

Compact hyperbolic universe and singularities

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Recently many people have discussed the possibility that the universe is hyperbolic and was in an inflationary phase at an early stage. Under these assumptions, it is shown that the universe cannot have compact hyperbolic time slices. Though the universal covering space of the universe has a past Cauchy horizon and can be extended analytically beyond it, the extended region has densely many points which correspond to singularities of the compact universe. The result is essentially attributed to the ergodicity of the geodesic flow on a compact negatively curved manifold. The relationship with *the strong cosmic censorship conjecture* also is discussed. [S0556-2821(96)01524-X]

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I. INTRODUCTION

Recent observations of the density parameter $\Omega_0 \approx 0.1$ [1] suggest that the spatial curvature of the universe is negative. In this paper we call such a universe with a negative curvature a hyperbolic universe rather than a conventional open universe because an *open* universe is meant to be noncompact while we shall consider a *compact* negative curvature space.

On the other hand, it is believed that inflation occurred in an early stage of the universe because the inflationary scenario is, so far, the only viable theory which can solve such cosmological problems as the flatness, horizon, and monopole problems [2] without appealing to fine tuning. It is, therefore, of great interest to investigate hyperbolic inflationary universe scenarios.

During the inflation epoch, the universe is well approximated by a de Sitter space-time. Then, we notice that a hyperbolic universe has no curvature singularity as long as it is in the inflationary phase, even though its spatial volume approaches zero. The initial singularity is just a coordinate singularity. According to the scenarios of the birth of space-time by quantum tunneling from “nothing” [3], we consider that the (Lorentzian) space-time is nucleated in the de Sitter phase at its minimal radius in the whole chart. A hyperbolic inflationary universe is expected to be realized by taking hyperbolic time slices on a portion of the de Sitter space-time. In this context, any space-time which contains a hyper-

bolic inflationary universe has a past (Cauchy) horizon¹ and continues further to the past because a hyperbolic chart cannot cover the whole de Sitter space-time. This also is applied to the case of the one-bubble inflationary model [5].

Observations normally give us only local information such as the local spatial curvature of our universe, because local information is, in general, independent of global properties such as the topology of the universe. Some observational effects are, however, expected, if the periodicities due to the nontrivial topology are shorter than the horizon scale. Such possibilities have been studied in theoretical and observational cosmologies [6].

Compactness of the universe seems to be an acceptable and appealing notion, especially in the context of the canonical treatments of the universe (or quantum gravity). Compactness provides a finite value of the action integral and gives the natural boundary conditions for the matter and gravitational fields in the universe.

The notion of compactness of a given universe depends on how to take its spatial sections. We concentrate our interests on the case that hyperbolic hypersurfaces are compact. In particular, an intriguing question arises in the case that a compact hyperbolic universe is in an inflationary phase. We naturally ask whether or not such a universe has the past horizon and continues further to the past as the noncompact case. If it does, we expect that a topology change takes place and closed timelike curves appear, which is suggested by the work on the Lorentzian topology change of (2+1)-

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¹Otherwise such a universe must be born directly in the hyperbolic inflationary phase by some mechanism. Recent work by Cornish *et al.* [4] is related to such a scenario, though the mechanism itself is not discussed in detail.

dimensional compact black hole geometry by M. Siino [7]. If it does not, we may find restriction to possible topologies of the universe.

In this paper, we will study the spatially compact hyperbolic inflationary universe. We will find that the latter is indeed the case. Namely, we will show that a compact hyperbolic universe cannot be extended beyond the past horizon. Since a hyperbolic inflationary universe has a past horizon as is mentioned above, our result means that if the universe is hyperbolic and was in an inflationary phase in the early stage, the universe is not spatially compact.

In the next section, we construct models of a compact hyperbolic inflationary universe. In Sec. III, we examine the extendibility of the universe. Section IV is for conclusions and discussions.

II. A COMPACT HYPERBOLIC INFLATIONARY UNIVERSE

A. Construction of a compact universe

In this section, we treat homogeneous isotropic models of a hyperbolic universe in an inflationary phase. Such a model is described by a hyperbolic chart on four-dimensional de Sitter space-time. In the hyperbolic chart, homogeneous isotropic spacelike hypersurface is a manifold H^3 of constant negative curvature with isometry group $SO(3,1)$. This universe has a past Cauchy horizon, and in the extended region a homogeneous hypersurface becomes three-dimensional de Sitter hypersurface dS^3 , which is an orbit of $SO(3,1)$.

We construct the compact universe model by identifying points of the space-time. A compact hyperbolic hypersurface is given as the quotient space of H^3 by the discrete subgroup Γ of its isometry group $SO(3,1)$. One of the simplest three-dimensional compact hyperbolic manifolds is known as the Seifert-Weber manifold [16], whose construction is shown explicitly in Appendix A.

In general, the action of a group Γ on a manifold M must be properly discontinuous so that the quotient space M/Γ be a Hausdorff manifold. We call an action of Γ on M *properly discontinuous* if it satisfies the following conditions [8,9]: (1) each point $q \in M$ has a neighborhood U such that $\gamma(U) \cap U = \emptyset$ for each $\gamma \in \Gamma$ which is not the identity element, and (2) if $q, r \in M$ are such that there is no $\gamma \in \Gamma$ with $\gamma(q) = r$, then there are neighborhoods U and U' of q and r , respectively, such that there is no $\bar{\gamma} \in \Gamma$ with $\bar{\gamma}(U) \cap U' \neq \emptyset$. Condition (1) implies that the quotient M/Γ is a manifold, and condition (2) implies that it is Hausdorff.

It is worth noting that, for a Riemannian manifold (M, g) , every *discontinuous* group Γ of $\text{Isom}(M, g)$ is properly discontinuous [9]. Here the action of Γ on a Riemannian manifold is called discontinuous if for every $p \in M$ and every sequence $\{\gamma_i\}$ of elements of Γ , where all γ_i are mutually distinct, the sequence $\{\gamma_i(p)\}$ does not converge to a point in M [9]. We can obtain the compact hyperbolic inflationary universe because the homogeneous hypersurface H^3 is a Riemannian manifold.

B. The Misner universe

As the simplest example of a compact universe and its extension, let us see the construction of the Misner universe, which is given as a quotient manifold in (1+1) dimensions.

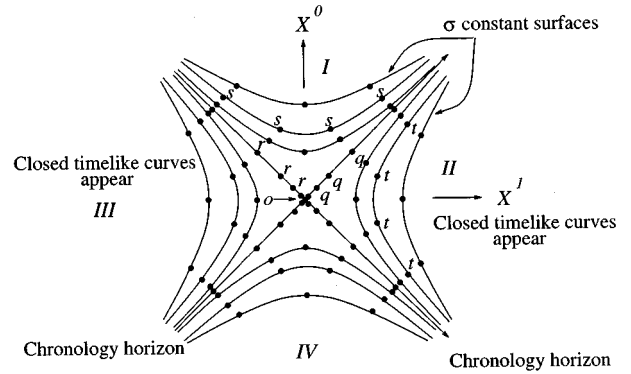


FIG. 1. The maximally extended Misner universe is illustrated. The coordinate origin o is an accumulation point of a sequence $\{\gamma^m(r)\}$. The region I is the original Misner universe region. Under the action of Γ , points s on the σ -constant surface are equivalent; similar for points r , points q , and points t , respectively. In the extended regions II , III , closed timelike curves appear.

The Misner universe (M, g_M) is a space-time diffeomorphic to $S^1 \times \mathbf{R}$. Its metric is given by

$$g_M = -t^{-1} dt^2 + td\psi^2, \quad t \in \mathbf{R}, 0 \leq \psi \leq 2\pi. \quad (2.1)$$

The universal covering manifold is the region $(I, \tilde{\eta})$ in Fig. 1, which is a part of two-dimensional Minkowski space-time $(\tilde{M}, \tilde{\eta})$. The isometry group of $(\tilde{M}, \tilde{\eta})$ is the Lorentz group $SO(1,1)$, whose orbits are the hyperbolas $(X^0)^2 - (X^1)^2 =: \sigma = \text{constant}$. The covering transformation group Γ of the Misner universe is a discrete subgroup of the Lorentz group consisting of γ^m , where m is an integer and γ maps $(X^0, X^1) \in \tilde{M}$ to

$$\begin{pmatrix} X^{0'} \\ X^{1'} \end{pmatrix} = \begin{pmatrix} \cosh \pi & \sinh \pi \\ \sinh \pi & \cosh \pi \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}. \quad (2.2)$$

This subgroup Γ acts on the region $(I, \tilde{\eta})$ properly discontinuously. However, if we extend maximally the Misner universe, for any point r on the null line $X^0 + X^1 = 0$, the sequence $\{\gamma^m(r)\}$ has an accumulation point $o := (0, 0)$ on $(\tilde{M}, \tilde{\eta})$. Hence the condition (1) of properly discontinuous is not satisfied and the quotient \tilde{M}/Γ fails to be a manifold. We, therefore, must remove the accumulation point o from the whole Minkowski space-time $(\tilde{M}, \tilde{\eta})$ so that we obtain the maximally extended Misner universe as a manifold $(\tilde{M} \setminus \{o\})/\Gamma$, which is illustrated in Fig. 1. Since the action of Γ on $\tilde{M} \setminus \{o\}$ does not satisfy the condition (2), $(\tilde{M} \setminus \{o\})/\Gamma$ is a non-Hausdorff manifold.

Because of removing the point o from $(\tilde{M}, \tilde{\eta})$, all causal curves toward o are incomplete. We define a singularity as an incomplete causal curve according to Hawking and Ellis [8]. Thus, the maximally extended Misner universe has a singularity. This indicates that the existence of the accumulation points of action of Γ causes singularities of the quotient space-time. In this paper, we call an accumulation point a *topological singularity*. The null boundaries of the original

region *I* and the extended regions *II*, *III* are compact Cauchy horizons, which are generated by closed null geodesics.

III. ANOMALOUS OCCURRENCE OF SINGULARITIES

In the case of the compact hyperbolic inflationary universe model, the discrete subgroup Γ of $\text{SO}(3,1)$ does not act on hypersurface dS^3 in the extended region properly discontinuously. Moreover, in contrast with the Misner universe, we will observe that topological singularities appear densely on the hypersurfaces dS^3 and the extended region is no longer a space-time manifold. Our main result is the following proposition.

Proposition. Suppose there exists an analytic extension of the universal cover of $(n+1)$ -dimensional compact hyperbolic universe, where $n \geq 2$. Then, for every neighborhood O_c of an arbitrary point c in the dS^n , there exist $\gamma \in \Gamma$ and point $s \in dS^n$ such that an accumulation point c' of the infinite sequence $\{\gamma^m(s)\}$ is contained in O_c .

Under the assumption of the analyticity, we naturally observe that there exists a neighborhood which contains a Cauchy horizon in the hyperbolic inflationary universe and that in a Milne universe which are analytically diffeomorphic to each other. Here the Milne universe is the Minkowski space-time $E^{n,1}$ with the hyperbolic chart. Thus it is convenient to examine the Milne universe instead of the hyperbolic inflationary universe.

Since the argument is the same for $(n+1)$ -dimensional universe with $n \geq 2$, we first present our argument in the $n=2$ case and comment on the $n \neq 3$ cases later.

Let us investigate what the action of Γ is like in the extended region of the $(2+1)$ -dimensional compact hyperbolic inflationary universe. As mentioned above, this is done in the three-dimensional Milne universe. Let $E^{2,1}$ be a three-dimensional Minkowski space-time with the metric

$$\eta = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2. \quad (3.1)$$

In the model the hyperbolic hypersurface is a two-hyperbolic space H^2 , which is embedded in $E^{2,1}$ as

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = -l^2, \quad (3.2)$$

by embedding $f: H^2 \rightarrow E^{2,1}$, where l is the curvature radius. The induced metric is given by $g_H := f^* \eta$ and whose isometry group is $\text{SO}(2,1)$.

In the extended region of the Milne universe the orbit of $\text{SO}(2,1)$ is dS^2 , which is a surface

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = l^2, \quad (3.3)$$

in $E^{2,1}$. Each element of the subgroup Γ must contain one of the boosts because otherwise the element consisted of only

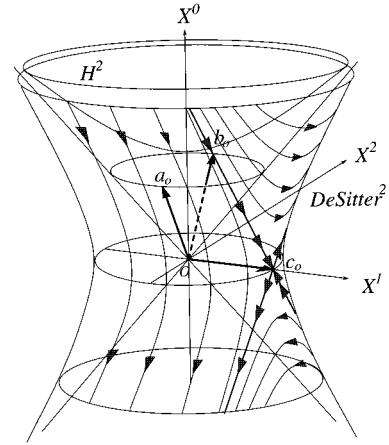


FIG. 2. Orbits of $\gamma_0^m(p), p \in dS^2$ are schematically depicted.

space-rotations would have fixed points.² We consider the action of the element $\gamma_0 \in \Gamma$ on dS^2 , where γ_0 is represented in the coordinate system (X^0, X^1, X^2) by the matrix

$$\gamma_0(\zeta) := \begin{pmatrix} \cosh \zeta & 0 & \sinh \zeta \\ 0 & 1 & 0 \\ \sinh \zeta & 0 & \cosh \zeta \end{pmatrix}, \quad \zeta = \text{const.} \quad (3.4)$$

γ_0 has three eigenvectors:

$$a_0 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad b_0 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad c_0 := l \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (3.5)$$

The two eigenvectors a_0 and b_0 are lying on the light cone,

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = 0, \quad (3.6)$$

and the point c_0 is in the dS^2 described by Eq. (3.3). Hereafter we use the same symbol to denote eigenvectors and the endpoints of the arrows of the eigenvectors. As in the maximally extended Misner universe, we observe that any point s on the null lines in the dS^2 through the point c_0 converges to c_0 by the action of γ_0^m as illustrated in Fig. 2. That is, the point c_0 is the accumulation point of the infinite sequence $\{\gamma_0^m(s)\}$. For any $\gamma (\neq \gamma_0) \in \Gamma$, $\gamma(c_0)$ is one of the eigenvectors of $\gamma_0 \gamma_0 \gamma^{-1} \in \Gamma$ and so the point $\gamma(c_0)$ in the dS^2 is also the accumulation point of the sequence $\{(\gamma_0 \gamma_0 \gamma^{-1})^m(\gamma(s))\}$. Thus, there exist many, countably infinite, accumulation points of the sequences $\{\gamma^m(s)\}$ in the dS^2 , if we consider all $\gamma \in \Gamma$.

²This implies that Γ comprises only hyperbolic elements. In the case that Γ contains parabolic elements, the quotient space H^2/Γ has a finite area while it is noncompact with cusps. Such a Γ is called geometrically finite [10]. It is known that for the geometrically finite group Γ , the geodesic flow on the unit tangent bundle of H^2/Γ is ergodic [11]. This fact ensures our arguments below holding for the case that Γ contains parabolic elements. Hereafter, for simplicity, we only discuss the case that Γ does not contain parabolic elements and H^2/Γ is compact.

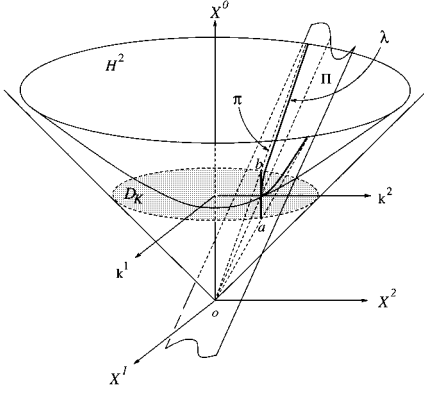


FIG. 3. Projection $\pi: E^{2,1} \rightarrow D_K$. The H^2 and D_K are identified by $\pi \circ f$.

Now we demonstrate that the accumulation points, that are topological singularities, occur densely in dS^2 .

First, for every point c in the dS^2 , there exists an element $\gamma_c \in \text{SO}(2,1)$ such that c is one of the eigenvectors of γ_c . If $\gamma_c \in \Gamma$, there exists $s \in dS^2$ such that c is the accumulation point of $\{\gamma_c^m(s)\}$.

In the case that $\gamma_c \notin \Gamma$, the other two eigenvectors, a and b , of γ_c are lying on the light cone [Eq. (3.6)]. These two vectors determine uniquely a geodesic in the H^2 , as depicted in Fig. 3. In fact, these two eigenvectors span a plane Π and the intersection $\Pi \cap H^2$ is a geodesic curve λ in the H^2 . The geodesic λ is projected to a straight line $\lambda_K := \Pi \cap D_K$ on the Klein Disk $D_K := \{(X^0, X^1, X^2) | X^0 = 1, (X^1)^2 + (X^2)^2 < 1\}$. The projection $\pi: E^{2,1} \ni (X^0, X^1, X^2) \mapsto (k_1, k_2) \in D_K$ is defined as

$$k_i := \frac{X^i}{X^0} \quad (i=1,2). \quad (3.7)$$

The induced metric takes the form

$$g_K = \frac{1}{(1-k^2)^2} [(1-k_2^2)dk_1^2 + (1-k_1^2)dk_2^2 + 2k_1k_2dk_1dk_2], \quad (3.8)$$

where $k := \sqrt{k_1^2 + k_2^2}$. We can identify the H^2 and D_K by the diffeomorphism $\pi \circ f$. This projection π maps a geodesic curve of the H^2 , which is a hyperbola in the $E^{2,1}$ to a straight line in the $E^{2,1}$. We also can identify the action of Γ on the H^2 and the action of Γ_K on D_K , where

$$\gamma_K := \pi \circ \gamma \circ \pi^{-1}, \quad \Gamma_K := \{\gamma_K | \gamma \in \Gamma\}. \quad (3.9)$$

We can see that the straight line from b to a on $\bar{D}_K := D_K \cup \partial D_K$ is the geodesic λ_K , where $\partial D_K := \{(X^0, X^1, X^2) | X^0 = 1, (X^1)^2 + (X^2)^2 = 1\}$.

As long as considering the case that $\gamma_c \notin \Gamma_K$, the geodesic λ_K is not closed on D_K/Γ_K .

It can be observed that $\Xi := \partial D_K \times \partial D_K \setminus \Delta_{\partial D_K}$ has one-to-one correspondence with the set of all geodesics on D_K , where $\Delta_{\partial D_K} := \{(p,p) | p \in \partial D_K\}$. Namely, a pair $(a,b) \in \Xi$ can be identified with the geodesic on D_K which has end-points $a, b \in \partial D_K$.

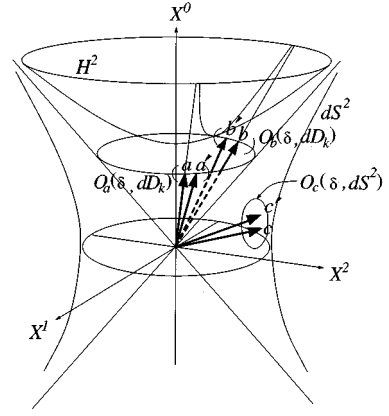


FIG. 4. Neighborhoods of points $a, b \in \partial D_K$, and $c \in dS^2$.

The eigenvectors a , b , and c are related to each other as follows. Choosing an appropriate coordinate system and a constant α , the eigenvector c can be represented as

$$c = l \begin{pmatrix} \sinh \alpha \\ \cosh \alpha \\ 0 \end{pmatrix}. \quad (3.10)$$

Then, the other two are determined uniquely as

$$a = \begin{pmatrix} \cosh \alpha \\ \sinh \alpha \\ -1 \end{pmatrix}, \quad b = \begin{pmatrix} \cosh \alpha \\ \sinh \alpha \\ 1 \end{pmatrix}, \quad (3.11)$$

except for the freedoms of the norms. Accordingly, for any point $c \in dS^2$, a geodesic λ on the H^2 (or a point in Ξ) is determined uniquely.³

Second, choose a Riemannian metric $\mathbf{e} := (dX^0)^2 + (dX^1)^2 + (dX^2)^2$ on $E^{2,1}$. Take an open ball, $B_c(\delta) := \{x \in E^{2,1} | \|x - c\|_{\mathbf{e}} < \delta\}$, of radius δ with respect to \mathbf{e} and define a neighborhood of c in the dS^2 as $O_c(\delta, dS^2) := B_c(\delta) \cap dS^2$. Similarly, define neighborhoods of a and b , respectively, in the ∂D_K as $O_a(\delta, \partial D_K) := B_a(\delta) \cap \partial D_K$ and $O_b(\delta, \partial D_K) := B_b(\delta) \cap \partial D_K$ corresponding to $O_c(\delta, dS^2)$ (see Fig. 4). Then, for any point $c' \in O_c(\delta, dS^2)$, there exist $a' \in O_a(\delta, \partial D_K)$, $b' \in O_b(\delta, \partial D_K)$, and $\gamma_{c'} \in \text{SO}(2,1)$ such that a' , b' , and c' are the eigenvectors of $\gamma_{c'}$.

Now, what we want to show is reduced to the following lemma;

Lemma. For any $(a,b) \in \Xi$ and $\delta > 0$, there exist $a' \in O_a(\delta, \partial D_K)$, $b' \in O_b(\delta, \partial D_K)$, $\gamma_K \in \Gamma_K$, and $p \in D_K$ such that

$$\lim_{n \rightarrow -\infty} \gamma_K^n(p) = a' \in O_a(\delta, \partial D_K),$$

$$\lim_{n \rightarrow -\infty} \gamma_K^n(p) = b' \in O_b(\delta, \partial D_K).$$

³If the point c is constrained on the part $\{(X^0, X^1, X^2) | X^0 > 0\} \cup \{(X^0, X^1, X^2) | X^0 = 0, (X^1)^2 + (X^2)^2 = 1, X^1 > 0\} \cup \{(0,0,1)\}$ of the dS^2 , the point c has one-to-one correspondence with a point in Ξ .

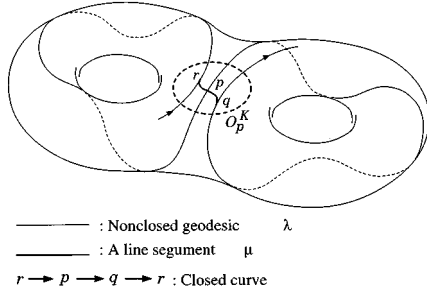


FIG. 5. A compact hyperbolic manifold D_K/Γ_K with genus 2 is illustrated. λ_K is a nonclosed geodesic and μ is a line segment which connects two points q and r on the λ_K . Then, one can see a closed curve ($r \rightarrow p \rightarrow q \rightarrow r$) composed of the line segment ($r \rightarrow p \rightarrow q$) of the geodesic λ_K and the line segment μ ($q \rightarrow r$).

Proof. We show that on the compact hyperbolic manifold D_K/Γ_K there exists a closed curve whose lift has endpoints $a' \in O_a(\delta, \partial D_K)$, $b' \in O_b(\delta, \partial D_K)$ in ∂D_K . We can make such a closed curve by using a nonclosed geodesic $\lambda_K(v) := \Pi \cap D_K$ affinely parametrized by v and ergodicity of geodesic flow on a compact manifold with negative curvature (see Appendix B).

Let p be a point on λ_K at $v=0$ and take an arbitrary small open neighborhood $O_p^K(\epsilon, D_K/\Gamma_K) := \{x \in D_K/\Gamma_K \mid d_K(x, p) < \epsilon\}$ of p , where d_K is the distance naturally defined by g_K . From the ergodicity of geodesic λ_K , for any large $N > 0$, there exist $v_1 > N$, $v_2 < -N$ such that $q := \lambda_K(v_1) \in O_p^K$ and $r := \lambda_K(v_2) \in O_p^K$ and the tangent vectors of λ_K at these points r, p, q are sufficiently parallel to each other as depicted in Fig. 5. Connecting the points r and q by a suitable line segment μ , we obtain a closed curve ($r \rightarrow p \rightarrow q \rightarrow r$) on the D_K/Γ_K . Corresponding to this closed curve, there exists an element $\gamma_K \in \Gamma_K$.

Let us fix a component \tilde{O}_p^K of the lift of O_p^K on D_K , which is diffeomorphic to O_p^K . Accordingly, $\tilde{p}, \tilde{q}, \tilde{r}, \lambda_K \cap \tilde{O}_p^K$, and $\tilde{\mu}$ denote the corresponding lifts. Any component of the lift of O_p^K is given by $\gamma_K^m(\tilde{O}_p^K)$. These are illustrated in Fig. 6. For any $N > 0$ there exist $m_1, m_2 > N$, $v_1 > 0$, and $v_2 < 0$ such that $\lambda_K(v_1) \in \gamma_K^{m_1}(\tilde{O}_p^K)$ and $\lambda_K(v_2) \in \gamma_K^{m_2}(\tilde{O}_p^K)$.

Let g_E denote the Euclidean metric on \tilde{D}_K induced by e . Then the radius of $\gamma_K^m(\tilde{O}_p^K)$ measured by g_E becomes smaller

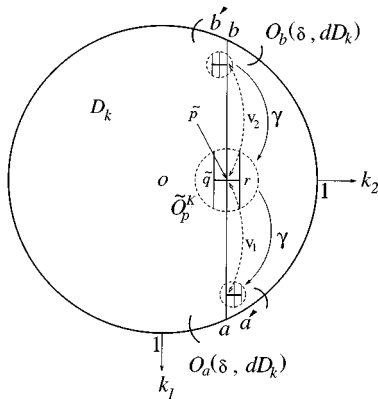


FIG. 6. A lift \tilde{O}_p^K and its images $\gamma_K^m(\tilde{O}_p^K)$ on \tilde{D}_K .

and smaller as $m \rightarrow \pm\infty$. We, therefore, have $\lim_{m \rightarrow \infty} \gamma_K^m(\tilde{O}_p^K) \subset O_a(\delta, \partial D_K)$ and $\lim_{m \rightarrow -\infty} \gamma_K^m(\tilde{O}_p^K) \subset O_b(\delta, \partial D_K)$. As a result, we obtain

$$a' := \lim_{m \rightarrow \infty} \gamma_K^m(p) \in O_a(\delta, \partial D_K),$$

$$b' := \lim_{m \rightarrow -\infty} \gamma_K^m(p) \in O_b(\delta, \partial D_K).$$

□

Proof of the Proposition. From the Lemma we have immediately the two eigenvectors a' and b' of $\gamma \in \Gamma$. Then we obtain the third eigenvector c' of γ so that the point c' , which is the accumulation point of $\{\gamma^m(s)\}$, is contained in $O_c(\delta, dS^2)$. This is the proof of $n=2$ case.

In $(3+1)$ -dimensional case, the discrete subgroup of $SO(3,1) \cong \text{Isom}(H^3)$ has four eigenvectors. Two of them are null vectors corresponding to a and b in the $(2+1)$ case. The other two corresponding to c are spacelike and direct to points in the hypersurface dS^3 . In addition, we also can observe the ergodicity of geodesic flows on three compact hyperbolic Riemannian manifold H^3/Γ [15] (see Appendix B). We obtain immediately the same result as the $(2+1)$ -dimensional case; topological singularities occur densely in the dS^3 .

The assertion is proven in a similar manner in the higher-dimensional cases. □

IV. CONCLUSIONS AND DISCUSSION

We have demonstrated that if a space-time with spatially compact hyperbolic hypersurfaces is extended analytically by extending both the universal cover and the action of $\Gamma \subset \text{Isom}(M)$ thereof, the topological singularities appear densely in the extended region and it is no longer a manifold. We conclude that the spatially compact hyperbolic inflationary universe cannot be extended beyond the Cauchy horizon of its universal cover. It follows that the universe cannot be compact, if the universe is hyperbolic and was in an inflationary phase. From observations, it is difficult to determine global properties of the universe. It is full of interest that our result answers theoretically to some global properties, for example, to the simple, but fascinating question whether our universe is spreading infinitely or compact.

Our result that the universe cannot be extended analytically across the Cauchy horizon obviously holds in the case that the spatial sections are $\mathbf{R} \times H^2/\Gamma$, which is noncompact, or its quotient manifold $S^1 \times H^2/\Gamma$. It is remarkable that this is true even if the metric is *inhomogeneous* along the \mathbf{R} (or S^1) factor. Consequently, the universe with such a topology does not have a Cauchy horizon.

As the scenario for a birth of hyperbolic universe, one-bubble inflationary universe scenario is an appealing one. However, it cannot be a universal covering manifold of a compact hyperbolic inflationary universe, because such a model has a past Cauchy horizon inside the bubble. Thus, one-bubble inflationary universe scenario is incompatible with a compact hyperbolic universe model. In other words, if the one-bubble inflationary scenario is verified by observations, our hyperbolic Friedmann-Robertson-Walker (FRW) universe is not compact. In general, any scenario which re-

alizes the hyperbolic inflationary universe by inducing a hyperbolic chart on a portion of de Sitter space-time has a Cauchy horizon for the hyperbolic hypersurfaces and, hence, it cannot be a universal cover of a compact hyperbolic inflationary universe.

As we show explicitly, our result holds for the (2+1)-dimensional case and anti-de Sitter space-time, compact three-dimensional black hole geometry does not realize Lorentzian topology change [7].

One may say that this anomaly is due to the high degrees of space-time symmetry. However we also can discuss the case that there is no symmetry. The result is essentially attributed to the ergodicity of geodesic flow on a compact manifold with negative curvature [15]. When we consider an inhomogeneous universe as a hyperbolic inflationary FRW universe model with perturbations on it, we can take a hypersurface whose sectional curvature is everywhere negative by taking a time-slice of sufficiently small scale factor, or l in Eq. (3.2) and Eq. (3.3), near the past Cauchy horizon of the background universe. If inhomogeneity due to fluctuations of matter fields is large enough, energy density of the matter fields dominates the universe and it is out of the context of inflationary universe. In such a case, an initial curvature singularity appears instead of the past Cauchy horizon. Thus, even in the case that the considering universe model is inhomogeneous, we expect that if the Cauchy horizon exists in its universal cover, there exists a neighborhood of the Cauchy horizon such that the hypersurfaces contained in it are everywhere negatively curved. The Cauchy horizon in the universal cover is considered a limit surface of a sequence of negatively curved time-slices. This would give restrictions on the possible shapes of the Cauchy horizon in the universal cover. Then if we observe that the neighborhood of the Cauchy horizon in the universal cover is homeomorphic to the neighborhood of the Cauchy horizon in the Milne universe, as considered in the previous section, we can obtain the same result of the Proposition even in the inhomogeneous case. The rigorous study of the above discussion is future work.

Our result that a compact hyperbolic universe does not admit a Cauchy horizon is closely related to the *strong cosmic censorship conjecture*, which states that physically realistic space-time is globally hyperbolic [12,13]. The case of spatially compact, locally homogeneous vacuum models has been investigated extensively by P. T. Chruściel and A. D. Rendall [14]. Our result is not restricted to the vacuum case and our approach would be useful to resolve the issue in the case of inhomogeneous universe.

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APPENDIX A: CONSTRUCTION OF A THREE-COMPACT HYPERBOLIC MANIFOLD

The three-dimensional hyperbolic space H^3 can be embedded in a four-dimensional Minkowski space-time $E^{3,1}$ as

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = -1,$$

where the curvature radius is normalized to unity. Taking the chart,

$$\begin{aligned} X^0 &= \cosh \xi, \\ X^1 &= \sinh \xi \cos \theta, \\ X^2 &= \sinh \xi \sin \theta \cos \psi, \\ X^3 &= \sinh \xi \sin \theta \sin \psi, \end{aligned} \tag{A1}$$

the induced metric takes the form

$$g_H = d\xi^2 + \sinh^2 \xi (d\theta^2 + \sin^2 \theta d\psi^2). \tag{A2}$$

$\text{Isom}(H^3)$ is $\text{SO}(3,1)$.

The simply connected Riemannian manifold H^3 can be compactified by quotienting by the subgroup Γ of its isometry $\text{SO}(3,1)$. It is known that H^3 is tessellated by hyperbolic dodecahedra. This means that the fundamental region of H^3/Γ is a hyperbolic dodecahedron [16]. Then the concrete representations of the generators of Γ are given as the following six matrices $\{T_{i=1\sim 6}\}$ under the coordinates (X^0, X^1, X^2, X^3) of $E^{3,1}$:

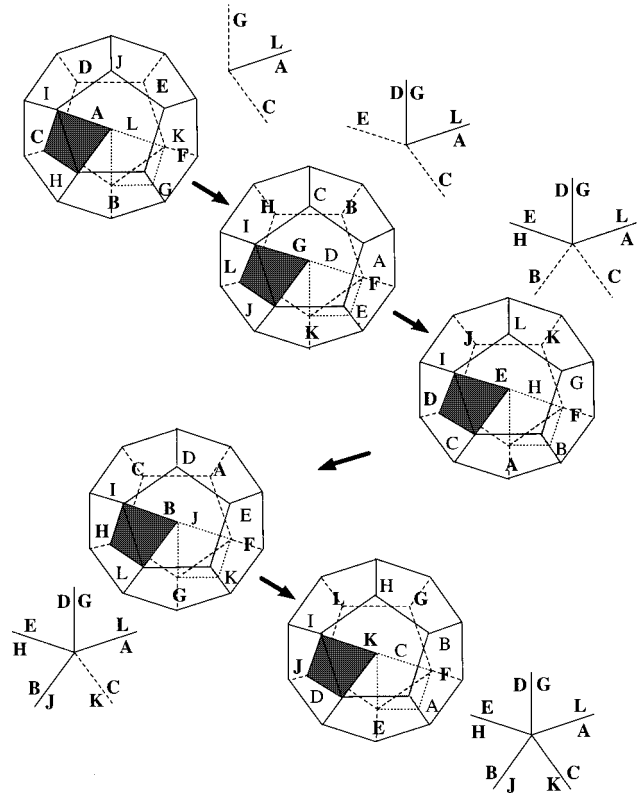


FIG. 7. Gray parts of surfaces of dodecahedra are dihedral pieces with dihedral angle $2\pi/5$. The front surfaces of a hyperbolic dodecahedron are marked with the boldfaced letters A–L. The surfaces marked with the letters A–L are behind. The dark gray triangular parts of the front surfaces are identified to the light gray triangular parts of the opposite side. Five dihedral pieces meet consistently at the identified edge.

$$T_1 := B_{0-3}(\alpha) \circ R_3 \left(\frac{3}{5} \pi \right), \quad (\text{A3})$$

$$T_2 := R_2(2\psi) \circ T_1 \circ R_2(-2\psi), \quad (\text{A4})$$

$$T_3 := R_3 \left(\frac{2}{5} \pi \right) \circ T_2 \circ R_3 \left(-\frac{2}{5} \pi \right), \quad (\text{A5})$$

$$T_4 := R_3 \left(\frac{4}{5} \pi \right) \circ T_2 \circ R_3 \left(-\frac{4}{5} \pi \right), \quad (\text{A6})$$

$$T_5 := R_3 \left(-\frac{4}{5} \pi \right) \circ T_2 \circ R_3 \left(\frac{4}{5} \pi \right), \quad (\text{A7})$$

$$T_6 := R_3 \left(-\frac{2}{5} \pi \right) \circ T_2 \circ R_3 \left(\frac{2}{5} \pi \right), \quad (\text{A8})$$

where

$$B_{0-3}(\alpha) := \begin{pmatrix} \cosh \alpha & 0 & 0 & \sinh \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \alpha & 0 & 0 & \cosh \alpha \end{pmatrix}, \quad (\text{A9})$$

$$R_2(2\psi) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & 0 & \sin 2\psi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin 2\psi & 0 & \cos 2\psi \end{pmatrix}, \quad (\text{A10})$$

$$R_3(\gamma) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma & 0 \\ 0 & \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A11})$$

$$\tanh \frac{\alpha}{2} := \frac{\sqrt{40\sqrt{5}-75}}{5}, \quad \tan 2\psi := 2. \quad (\text{A12})$$

The compact hypersurfaces H^3/Γ are constructed by the identifications;

$$X^a \stackrel{\text{identify}}{\leftrightarrow} X'^b = (T_i)^a_b X^b. \quad (\text{A13})$$

The value of the boost angle α is determined such that the H^3/Γ is a regular compact manifold. Each of the T_i transforms each surface of a dodecahedron to an opposite side after rotating by $3\pi/5$. The rotations are necessary so that five dodecahedra with dihedral angle $2\pi/5$ meet consistently at the identified edge and add up to 2π as depicted in Fig. 7.

APPENDIX B: ERGODIC THEORY

A triplet (M, μ, ϕ_t) is called an *abstract dynamical system*, where (M, μ) is a measure space and $\phi_t: M \rightarrow M$ is a one-parameter group of transformations which preserve measure μ .

Let f be a function on M . The *time-average* $f^*(x)$ of f at $x \in M$ is defined as

$$f^*(x) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(\phi_t(x)) dt,$$

which exists for almost every x [15]. The *space-average* \bar{f} is defined as

$$\bar{f} := \frac{1}{\mu(M)} \int_M f(x) d\mu.$$

A dynamical system (M, μ, ϕ_t) is *ergodic* if almost everywhere $f^*(x) = \bar{f}$, for any f which is integrable with respect to μ [i.e., $f \in L_1(M, \mu)$].

Let us derive a geometrical property implied by ergodicity. Let (M, μ, ϕ_t) be an ergodic abstract dynamical system on a compact, connected Riemannian manifold M . Let A be an open subset of M . Define a function f_A as

$$f_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Then, there exists a time-average

$$f_A^*(x) = \lim_{T \rightarrow \infty} \frac{I_A(T)}{T} \quad (\text{B1})$$

for almost every x , where $I_A(T)$ is a total length of t of the intersections of $\{\phi_t(x) | 0 \leq t \leq T\}$ and A . The space-average is given by

$$\bar{f}_A = \frac{\mu(A)}{\mu(M)}. \quad (\text{B2})$$

The ergodicity implies $f_A^*(x) = \bar{f}_A$ for any A and almost every x . From these observations, we have the following lemma.

Lemma. Let (M, μ, ϕ_t) be an ergodic dynamical system. For any $T_0 > 0$, and for almost every x and any open neighborhood A of x , there exist $T > T_0$ such that $\phi_T(x) \in A$.

Proof. Suppose there does not exist $T > T_0$ such that $\phi_T(x) \in A$, for almost every x . It follows from Eq. (B1) that $f_A^*(x) = 0$ for almost every x . On the other hand, Eq. (B2) implies that $\bar{f}_A > 0$. Thus we have $f_A^* \neq \bar{f}_A$ for almost every x , which contradicts the ergodicity. \square

Let us consider a geodesic flow on a Riemannian manifold. Let (M, g) be a compact, connected Riemannian manifold, and $T_1 M$ be its unit tangent bundle. There is a natural Riemannian metric \hat{g} on $T_1 M$ induced by g and a natural measure μ on $T_1 M$ induced by \hat{g} . Consider a geodesic λ_x on M parametrized by length τ and, which determined by $x = (p, v) \in T_1 M$ by the condition $\lambda_x(0) = p \in M$, $\dot{\lambda}_x(0) = v \in T_{1p} M$. Each λ_x has a unique lift on $T_1 M$. Considering all geodesics we can define a *geodesic flow* ϕ_τ on $T_1 M$ by $\phi_\tau(x) := (\lambda_x(\tau), \dot{\lambda}_x(\tau))$.

We essentially used the following theorem in the proof of the main proposition of this paper.

Theorem. (Lobatchewsky-Hadamard-Anosov [15]) Let M be a compact, connected Riemannian manifold with a negative curvature and T_1M be a unit tangent bundle of M , then the geodesic flow on T_1M is ergodic. From the Theorem and the Lemma above, we have the following, which is actually used in our proof.

Proposition. Let M be a compact, connected Riemannian manifold with a negative curvature and let λ_x be a geodesic defined above. For any $T_0 > 0$, and for almost every $x = (p, v)$ and any open neighborhood O of p in M , there exists $T > T_0$ such that $\lambda_x(T) \in O$ and the (unit) tangent vector $\dot{\lambda}_x(T)$ is arbitrarily close to $\dot{\lambda}_x(0)$.

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