

## Charge scaling and universality in critical collapse

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Consider any one-parameter family of initial data such that data with a parameter value  $p > p_*$  form black holes, and data with  $p < p_*$  do not. As  $p \rightarrow p_*$  from above ("critical collapse"), the black hole mass scales as  $M \sim (p - p_*)^\gamma$ , where the critical exponent  $\gamma$  is the same for all such families of initial data. So far critical collapse has been investigated only for initial data with zero charge and zero angular momentum. Here, we allow for U(1) charge. In scalar electrodynamics coupled to gravity, with action  $R + |(\partial + iqA)\phi|^2 + F^2$ , we consider initial data with spherical symmetry and a nonvanishing charge. From dimensional analysis and a previous calculation of Lyapunov exponents, we predict that in critical collapse the black hole mass scales as  $M \sim (p - p_*)^\gamma$ , and the black hole charge as  $Q \sim (p - p_*)^\delta$ , with  $\gamma = 0.374 \pm 0.001$  (as for the real scalar field) and  $\delta = 0.883 \pm 0.007$ . We conjecture that, where there is no mass gap, this behavior generalizes to other charged matter models, with  $\delta \geq 2\gamma$ . We suggest the existence of universality classes with respect to parameters such as  $q$ . [S0556-2821(96)03924-0]

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### I. INTRODUCTION

Consider any one-parameter family of initial data for general relativity such that data with a parameter value  $p > p_*$  form black holes, and data with  $p < p_*$  do not. Recently, it has been discovered that as  $p \rightarrow p_*$  from above ("critical collapse"), and unless there is a mass gap, the black hole mass scales as  $M \sim (p - p_*)^\gamma$ , where the critical exponent  $\gamma$  is the same for all such families of initial data.

This behavior was first discovered for a real scalar field in spherical symmetry [1], has been observed in a variety of other matter models [2–5], and is strongly expected to occur in two larger classes [6,7], all in spherical symmetry. Critical phenomena were also found in axisymmetric pure gravity [8]. Historically, this being the second case to be discovered, it has so far remained alone in going beyond spherical symmetry, and in being vacuum. No counter-examples have been found, except that when the field equations allow for a metastable static solution this solution creates a mass gap for certain initial data regimes [5].

The critical scaling of the mass can now be considered as understood [2,9,10] in terms of general relativity as a dynamical system, and of Lyapunov exponents, at least in spherical symmetry. For reasons of space we assume here that the reader is already familiar with these arguments. Our notation here is compatible with [10]. An introductory review is [11].

Black holes are characterized by mass  $M$ , U(1) charge  $Q$ , and angular momentum  $L$ , but so far critical collapse has been investigated only for initial data with zero charge and zero angular momentum. [Linear perturbations of extremal charged black holes ( $Q=M$ ) were studied in [12].] Here, as a first step towards the generic situation, we allow for charge in the initial data and hence in the resulting black hole. We keep the restriction to spherical symmetry (and, therefore, exclude angular momentum).

### II. SCALAR ELECTRODYNAMICS

As a matter model with U(1) charge, we choose scalar electrodynamics coupled to gravity. This has the advantage of being a simple field theory, and of allowing arbitrary and independent initial distributions of energy momentum and charge. Furthermore, it is a generalization of two other models in which critical collapse has already been studied, namely, the real and complex massless scalar fields (without electromagnetism). The action, in units where  $c = 2\pi G = 1$ , is

$$S = \int \sqrt{g} \left[ \frac{1}{8} R - \frac{1}{2} g^{ab} (\nabla_a + iqA_a) \phi (\nabla_b - iqA_b) \phi^* - \frac{1}{4} g^{ab} g^{cd} F_{ac} F_{bd} \right], \quad (1)$$

where  $F_{ab} \equiv \nabla_a A_b - \nabla_b A_a$ . We restrict to spherical symmetry. For the metric we choose

$$ds^2 = -\alpha(r,t)^2 dt^2 + a(r,t)^2 dr^2 + r^2 d\Omega^2 \quad (2)$$

with remaining gauge condition  $\alpha = 1$  and regularity condition  $a = 1$  (or  $a_{,r} = 0$ ) at the origin  $r = 0$  (before a singularity forms there). For the electromagnetic field we choose the gauge  $A_r = 0$ , and  $A_t = 0$  at  $r = 0$ .

In order to use dimensional analysis and scaling arguments later, we need the field equations in first-order and dimensionless form. For this purpose we introduce the dimensionless fields

$$X \equiv \frac{r}{a} \phi_{,r}, \quad Y \equiv \frac{r}{a} (\phi_{,t} + iqA_t \phi), \quad X_{\pm} \equiv X \pm Y, \\ A \equiv A_t, \quad F \equiv \frac{r}{a\alpha} A_{t,r}, \quad g \equiv \frac{a}{\alpha}. \quad (3)$$

With the notation  $Z \equiv \{X_+, X_-, \phi, a, g, A, F\}$ , we can write the field equations formally as

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$$\mathbf{E}(rZ_{,r}, rZ_{,t}, Z, qr) = 0, \quad (4)$$

where the vector  $\mathbf{E}$  of equations is polynomial in its arguments. (The matter stress-energy tensor and the matter equations of motions are manifestly polynomial because they have been derived from a polynomial matter Lagrangian. The components of the Ricci tensor can be brought into polynomial form after multiplication by certain metric coefficients.) Furthermore, it is linear in the first two arguments, because the field equations are now first order.

The arguments of  $\mathbf{E}$  form a complete set of independent dimensionless quantities. To make dimensionless quantities out of  $q$  and the derivatives, we have combined them with  $r$ , taking advantage of the spherical symmetry. (The Ricci scalar  $R$  has dimension  $L^{-2}$  and therefore,  $F_{ab}F^{ab}$  must have the same dimension. This means that  $A_a$  is dimensionless. As the minimal coupling of matter to electromagnetism is via  $\nabla_a \rightarrow \nabla_a + iqA_a$ ,  $q$  has dimension  $L^{-1}$ .)

Before giving the equations explicitly, we introduce new dimensionless coordinates

$$\tau \equiv \ln\left(\frac{t}{r_0}\right), \quad \zeta \equiv \ln\left(\frac{r}{t}\right) - \xi_0(\tau), \quad (5)$$

where  $r_0$  is an arbitrary fixed scale, and  $\xi_0(\tau)$  is a periodic function with period  $\Delta$ . This choice of coordinates is adapted to the self-similar solution we shall be looking for [10]. In these coordinates, discrete self-similarity corresponds to all fields being periodic in  $\tau$ . (As a degenerate case, continuous self-similarity corresponds to all fields being independent of  $\tau$ .)

Rewriting the arguments of  $\mathbf{E}$  in Eq. (4), we obtain

$$\mathbf{E}\{Z_{,\zeta}, e^{\zeta+\xi_0}[Z_{,\tau} - (1+\xi'_0)Z_{,\zeta}], Z, qr_0 e^{\tau+\zeta+\xi_0}\} = 0. \quad (6)$$

By shifting the origin of  $\tau$ , one may set  $r_0 = q^{-1}$  to simplify the notation, but we want to stress here that  $q$  is always accompanied by  $e^\tau$  in the field equations, and vice versa. This will be crucial later on.

The complete field equations in these fields and coordinates, written out explicitly, are

$$X_{\pm,\zeta} = \frac{[\frac{1}{2}(1-a^2) - a^2|X_{\mp}|^2 + a^2F^2]X_{\pm} - X_{\mp} \pm e^{\zeta+\xi_0}[gX_{\pm,\tau} + iqr_0 e^{\tau}(gAX_{\pm} + aF\phi)]}{1 \pm (1+\xi'_0)e^{\zeta+\xi_0}g}, \quad (7)$$

$$g_{,\zeta} = (1 - a^2 + 2a^2F^2)g, \quad (8)$$

$$a_{,\zeta} = \frac{1}{2}a[1 - a^2 + a^2(|X_+|^2 + |X_-|^2 + 2F^2)], \quad (9)$$

$$\phi_{,\zeta} = aX, \quad (10)$$

$$F_{,\zeta} = -F + iqr_0 e^{\tau+\zeta+\xi_0}a \frac{1}{2}(\phi Y^* - \phi^* Y), \quad (11)$$

$$A_{,\zeta} = g^{-1}a^2F, \quad (12)$$

$$a_{,\tau} = e^{-(\zeta+\xi_0)} \frac{1}{2}a^3g^{-1}(|X_+|^2 - |X_-|^2) + (1+\xi'_0) \frac{1}{2}a[1 - a^2 + a^2(|X_+|^2 + |X_-|^2 + 2F^2)], \quad (13)$$

$$F_{,\tau} = iqr_0 e^{\tau}ag^{-1} \frac{1}{2}(\phi X^* - \phi^* X) + (1+\xi'_0) \left[ -F + iqr_0 e^{\tau+\zeta+\xi_0}a \frac{1}{2}(\phi Y^* - \phi^* Y) \right], \quad (14)$$

$$\phi_{,\tau} = a[(1+\xi'_0)X + g^{-1}e^{-(\zeta+\xi_0)}Y] - iqr_0 e^{\tau}A\phi, \quad (15)$$

where  $\xi'_0 \equiv d\xi_0/d\tau$ . The last three equations, which contain only  $\tau$  derivatives, can be considered as constraints (at constant  $\zeta$ ) which are propagated in  $\zeta$  by the first six equations.

### III. CRITICAL SOLUTION

Critical collapse as we know it is dominated by a “critical solution” with two crucial properties: it is an attractor of co-dimension one, thus serving as an intermediate asymptotic, and it is self-similar, thus linking the large scale of the initial data with the small scale of the final black hole. The

critical solution acts as an attractor within the hypersurface which separates black hole data from non-black-hole data (the “critical surface”). It thus funnels all initial configurations on either side of the surface into two unique channels; one side goes to a black hole, and the other to dispersion. This explains the universality of the critical exponent. The power-law behavior of the black hole mass follows from the self-similarity of the critical solution.

Finding a critical solution is a two-step process: one has to find a self-similar solution, and then check explicitly the number of its unstable perturbations.

### A. Asymptotically self-similar solutions

Self-similarity corresponds to periodicity of the solution in the coordinate  $\tau$ , while the field equations contain explicit factors of  $e^\tau$ . Therefore, no self-similar solution exists. The physical reason is that the coupling to electromagnetism introduces the scale  $q^{-1}$  into the action and field equations, thus excluding scale-invariant solutions.

Does the absence of an exactly self-similar solution rule out critical phenomena? No, as we really only need a solution which approaches a self-similar spacetime as the latter approaches its singularity, that is in the limit of small space-time scale. (Any spherical self-similar spacetime has a curvature singularity, with  $R \sim e^{-2\tau}$ .) This leads us to the ansatz

$$Z(\zeta, \tau) = \sum_{n=0}^{\infty} e^{n\tau} Z_n(\zeta, \tau), \quad (16)$$

where each  $Z_n(\zeta, \tau)$  is periodic in  $\tau$ . As  $\tau \rightarrow -\infty$ ,  $Z$  is dominated by  $Z_0$ , which is by assumption periodic in  $\tau$ . Therefore, the form (16) guarantees that the solution  $Z$  is asymptotically self-similar as  $\tau \rightarrow -\infty$ . Like a self-similar solution, it has a singularity, at  $(t=0, r=0)$ , which corresponds to  $\tau = -\infty$  for any  $\zeta$ .

Now we argue that there are solutions of this form at least up to some distance from the singularity. We obtain equations for the  $Z_n$  by substituting the ansatz into the field equations and separating powers of  $e^\tau$ .  $Z_0$  obeys an equation independent of the other  $Z_n$ , while all the other  $Z_n$  are coupled to  $Z_{n-1}$  and lower. This allows us to determine the  $Z_n$  recursively, starting with  $Z_0$ . Therefore, solutions  $Z$  of the full problem are mapped one to one to the solutions  $Z_0$  of the zeroth order problem, which we shall see is simpler.

We know from the study of self-similar solutions that  $Z_0$  is locally uniquely given by a nonlinear eigenvalue problem, arising from the boundary conditions of (i) periodicity in  $\tau$  and (ii) regularity at (a)  $r=0$  (for  $t \neq 0$ , before the singularity forms) and (b) the past light cone of the singularity. By analogy, it is plausible that  $Z_1$  exists and is uniquely determined by the source term  $Z_0$  and similar boundary conditions, and so on for all the  $Z_n$ .

Does the sum converge? The  $Z_n$  are periodic in  $\tau$ . As  $\zeta \rightarrow -\infty$ , which corresponds to  $r \rightarrow 0$  (for  $t \neq 0$ ), the  $Z_n$  are bounded by the regularity conditions we impose.  $\zeta \rightarrow \infty$  corresponds to  $t \rightarrow 0$  (for  $r \neq 0$ ) and is only a coordinate singularity. The two examples for  $Z_0$  discussed below have been continued, by a change of coordinates, up to the future light cone, and have been verified explicitly to be bounded. As the higher  $Z_n$  obey equations similar in type to  $Z_0$ , we assume that all  $Z_n(\zeta, \tau)$  are bounded. As for the growth of the  $Z_n$  with  $n$ , it seems very likely that it is bounded as  $Z_n < A e^{-\tau_1 n}$  for some  $\tau_1$ , so that the solution converges for  $\tau < \tau_1$ . The fact that it does not converge for all  $\tau$  is not crucial to its usefulness. Even when the critical solution is exactly self-similar and, therefore, known to exist for all  $\tau$ , an actual collapse spacetime is not even approximately self-similar outside a bounded spacetime region.

What is the meaning of the expansion (16) in powers of  $e^\tau$ ? As  $e^\tau$  always appears together with  $q$  in the field equations, it is also an expansion in powers of  $q$ , that is in the coupling of the matter to electromagnetism. For example,

while  $A_0$  vanishes,  $A_1$  is the electromagnetic field created by  $\phi_0$ , and  $\phi_2$  is the reaction of  $\phi$  to that electromagnetic field, and so on. In particular, the equation for  $Z_0$  is obtained by writing  $Z_0$  for  $Z$  in the field equations while setting all appearances of  $q$  equal to zero. In effect,  $Z_0$  obeys the field equations for a complex scalar field coupled only to gravity, but not to electromagnetism, which, as we anticipated, is a simpler problem. Exactly two self-similar solutions of that problem, or candidates for  $Z_0$ , are known.

(1) The critical solution found by Choptuik [1,13,10] has no charge, because the scalar field is real [up to the remaining  $U(1)$  gauge freedom  $\phi \rightarrow e^{ic} \phi$  with  $c$  a constant]. This means that all the  $Z_n$  for  $n \geq 1$  have vanishing source terms and, therefore, vanish, or in other words that the Choptuik solution is already a solution of the full problem. [Strictly speaking, we have to complete the real Choptuik solution by a vanishing  $A$  and  $F$  and a real  $\phi$  calculated from Eq. (15).  $\phi$  is periodic because in the Choptuik solution  $X$  and  $Y$  have vanishing mean values. This is essential, but was not automatically guaranteed.]

(2) The continuously self-similar, charged solution found by Hirschmann and Eardley [14] does have charge. This means that it gives rise to a solution  $Z$  of the full problem, including electromagnetism, only when all the terms  $Z_n$  are added up.

### B. Perturbation spectrum

Now, we come to the second criterion for a solution to be a critical solution, namely, the presence of exactly one-linear perturbation mode that is growing as  $\tau \rightarrow -\infty$ . Anticipating the next section, we note that the general linear perturbations  $\delta_i Z$  of  $Z$  ( $i$  labels the independent modes) have a similar series form to the background critical solution  $Z$  itself. As for the background expansion (where  $Z$  is completely determined by  $Z_0$ ),  $\delta_i Z$  is specified by giving its leading term  $\delta_{i0} Z$ . This obeys an equation where (a) once more, the terms with  $q$  have been set equal to zero, and where (b) the background dependence is only through  $Z_0$ . This means  $\delta_{i0} Z$  on its own is a complete perturbation, in the theory without electromagnetism, of  $Z_0$ , which is itself a complete background solution of that theory.

The spectrum of the perturbations is determined, in a linear eigenvalue problem, together with the  $\delta_{i0} Z$  (and is not influenced by the higher  $\delta_{in} Z$ ). In consequence, the spectrum is the same in the presence of the coupling to electromagnetism, or in its absence. We can, therefore, recycle earlier work, where the perturbation spectra of the Choptuik and Hirschmann and Eardley solutions were determined in the theory without electromagnetism: their spectrum in scalar electrodynamics is just the same. As, in particular, the number of unstable modes is the same, we immediately conclude that the Choptuik solution is an attractor of co-dimension one [10], that is a genuine critical solution, even in the full theory with electromagnetism, while the complex solution based on (but not identical with) the Hirschmann and Eardley solution has three unstable modes [15]. (To state it once more, its perturbation mode functions are different in the full theory, but their Lyapunov exponents are the same.)

### C. Uniqueness of the critical solution

Collapse simulations [16] suggest that the Choptuik solution is, in fact, the *only* critical solution, and a global intermediate attractor, for the free complex scalar field *not* coupled to electromagnetism. This type of evidence can never be complete, because the entire phase space can never be probed. It only suggests that any other other critical solutions have rather small basins of attraction. As we have seen, self-similar solutions and their perturbations are mapped one to one between the free complex scalar field and the theory with electromagnetism. This means that critical solutions are mapped one to one between the theories. Their basins of attractions, however, could be very different in size, so that the counterpart of a hypothetical critical solution for the free complex scalar, with a basin of attraction so small that it would have been overlooked so far, could play an important role in scalar electrodynamics.

We must also consider the possibility of critical solutions not of the form (16). Two such possibilities have occurred to us, but can be definitely ruled out.

Any critical solution of the form (16) describes a situation where charge becomes less and less important in the final black hole as one fine tunes along some one-parameter family of initial data with the aim of making black holes of ever smaller mass. (This does not exclude *initial data* with an appreciable charge-to-mass radius, but most of that charge must be radiated away along with most of the mass.) The other possibility for the charge allowed by the limit  $Q < M$  for black holes is that  $Q/M$  approaches a constant as  $M \rightarrow 0$  in the process of fine tuning. This is excluded: The black hole charge  $Q$  for a given scalar field evolution must change sign when  $q$  changes sign. As  $q$  always appears in the company of  $e^\tau$ ,  $Q$  must be suppressed with respect to  $M$  by at least one power of  $e^\tau$  as small scales are approached.

A second possibility (suggested by [5]) is that of another critical solution which is not self-similar but static, giving rise to a mass gap in the one-parameter families of data. (We come back to this possibility below, when we consider other matter models with charge.) But massless scalar electrodynamics has no static solutions, even unstable. “Charged boson stars” exist only if one adds a mass term  $m^2\phi^2$  to the action, with  $4\pi Gm^2 > q^2$  [17].

We come to the surprising conclusion that the Choptuik solution is a critical solution in scalar electrodynamics, and, as far as we can see, the only one, even though it has no charge itself. (In particular, the Hirschmann and Eardley solution, although charged, is not a critical solution.)

Ongoing work of C.G. shows that the critical solution for spherical SU(2) Einstein-Yang-Mills collapse [5] is of the form (16), with all the terms  $Z_n$  nonvanishing. This indicates that if they vanish for  $n > 0$  in scalar electrodynamics, this is accidental. Details will be published elsewhere.

### IV. LINEAR PERTURBATIONS

We now consider the real critical solution of Choptuik and its linear perturbations. Because the background is real, these split naturally into two kinds. The first are perturbations  $\delta Z$  with  $\delta\phi$  purely real. In the following we call these

the “real” perturbations. The general perturbation of this kind is of the form

$$\delta Z_{\text{real}}(\zeta, \tau) = \sum_{i=1}^{\infty} C_i e^{\lambda_i \tau} \delta_i Z(\zeta, \tau), \quad (17)$$

where each  $\delta_i Z$  is periodic in  $\tau$  with period  $\Delta$ , and where the  $C_i$  are free parameters. The  $\delta_i Z$  were already calculated within the real scalar field model in [10]. In particular, there is precisely one eigenvalue  $\lambda$  with  $\text{Re}\lambda < 0$  in the spectrum  $\{\lambda_i\}$ , namely,  $\lambda_1 \approx -2.674$ .

The linearized equations for the remaining, not purely real, perturbations are of the form

$$\delta Z_{,\zeta} = A \delta Z + B \delta Z_{,\tau} + e^\tau C \delta Z, \quad (18)$$

where the dimensionless coefficients  $A$ ,  $B$ , and  $C$  are periodic in  $\tau$ . The “real” perturbations obey the same equations, but the term proportional to  $e^\tau$  (or, as always, proportional to  $q$ ) vanishes for the real perturbations because they do not carry charge. Its presence means that the  $\delta_i Z$  cannot be periodic in  $\tau$ . Instead, we have

$$\delta Z_{\text{imaginary}}(\zeta, \tau) = \sum_{i=1}^{\infty} D_i e^{\mu_i \tau} \delta_i Z(\zeta, \tau), \quad (19)$$

where each independent mode  $\delta_i Z$  is now expanded as

$$\delta_i Z(\zeta, \tau) = \sum_{n=0}^{\infty} e^{n\tau} \delta_{in} Z(\zeta, \tau), \quad (20)$$

where only the  $\delta_{in} Z$  are periodic in  $\tau$ . This expansion is exactly analogous to that of the asymptotically self-similar background solution, with the difference that, due to the linearity, the ansatz (19), (20) is generic, while Eq. (16) is not. The  $\delta_{in} Z$  obey the coupled equations

$$\delta_{i0} Z_{,\zeta} = (A + \mu_i B) \delta_{i0} Z + B \delta_{i0} Z_{,\tau}, \quad (21)$$

$$\delta_{in} Z_{,\zeta} = (A + \mu_i B + nB) \delta_{in} Z + B \delta_{in} Z_{,\tau} + C \delta_{in-1} Z, \quad n \geq 1. \quad (22)$$

The equation for  $\delta_{i0} Z$  describes purely imaginary perturbations of the scalar field, not coupled to the electromagnetic field or the metric. It is homogeneous in  $\delta_{i0} Z$ , and is complemented by periodic boundary conditions (with period  $\Delta$ ) in  $\tau$ , and by regularity conditions at  $\zeta = -\infty$  and  $\zeta = 0$ . A solution exists only for discrete values of  $\mu$ . From this equation one obtains the spectrum  $\{\mu_i\}$  of perturbations. This equation is, in fact, that for the perturbations, around the real solution, of the complex scalar field model without electromagnetism. They have already been considered in [10], with the result that  $\text{Re}\mu_i > 0$  for all  $i$ .

The terms  $\delta_{in} Z$  for  $n \geq 1$  are determined both by boundary conditions and by the source terms  $\delta_{in-1} Z$ . As the value of  $\mu_i$  has already been fixed as an eigenvalue in the equation for  $\delta_{i0} Z$ , the boundary conditions admit no homogeneous solution for  $n \geq 1$ , and the solution for  $n \geq 1$  is proportional to the source term  $\delta_{in-1} Z$ . The electromagnetic field comes in to order  $e^\tau$ , that is  $q$ , because the purely imaginary perturbation  $\delta_{i0} \phi$  of the real background scalar field  $\phi_*$  gives rise to a charge distribution and, therefore, generates an elec-

tromagnetic field. This charge distribution is ultimately responsible for the charge of the final black hole. To order  $e^{2\tau}$ , or  $q^2$ , the scalar field perturbations acquire a real part and in consequence couple to metric perturbations, thus also creating a gravitational field. Nevertheless, we shall refer to these perturbations as the “imaginary” perturbations, because to leading order they are purely imaginary perturbations of the scalar field.

## V. MASS SCALING

For the generic solution close to the real self-similar solution  $Z_*$ , we now have the form

$$Z(\zeta, \tau) \approx Z_*(\zeta, \tau) + \sum_{i=1}^{\infty} C_i(p) e^{\lambda_i \tau} \delta_i Z(\zeta, \tau) + \sum_{i=1}^{\infty} D_i(p) e^{\mu_i \tau} \sum_{n=0}^{\infty} e^{n\tau} \delta_{in} Z(\zeta, \tau), \quad (23)$$

where the first term is the critical solution, and the second and third terms its “real” and “imaginary” perturbations, as discussed above. The amplitudes  $C_i$  and  $D_i$  of the perturbations depend on the initial data in general and hence on the parameter  $p$  of a given one-parameter family of initial data in particular.

As  $\tau \rightarrow -\infty$ , we can neglect all perturbations but the one growing mode, associated with  $\lambda_1$ . As we are interested in the asymptotic behavior of the charge, however, we also keep the most slowly decaying of the imaginary perturbations (which alone carry charge), associated with  $\mu_1$ . By definition we obtain the precisely critical solution for  $p = p_*$ , and so we must have  $C_1(p_*) = 0$ . Expanding  $C_1(p)$  and  $D_1(p)$  to leading order, we obtain

$$Z(\zeta, \tau) \approx Z_*(\zeta, \tau) + (p - p_*) \frac{\partial C_1(p_*)}{\partial p} e^{\lambda_1 \tau} \delta_1 Z(\zeta, \tau) + D_1(p_*) e^{\mu_1 \tau} \sum_{n=0}^{\infty} e^{n\tau} \delta_{1n} Z(\zeta, \tau). \quad (24)$$

To keep the notation compact, we define, following [10],

$$\tau_*(p) \equiv -\frac{1}{\lambda_1} \ln \left( \frac{p - p_*}{p_*} \right), \quad \tau_1 \equiv -\frac{1}{\lambda_1} \ln \left[ \epsilon^{-1} \frac{\partial C_1}{\partial \ln p}(p_*) \right], \quad \tau_0 \equiv \tau_1 + \tau_*(p), \quad (25)$$

where  $\epsilon$  is an arbitrary small constant (independent of  $p$ ). If we now fix  $\tau = \tau_0$  in the approximate solution (24), we obtain a  $p$ -dependent family of Cauchy data: namely,

$$Z_p(r) \equiv Z(\zeta, \tau_0) = Z_* \left( \ln \frac{r}{r_p}, \tau_0 \right) + \epsilon \delta_1 Z \left( \ln \frac{r}{r_p}, \tau_0 \right) + K(p) \sum_{n=0}^{\infty} e^{n\tau_0} \delta_{1n} Z \left( \ln \frac{r}{r_p}, \tau_0 \right), \quad (26)$$

where

$$r_p \equiv r_0 e^{\tau_0 + \xi_0(\tau_0)}, \quad K(p) \equiv D_1(p_*) e^{\mu_1 \tau_0}. \quad (27)$$

$K(p)$  is small, even compared to  $\epsilon$ , if  $e^{\tau_0}$  is sufficiently small, that is, if  $(p - p_*)$  is sufficiently small. We, therefore, treat the terms proportional to  $K(p)$  as a linear perturbation throughout.

But now we consider the exact, nonlinear evolution of the data  $Z_* + \epsilon \delta_1 Z$ , without treating  $\epsilon \delta_1 Z$  as a perturbation any longer. This is necessary because its presence makes a qualitative difference at late times. If it has one sign, a black hole is formed. If it has the other, the matter disperses. Because the solution at late times is no longer even approximately self-similar, we go back to the coordinates  $r$  and  $t$ .

The data  $Z_* + \epsilon \delta_1 Z$  are purely real, and evolve to a purely real solution, with vanishing electromagnetic field. Consequently, the equations determining the solution do not contain the term  $qr$ , and are scale invariant. Therefore, the entire solution depends on  $r_p$  in the simple way

$$Z(r, t) = f(\bar{r}, \bar{t}, \tau_0), \quad \text{where } \bar{r} \equiv \frac{r}{r_p} \quad \text{and} \quad \bar{t} \equiv \frac{t - t_p}{r_p}, \quad (28)$$

where the (irrelevant) shift in  $t$  is  $t_p \equiv r_0 e^{\tau_0}$ .  $f$  obeys the equation

$$\mathbf{E}(\bar{r} \bar{f}_{,\bar{r}}, \bar{r} \bar{f}_{,\bar{t}}, \bar{f}, 0) = 0, \quad (29)$$

with initial data

$$f(\bar{r}, \bar{t} = 0, \tau_0) = Z_*(\ln \bar{r}, \tau_0) + \epsilon \delta_1 Z(\ln \bar{r}, \tau_0). \quad (30)$$

In particular, we know that the mass of the black hole that forms in this solution must be a multiple of the underlying scale  $r_p$ , namely,

$$M = r_p e^{\mu(\tau_0)}, \quad (31)$$

where  $\mu(\tau)$  is a function that we do not know, but which is periodic with period  $\Delta$  in  $\tau$ . The black hole mass as a function of the family of initial data and the parameter value  $p$  is then

$$M(p) = r_0 \left( \frac{p - p_*}{p_*} \right)^{-1/\lambda_1} \exp\{\tau_1 + \nu[\tau_*(p) + \tau_1]\}, \quad (32)$$

where  $\nu(\tau) \equiv \mu(\tau) + \xi_0(\tau)$  is a universal wiggle superimposed on the basic power-law behavior, and the constant  $\tau_1$  (defined above) depends on the initial data family.

## VI. CHARGE SCALING

So far we have only repeated [10] (which itself is a generalization of [9]). Now we consider the effect of the perturbation proportional to  $K(p)$  in the initial data for  $f$ . This linear perturbation, say  $\delta f$ , obeys the linearization of Eq. (4), which is of the form

$$r \delta f_{,t} = A r \delta f_{,r} + B \delta f + q r C \delta f. \quad (33)$$

Rewriting this in terms of  $\bar{r}$  and  $\bar{t}$ , we obtain [using the definition (27) of  $r_p$ ]

$$\bar{r} \delta f_{,\bar{t}} = A \bar{r} \delta f_{,\bar{r}} + B \delta f + q r_0 e^{\tau_0 + \xi_0(\tau_0)} C \bar{r} \delta f, \quad (34)$$

where the dimensionless coefficients  $A$ ,  $B$ , and  $C$  are functions only of  $\bar{r}$  and  $\bar{t}$  (as well as of the parameter  $\tau_0$  characterizing the background solution  $f$ ). To obtain a solution for all small values of the parameter  $e^{\tau_0}$  at once, we expand in powers of it. For the critical solution plus linear perturbation we then have

$$Z_p(r, t) = f(\bar{r}, \bar{t}, \tau_0) + K(p) \sum_{n=0}^{\infty} e^{n\tau_0} \delta_n f(\bar{r}, \bar{t}, \tau_0). \quad (35)$$

The expansion coefficients  $\delta_n f$  obey the equations

$$\bar{r} \delta_{0f, \bar{t}} = A \bar{r} \delta_{0f, \bar{r}} + B \delta_{0f}, \quad (36)$$

$$\bar{r} \delta_{nf, \bar{t}} = A \bar{r} \delta_{nf, \bar{r}} + B \delta_{nf} + q r_0 e^{\xi_0(\tau_0)} C \bar{r} \delta_{n-1f}, n \geq 1, \quad (37)$$

with initial data given by

$$\delta_n f(\bar{r}, \bar{t} = 0, \tau_0) = \delta_{1n} Z(\ln \bar{r}, \tau_0). \quad (38)$$

The perturbation  $\delta f$  gives rise to a perturbation of the black hole mass  $M$ , but we ignore this here as a subdominant effect. We are, however, interested in the black hole charge  $Q$ , which *only* comes in through  $\delta f$ . We do not need to calculate  $\delta f$  to see how  $Q$  scales. It is sufficient to note that the charge-to-mass ratio is dimensionless and must be odd in the coupling constant  $q$ , and hence  $e^{\tau_0}$  and, therefore, must go as

$$\frac{Q}{M} = K(p) \left[ e^{\tau_0} \left( \frac{Q}{M} \right)_1(\tau_0) + e^{3\tau_0} \left( \frac{Q}{M} \right)_3(\tau_0) + O(e^{5\tau_0}) \right]. \quad (39)$$

Taking only the dominant term, and putting it all together, we obtain

$$Q(p) = M(p) K(p) e^{\tau_0} \left( \frac{Q}{M} \right)_1(\tau_0). \quad (40)$$

After regrouping terms, we obtain the final result

$$Q(p) = r_0 \left( \frac{p - p_*}{p_*} \right)^{-(\mu_1 + 2)/\lambda_1} \exp\{\tau_2 + \pi[\tau_1 + \tau_*(p)]\}, \quad (41)$$

where  $\tau_2 \equiv (\mu_1 + 2)\tau_1 + \ln D_1(p_*)$  is a new family-dependent constant,  $\tau_1$  is the same family-dependent constant as before, and  $\pi(\tau) \equiv \nu(\tau) + \ln(Q/M)_1(\tau)$  is a new universal wiggle.

Numerical values for the Lyapunov exponents are  $\lambda_1 = -2.674 \pm 0.009$  [10] and  $\mu_1 = 0.362 \pm 0.012$ .  $\mu_1$  has been calculated by the same method [10] as  $\lambda_1$ , but converges somewhat more slowly with decreasing step size, so that the estimated numerical error in  $\mu_1$ , and in consequence in  $\delta$ , is somewhat larger. We obtain critical exponents  $\gamma = -1/\lambda_1 = 0.374 \pm 0.001$  for the mass and  $\delta = -(\mu_1 + 2)/\lambda_1 = 0.883 \pm 0.007$  for the charge.

## VII. OTHER CHARGED MATTER MODELS

Given that our arguments did not rely on the exact form of the field equations, we can generalize them to any matter

model in spherical symmetry where the matter is coupled to electromagnetism only via the U(1)-covariant derivative  $D_a \equiv \nabla_a + iqA_a$ . Then it follows from dimensional analysis that whenever one casts the field equations in first-order, dimensionless form, they must be of the form (4).  $Z$  would stand for another set of fields, and  $\mathbf{E}$  need not be a polynomial. The argument showing that the electromagnetic interaction can be neglected asymptotically in the strong field or small-scale regime of the critical solution would go through as before, even if the critical solution itself carries charge. (Strictly speaking, we require that a polynomial form of the field equations exists for our arguments to apply directly, but we hope that this technical requirement can be relaxed.) Now we need to consider three cases separately:

(1) The critical solution does not carry charge, only its perturbations do, which brings in Lyapunov exponents. Let  $\lambda_1$  be the one eigenvalue with  $\text{Re} \lambda < 0$ : (a) The unstable mode itself carries no charge. This case is similar to scalar electrodynamics. Let  $\mu_1$  be the eigenvalue associated with charge which has the smallest real part. We must have  $\text{Re} \mu_1 > 0$ , because the critical solution, by definition, has only one unstable mode. From  $\gamma = -1/\lambda_1$  and  $\delta = -(\mu_1 + 2)/\lambda_1$  we then find the relation  $\delta > 2\gamma$ . (b) The unstable mode itself carries charge. In this case, we obtain the charge, as well as the mass, from the nonlinear evolution of the data  $Z_* + \epsilon \delta_1 Z$ , neglecting *all* the other perturbations. The solution arising from these data is no longer of the simple form (28), due to the presence of  $qr$  in the Eq. (4). But we can obtain the solution for all small  $r_p$  by expanding in  $qr_p$ , using the fact that  $qr = \bar{r} q r_p$ . The solution (28) is now replaced by the more general

$$Z(r, t) = \sum_{n=0}^{\infty} (qr_p)^n f_n(\bar{r}, \bar{t}, \tau_0). \quad (42)$$

The mass  $M$  is again proportional to  $r_p$ , but this time only to leading order in  $qr_p$  (or  $e^{\tau_0}$ ). The charge-to-mass ratio is now  $Q/M = (qr_p)(Q/M)_1(\tau_0) + O(qr_p)^3$ , without the prefactor  $K(p)$ , so that the charge to leading order is simply proportional to  $r_p^2$ , or  $\delta = 2\gamma$ , with  $\gamma = -1/\lambda_1$ .

(2) Finally, the critical solution itself may carry charge. Then the higher terms in the expansion (16) do not vanish identically. This would have been the case for example for the solution [14]. There the charge would have appeared to order  $e^{\tau}$  of the background expansion (16). We should still need to consider the data  $Z_0 + \epsilon \delta_1 Z$ , where  $\delta_1 Z$  is the one growing perturbation (charged or not), in order to determine  $r_p$ , the spacetime scale on which the solution leaves the intermediate attractor. But the electromagnetic field giving rise to the black hole charge  $Q$  is now dominated by  $A_1$  (the component  $A$  of  $Z_1$ ), which is once more down a factor of  $r_p$  from  $Z_0$ , so that  $Q$  is again proportional to  $r_p^2$ , or  $\delta = 2\gamma$ .

We must make one proviso, namely, that a mass gap may exist in critical collapse. Recent work on critical collapse of the Einstein-Yang-Mills system has illustrated this [5]: If the model admits a static or oscillating solution with precisely one unstable mode (such as the Bartnik-McKinnon solution in Einstein-Yang-Mills theory), this solution acts as an alternative intermediate attractor. As this intermediate asymptotic

is not self-similar, but instead has a finite mass, it gives rise to a mass gap in some region of initial data space. Our results then only hold for that part of initial data space where formation of the static solution is avoided. For a mass gap to occur when a critical solution also exists, it is essential that a static or oscillating solution not only exists, but has exactly one unstable mode. If it was unconditionally stable, there would be a third possible outcome in addition to black hole formation or dispersion, namely the formation of a star. (This is, of course, the situation in astrophysics.) If it had more than one unstable mode, it would be a lesser attractor than the critical solution and would be “missed” by almost all one-parameter families of initial data.

### VIII. UNIVERSALITY CLASSES AND THE RENORMALIZATION GROUP

Finally, we note a consequence of our work that does not concern charge. By exactly the same argument which shows that in the critical solution with charge electromagnetism can be neglected asymptotically, one can show that the mass term can be neglected asymptotically for the massive scalar field. (One only has to replace  $qr$  by  $mr$  in Eq. (4) and the following equations derived from it.) As the critical exponent for the black hole mass is unchanged by the coupling to electromagnetism, so it is, by the same argument, when a mass term or a more general polynomial scalar field self-interaction is added to the action. (This was already known [18].)

Our argument generalizes this to *any* parameter  $m$  of dimension  $(\text{length})^{-1}$  in geometrical units  $G=c=1$  appearing in a polynomial form of the field equations. [Note that as we insist on the equations being polynomial in the fields  $Z$  and parameter  $m$ , we implicitly also allow parameters of dimension  $(\text{length})^{-n}$ , but only for positive integer  $n$ .]

In the language of critical phenomena, theories with different values of such a dimensionful parameter form “universality classes” [11]. The analogy with critical phenomena in statistical mechanics seems good enough to use this term deliberately.

The renormalization group in statistical mechanics acts on the phase space by blocking degrees of freedom, and a change of scale. If one demands that the partition function remain invariant, this induces an equivalent action on (the parameters of) the Hamiltonian [19]. The equivalents of the spins and their Hamiltonian in statistical mechanics are the fields  $Z(r)$  and their equations of motion in critical collapse. The renormalization group in critical collapse acts on the phase space by a time evolution (in  $t$ ) followed by change of scale (in  $r$ ), as  $Z(r, t) \rightarrow Z(e^{-\Delta}r, e^{-\Delta}t)$ . (One can combine the time evolution and rescaling into a time evolution in  $\tau$ , at constant  $\zeta$ , using the freedom of lapse and shift in general relativity. The critical solution is by definition invariant under this transformation.) If one demands that the rescaled data evolve in the same way, this action induces an equivalent action on (the parameters of) the field equations.

In the case of scalar electrodynamics, this action is rather simple: In the field equations,  $q$  and  $r$  appear together as  $qr$ . If one changes the scale as  $r \rightarrow e^{-\Delta}r$ , this has the same effect as  $q \rightarrow e^{-\Delta}q$ .

Critical phenomena in gravitational collapse correspond

to very small scales (compared to the scale of the initial data and the scale  $q^{-1}$ ), in contrast with critical phenomena in statistical mechanics, where they correspond to very large scales (compared to the scale of the microscopic physics). In the limit of small scales, the parameter  $q$  becomes “irrelevant,” to borrow another term from statistical mechanics.

If there are two (or more) parameters such as  $m$  or  $q$ , universality does not hold, as  $m_1/m_2$  is a dimensionless parameter, which generically has qualitative effects. In massive scalar electrodynamics, for example, the value of  $|q|/m$  determines if static solutions (“charged Boson stars”) exist. Universality may be recovered if the two scales are very different, that is for  $m_1/m_2 \rightarrow 0$  or  $m_2/m_1 \rightarrow 0$ , but this limit need not be regular.

### IX. CONCLUSIONS

We predict that in critical collapse of massless scalar electrodynamics in spherical symmetry, the black hole mass scales as  $M \sim (p - p_*)^\gamma$ , and the black hole charge as  $Q \sim (p - p_*)^\delta$ , each overlaid with a universal wiggle, with  $\gamma = 0.374 \pm 0.001$  (as for the real scalar field) and  $\delta = 0.883 \pm 0.007$ .

We have gone beyond the restriction to uncharged initial data (and hence black holes), but have kept the restriction to vanishing angular momentum. We predict the appearance of a new critical exponent  $\delta$ , and give its numerical value. Verification of our predictions in numerical collapse simulations should be straightforward.

Furthermore, we predict that in spherical critical collapse of matter models with minimal coupling  $\nabla \rightarrow \nabla + iqA$  to electromagnetism, both mass and charge scale as universal power laws (times a universal wiggle), with  $\delta \geq 2\gamma$ , so that the black hole charge always disappears faster than the black hole mass. Depending on the matter (not in massless scalar electrodynamics) this behavior may hold only for parts of the initial data space, with a mass gap in other parts.

We also find that adding terms with a parameter  $m$  of dimension  $(\text{length})^{-1}$  in units  $c=G=1$  (such as a scalar field mass, or the electromagnetic coupling in the present paper), in a form of the field equations which is polynomial in the fields  $Z$  and parameter  $m$ , does not change the critical exponent for the black hole mass. In this sense models with different values of such a parameter form universality classes.

*Note added.* After this paper had been submitted, our value of the critical exponent for the charge was confirmed in numerical collapse situations by Hod and Piran [20]. Universality classes in critical collapse were independently described by Hara, Koike, and Adachi [21].

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