Quantum stability of the time machine

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In a number of papers it has been claimed that the time machine is quantum unstable, which manifests itself in the divergence of the vacuum expectation value of the stress-energy tensor $\langle \mathbf{T} \rangle$ near the Cauchy horizon. The expression for $\langle \mathbf{T} \rangle$ was found in these papers on the basis of some specific approach. We show that this approach is untenable in that the above expression, first, is not derived from some more fundamental and undeniable premises, as is claimed, but rather postulated, and second, contains undefined terms, so that one can neither use nor check it. From this we conclude that at the moment there is not a grain of evidence to suggest that the time machine must be unstable. As an illustration we cite a few cases of (two-dimensional) spacetimes containing time machines with $\langle \mathbf{T} \rangle$ bounded near the Cauchy horizon. [S0556-2821(96)01822-X]

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I. INTRODUCTION

Since the wormhole-based time machine was proposed [3] much effort has been directed towards finding a mechanism that could "protect causality" [4] and destroy such a time machine. One of the most popular ideas (see [1,2,4–6]) is that the creation of the time machine might be prevented by quantum effects since as it is claimed in [2] "at any event in spacetime, which can be joined to itself by a closed null geodesic, the vacuum fluctuations of a massless scalar field should produce a divergent renormalized stress-energy tensor." The considerations leading to such a claim I shall call hereafter the "Frolov-Kim-Thorne (FKT) approach."[1]

In essence, the FKT approach amounts to the following [1,2]. The vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$ of the field ϕ in the (multiply connected) spacetime M containing a time machine is found by applying some differential operator $D_{\mu\nu}$ to the Hadamard function

$$G^{(1)}(X,X') \equiv \langle \{ \phi(X), \phi(X') \} \rangle. \tag{1}$$

To find $G^{(1)}$, it is proposed to use the formula

$$G^{(1)}(X,X') = G^{\Sigma} \equiv \sum_n \widetilde{G}^{(1)}(X, \gamma^n X'). \tag{2}$$

Here $\widetilde{G}^{(1)}$ is the Hadamard function of ϕ in the spacetime $\widetilde{M}$, which is the universal covering space for M, and $\gamma^n X \in \widetilde{M}$ is the nth inverse image of $X \in M$ [$\gamma^0 X$ is identified in Eq. (2) with X]. The advantage of the use of Eq. (2) is that $\widetilde{G}^{(1)}$ is supposed to have the Hadamard form

$$\widetilde{G}^{(1)}(X,X') = \widetilde{u}\sigma^{-1} + \widetilde{v}\ln|\sigma| + \text{nonsingular terms} \tag{3}$$

where σ is half the square of the geodesic distance between X and X', and $\widetilde{u},\widetilde{v}$ are some smooth functions. We might think thus that

$$\langle T_{\mu\nu} \rangle^{\text{ren}}_M = \langle T_{\mu\nu} \rangle^{\text{ren}}_{\widetilde{M}} + \sum_{n\neq 0} \lim_{X'\to X} D_{\mu\nu} \widetilde{G}^{(1)}(X, \gamma^n X')$$

$$\to \langle T_{\mu\nu} \rangle^{\text{ren}}_{\widetilde{M}} + \sum_{n\neq 0} \lim_{\substack{X'\to X \\ X\to \text{ horizon}}} D_{\mu\nu}(\widetilde{u}\sigma_n^{-1} + \widetilde{v}\ln|\sigma_n|). \tag{4}$$

Here, $\sigma_n \equiv \sigma(X, \gamma^n X')$, and the superscript "ren" (renormalized) has appeared because renormalization of $\langle \mathbf{T} \rangle_{\widetilde{M}}$ and of $\langle \mathbf{T} \rangle_M$ requires subtraction of the same terms. The last series in Eq. (4) diverges (since $\sigma_n \to 0$, when X approaches the horizon), so the conclusion is made that the appearance of a closed timelike curve must be prevented (unless some effects of quantum gravity remedy the situation) by the infinite increase of the energy density.[2]

The goal of the present paper is to show that there is actually *no* reason to expect that the energy density diverges at the Cauchy horizon in the general case.

Section II is an extensive discussion of two relevant points. We argue the following.

(1) The divergency of the last series in Eq. (4) does not imply the divergency in the vacuum stress-energy tensor since the following is true.

(a) The value of the term $\langle T_{\mu\nu} \rangle^{\text{ren}}_{\widetilde{M}}$ in Eq. (4) depends on which vacuum in $\widetilde{M}$ we consider. Choosing different vacua we can change the value of the right-hand side of Eq. (4) almost arbitrarily. In particular, we can make it finite or infinite at will. Note that, as long as Eq. (2) holds, none of these vacua is better than any other (as in $\widetilde{M}$ itself, they are

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[1]This approach is essentialy an application of what is sometimes called "the method of images" (see [7], for example) to the wormhole-based time machine. So the bulk of this paper concerns this method too.

[2]This effect differs from what can be called "classical instability" [8]. The latter arises from *infinite* returning of a blueshifted particle in some compact region, while in Eq. (4) *each* term gives the divergent contribution.

auxiliaries without any pronounced physical meaning). So, we come to the conclusion that the right-hand-side of Eq. (4) is not fully defined.

(b) The transition from Eq. (2) to Eq. (4) includes a few lacunas important from the mathematical point of view. So, it is not clear whether in the general case Eq. (4) is valid for any vacuum at all.

(2) It was implicitly assumed so far that it is the behavior of the *vacuum* expectation $\langle \mathbf{T} \rangle$ that is conclusive for whether or not a time machine is quantum stable. However, trying to create a time machine, one need not restrict oneself just to the vacuum. It is one's right to choose the most appropriate quantum state (cf. [4]). And it turns out that the energy density may remain bounded for some nonvacuum states even when it diverges for the vacuum.

In Sec. III we construct a few two-dimensional examples. Of course, the two- and four-dimensional cases differ in many respects (for example, the wave equation on the two-dimensional cylinder has no solutions, except for constant, continuous at the Cauchy horizon). So, we do not intend to prove anything peculiar to the four-dimensional one by these examples. In particular, the time machines from Sec. III in no way disprove Hawking's ''chronology protection conjecture'' though the energy density is bounded in their causal regions. The only intention of Sec. III is to provide illustrations to corresponding statements from Sec. II. For example, we state in Sec. II that the neglecting of the regular part of $\widetilde{G}^{(1)}$ is unsound. So, in Sec. III the situation is adduced when this neglection leads to an absurd result.

Also, two-dimensional time machines can be a useful toy model. The energy density can be easily found for them, while for four-dimensional ones, not in a single case it has been found so far without use of the ''method of images.''

II. ANALYSIS

Our main purpose in this section is to prove that the behavior of the last series in Eq. (4) has nothing to do with the behavior of the vacuum energy density. So, it would have sufficed to restrict the consideration to say Sec. II B. But with an eye to other possible applications of the ideas cited above [quantum field theory on multiply-connected spaces] we try to give a detailed analysis of the FKT approach.

A. Going to the universal covering

Formula (2), combined with some implicit assumptions, serves as a basis for the overall FKT approach since one cannot use Eq. (3) in multiply connected spacetimes, where σ is not defined. To discuss Eq. (2) and to reveal these assumptions, let us first state the simple fact that most properties of the Hadamard function, *including* the validity of Eq. (3), depend on the choice of the vacuum appearing in definition (1). So, formulas such as Eq. (2) are meaningless until we specify the vacua $|0\rangle$ and $|\widetilde{0}\rangle$ in M and $\widetilde{M}$, respectively. We come thus to the problem of great importance in our consideration: how, given $|0\rangle$, one could determine corresponding $|\widetilde{0}\rangle$? The above-mentioned assumptions concern just this problem. They must be like the following: (1) For any vacuum on M there exists a vacuum $|\widetilde{0}\rangle$ on $\widetilde{M}$ such that Eq. (2) holds; (2) the function $\widetilde{G}^{(1)}$ corresponding to $|\widetilde{0}\rangle$ has the ''Hadamard form'' (3); (3) $G^{(1)}$ determines $\widetilde{G}^{(1)}$ uniquely.

The validity of assumption 1 is almost obvious in the simplest cases (see below), but it was not proven in the general case. [One can meet the references to [9] in this connection. Note, however, that the functions K_C which stand there in the analogue of our formula (2), are actually not defined[3] in our case, i.e., when $|\Gamma| = \infty$.]

Assumption 2 seems still more arbitrary. The validity of Eq. (3) was proven not for *any* state, but only for some specific class of states (see [10], Sec. 2c), and there is no reason to believe that our $|\widetilde{0}\rangle$ belongs just to this class. Note that even if one accounts the states $|\widetilde{0}\rangle$ violating Eq. (3) to be ''unphysical'' [10] this does not allow one to rule them out since it is $|0\rangle$ which should be ''physical.''

Assumption 3 is definitely untrue. In the following section we construct as an example a class of vacua $|\widetilde{0}\rangle_f$ such that Eq. (2) is satisfied for any f while $\widetilde{G}_f$ differ for different f. This nonuniqueness is far from harmless. As we argue below it makes, in fact, expression (4) meaningless.

B. The expression for the stress-energy tensor

Expression (4) is the main result of the FKT approach. It is Eq. (4) that accomplishes changing to the universal covering, and Eq. (2) is needed only to justify it. So, let us state first that Eq. (4) does not follow (or, at least, does not follow immediately) from Eq. (2), since the following is true.

(1) To write $\lim D_{\mu\nu} \Sigma \widetilde{G}^{(1)} = \Sigma \lim D_{\mu\nu} \widetilde{G}^{(1)}$ without a special proof, one must be sure that the series $\Sigma \widetilde{G}^{(1)}$ and $\Sigma D_{\mu\nu} \widetilde{G}^{(1)}$ converge uniformly, while it is clear that they do not [at least as long as Eq. (3) holds].

I would like to stress that this is not a matter of pedantry. This nonuniformity manifests itself, in particular, in the fact that, in general, one cannot drop the nonsingular terms in $\widetilde{G}^{(1)}$. In Sec. III B we shall show that the last series in Eq. (4) can diverge *off* the Cauchy horizon even though Eqs. (3) and (2) hold and $G^{(1)\text{ren}}$ (and $\langle \mathbf{T} \rangle^{\text{ren}}$) are smooth there. The matter is that the representation of a function by a nonuniformly convergent series deceives our intuition. *One cannot judge a function from asymptotic behavior of terms of such a series.*

(2) Even when $|\widetilde{0}\rangle$ belongs to the above-mentioned class, Eq. (3) is proven not for *any* X, X', but only for X' lying in the ''sufficiently small'' neighborhood of X. It is necessary, in particular, that $\sigma(X, X')$ would be defined uniquely. To provide this, in Ref. [10], for example, X and X' are required not to lie, respectively, near points x and y connected by a null geodesic with a point conjugate to x before y. To violate this condition for the points X' and $\gamma X'$, it suffices to separate the mouths of the wormhole widely enough and to fill the space between them with the conventional matter [11, Proposition 4.4.5.].

Thus, we see that Eq. (4) must be regarded as an independent assumption. We can, however, neither use nor check it

[3] I am grateful to Dr. G. Parfyonov, who explained this issue to me.

in view of the aforementioned ambiguity. Indeed, in common with the Hadamard function, $\langle T_{\mu\nu}\rangle^{\rm ren}_{\widetilde M}$ depends on which vacuum we choose, while from the FKT standpoint all vacua $|\widetilde{0}\rangle$ satisfying Eq. (2) are equivalent. This equivalence is of a fundamental nature, the only physical object is the spacetime M, while $\widetilde M$ and $|\widetilde{0}\rangle$ are some auxiliary matters, and as long as Eq. (2) holds we cannot apply any extraneous criteria to distinguish $|\widetilde{0}\rangle$. So, we have no way of determining what to substitute in Eq. (4) as $\langle T_{\mu\nu}\rangle^{\rm ren}_{\widetilde M}$. In Sec. III B we shall see that choosing different $|\widetilde{0}\rangle$ (even when $\widetilde M$ is a part of the Minkowski plane), one can, at will, make $\langle T_{\mu\nu}\rangle^{\rm ren}_{\widetilde M}$ finite or infinite at the horizon.

Let me note in passing that there is no point in using Eq. (4) unless we decide that $|\widetilde{0}\rangle$ is among the very ''good'' and convenient vacua. For an arbitrary $|\widetilde{0}\rangle$, it is not a bit easier to find $\langle \mathbf{T}\rangle^{\rm ren}_{\widetilde M}$ than $\langle \mathbf{T}\rangle^{\rm ren}_{M}$.

C. Interpretation

Suppose that $\langle\Psi|\mathbf{T}|\Psi\rangle^{\rm ren}_M$ for some $|\Psi\rangle$ does diverge at the Cauchy horizon. Suppose further that it is $\langle\mathbf{T}\rangle^{\rm ren}_M$ that stands in the right-hand side of the Einstein equations (though it is not obvious, see [12] for the literature and discussion). Does this really mean that owing to the quantum effects the time machine M cannot be created? I think that the answer is negative. It well may be that $\langle\Phi|\mathbf{T}|\Phi\rangle^{\rm ren}_M$ does not diverge for some other state $|\Phi\rangle$. Roughly speaking, the infinite positive energy of infinitely blueshifted particles can cancel out the infinite negative energy of vacuum (for examples see the end of Sec. III A). So why must we restrict ourselves to the state $|\Psi\rangle$? To prove that the Einstein equations and QFT are incompatible in M (by M here is meant the manifold plus its causal structure), one must have proven that there is no self-consistent solutions on M at all, or, at least, that there is no solutions satisfying some reasonable physical conditions (say, stability).

III. EXAMPLES

Let us find the expectation value of the stress-energy tensor in a few specific cases. We restrict our consideration to the two-dimensional cylinder M obtained from the plane (τ,χ) by identifying $\chi \rightsquigarrow \chi+H$ and endowed with the metric

$$ds^2=C(-d\tau^2+d\chi^2)=Cdudv. \tag{5}$$

Here, $u\equiv\chi-\tau, v\equiv\chi+\tau$; C is a smooth function on M. To find in the ordinary way $\langle\mathbf{T}\rangle$ for the free real scalar field ϕ,

$$\Box\phi=0,\quad \phi(\chi+H,\tau) = \begin{cases} \phi(\chi,\tau) & \text{for the nontwisted field,} \\ -\phi(\chi,\tau) & \text{for the twisted field,}\end{cases} \tag{6}$$

we must first of all specify the vacuum we consider. That is, we must choose a linear space of solutions of Eq. (6) and an ''orthonormal'' basis [12] $U=\{u_n\}$ in it. In particular, this will define the Hadamard function:

$$G^{(1)}(X,X')=\sum_n u_n(X)u_n^*(X') + \text{ complex conjugate.}$$

A possible choice of U for the nontwisted field is

$$u_n=|4\pi n|^{-1/2}e^{2\pi i H^{-1}(n\chi-|n|\tau)},\quad n=\pm1,\pm2,\ldots. \tag{7}$$

The vacuum $|0\rangle_C$ defined by Eq. (7) (the ''conformal'' vacuum) is especially attractive as the expressions for the Hadamard function $G^{(1)}_C$ and for the stress-energy tensor $\langle\mathbf{T}\rangle_C$ are already obtained [see [12], the neighborhood of formula (6.211)]:

$$\langle T_{ww}\rangle^{\rm ren}_C=-\frac{\pi\epsilon}{12H^2}+\frac{1}{24\pi}\left[\frac{C_{,ww}}{C}-\frac{3}{2}\frac{{C_{,w}}^2}{C^2}\right],\quad w=u,v, \tag{8}$$

$$\langle T_{uv}\rangle^{\rm ren}_C=\langle T_{vu}\rangle^{\rm ren}_C=-RC/(96\pi).$$

Here, $\epsilon=-1/2$ or 1 depending on whether ϕ is twisted or untwisted, and R is the curvature of M. Though the absence of a solution corresponding to $n=0$ in Eq. (7) may seem artificial, it is, in fact, an inherent feature of $|0\rangle_C$, which is to describe the vacuum of ϕ as a massless limit of the ''natural'' vacuum of a massive field [cf. [12], below (4.220)]. One could start, however, from another vacuum for the massive field and arrive at another theory (see below) with the basis U':

$$U'=U\cup u_0\equiv(2H)^{-1/2}(F\tau+i/F),$$

where the real constant F is a free parameter. Choosing different F, we obtain different vacua $|0\rangle_F$ and Hadamard functions $G^{(1)}_F$. It is easy to see that

$$G^{(1)}_F=G^{(1)}_C+\frac{F^2}{H}\tau\tau'+\text{const.} \tag{9}$$

A. Two-dimensional time machines in the conformal vacuum state

As a first example, let us consider the Misner spacetime, which is the quadrant $\alpha<0$, $\beta>0$ of the Minkowski plane $ds^2=d\alpha d\beta$ with points identified by the rule $(\alpha_0,\beta_0) \rightsquigarrow (A\alpha_0,\beta_0/A)$. The coordinate transformation

$$u=-W^{-1}\ln|W\alpha|,\quad v=W^{-1}\ln(W\beta)$$

delivers the isometry between Misner space and M with

$$C=e^{2W\tau},\quad H=W^{-1}\ln A.$$

W here is an arbitrary parameter with dimension of mass. Substituting this in Eq. (8), we immediately find

$$\langle T_{ww'}\rangle^{\rm ren}_C=-W^2\left(\frac{\epsilon\pi}{12\ln^2 A}+\frac{1}{48\pi}\right)\delta_{ww'}.$$

The coordinates u,v are handy in evaluating $\langle\mathbf{T}\rangle^{\rm ren}$, but there is a coordinate singularity near the horizon. By contrast, in the coordinates α,β the metric is ''good'' (smooth, nondegenerate) near the Cauchy horizons $\alpha=0$ or $\beta=0$. So, the proper basis of an observer approaching one of them with a finite acceleration is related to the basis $D\equiv\{\partial_\alpha,\partial_\beta\}$ by a finite Lorentz transformation. Thus, the quantities we are to examine are, in fact, the components of $\langle\mathbf{T}\rangle_C^{\rm ren}$ in the basis D, which are

$$\langle T_{\alpha\alpha}\rangle_C^{\rm ren}=T\alpha^{-2},\quad \langle T_{\beta\beta}\rangle_C^{\rm ren}=T\beta^{-2},$$

$$\langle T_{\alpha\beta}\rangle_C^{\rm ren}=\langle T_{\beta\alpha}\rangle_C^{\rm ren}=0,$$

$$T\equiv-\left(\frac{\pi\epsilon}{12\ln^2 A}+\frac{1}{48\pi}\right).$$

Now, let us use the above simple method to find $\langle\mathbf{T}\rangle^{\rm ren}$ for two more time machines (see also [13]). Consider first the cylinder S obtained from the strip

$$ds^2=W^{-2}\xi^{-2}(-d\eta^2+d\xi^2)=\xi^{-2}d\alpha d\beta, \tag{10}$$

where $\alpha\equiv(\xi-\eta)/W,\beta\equiv(\xi+\eta)/W;\quad \eta\in(-\infty,\infty),\quad \xi\in[1,A]$ by gluing points $\eta=\eta_0,\xi=1$ with the points $\eta=A\eta_0,\xi=A$. This spacetime was considered in detail in [1] where it was called the ''standard model.'' A simple investigation shows that the Cauchy horizons $\alpha=0$ and $\beta=0$ divide S into three regions. Causality holds in the ''inner'' region $\tilde{S}:\alpha,\beta>0$ and violates in $I^{\pm}(\tilde{S})$. Introducing new coordinates u,v:

$$u\equiv W^{-1}\ln\alpha,\quad v\equiv W^{-1}\ln\beta,$$

we find that $\tilde{S}$, as well as the Misner space,[4] is isometric to M. This time

$$C=\cosh^{-2}W\tau,\quad H=W^{-1}\ln A,$$

which yield

$$\langle T_{\alpha\alpha}\rangle_C^{\rm ren}=T\alpha^{-2},\quad \langle T_{\beta\beta}\rangle_C^{\rm ren}=T\beta^{-2}, \tag{11}$$

$$\langle T_{\alpha\beta}\rangle_C^{\rm ren}=\langle T_{\beta\alpha}\rangle_C^{\rm ren}=(1/12\pi)(\alpha+\beta)^{-2}.$$

Consider lastly, the spacetime obtained by changing $\xi^{-2}\to\eta^{-2}$ in Eq. (10). This spacetime is similar to the standard model, but has a somewhat more curious causal structure; there are two causally nonconnected regions separated by the time machine. $\langle\mathbf{T}\rangle_C^{\rm ren}$ differs from that in Eq. (11) by the off-diagonal (bounded) terms

$$\langle T_{\alpha\beta}\rangle_C^{\rm ren}=\langle T_{\beta\alpha}\rangle_C^{\rm ren}=-(1/12\pi)(\alpha-\beta)^{-2}.$$

So, we see that in all three cases the vacuum energy density (associated with the conformal vacuum state) does grow infinitely as one approaches the Cauchy horizon. A few comments are necessary, however.

(1) The divergence in discussion is not at all something peculiar to the time machine: the passage to the limit $A\to\infty$ shows that precisely the same divergence [with $T=-1/(48\pi)$] takes place in $\tilde{M}$ though (in the case of Misner space) $\tilde{M}$ is merely a part of the Minkowski plane. This suggests that for the time machine too, the divergence of the stress-energy tensor is a consequence not of its causal or topological structure but rather of the unfortunate choice of the quantum state.

(2) The twisted field at $A=e^{\sqrt{2}\pi}$ has the bounded $\langle\mathbf{T}\rangle_C^{\rm ren}$ (cf. [6]).

(3) Let us consider nonvacuum states now (see Sec. II C). The first example is a two-particle state $|1_n 1_{-n}\rangle$ with the particles corresponding to the nth and $-n$th modes of Eq. (7). $\langle 1_{-n}1_n|\mathbf{T}|1_n 1_{-n}\rangle^{\rm ren}$ is readily found using [12], Eq. (2.44):

$$\langle 1_{-n}1_n|T_{\gamma\gamma}|1_n 1_{-n}\rangle^{\rm ren}=T'\gamma^{-2},$$

$$\langle 1_{-n}1_n|T_{\alpha\beta}|1_n 1_{-n}\rangle^{\rm ren}=\langle T_{\alpha\beta}\rangle_C^{\rm ren}$$

with $T'\equiv T+2\pi nH^{-2}$, and $\gamma\equiv\alpha,\beta$. Thus, we see that there *are* states with the bounded energy density of the untwisted field.

Yet another example is the equilibrium state at a nonzero temperature $|t\rangle$. Expression (4.27) of [12] gives

$$\langle t|T_{ww}|t\rangle^{\rm ren}=\langle T_{ww}\rangle_C^{\rm ren}+\frac{\pi}{2H^2}\sum_{m=1}^{\infty}\sinh^{-2}\frac{\pi m}{k_B tH}.$$

So, for any H there exists such temperature t that $\langle t|T_{\gamma\gamma}|t\rangle^{\rm ren}$ does not diverge at the horizon.

B. Another vacuum

The conformal vacuum is not suited for verifying or exemplifying most of statements made in Sec. II, since the Hadamard function does not exist in this state. So, consider now the new vacuum $|0\rangle_f$ on the plane (τ,χ) defined by the modes

$$u'_p\equiv\begin{cases}\dfrac{1}{2\sqrt{\pi\omega}}e^{ip\chi-i\omega\tau}, & \omega\geqslant\delta,\\[2mm] \dfrac{1}{2\sqrt{\pi}}e^{ip\chi}(f^{-1}\cos\omega\tau-i\omega^{-1}f\sin\omega\tau), & \omega<\delta,\end{cases} \tag{12}$$

where $\omega\equiv|p|$, δ is an arbitrary positive constant: $\delta<1$, and f is an arbitrary smooth, positive function: $f(\omega\geqslant\delta)=\sqrt{\omega}$. The modes (12) are obtained from that defining the conformal vacuum on the plane by a Bogoliubov transformation of the low-frequency modes so as to avoid the infrared divergence without affecting the ultraviolet behavior of $\langle\mathbf{T}\rangle$. The asymptotic form of $\tilde{G}_f^{(1)}$ does not depend on f:

$$\forall f\quad \tilde{G}_f^{(1)}=-1/(2\pi)\ln|\Delta u\Delta v| + \text{smooth, bounded function}. \tag{13}$$

If we retain only the first term, we obtain (in the flat case)

$$\lim_{X'\to X}D_{\alpha\alpha}\tilde{G}^{(1)}(X,\gamma^n X')=\ln^2(A)\alpha^{-2}A^{-n}n^{-2}.$$

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[4]In spite of their apparent similarity, these spaces are significantly distinct. For example, the Misner spacetime is geodesically incomplete [11], while the standard model is not [14]. This may be of importance if one would like to separate X and X' ''widely enough'' (see item 2 in Sec. II B).

So, the last series in Eq. (4) diverges not only at the horizon, but everywhere on M (cf. Sec. II B).

$\langle \mathbf{T} \rangle_f$ can be found from Eq. (9) (see [12], Sec. 6.4). For any C we have

$$\langle T_{ww} \rangle_f^{\rm ren} = \frac{1}{8\pi}\left[-\frac{W^2}{6} + \int_0^\delta (f^{-2}\omega^2 + f^2 - \omega) d\omega \right],$$

$$\langle T_{uv} \rangle_f^{\rm ren} = \langle T_{uv} \rangle_C^{\rm ren}.$$

Having taken an appropriate $f(\omega)$, one can make $\langle T_{\alpha\alpha} \rangle_f^{\rm ren}$ infinite or zero at the horizon, as we have stated in Sec. II B.

To illustrate some more statements from Sec. II, let us first find G^Σ. To this end note that it has the form

$$G^\Sigma = \sum_n \int_{-\infty}^{\infty} h(p) e^{inHp} dp + \text{c.c.} \tag{14}$$

with

$$h \equiv \begin{cases} \dfrac{1}{4\pi\omega} e^{ip\Delta\chi - i\omega\Delta\tau}, \quad \omega \geqslant 1/2, \\ \dfrac{1}{4\pi} e^{ip\Delta\chi} (f^{-2}\cos\omega\tau\cos\omega\tau' + \omega^{-2} f^2 \sin\omega\tau\sin\omega\tau'), \\ \omega < 1/2. \end{cases}$$

The function $h(p)$ can be written as a sum: $h = (h - h_0) + h_0$, where

$$h_0 \equiv \frac{1}{4\pi\sqrt{1+p^2}} e^{ip\Delta\chi} e^{-i\sqrt{1+p^2}\Delta\tau}.$$

The first summand is a smooth function falling off at infinity like p^{-2}, and the second summand (h_0) is a holomorphic (but for $p = \pm i$) function admitting the estimate

$$|h_0| \leqslant C|x|^{-1/2} e^{|(\Delta\chi - \Delta\tau) y|}.$$

Hence [15], we can apply the Poisson formula to Eq. (14) and obtain

$$G^\Sigma = 2\pi H^{-1} \sum_n h(2\pi H^{-1} n) + \text{c.c.}$$

We see thus that G^Σ is indeed the Hadamard function, and it corresponds to the vacuum $|0\rangle_F$ with $F = f(0)$.

Remark 1. This does not mean, however, that G^Σ will be a Hadamard function of some reasonable state for *any* $\tilde{G}^{(1)}$. One can easily construct, for example, such a vacuum that $\tilde{G}^{(1)}(\chi, \chi')$ will not be invariant under translations $\chi, \chi' \mapsto \chi + H, \chi' + H$ and $G^\Sigma(\chi, \chi')$, as a consequence, will not even be symmetric.

Remark 2. For all $G_f^{(1)}$ with the same $f(0)$ the Hadamard functions G^Σ are the same. This proves our statement from Sec. II A.

To find $\langle \mathbf{T} \rangle_F$, note that it differs from $\langle \mathbf{T} \rangle_C$ only by the term arising from the second summand in Eq. (9) [cf. [12], Eqs. (4.20) and (6.136)]:

$$\Delta \langle T_{ww'} \rangle^{\rm ren} = \frac{F^2}{2H} (\tau_{,w} \tau_{,w'} - 1/2\, \eta_{ww'} \eta^{\lambda\delta} \tau_{,\lambda} \tau_{,\delta}).$$

That is

$$\langle T_{ww} \rangle_F^{\rm ren} = \frac{F^2}{8H} - W^2 \left(\frac{\epsilon\pi}{12 \ln^2 A} + \frac{1}{48\pi} \right),$$

$$\langle T_{uv} \rangle_F^{\rm ren} = \langle T_{vu} \rangle_F^{\rm ren} = \langle T_{uv} \rangle_C^{\rm ren}.$$

So, for all three time machines considered here, there exists a vacuum, such that the expectation value of the stress-energy tensor is bounded in the causal region.

IV. CONCLUSION

Thus, we have seen that one cannot obtain any information about the energy density near the Cauchy horizon employing the FKT approach. Of course, it may well be that the vacuum fluctuations do make the time machine unstable, but nothing at present suggests this. All we have is a few simple examples. In some of them the energy density diverges at the horizon and in some does not. So, the time machine perhaps is stable and perhaps is not. This seems to be the most strong assertion we can make now.

Note added. A general approach different from FKT was developed recently in [16]. It was rigorously shown there that if some postulate holds in the acausal region, then $\langle \mathbf{T} \rangle$ must have singularity on the Cauchy horizon provided it is compactly generated. The character of this singularity is not specified though. Our examples show that it may well be that the energy behaves well up to the horizon. It is possible to suppose then that the singularity arises from the improper calculation of the vacuum polarization in the acausal region. In other words, it seems not unlikely that the results of [16] suggest violation of the above-mentioned postulate rather than the impossibility of time machine creation.

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