay rate.

The calculation has thus been reduced down to performing the integrals for σ_L, ρ_L, and S. These are straightforward to do; they can be done analytically for small oscillations about the false vacuum, as shown in Sec. IV B, although in the general case numerical integration is required. The result is an expression for the time-dependent decay rate:

$$\Gamma(t_0)\approx\exp\{-2\,{\rm Im}[S(t_0)]\}.$$

A. Structure of the oscillating bubble

To analyze the case when the field is initially oscillating around the false vacuum, $\phi = \phi_f(t)$, we first need to determine how the structure of the bubble is altered. We will now present an example calculation of the bubble solution $\phi_{\rm bubble}(r,t)$, which interpolates between the true vacuum ϕ_t and the oscillating initial state $\phi_f(t)$. We shall refer to this as the oscillating bubble.

The field equation that we need to study is

$$\ddot{\phi} - \frac{1}{r^2}(r^2\phi')' = -\frac{dV}{d\phi}.$$

As an example, we consider the potential discussed by Coleman,

$$V(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2 + \frac{\epsilon}{2a}(\phi - a),$$

where $\epsilon > 0$. The true vacuum is located at

$$\phi_- \approx -a - \frac{\epsilon}{8\lambda a^3} + O(\epsilon^2),$$

where the potential has the value $V(\phi_-) \approx -\epsilon + O(\epsilon^2)$. The false vacuum is located at

$$\phi_+ \approx a - \frac{\epsilon}{8\lambda a^3} + O(\epsilon^2),$$

where $V(\phi_+) \approx 0 + O(\epsilon^2)$.

In the standard scenario of decay from false vacuum to true vacuum, the structure of the bubble is obtained from the static solution $\phi_0(r)$ of the field equation:

$$\phi_0'' + \frac{2}{r}\phi_0' = \frac{dV}{d\phi}(\phi_0). \tag{21}$$

ϕ_0 interpolates between the true vacuum ϕ_- inside the bubble and the false vacuum ϕ_+ outside the bubble, with the nontrivial r dependence concentrated in the bubble wall. Without going into the mathematics of the exact form of the solution, we recall that the qualitative behavior of the solution is as depicted in Fig. 8.

The radius of the bubble is $R = 3\sigma_0/\epsilon$ [see Eq. (13)], and the thickness L of the bubble wall is of the order $L \sim 1/\mu$, where

$$\mu \sim \sqrt{\frac{d^2V}{d\phi^2}(\pm a)} \sim a\sqrt{\lambda}.$$

An approximation to the static solution in the vicinity of the wall can be obtained by dropping the term $(2/r)\phi_0'$ in the static field equation (since $r \sim R \gg 0$) and dropping the small constant term $\epsilon/2a$. The solution of the approximate field equation is the ''kink''

$$\phi_0^{\rm approx}(r) = a\ \tanh\mu(r - R),$$

where $\mu = a\sqrt{\lambda}$. Its behavior is similar to that depicted in Fig. 8.

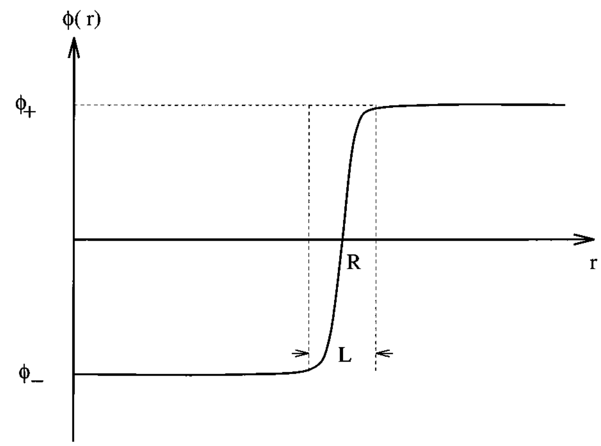

FIG. 8. Picture of the qualitative behavior of the static solution $\phi_0(r)$.

Now we try to find a time-dependent solution $\phi_{\rm bubble}(r,t)$, which reduces to the coherently oscillating field $\phi_f(t) = \phi_+ + \alpha_0 \sin\omega t$ about the false vacuum as $r \to \infty$. We assume that the amplitude α_0 of the oscillations is small. The frequency ω of the oscillations is given by

$$\omega^2 = \frac{d^2V}{d\phi^2}(\phi_+) = 4\lambda a^2 - \frac{3\epsilon}{2a^2} + O(\epsilon^2). \tag{22}$$

We make the following ansatz for $\phi_{\rm bubble}(r,t)$:

$$\phi_{\rm bubble}(r,t) = \phi_0(r) + \alpha(r)\sin\omega t.$$

Substituting this ansatz into the full field equation and using the fact that ϕ_0 is the static solution, we obtain a linearized differential equation for the profile function $\alpha(r)$:

$$\alpha''(r) + \frac{2}{r}\alpha'(r) + \left[\omega^2 - \frac{d^2V}{d\phi^2}(\phi_0)\right]\alpha(r) = 0. \tag{23}$$

Without knowing the exact form of the static solution ϕ_0, we can still proceed by using its known asymptotic properties. In the region outside of the bubble, $r > R$, ϕ_0 reduces to ϕ_+, and so $(d^2V/d\phi^2)(\phi_0) \to (d^2V/d\phi^2)(\phi_+)$. We see that since the frequency ω^2 is equal to $(d^2V/d\phi^2)(\phi_+)$, the profile function $\alpha(r)$ must be constant in this region. So $\phi_{\rm bubble}(r,t)$ correctly reduces to the oscillating initial configuration $\phi_f(t)$ in this region:

$$\phi_{\rm bubble}(r,t) \to \phi_+ + \alpha_0 \sin\omega t \quad \text{as} \quad r > R.$$

In the region inside the bubble, $r < R$, ϕ_0 reduces to the true vacuum value ϕ_-. Thus $(d^2V/d\phi^2)(\phi_0) \to (d^2V/d\phi^2)(\phi_-)$. Using

$$\frac{d^2V}{d\phi^2}(\phi_-) = 4\lambda a^2 + \frac{3\epsilon}{2a^2} = \omega^2 + \frac{3\epsilon}{a^2}$$

and denoting $k^2 \equiv 3\epsilon/a^2$, we see that the differential equation (23) reduces to a familiar equation

$$\alpha''(r) + \frac{2}{r}\alpha'(r) - k^2\alpha(r) = 0.$$

The solution for $\alpha(r)$ is,

$$\alpha(r)=A\,\frac{\sinh(kr)}{kr}.$$

We fix the constant A by matching $\alpha(r)$ with α_0 at $r=R$:

$$A=\alpha_0\,\frac{kR}{\sinh(kR)}.$$

The solution for $\alpha(r)$ tells us that the oscillations $\alpha(r)\sin\omega t$ decay to zero inside the bubble, in a region of thickness $\Delta=1/k$. Thus there are three scales that characterize the structure of the oscillating bubble: (1) the radius of the bubble $R\sim 3\sigma_0/\epsilon$; (2) the thickness of the bubble wall $L\sim 1/(a\sqrt{\lambda})$; (3) the thickness of the region inside the bubble where the oscillations decay $\Delta\sim a/(3\sqrt{\epsilon})$. The relative sizes of these scales are

$$\frac{\Delta}{R}\sim\frac{a\sqrt{\epsilon}}{\sigma_0}\rightarrow 0 \quad \text{as} \quad \epsilon\rightarrow 0,$$

$$\frac{L}{\Delta}\sim\frac{\sqrt{\epsilon}}{a^2\sqrt{\lambda}}\rightarrow 0 \quad \text{as} \quad \epsilon\rightarrow 0.$$

Thus

$$L\ll\Delta\ll R.$$

Also,

$$kR\sim\frac{\sigma_0}{\sqrt{\epsilon}}\gg 1.$$

Hence the solution for $\alpha(r)$ can be rewritten as

$$\alpha(r)\approx\alpha_0\,\frac{R}{r}\,e^{(r-R)/\Delta}$$

to a good approximation.

Finally, we would like to obtain at least an approximate solution for $\alpha(r)$ in the bubble wall region $r\sim R$. We do this by replacing $\phi_0(r)$ in Eq. (23) with the approximate solution $a\tanh\mu(r-R)$ and dropping the term $(2/r)\alpha'(r)$. Further, we approximate the frequency (22) by $\omega^2=4\lambda a^2$. Then Eq. (23) reduces to

$$\alpha''(x)+\frac{6}{\cosh^2 x}\,\alpha(x)=0,$$

where $x\equiv\mu(r-R)$. This has the solution

$$\alpha(r)=\frac{B}{2}\,\{3\tanh^2[\mu(r-R)]-1\}.$$

Since $\alpha(r)\rightarrow B$ in regions $r\gtrsim R$, where we know that $\alpha(r)=\alpha_0$, we set $B=\alpha_0$.

To summarize, we have found an approximate solution for the oscillating bubble: $\phi_{\text{bubble}}(r,t)=\phi_0(r)+\alpha(r)\sin\omega t$, where $\phi_0(r)$ is the solution in the static case, modified by small oscillations with a profile function $\alpha(r)$ given by

$$\alpha(r)=\begin{cases}\alpha_0 & r\gtrsim R+L/2\\ \frac{\alpha_0}{2}\,\{3\tanh^2[(r-R)/L]-1\}, & R-L/2\lesssim r\lesssim R+L/2,\\ \alpha_0\,\frac{R}{r}\,e^{(r-R)/\Delta}, & r\lesssim R-L/2.\end{cases} \tag{24}$$

B. Calculation of the bubble nucleation rate

Now that we have found the oscillating bubble solution $\phi_{\text{bubble}}(r,t)$, we can proceed to calculate the instantaneous bubble nucleation rate $\Gamma(t_0)\approx\{2\,\text{Im}[S(t_0)]\}$, where t_0 is the time immediately after the bubble has nucleated.

First, we need to find the quantities σ_E and ρ_E that appear in the equation of motion of the bubble. They are

$$\sigma_E\approx\sigma_0-0.42\omega^2\alpha_0^2L,$$
$$\rho_E\approx\epsilon+\tfrac{1}{2}\omega^2\alpha_0^2. \tag{25}$$

For some calculational details, see Appendix B. Since σ_E and ρ_E are constant, we could use energy conservation which gave us the trajectory of the bubble (17):

$$R=\sqrt{R_0^2+(t-t_0)^2},$$

where $R_0=3\sigma_E/\rho_E$.

For bubble nucleation, we need a time path which shrinks the bubble to zero radius. Thus the time path has an imaginary segment

$$t=t_0+i\sqrt{R_0^2-R^2}$$

for $R<R_0$. The above branch choice gives the correct exponential behavior for the nucleation rate.

To calculate the imaginary part of the action, it is first convenient to divide the action (18) into two parts: $S=S_0+S_1$, where

$$S_0=-\int_{t_0+iR_0}^{t_0}dt\left\{4\pi\sigma_E R^2\sqrt{1-\dot{R}^2}-\frac{4\pi}{3}\,\rho_E R^3\right\}$$

and

$$S_1=\int_{t_0+iR_0}^{t_0}dt\left\{4\pi R^2\sqrt{1-\dot{R}^2}\int_{R-L/2}^{R+L/2}dr(\dot{\phi}_{\text{bubble}}^2-\dot{\phi}_f^2)-\frac{4\pi}{3}\,R^3\dot{\phi}_f^2\right\}.$$

For the first term S_0, the calculation proceeds as in the ‘‘static’’ case. The result is

$$\text{Im}S_0=\frac{\pi^2}{12}\,\rho_E R_0^4.$$

For the second term, after a bit longer calculation we find

$$\mathrm{Im}S_1 \approx \alpha_0^2 \pi^2 R_0^2 \left\{ \frac{(2\omega R_0)^2}{32} + [0.84 I_1(2\omega R_0) + \tfrac{1}{4} I_2(2\omega R_0)] \cos(2\omega t_0) \right\}.$$

(We have used $L \approx 2/\omega$ and dropped a negligible subleading term.) The functions $I_n(2\omega R_0)$ are modified Bessel functions. The total instantaneous bubble nucleation rate is then

$$\Gamma(t_0) \approx \exp\left\{ -\frac{\pi^2}{6} \rho_E R_0^4 - \alpha_0^2 \pi^2 R_0^2 \left[\frac{(2\omega R_0)^2}{16} + [1.68 I_1(2\omega R_0) + \tfrac{1}{2} I_2(2\omega R_0)] \cos(2\omega t_0) \right] \right\}.$$

The decay rate is oscillatory, with the leading correction to the static decay rate coming from the oscillatory term in the exponent. Recall that $R_0\omega \sim$radius of the bubble/thickness of the bubble wall, and so $R_0\omega \gg 1$. Thus the leading order correction is of the order $\alpha_0^2 R_0^2 e^{2\omega R_0}/2\omega R_0$: This is much larger than what one might have anticipated. However, this is reminiscent of what happens when a particle tunnels through an oscillating barrier. Büttiker and Landauer showed [10] that the tunneling particle absorbs quanta from the oscillating barrier; the net effect is that tunneling becomes easier. In the leading order correction to the tunneling probability, the amplitude of the oscillations of the barrier is multiplied by an exponential term, and so the correction is much larger. As we can see in our case, small oscillations about the false vacuum also render the state more unstable.

V. CONCLUSION

In this paper we have presented a rather general approach to treating time-dependent tunneling problems and have illustrated the method with some concrete examples. Through the use of the WKB approximation, we found that we could reduce such problems to solving classical equations of motion along complex time contours. Even when such equations of motion are analytically intractable, we have shown that they are amenable to straightforward numerical analysis. The standard approach to tunneling problems in field theory, using instantons, is seen as a special case of the complex time method. In particular, the familiar procedure of evolving along the imaginary time direction is valid when there is no nontrivial time dependence in the problem, but more general problems require more general complex time contours.

The most straightforward and elegant application of our methods is to the case of pair production by a time-dependent electric field. As discussed previously, this problem has been analyzed by a number of workers over the years by a variety of methods; we derived a slightly more general result than had been previously been obtained, but our main interest in the problem was as a prototype for more complicated time-dependent systems. The qualitative features of the electric field problem carry over to these systems, the only difference being that several steps must be performed numerically rather than analytically.

As an example of an interesting process which can be tackled by the methods developed here, we mention again the quantum nucleation of defects in an expanding universe. The form of the expansion can have an important effect on the resulting distribution of the defects. Consider cosmic string nucleation, for instance. The strings nucleate with a size equal to the horizon, or inverse Hubble constant, which thus also sets the scale for the duration of the complex time evolution. If the expansion rate of the universe varies appreciably on this time scale, then the nucleation rate will depart from what a naive quasistatic analysis would indicate. Depending on the cosmological model, the resulting distribution of strings is potentially relevant.

To illustrate another source of time dependence—that arising from initial conditions, rather than external sources—we considered the problem of false vacuum decay in field theory, where the initial state consists of coherent field oscillations about the false vacuum. Again, instanton techniques are not directly applicable to this system. Although the analysis was rather involved, we were able to obtain an expression for the time-dependent decay rate in the case of small oscillations. Of course, to see large time-dependent effects one must allow for large oscillations, but this would require a rather intricate computation which we have not attempted. But again, we stress that the steps are, in principle, straightforward.

ACKNOWLEDGMENTS

The work of E.K. was supported in part by DOE Grant No. DE-FG03-92-ER40701. The work of P.K. was supported in part by DOE Grant No. DE-FG03-92-ER40701 and by the DuBridge Foundation.

APPENDIX A

In this appendix, based on Ref. [4], we discuss how the problem of pair production in a homogeneous time-dependent electric field is related to overbarrier scattering in nonrelativistic quantum mechanics; this connection is the basis for the first quantized approach presented in Sec. III.

We start from a relativistic field theory formulation, in which a charged scalar in an electric field has the equation of motion

$$\{(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2\}\phi = 0. \tag{A1}$$

In the canonical quantization approach, we seek ''natural'' mode solutions of Eq. (A1) with suitable asymptotic properties, which we then use in the oscillator expansion of the field operator to identify the asymptotic states. If we consider the background of a classical time-dependent homogeneous electric field $\vec{E} = E(t)\hat{e}_x$, it is convenient to take the gauge choice $A^\mu = (A^0, \vec{A}) = (0, A(t), 0, 0)$, where $A(t) = -\int dt\, E(t)$. Then we can use the separable ansatz

$$\phi = \frac{1}{(2\pi)^{3/2}} f_{\vec{p}}(t) e^{i\vec{p}\cdot\vec{x}}$$

in the field equation, to reduce it to a time-dependent generalized oscillator equation

$$\ddot{f}_{\vec{p}} + \omega_{\vec{p}}^2(t) f_{\vec{p}} = 0, \tag{A2}$$

where $\omega_{\vec{p}}^2(t)=m^2+[p_x-eA(t)]^2+p_y^2+p_z^2\equiv\mu^2+[p_x-eA(t)]^2$. In addition, the relativistic Klein-Gordon scalar product reduces to a simple form

$$(f_1,f_2)=i(f_1^*\dot{f}_2-\dot{f}_1^*f_2). \tag{A3}$$

Equation (A2) is identical to a nonrelativistic time-independent Schrödinger equation

$$\psi''+k^2(x,E)\psi=0,$$

via the identifications

$$t\leftrightarrow x, \quad f_{\vec{p}}\leftrightarrow\psi, \tag{A4}$$

$$\omega_{\vec{p}}^2(t)=\mu^2+[p_x-qA(t)]^2\leftrightarrow k^2(x,E)=2mE-2mV(x). \tag{A5}$$

Also, the scalar product (A3) can be seen to match with the inner product for wave functions ψ_1,ψ_2. If we make the more precise identifications

$$\mu^2\leftrightarrow 2mE, \tag{A6}$$

$$[p_x-qA(t)]^2\leftrightarrow -2mV(x), \tag{A7}$$

we notice that in quantum-mechanics language we must take $E>0$ and $V\leqslant 0$. This suggests that we are dealing with a quantum-mechanical overbarrier scattering problem.

To complete this connection, we first outline the pair production calculation in the field theory picture. If we switch the electric field on in the far past and off in the far future, the gauge potential is asymptotically constant: $A(t)\rightarrow A^{\text{in,out}}$ as $t\rightarrow\pm\infty$. Then the oscillator equation has two natural linearly independent solutions $f^{\text{in,out}}$ with the asymptotic properties $f_{\vec{p}}^{\text{in,out}}(t)\rightarrow\exp(-i\omega_{\vec{p}}^{\text{in,out}}t)$, $t\rightarrow\pm\infty$. These give two bases for the oscillator expansion of the field operator, corresponding to the ''in'' and ''out'' Fock spaces, as in a quantum field theory in curved space [11]. In particular, the definition of an initial vacuum state is different from a final vacuum state; this gives rise to particle production. To compute the particle production rate, one must find a Bogoliubov transformation which relates the two bases:

$$f_{\vec{p}}^{\text{in}}=\alpha_{\vec{p}}f_{\vec{p}}^{\text{out}}+\beta_{\vec{p}}f_{\vec{p}}^{*\text{out}}. \tag{A8}$$

Then the time-averaged pair production probability Γ is given by

$$\Gamma=\left|\frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}}\right|^2.$$

To see the connection to the quantum-mechanical scattering problem, we first rewrite Eq. (A8) as

$$f_{\vec{p}}^{\text{out}}+\frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}}f_{\vec{p}}^{*\text{out}}=\frac{1}{\alpha_{\vec{p}}}f_{\vec{p}}^{\text{in}}.$$

Then, asymptotically, as $t\rightarrow-\infty$, the left-hand side (LHS) reduces to

$$e^{-i\omega_{\vec{p}}^{\text{out}}t}+\frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}}e^{i\omega_{\vec{p}}^{\text{out}}}\leftrightarrow e^{-ikx}+Re^{ikx}$$

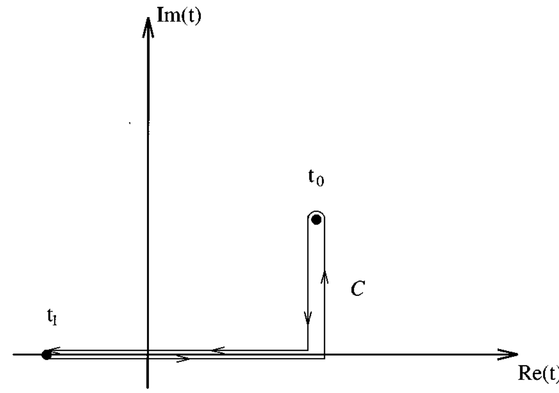

FIG. 9. The integration contour C.

and the RHS reduces to (as $t\rightarrow\infty$)

$$\frac{1}{\alpha_{\vec{p}}}e^{-i\omega_{\vec{p}}^{\text{out}}t}\leftrightarrow Te^{-ik'x}.$$

We see that the analysis of the oscillator equation (A2) in the field theory picture exactly corresponds to an overbarrier scattering problem in a quantum-mechanical picture, with the reflection and transmission coefficients being related to the Bogoliubov coefficients by

$$|R|=\left|\frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}}\right|,$$

$$|T|=\left|\frac{1}{\alpha_{\vec{p}}}\right|.$$

The time-averaged pair production probability is

$$\Gamma=|R|^2.$$

If the electric field $E(t)$ is not very rapidly varying, we can simplify the problem further with the use of WKB approximation. We use the WKB solution of the oscillator (or Schrödinger) equation

$$f\approx\exp\left\{-i\int\omega\, dt\right\}\leftrightarrow\exp\left\{-i\int k\, dx\right\}. \tag{A9}$$

The treatment of semiclassical reflection above a barrier requires an extension of the WKB method [12]. One method is to first find the complex turning points off the real axis where $\omega\sim k=0$. Then the leading contribution to the reflection coefficient can be found by computing the integral $\int\omega\, dt$ along a complex time contour C, traveling from some initial time t_1 on the real axis to the closest complex turning point t_0 and back (see Fig. 9).

This procedure yields the reflection coefficient

$$|R|^2=\exp\left\{-2\,\text{Im}\int_C\omega\, dt\right\}. \tag{A10}$$

Alternatively, we can derive the exponent of the WKB wave function (A9) by starting from the action

$$S=-\int dt\{m\sqrt{1-\dot{x}^2}-qA(t)\dot{x}\} \tag{A11}$$

for the relativistic charged particle in the background electric field (using the same gauge choice as before). Using the equation of motion

$$v\equiv\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}=-\frac{qA}{m}+p_x,$$

and substituting $\dot{x}$, gives

$$S=-\int dt\sqrt{1+v^2}=-\int dt\sqrt{(p_x-qA)^2+m^2}=-\int dt\ \omega.$$

The WKB wave function (A9) is e^{iS} as expected. Therefore we can use the one-particle action to calculate the reflection coefficient (A10); this is the basis for the first quantized analysis of the pair production rate presented in Sec. III.

APPENDIX B

We now calculate the quantities σ_E and ρ_E that appear in the equation of motion of the bubble. The energy density ρ_E is

$$\begin{aligned}\rho_E&=\tfrac{1}{2}[\dot{\phi}_f(t)]^2+V(\phi_f(t))-V(\phi_f)\\&=\tfrac{1}{2}\omega^2\alpha_0^2\cos^2\omega t+V(\phi_+)\\&\quad+\frac{1}{2}\frac{d^2V}{d\phi^2}(\phi_+)\alpha_0^2\sin^2\omega t+\epsilon\\&=\tfrac{1}{2}\omega^2\alpha_0^2+\epsilon.\end{aligned}$$

For the surface tension, we first need

$$\sigma_E^{\mathrm{bubble}}=\int_{\mathrm{wall}}dr\{\tfrac{1}{2}(\dot{\phi}_{\mathrm{bubble}})^2+\tfrac{1}{2}\phi'_{\mathrm{bubble}}+V(\phi_{\mathrm{bubble}})\}.$$

Substituting $\phi_{\mathrm{bubble}}(r,t)=\phi_0(r)+\delta(r,t)=\phi_0(r)+\alpha(r)\sin\omega t$ and expanding to second order in δ, we find that the leading order term gives the surface tension σ_0 in the traditional false vacuum decay,

$$\sigma_0=\int_{\mathrm{wall}}dr[\tfrac{1}{2}(\phi_0')^2+V(\phi_0)].$$

The contribution from the linear order in δ vanishes after integration by parts and using the equation of motion for ϕ_0. The contribution from the second order in δ is

$$\sigma_2=\frac{1}{2}\int_{\mathrm{wall}}dr\left\{\left[\alpha^2\left(\frac{d^2V}{d\phi^2}(\phi_0)-\omega^2\right)+(\alpha')^2\right]\sin^2\omega t+\omega^2\alpha^2\right\}.$$

Using the differential equation (23) for α (dropping the $2\alpha'/r$ term) and integrating by parts, we find

$$\sigma_2=\frac{\omega^2}{2}\int_{R-L/2}^{R+L/2}dr\ \alpha^2(r).$$

Substituting

$$\alpha(r)=\frac{\alpha_0}{2}\{3\tanh^2[(r-R)/L]-1\}$$

and combining the contributions,

$$\sigma_E^{\mathrm{bubble}}\approx\sigma_0+0.16\tfrac{1}{2}\omega^2\alpha_0^2L.$$

There is also the contribution from the initial state,

$$\sigma_E^{\mathrm{FV}}=\int_{\mathrm{wall}}dr[\tfrac{1}{2}\dot{\phi}_f^2+V(\phi_f)]=\tfrac{1}{2}\omega^2\alpha_0^2L.$$

Then, finally, the surface tension is

$$\sigma_E=\sigma_E^{\mathrm{bubble}}-\sigma_E^{\mathrm{FV}}\approx\sigma_0-0.42\omega^2\alpha_0^2L.$$

[1] For recent work on this topic, see, e.g., K. Ergenzinger, Phys. Rev. A (to be published).
[2] E. Brezin and C. Itzykson, Phys. Rev. D **2**, 1191 (1970).
[3] V. S. Popov, Sov. Phys. JETP **34**, 709 (1972); M. S. Marinov and V. S. Popov, Fortschr. Phys. **25**, 373 (1977).
[4] J. Audretsch, J. Phys. A **12**, 1189 (1979); Gen. Relativ. Gravit. **10**, 725 (1979).
[5] R. Basu, A. H. Guth, and A. Vilenkin, Phys. Rev. D **44**, 340 (1991).
[6] L. M. Widrow, Phys. Rev. D **44**, 2306 (1991).
[7] J. Schwinger, Phys. Rev. **82**, 664 (1951); **93**, 615 (1954).
[8] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, 1985).
[9] S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callan and S. Coleman, *ibid.* **16**, 1762 (1977).
[10] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982); Phys. Scr. **32**, 429 (1985).
[11] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[12] M. V. Berry and K. E. Mount, Rep. Prog. Phys. **35**, 315 (1972); L. D. Landau and E. M. Lifschitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon, Oxford, 1965); V. L. Pokrovskii, S. K. Savvinykh, and F. R. Ulinich, Sov. Phys. JETP **34**, 879 (1958); **34**, 1119 (1958).