Limits on a stochastic background of gravitational waves from gravitational lensing

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We compute the effects of a stochastic background of gravitational waves on multiply imaged systems or on weak lensing. There are two possible observable effects: a static relative deflection of images or shear, and an induced time-dependent shift or proper motion. We evaluate the rms magnitude of these effects for a COBE normalized, scale-invariant spectrum, which is an upper limit on spectra produced by inflation. Previous work has shown that large-scale structure may cause a relative deflection large enough to affect observations, but we find that the corresponding effect of gravity waves is smaller by $\sim 10^4$ and so cannot be observed. This results from the oscillation in time as well as the redshifting of the amplitude of gravity waves. We estimate the magnitude of the proper motion induced by deflection of light due to large-scale structure, and find it to be $\sim 10^{-8}$ arcsec per year. This corresponds to ~ 50 km/s at cosmological distances, which is quite small compared to typical peculiar velocities. The COBE normalized gravity wave spectrum produces motions smaller still by $\sim 10^2$. We conclude that light deflection due to these cosmological perturbations cannot produce observable proper motions of lensed images. On the other hand, there are only a few known observational limits on a stochastic background of gravity waves at shorter, astrophysical wavelengths. We calculate the expected magnitudes of the effects of lensing by gravity waves of such wavelengths, and find that they are too small to yield interesting limits on the energy density of gravity waves. [S0556-2821(96)04724-8]

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I. INTRODUCTION

Events in the early Universe may have left a stochastic background of gravitational waves (GW's). In particular, a generic prediction of inflation is a relic spectrum of GW's [1]. Detecting these elusive remnants would not only establish this prediction of general relativity, but also serve as a critical test for inflation. While the predicted background may be too weak for direct detection [2], it could be detected indirectly through its effect on light propagation in the Universe. Even if the effects of GW's cannot be distinguished observationally from other effects, observers who assume no GW's might reach incorrect conclusions about the distribution of matter in the Universe.

Gravitational lensing is one of the most promising methods of mapping the distribution of matter at cosmological distances. Detailed observations of multiple images of quasars have been used to try to reconstruct the lensing mass distribution (e.g., [3]). It has also long been recognized that measurements of the time delay between images can be used to determine the Hubble constant [4]. Gravitational lenses and sources, however, typically lie at significant redshifts. Light rays are thus deflected by large-scale structure (LSS) and GW's as they traverse the cosmological distance to the observer, and these deflections may change the simple lensing picture.

GW's may be produced by many sources. Astrophysical sources, such as close binary systems which include a neutron star or black hole, radiate GW's, and numerous individual sources may superpose to create a stochastic background. At the Planck time, quantum fluctuations in the metric are significant and may produce gravitons. Phase transitions in the universe may lead to topological defects such as cosmic strings, which generate GW's. A period of inflation may leave behind a significant amount of GW's. Whatever the source, any spectrum which extends over wavelengths comparable to the present horizon would contribute to the quadrupole anisotropy of the cosmic microwave background (CMB) [5]. Such a spectrum is therefore limited by the anisotropy measured by the cosmic background explorer (COBE) Differential Microwave Radiometer (DMR) experiment [6]. For our calculations we adopt a scale-invariant primordial spectrum, i.e., one which has constant energy density per logarithmic frequency, which we assume produces the entire measured quadrupole anisotropy. Inflationary models predict slightly tilted spectra which are responsible only for some fraction of the anisotropy [5,2], and so are generally weaker than our adopted case.

In inflation, GW's are produced in conjunction with density fluctuations. The initial nearly-scale-invariant power spectrum of density fluctuations evolves as modes reenter the horizon after inflation, and as structure later forms in a universe dominated by dark matter. The present spectrum is strongly constrained by galaxy and cluster surveys, and can be used to study the effects of LSS on lensing. The induced effects are small but potentially observable. In weak lensing, the effect is a coherent distortion of background galaxies by an ellipticity of the order of a few percent [7,8]. In strong lensing, the primary effect is an external shear which may be significant for observed four-image systems [9,10].

In general, the influence at a given time of a weak metric perturbation on light propagation is simply described by two effects. Their magnitudes were estimated for LSS in Ref. [10], which we summarize here. The first effect is a constant deflection, the same for all nearby light rays. This deflection simply displaces the ''true'' angular position of an observed lens or source, and is not directly observable. In the case of LSS, deflections from coherent structures of size ~ 1 Mpc add up in a random walk, giving an overall deflection of

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order a few arcminutes at redshift 1, which scales as the square root of comoving distance r. The second effect is a *relative* deflection between nearby light rays, which produces a focusing and shear with observable effects on weak and strong lensing. For two rays at initial angle θ, each coherent structure at a distance r causes a relative deflection proportional to their separation of $\approx r\theta$. The additional random walk gives a relative angular fluctuation of $\approx 0.07\theta$ at redshift 1, which scales as $r^{3/2}$.

It was suggested in Ref. [11] that gravity waves could significantly affect the time delays in a multiply imaged system. It was later pointed out [12] that a correct analysis must include the lensing constraint, i.e., the fact that image rays in the presence of GW's follow different paths than for no GW's, so that all rays go from the source to a common destination, the observer. These later authors also showed that both LSS and GW's have no observable effects on lensing, to lowest order. However, in their lowest order expansion they assumed that two image rays that are observed at an angular separation θ are separated by a distance of exactly $r\theta$ on the lens plane at a distance r. In other words, they neglected the relative deflection between light rays, and therefore only included an overall, constant deflection due to LSS or GW's.

We can easily see why this assumption leads to no observable effects. In the absence of metric perturbations, we can write the lens equation for a thin lens as (e.g., [13]) $\vec{\beta}=\vec{\theta}-\vec{\alpha}_{\rm lens}(\vec{\theta})$, where $\vec{\theta}$ and $\vec{\beta}$ are the image and source angles, respectively, and $\vec{\alpha}_{\rm lens}$ is the scaled deflection angle, which is determined by the mass distribution of the lens. If we neglect relative deflections, then LSS or GW's can only cause an angular shift $\vec{\alpha}_L$ between the observer and the lens, and a shift $\vec{\alpha}_{LS}$ between the lens and the source. Then the lens equation becomes $\vec{\beta}=\vec{\theta}-\vec{\alpha}_{\rm lens}(\vec{\theta})+\vec{\alpha}_S$, where $\vec{\theta}$ is now measured relative to the *observed* (and shifted) lens position, and $\vec{\alpha}_S$ involves $\vec{\alpha}_L$ and $\vec{\alpha}_{LS}$ (see Sec. III for the full details). The constant (i.e., $\vec{\theta}$-independent) deflection $\vec{\alpha}_S$ has no effect on any observables of the lens system (e.g., [13]), since $\vec{\beta}$ is not directly observable. Fermat's principle then implies that the lens equation must be equivalent to $\partial\Delta t/\partial\vec{\theta}=0$ at fixed $\vec{\beta}$, where Δt is the relative time delay. There is thus no observable effect on the time delay, either, since it can be derived from the lens equation, up to (unobservable) $\vec{\theta}$-independent terms.

This approximation of neglecting the relative deflection may not be a good one. Indeed, such deflection can have observational consequences, which may be sufficiently large to detect in the case of LSS [9,10]. In this paper, we compute the rms total and relative deflections between light rays induced by a scale-invariant stochastic background of GW's. Unlike LSS, GW's oscillate with time, and so the effect of short wavelength modes does not amplify, as light rays deflect one way in crests and the opposite way in troughs. In addition, the energy density and thus also the amplitude of subhorizon GW's redshift away as the universe expands. The lensing effect is thus dominated by wavelengths on the scale of the distance to the source. Each such mode acts as a single coherent structure, and so both the total and relative deflections due to GW's scale approximately linearly with distance. The effect of different modes must be convolved with a particular power spectrum and include the above-mentioned decay of each mode as the universe expands. We find simple integral expressions for the scale-invariant spectrum. The total and relative deflections are smaller than those caused by large-scale structure by factors of the order of 10^2 and 10^4, respectively. We do not need to explicitly set up the lens equation, since the rms shear in the lens equation is directly related to the rms relative deflection of light rays, which we calculate. This fact was demonstrated for LSS in Ref. [9], and we give a general proof in Sec. III below. Our results imply that the static effects of the GW spectrum on lensing are negligible compared to those of LSS, and cannot be detected in practice.

In addition to the static effects of LSS and GW's on lensing, it is possible that the fluctuation in the induced deflection with time would be directly manifested as an observed proper motion of images. In other words, the sources do not really move but the light rays from the sources are deflected and so the sources appear to move. We find that even LSS can only produce motions of order 10^{-8} arcsec per year from this effect. This corresponds to ~ 50 km/s at a distance of a Gpc, and the effect of GW's is smaller still by a factor of $\sim 10^2$. Since typical peculiar velocities are much larger, the proper motion induced by deflection of light due to LSS is unobservable, and the same is true for the COBE-normalized scale-invariant spectrum of GW's.

However, we may try to use shear or proper motions of imaged sources to improve existing limits on stochastic GW's at a range of astrophysical wavelengths. There are only a few such limits known: Single-pulsar timing yields $\Omega_\lambda < 1\times 10^{-8}$ at $\lambda\approx 2$ pc [14,15], binary pulsar timing implies $\Omega_\lambda < 0.04$ over $\lambda\approx 2$ pc to 1 kpc and $\Omega_\lambda < 0.5$ up to 10 kpc [14,16], and the observed angular correlation function of galaxies sets a limit of $\Omega_\lambda < 10^{-3}$ over $\lambda\approx 100$ kpc to 100 Mpc [17]. These limits apply to any stochastic background of GW's, whether cosmological in origin or generated at low redshift as a superposition of many discrete sources. For a cosmological spectrum that existed at early times, there are also big bang nucleosynthesis constraints of $\Omega_\lambda < 10^{-4}$ for $\lambda < 100$ pc [18] and CMB limits of $\Omega_\lambda < 10^{-12}$ at horizon wavelengths (from COBE) and $\Omega_\lambda < 10^{-8}$ for $\lambda > 1$ Mpc from small-scale anisotropy [19].

In Ref. [17] it was suggested that highly magnified lensed sources could increase the sensitivity to detecting proper motions due to GW's. The angular deviations induced by GW's produced by an individual source were discussed in Ref. [20]. Reference [21] considered detecting proper motions (of unlensed sources) due to GW's through very long baseline interferometry (VLBI) measurements, but our approach is simpler than theirs. For an image of a lensed source, only an angular deflection of the source *relative* to the lens is easily observed, and we find that this relative motion is small when we assume an isotropic GW background. Thus we do not find an interesting limit on the energy density.

II. FORMALISM

In this section we review the formalism describing gravity waves, their cosmological evolution, and their effect on lensing, as well as the usual formalism of gravitational lensing.

We work in the framework of a flat Robertson-Walker metric with small-amplitude tensor metric fluctuations. For weak perturbations, we can consider the effect of GW's without including LSS, since the cross terms between them would be of higher order. In comoving coordinates we can write the line element as

$$ds^2=a^2(\tau)[-d\tau^2+(\delta_{ij}+h_{ij})dx^i dx^j]. \tag{1}$$

Here τ is the conformal time, $a(\tau)$ the expansion factor, and we have set $c=1$. We expand the metric perturbation in plane waves ($k=2\pi/\lambda$):

$$h_{lm}(\vec{x},\tau)=\int d^3k\, h^n(\vec{k},\tau)\epsilon^n_{lm}(\hat{k})e^{-i\vec{k}\cdot\vec{x}}, \tag{2}$$

where ϵ^n_{lm} is the polarization tensor which depends on the direction $\hat{k}$ (l and m are spatial indices ranging from 1 to 3, while n ranges over the polarization components $+,\times$). For a wave propagating in the z direction, the nonvanishing components are, in the x and y rows and columns,

$$\epsilon^+_{lm}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \quad \epsilon^\times_{lm}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

For other propagation directions $\vec{k}$, we rotate $\epsilon\rightarrow R\epsilon R^{\mathrm{T}}$, with R the standard 3×3 rotation matrix.

GW's with a given wave vector $\vec{k}$ are produced during inflation and then stretched outside the horizon. The amplitude is constant outside the horizon, but once a mode reenters its energy redshifts as a^{-4}. For the inflationary spectrum the effect of very short wavelength modes is negligible, and so we can assume that all modes enter during the matter-dominated era, for which the exact time evolution is given [1,2] by a spherical Bessel function, $3j_1(k\tau)/(k\tau)$. This time evolution is also correct for all modes long after matter-radiation equality. Inflation produces Gaussian, stochastic perturbations. The Fourier components have zero ensemble mean and a covariance

$$\langle h^i(\vec{k},\tau_1)h^j(\vec{q},\tau_2)\rangle=A_{\mathrm{T}}k^{-3}\left[\frac{3j_1(k\tau_1)}{k\tau_1}\right]\left[\frac{3j_1(k\tau_2)}{k\tau_2}\right] \times\delta^3(\vec{k}+\vec{q})\,\delta_{ij}, \tag{3}$$

for the scale-invariant k^{-3} spectrum. Note that we do not assume the short wavelength approximation $h^i(\vec{k},\tau) \propto a^{-1}(\tau)e^{ik\tau}$. The contribution to Ω at the present (averaged over several periods) is

$$\Omega_\lambda=\frac{d\Omega_{\mathrm{GW}}}{d\ln k}=\frac{3\pi}{2}A_{\mathrm{T}}(k\tau_0)^{-2}, \tag{4}$$

where $\tau_0=2H_0^{-1}$ is the present value of τ, and throughout we set $H_0=75$ km sec^{-1} Mpc^{-1}. Normalization to the full CMB quadrupole anisotropy gives $A_{\mathrm{T}}=6\times 10^{-11}$.

Consider a photon emitted from a source toward an observer at the origin, with the photon's final direction defined as (minus) the z axis. We use r to denote values of the z coordinate (with z_S denoting the source redshift, not its z coordinate). GW's affect the distance-redshift relation, but this effect is separate from that of the angular deflections which we are interested in, and it introduces only small additional corrections in these quantities [17]. We can thus neglect this effect, and assume that the photon path obeys $r(\tau)=\tau_0-\tau$. In a flat, matter-dominated universe, $r_S=2H_0^{-1}[1-(1+z_S)^{-1/2}]$. The components perpendicular to the z axis of the photon direction obey [17]

$$\frac{dx^i}{d\tau}(\tau)-\frac{dx^i}{d\tau}(\tau_0)=h_{zi}(\tau)-h_{zi}(\tau_0)-\frac{1}{2}\int_\tau^{\tau_0}\nabla_i h_{zz}(\tau')d\tau'. \tag{5}$$

Integrating this we find, for the perpendicular components of the position [with respect to $\vec{x}(\tau_0)=0$],

$$x^i(\tau)=\int_\tau^{\tau_0}\left[\frac{1}{2}(\tau'-\tau)\nabla_i h_{zz}(\tau')+h_{zi}(\tau_0)-h_{zi}(\tau')\right]d\tau'. \tag{6}$$

We define a (two-component) angle $\beta^i=x^i(\tau)/r(\tau)$.

In gravitational lensing with a primary thin lens at a distance r_L (but no LSS or GW) the lens equation is (e.g., [13])

$$\vec{\beta}=\vec{\theta}-\vec{\alpha}_{\mathrm{lens}}(\vec{\theta}), \tag{7}$$

where $\vec{\theta}$ is the observed image angle, $\vec{\beta}$ is the source angle (defined as $\vec{x}_S/r_S$, in terms of the perpendicular position of the source), and $\vec{\alpha}_{\mathrm{lens}}$ is the deflection angle scaled by r_{LS}/r_S (we define $r_{LS}=r_S-r_L$). In this case, the fiducial z axis is defined to be in the observed direction of the lens. The distortion of the image of a small source is given by the inverse of the Jacobian matrix:

$$\frac{\partial\beta^i}{\partial\theta^j}=\delta^{ij}-\Psi^{ij}, \tag{8}$$

where Ψ^{ij} is also termed the shear tensor of the lens.

III. SHEAR INDUCED BY GW's ON LENSING

In this section we follow the approach used for LSS in Ref. [10]; i.e., we compute some of the same quantities for GW's and compare the results. As stated in Sec. I, we do not need to include a lens explicitly, as we now justify. In the presence of a metric perturbation, but without a primary lens, the lens equation has the form $\vec{\beta}=\vec{\theta}-\vec{\alpha}_{\mathrm{OS}}(\vec{\theta})$, where $\vec{\alpha}_{\mathrm{OS}}$ results from the accumulated deflection between the observer and the source. As defined in Sec. II, the shear tensor for an image at $\vec{\theta}$ due to the perturbation is $F^{ij}=\partial\alpha^i_{\mathrm{OS}}(\vec{\theta})/\partial\theta^j$. On the other hand, the relative deflection at $\vec{\theta}$ between two rays separated by a tiny angle $\vec{\gamma}$ is $\vec{\alpha}_{\mathrm{OS}}(\vec{\theta}+\vec{\gamma})-\vec{\alpha}_{\mathrm{OS}}(\vec{\theta})$. We denote the rms of this quantity by $\sigma_{\Delta\beta}$. We average over directions of $\vec{\gamma}$ (which for this calculation is equivalent to fixing $\vec{\gamma}$ and assuming that F^{ij} is isotropic) and take $\gamma\rightarrow 0$, obtaining the relation[1]

[1]Repeated indices are summed over the x and y directions. There is no distinction between upper and lower indices.

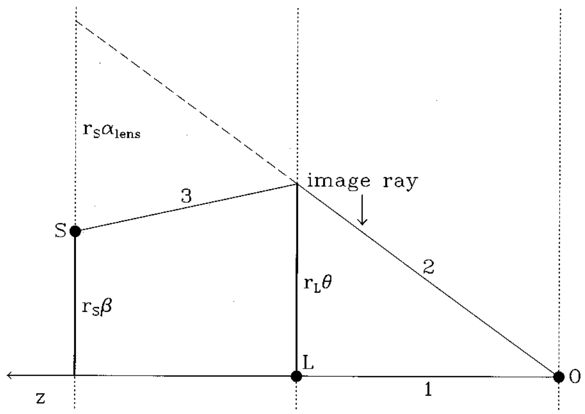

FIG. 1. Sketch showing positions of the observer, lens, and source, as well as an image ray and several comoving distances.

$$\left(\frac{\sigma_{\Delta\beta}}{\gamma}\right)^2=\frac{1}{2}\langle F^{ij}F_{ij}\rangle, \tag{9}$$

all evaluated at position $\vec{\theta}$. Thus, $\sigma_{\Delta\beta}/\gamma$ yields an estimate of the magnitude of the shear tensor. Indeed, it fully characterizes rms values of F^{ij}, since, for an isotropic field,

$$\langle F_{ij}F_{kl}\rangle=\tfrac{1}{8}[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}]\langle F^{mn}F_{mn}\rangle. \tag{10}$$

If we also include a primary lens, in the lens equation we simply add up all deflections linearly, assuming all deflections are small. For the primary lens alone, we have Eq. (7). Figure 1 shows this setup schematically. In the presence of a metric perturbation, we trace a light ray that is observed at $r=0$ to come from the direction $\vec{\theta}$, back to the source. We find a different form for the lens equation:

$$\vec{\beta}=\vec{\theta}-\vec{\alpha}_{\rm OS}^{(2,3)}-\vec{\alpha}_{\rm lens}(\vec{\theta}-\vec{\alpha}_{\rm OL}^{(2)}). \tag{11}$$

Here $\vec{\alpha}_{\rm OS}^{(2,3)}$ refers to the integrated deflection caused by LSS or GW's along the paths labeled 2 and 3 in Fig. 1, defined so that the total induced change in $\vec{x}(\tau_S)$ equals $r_S\vec{\alpha}_{\rm OS}^{(2,3)}$. Similarly, $r_L\vec{\alpha}_{\rm OL}^{(2)}$ is the induced change in $\vec{x}(\tau_L)$. In integrating the deflections along the unperturbed paths 2 and 3, we are assuming that the relative deflections due to LSS or GW's are small compared to θ and $\alpha_{\rm lens}$, which is true for the cases which we consider below.

When the perturbation is included, $\vec{\theta}$ is no longer an observable, since it is measured with respect to the unperturbed position of the lens. The observed position of the lens (whose actual position has not changed) is now $\vec{\theta}_{\rm lens}=\vec{\alpha}_{\rm OL}^{(1)}$, and so the lens equation in terms of the observable $\vec{\theta}'=\vec{\theta}-\vec{\theta}_{\rm lens}$ is

$$\vec{\beta}=\vec{\theta}'+\vec{\alpha}_{\rm OL}^{(1)}-\vec{\alpha}_{\rm OS}^{(2,3)}-\vec{\alpha}_{\rm lens}(\vec{\theta}'+\vec{\alpha}_{\rm OL}^{(1)}-\vec{\alpha}_{\rm OL}^{(2)}). \tag{12}$$

If we now calculate the shear tensor resulting from Eq. (12), it will contain the shear of the primary lens, shear terms from the perturbation, and also cross terms. For simplicity, in the case of GW's we only estimate one characteristic magnitude, that of the shear resulting from $\vec{\alpha}_{\rm OS}^{(2,3)}$, by evaluating the corresponding $\sigma_{\Delta\beta}/\gamma$. In Ref. [9] all the different shear terms were studied for LSS, showing that $\sigma_{\Delta\beta}/\gamma$ indeed estimates the relative magnitude of the various corrections due to LSS. Since we find that $\sigma_{\Delta\beta}/\gamma$ is much smaller for GW's, we do not have any motive to explore Eq. (12) further in this case. Instead of the path (2),(3), we may use a straight path from $r=0$ to $r=r_S$ to evaluate the rms of various quantities in this section, since $\alpha_{\rm lens}\ll 1$ and so the components of vectors and tensors as well as the relative distances of points along the path (both of which enter into the rms calculations) are unchanged [except for $O(\alpha_{\rm lens})$ corrections]. Thus we only need to consider the effect of GW's in the absence of a primary lens.

Consider first a single light ray with $\vec{\theta}=0$. In the absence of GW's (or LSS) it would follow the straight line $x^i(\tau)=0$ for all τ. We now include the effect of GW's, and compute the rms fluctuation in the photon's perpendicular displacement at the source, $\sigma_\beta=\langle\vec{\beta}(\tau_S)\cdot\vec{\beta}(\tau_S)\rangle^{1/2}$. This is a measure of the common deflection of all image rays, and is therefore not observable, but it is useful for the calculations that follow. We use Eq. (6) and convert the expression to Fourier space. Consider first only the h_{zz} term, whose contribution to σ_β^2 we denote $\sigma_{\beta,a}^2$. The polarization gives $(\epsilon_{zz}^+)^2+(\epsilon_{zz}^\times)^2=\sin^4\theta_k$, where $\vec{k}=(k,\theta_k,\phi_k)$ in spherical coordinates. Performing the angular $\vec{k}$ integrations then yields

$$\sigma_{\beta,a}^2=\frac{432\pi}{r_S^2}A_{\rm T}\int_{\tau_S}^{\tau_0}d\tau_1\int_{\tau_S}^{\tau_0}d\tau_2\int_0^\infty dk\,k(\tau_1-\tau_S)(\tau_2-\tau_S)$$
$$\times\frac{j_1(k\tau_1)}{k\tau_1}\frac{j_1(k\tau_2)}{k\tau_2}\frac{j_3[k(\tau_1-\tau_2)]}{[k(\tau_1-\tau_2)]^3}.$$

The $j_3[k(\tau_1-\tau_2)]/[k(\tau_1-\tau_2)]^3$ term represents a further suppression of short wavelength modes due to phase cancellations among different waves in the assumed isotropic stochastic background. Letting $s=k\tau_1$ and $q=\tau_2/\tau_1$, we can simplify this expression to a double integral,

$$\sigma_{\beta,a}^2=\frac{864\pi}{r_S^2}A_{\rm T}\int_{q_S}^1\left\{q\tau_0\left(\frac{1}{2}\tau_0-\tau_S\right)-\tau_S r_S\right.$$
$$\left.+\tau_S^2\left[\frac{1}{2q}+\ln(q/q_S)\right]\right\}W(q)dq, \tag{13}$$

where $q_S=\tau_S/\tau_0$, and with $u\equiv(1-q)s$ we define

$$W(q)=\int_0^\infty\frac{j_1(s)}{s}\frac{j_1(qs)}{qs}\frac{j_3(u)}{u^3}s\,ds. \tag{14}$$

Similarly, the contribution of the h_{zi} terms of Eq. (6) is

$$\sigma_{\beta,b}^2=\frac{144\pi}{r_S^2}A_{\rm T}\int_{q_S}^1\left[\frac{\tau_S^2}{2q^2}+\tau_0\left(\frac{1}{2}\tau_0-\tau_S\right)\right]G(q)dq, \tag{15}$$

where

$$G(q)=\int_0^\infty\left\{\frac{1}{5}\left[\frac{j_1(s)}{s}\right]^2+\frac{1}{5}\left[\frac{j_1(qs)}{qs}\right]^2\right.$$
$$\left.-2\frac{j_1(s)}{s}\frac{j_1(qs)}{qs}\left[\frac{j_1(u)}{u}-2\frac{j_2(u)}{u^2}\right]\right\}\frac{ds}{s}. \tag{16}$$

Integrating over angles gives a zero cross term, and so $\sigma_\beta^2=\sigma_{\beta,a}^2+\sigma_{\beta,b}^2$ Numerically, we find that $\sigma_\beta=5\times10^{-6}$ $(z_S=1)$, $9\times10^{-6}(z_S=3)$. This is much smaller than the estimates for LSS [10], 6×10^{-4} $(z_S=1)$, 7×10^{-4} $(z_S=3)$.

To estimate the relative deflection between rays at $\vec{\theta}=0$, we choose two directions (labeled A and B) separated at the observer by an infinitesimal angle γ, and find the rms difference between the deflections due to GW's in these two directions, $\sigma_{\Delta\beta}=\langle[\vec{\beta}_A(\tau_S)-\vec{\beta}_B(\tau_S)]^2\rangle^{1/2}$. We cannot evaluate this with the method used for LSS, which assumes that horizon size modes are negligible [8]. Instead we must calculate $\sigma_{\Delta\beta}^2$ explicitly and keep all the terms to lowest order in γ, i.e., quadratic order. These include terms which come from multiplying polarization components for the different directions A and B. The final result is

$$(\sigma_{\Delta\beta}/\gamma)^2=4\sigma_{\beta,a}^2+\sigma_{\Delta\beta,a}^2+\sigma_{\Delta\beta,b}^2+\sigma_{\Delta\beta,c}^2, \tag{17}$$

where

$$\sigma_{\Delta\beta,a}^2=\frac{576\pi}{r_S^2}A_{\mathrm{T}}\int_{q_S}^1\left[\frac{\tau_0\tau_S}{q}(1+q)^2-\left(\tau_0^2+\frac{\tau_S^2}{q}\right)(1+q)\right.$$
$$\left.+(\tau_0+\tau_S)r_S\ln(q/q_S)\right]I_1(q)dq,$$

$$\sigma_{\Delta\beta,b}^2=\frac{1728\pi}{r_S^2}A_{\mathrm{T}}\int_{q_S}^1\{(\tau_0+\tau_S)^2-\tau_0^2(1+q)^2$$
$$+2q\tau_S(\tau_S-q\tau_0)+[\tau_S\tau_0(1+q^2)$$
$$+q(\tau_0+\tau_S)^2]\ln(q/q_S)\}I_2(q)dq,$$

$$\sigma_{\Delta\beta,c}^2=\frac{288\pi}{r_S^2}A_{\mathrm{T}}\int_{q_S}^1\left[\frac{\tau_S^2}{2q^2}+\tau_0\left(\frac{1}{2}\tau_0-\tau_S\right)\right]I_3(q)dq,$$

$$I_1(q)=\int_0^\infty\frac{j_1(s)}{s}\,\frac{j_1(qs)}{qs}\left[\frac{j_2(u)}{u^2}-3\frac{j_3(u)}{u^3}\right]s\,ds,$$

$$I_2(q)=\int_0^\infty\frac{j_1(s)}{s}\,\frac{j_1(qs)}{qs}\,\frac{j_4(u)}{u^4}s^3\,ds,$$

$$I_3(q)=\int_0^\infty\left\{\frac{2}{15}\left[\frac{j_1(s)}{s}\right]^2+\frac{2}{15}\left[\frac{j_1(qs)}{qs}\right]^2\right.$$
$$\left.-2\frac{j_1(s)}{s}\,\frac{j_1(qs)}{qs}\left[\frac{j_1(u)}{u}-3\frac{j_2(u)}{u^2}\right]\right\}\frac{ds}{s}. \tag{18}$$

Then $\sigma_{\Delta\beta}/\gamma=7\times10^{-6}$ $(z_S=1)$, 1.3×10^{-5} $(z_S=3)$. By contrast, LSS gives a $\sigma_{\Delta\beta}/\gamma=0.07$ $(z_S=1)$,0.14 $(z_S=3)$. For LSS, the relative deflection is greatly increased by coherent deflections for short wavelength modes, but for GW the effect of these modes is cut off by the redshifting as well as the temporal oscillations. We also used the relation $kr_S\gamma\ll1$ in the calculation of $\sigma_{\Delta\beta}$. The reason we find a $\sigma_{\Delta\beta}$ of order $\gamma\sigma_\beta$ is that long wavelength modes overlap over the two light rays, and the relative deflection is small compared to the total deflection. Indeed, a Taylor expansion suggests that in general $\sigma_{\Delta\beta}/\gamma\sim kr_S\sigma_\beta$, and $kr_S\sim1$ is dominant for this GW spectrum. As shown above, the shear tensor (which is also used in weak lensing) is closely related to $\sigma_{\Delta\beta}/\gamma$, and so the mean square ellipticity at a point induced by GW's is of order 10^{-5}, again negligible compared to the few percent expected from LSS (e.g., [8]).

We can also try to derive general limits on GW's at astrophysical wavelengths from the induced shear. To obtain a limit on Ω_λ, we compute the $\sigma_{\Delta\beta}/\gamma$ produced by an isotropic background of GW's at a single wave number k. Note that for modes at a given k, we can use Eqs. (3) and (4) even for short wavelengths (with A_{T} a normalization factor, separate for each k), for times τ long after matter-radiation equality. Since GW's at horizon wavelengths are already strongly constrained by the CMB as noted above, we restrict our calculation to the case $kr_S\gg1$, in which case the h_{zi} terms in Eq. (6) can be neglected. We can estimate from Eq. (6) that in order of magnitude σ_β^2 should equal $A_{\mathrm{T}}/(k\tau_0)^4$, and thus that $\sigma_{\Delta\beta}^2/\gamma^2\sim A_{\mathrm{T}}/(k\tau_0)^2$. However, we find from the exact calculation that there is no term this large, only higher order terms in $1/(k\tau_0)$. We outline in the Appendix a mathematical argument showing this cancellation at small wavelengths. This result requires both the phase cancellations that come in averaging over an isotropic background and also the oscillation with time of the GW's. With different assumptions, e.g., if we analyzed GW's from a particular source, which are then not isotropic, stronger limits may be possible.

IV. PROPER MOTIONS INDUCED BY LSS AND GW's

We now consider the fluctuation of the angular deflection of image rays with time and the resulting proper motion. If the deflection of image rays induced by LSS or GW's changes significantly during an observation of a lens system, then the slow shift in alignment between the lens and the source will change the impact parameter at the lens of a given ray from the source. The images will therefore move, and even tiny motions may be detected since the source motion is magnified if it is lensed by a primary lens. We first show that this effect is still expected to be too small to measure for LSS and for the GW power spectrum that we have considered above. However, given the weakness of existing limits on GW's at astrophysical wavelengths (Sec. I), we also consider possible limits on a general GW spectrum.

Again we consider a single light ray from the observer out to some distance r_S, in the absence of a primary lens (we consider the effect of a lens below). Given a ray with a fixed direction at the observer, its position $x^i(r_S)$ at r_S moves with time, and it is this motion which we evaluate. In practice, we are interested in a fixed source at r_S, in which case its *apparent* position will drift with the same speed but in the opposite direction. For LSS we have (e.g., [10])

$$x^i(r_S)=-2\int_{r=0}^{r_S}(r_S-r)\nabla_i\phi(\tau=\tau_0-r)dr, \tag{19}$$

in terms of the Newtonian potential (or scalar metric perturbation) ϕ. We are now using the parameter r rather than τ, since as time changes all comoving distances remain fixed. The only change is the time of evaluation of ϕ, and so to find $dx^i(r_S)/d\tau_0$ from $x^i(r_S)$ we simply replace $\phi(\tau=\tau_0-r)$ by $\dot{\phi}(\tau=\tau_0-r)$, with the partial time derivative in $\dot{\phi}$ taken at a fixed position. The rms value of $dx^i(r_S)/d\tau_0$ depends on the power spectrum of $\dot{\phi}$, a quantity which has been estimated by various authors in connection with the Rees-Sciama effect on the CMB (e.g., [22]). While the integrated deflection is dominated by short (~ 1 Mpc) wavelengths, the LSS potential only evolves on a cosmological time scale. In an Einstein–de Sitter universe, ϕ is time independent in the linear regime of small density perturbations, but in this case too $\dot{\phi}$ becomes nonzero when nonlinear structure forms. In general, therefore, the proper motion induced by LSS is of order $\sigma_\beta/\tau_0 \simeq 10^{-8}$ arcsec per year. For the gravity wave spectrum considered above, horizon size modes are dominant, and so here too the induced proper motion is of order σ_β/τ_0, with a σ_β smaller by $\sim 10^2$ than for LSS. Any observed proper motion will thus be dominated by peculiar velocities of hundreds of km/s generated, e.g., by the velocity dispersion of stars in a galaxy or galaxies in a galaxy group or cluster.

We now estimate the lensing limit on stochastic GW's in general, at any wave number k. VLBI observations can directly measure or limit proper motions, and this then implies a limit on GW's. Again we restrict ourselves to wavelengths with $kr_S \gg 1$, and consider first the apparent motion of a source that is not lensed by a primary lens. The apparent motion due to GW's of a fixed object at distance r_S is $-d\vec{\beta}(r_S)/d\tau_0$. Up to corrections of order $1/kr_S$, the mean square of this motion is

$$\left\langle \left[\frac{d}{d\tau_0}\beta(r_S) \right]^2 \right\rangle = \frac{18\pi A_{\mathrm{T}}}{5k^2\tau_0^4}\left[1+\frac{2}{3}\cos(2k\tau_0) \right]. \tag{20}$$

However, when there are both a lens and a source, a GW background will produce correlated proper motions in both, and the relative motion may be small. Limits from VLBI on proper motions in gravitational lenses were recently considered in Ref. [23], and we proceed similarly. We may hope for strong limits because, in the presence of lensing, a proper motion of the source relative to the lens is magnified into a larger proper motion of the images. Furthermore, only a *relative* motion between images needs to be detected, as opposed to a more difficult measurement of motion with respect to an external reference frame, since if the source moves (relative to the lens), the different images do *not* all move together. In general, different values of the magnification matrix at the different image positions will produce relative motions between images of the same order of magnitude as the absolute motions. Moreover, pairs of highly magnified images generally have antiparallel motions [23].

To analyze how proper motion due to GW's may be magnified, we start from Eq. (12), and consider the same equation a time Δt later, when the deflections from GW's have changed. E.g., $\vec{\alpha}_{\mathrm{OL}}^{(1)}$ has changed to $\vec{\alpha}_{\mathrm{OL}}^{(1)}+\Delta^{(1)}$, and a total change $\vec{\Delta}$ in the observed $\vec{\theta}'$ has been induced. Expanding the lens equation to first order in the small changes and solving for $\vec{\Delta}$, we obtain

$$\Delta_i = \Delta_i^{(2)} - \Delta_i^{(1)} + M_i^j[\Delta_j^{(2,3)} - \Delta_j^{(2)}], \tag{21}$$

where the magnification matrix M_i^j equals the inverse of $\delta_i^j - \partial_i\alpha_{\mathrm{lens}}^j$ (and is evaluated at $\vec{\theta}' + \vec{\alpha}_{\mathrm{OL}}^{(1)} - \vec{\alpha}_{\mathrm{OL}}^{(2)}$). Consider first the magnified term $\Delta_j^{(2,3)} - \Delta_j^{(2)}$. Averaging over directions of $\vec{\Delta}^{(2,3)} - \vec{\Delta}^{(2)}$ we obtain a result analogous to Eq. (9) for the mean square. Since M^{ij} is symmetric for a thin lens [13], it has two real eigenvalues m_a and m_b (where the magnification $M = |m_a m_b|$). Letting

$$\widetilde{M} = [\tfrac{1}{2}(m_a^2 + m_b^2)]^{1/2}, \tag{22}$$

we find that

$$\mathrm{rms}|M_i^j[\Delta_j^{(2,3)} - \Delta_j^{(2)}]| = \widetilde{M} \times \mathrm{rms}|\vec{\Delta}^{(2,3)} - \vec{\Delta}^{(2)}|. \tag{23}$$

In Eq. (23) we may evaluate the rms on the right-hand side using a straight path (as in Sec. III). Letting

$$\frac{d}{d\tau_0}\vec{\beta}_{LS} = -\frac{d}{d\tau_0}\vec{\beta}(r_S) + \frac{d}{d\tau_0}\vec{\beta}(r_L), \tag{24}$$

we find that in $\langle \dot{\beta}_{LS}^2 \rangle$ there is no term of order $A_{\mathrm{T}}/(k^2\tau_0^4)$ [as in Eq. (20)], but only higher order terms in $1/(k\tau_0)$. Once again this small wavelength cutoff results from combining the time oscillation of GW's and the phase cancellations in averaging over an isotropic background (see the Appendix), and as a result there is only a very weak limit on Ω_λ.

V. CONCLUSIONS

Gravitational lensing is affected by perturbations to the homogeneous and isotropic background metric. Such perturbations, whether they are caused by LSS or GW's, may produce a number of effects on light propagation. One such effect is an overall shift in the angular positions of nearby objects, which is not observable. Another is a relative difference between the induced shifts in nearby light rays. This relative deflection manifests itself as a shear which may cause weak lensing and also affect strong lensing. A third effect is a fluctuation of the angular position of distant objects with time, leading to a directly observable proper motion.

The actual amplitude of long wavelength modes of LSS and GW's is limited by the quadrupole anisotropy of the CMB. Even if both make comparable contributions to the anisotropy, LSS is dominant in its effects on lensing. This results from cancellations due to the time oscillation of short wavelength gravity waves, as well as the redshifting of their amplitude. For LSS, on the other hand, the effect of small coherent structures is amplified as the deflection executes a random walk. We find that the relative deflection due to GW's is four orders of magnitude smaller than that of LSS, and is therefore not observable.

The induced proper motions expected for LSS or for GW's generated in inflation are small compared to typical peculiar velocities, and thus are not observable. The motions are also too small to yield interesting limits on the energy density of GW's at shorter wavelengths.

After this paper was submitted for publication, the bending of light by gravity waves was analyzed differently in Ref. [24], for the case of short (subhorizon) wavelengths, in a nonexpanding flat space (i.e., neglecting the redshifting of the amplitude of GW's). That simplified analysis shows that the relative proper motion between two sources is small not only if they are at different redshifts along the same line of sight (in agreement with our calculation of $\langle \dot{\beta}_{LS}^2 \rangle$ in Sec. IV), but also if they are separated on the sky by a small angle. The treatment presented in Ref. [24] changes quantitatively if expansion is included, but not qualitatively for GW's with a period short compared to the redshifting time scale (i.e., a Hubble time).

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APPENDIX

In trying to set limits on GW's at short wavelengths $kr_S \gg 1$, we twice encountered a weaker limit than simple dimensional analysis would suggest: Once in calculating $\sigma_{\Delta\beta}/\gamma$ or shear in Sec. III, and then in estimating the magnified proper motion in Sec. IV. In this appendix we outline the first of these calculations and show how this result emerges. The second calculation can be done similarly.

From Eq. (6) in the limit of short wavelengths (compared with the present horizon), we derive

$$\sigma_{\Delta\beta}^2/\gamma^2 = \frac{1728\pi}{r_S^2} A_{\mathrm{T}} k^4 \int_{\tau_S}^{\tau_0} d\tau_1 \int_{\tau_S}^{\tau_0} d\tau_2 (\tau_1 - \tau_S)(\tau_2 - \tau_S) \times (\tau_0 - \tau_1)(\tau_0 - \tau_2) \frac{\cos(k\tau_1)}{(k\tau_1)^2} \frac{\cos(k\tau_2)}{(k\tau_2)^2} \times \frac{j_4[k(\tau_1 - \tau_2)]}{[k(\tau_1 - \tau_2)]^4}.$$

The $j_4[k(\tau_1-\tau_2)]/[k(\tau_1-\tau_2)]^4$ term comes from the angular $\vec{k}$ integrations, including the angular dependence of the polarizations and assuming an isotropic background. Letting $x = k\tau_1$ and $u = k(\tau_1 - \tau_2)$ (also $x_0 = k\tau_0$, etc.) leads to

$$\frac{864\pi}{k^2 r_S^2} A_{\mathrm{T}} \int_{x_S}^{x_0} dx \frac{(x-x_S)(x_0-x)}{x^2} \times \int_{x_S-x}^{x_0-x} du \frac{(x+u-x_S)(x_0-x-u)}{(x+u)^2} \frac{j_4(u)}{u^4} \times [\cos u(1+\cos 2x) - \sin u \sin 2x].$$

We evaluate only the first $\cos u$ term here, since the other terms in the square brackets can be evaluated similarly. Note that dimensional analysis suggests that the x and u integrals should give a term of order 1 (not larger, because of the oscillating integrand).

To do the u integral we separate the smooth and oscillating parts and then repeatedly integrate by parts: We let $w^{[n]}(u)$ be the nth indefinite integral of $[\cos u] j_4(u)/u^4$ with respect to u, and $v^{[n]}(u)$ the nth derivative of $(x+u-x_S)(x_0-x-u)/(x+u)^2$ with respect to u. For each n such that $w^{[n]}(u)$ converges for $u \to \pm\infty$, we fix the arbitrary constant by $w^{[n]}(\infty) + w^{[n]}(-\infty) = 0$ (any constant will do for other n). Then the u integral equals a series of terms evaluated at the two limits of integration:

$$\sum_{n=0}^{\infty} (-1)^n \{ w^{[n+1]} v^{[n]}|_{u=x_0-x} - w^{[n+1]} v^{[n]}|_{u=x_S-x} \}.$$

Since the two series of terms can be handled similarly, we evaluate here only the $u = x_0 - x$ terms. We do the x integration in the same way as the u integration. Thus we continue to integrate $w^{[n+1]}(x)$ with respect to x, and let $v^{[n,m]}(x)$ be the mth derivative of $(x-x_S)(x_0-x)v^{[n]}(x_0-x)/x^2$ with respect to x. The contribution to $\sigma_{\Delta\beta}^2/\gamma^2$ from the terms we have kept is

$$\frac{864\pi}{x_S^2} A_{\mathrm{T}} \sum_{n,m=0}^{\infty} (-1)^{n+1} \{ w^{[n+m+2]}(0) v^{[n,m]}(x_0) - w^{[n+m+2]}(x_0 - x_S) v^{[n,m]}(x_S) \}.$$

Now, $v^{[n,m]}(x)$ at $x \gg 1$ is of order $x^{-(n+m)}$, $w^{[n]}(0)$ is 0 or a constant, and we find that $w^{[n]}(x)$ at $|x| \gg 1$ is of order $|x|^{n-5}$. This last fact, that $w^{[n]}(\pm\infty)$ converges for the first few n, depends on the specific function $w^{[0]}(x)$ which in turn is determined by the two physical assumptions of time oscillation and angular averaging. The only term from the final sum that could give a contribution of order A_{T}/x_S^2 is the $n=m=0$ term. We find that $w^{[2]}(0)$ is a nonzero constant, but since $v^{[0,0]}(x) = 0$ identically, there is no term of this lowest order.

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