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Wigner inequalities for a black hole

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Wigner inequalities for the minimum size and maximum running time of a clock are applied to a black hole. They give the Hawking lifetime of a black hole as the maximum time that a black hole could be used to measure and identify the information content of a black hole. [S0556-2821(96)03022-6]

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There is a smallest clock. Wigner [1] was the first to consider the fundamental limits that govern the mass and size of any physical entity that functions as a time-registering device. The limit is more severe than that imposed by the energy-time form of Heisenberg's uncertainty principle because it requires that a clock still show proper time after being read: the quantum uncertainty in its position must not introduce significant inaccuracies in its measurement of time over long periods. If a clock of mass M has quantum position uncertainty of λ then its momentum will have a spread $\hbar\lambda^{-1}$, and so after a time T its position spread will grow to $\Delta\lambda=\lambda+\hbar TM^{-1}\lambda^{-1}$. If the mass is fixed then this will be a minimum when $\lambda=(\hbar T/M)^{1/2}$. Hence, if the linear spread in the dimension of the clock is λ, and its mass is M, then the total running time over which it can remain accurate is $T_{\max}$, where

$$\lambda>\left(\frac{\hbar T_{\max}}{M}\right)^{1/2}. \tag{1}$$

This is Wigner's first clock inequality [1].

If we are to read the clock repeatedly and reliably then the position uncertainty created by the measurement of time must be smaller than the minimum wavelength of the quanta used to read the clock; that is $\lambda\leqslant cT_{\min}$, where $T_{\min}$ is the smallest time interval that the clock is capable of resolving. Hence, the minimum size limit may be reexpressed as a bound on the minimum mass of a clock:

$$M>\frac{\hbar}{c^2T_{\min}}\left(\frac{T_{\max}}{T_{\min}}\right). \tag{2}$$

This is Wigner's second clock inequality [1]. We recognize this inequality as Heisenberg's energy-time uncertainty principle, but strengthened by the factor $(T_{\max}/T_{\min})>1$. The requirement that repeated measurement not disrupt the clock over the total running time $T_{\max}$ clearly imposes a stronger limit on its mass than does a single simultaneous measurement of both the energy Mc^2 and the time $T_{\min}$. There have been several attempts to limit the ultimate capability of computers using constraints from fundamental physics [2], like the uncertainty principle and finite light speed, but it is inequalities (1) and (2) that are likely to provide the strongest constraints on the ultimate capability of any nanotechnologies which require accurately time-ordered or synchronized activities. As an illustration, we can use Eq. (2) to obtain a limit on the power required by any information processor. If we denote the energy by E then Eq. (2) can be rewritten as

$$ET_{\max}>\hbar\left(\frac{T_{\max}}{T_{\min}}\right)^2. \tag{3}$$

The mean power generated by an information processor is $P\equiv E/T_{\max}$. Since $\nu=T_{\min}^{-1}$ is the fastest possible processing frequency, the maximum number of steps of information processing will be $T_{\max}/T_{\min}$, and there is a limit on the frequency of information processing possible with a mean input of power P:

$$P>\hbar\nu^2. \tag{4}$$

Now suppose we apply Wigner's size limit (1) to gravitating systems. If we use a black hole as a clock then the minimum clock size is the Schwarzschild radius

$$R_g=\frac{2GM}{c^2} \tag{5}$$

and so Eq. (1) gives the maximum running time of this gravitational clock as

$$T_{\max}\leqslant\frac{MR_g^2}{\hbar}=\frac{4G^2M^3}{\hbar c^4}\sim\frac{M^3}{m_{P^4}}t_P, \tag{6}$$

where $t_P=(G\hbar/c^5)^{1/2}$ and $m_P=(c\hbar/G)^{1/2}$ are the Planck units of time and mass. Thus the maximum clock running time is the Hawking black hole lifetime [3]. This result is surprising. If we had not known of the existence of black hole evaporation it would have implied that there is a maximum lifetime for a black hole state when quantum observers are introduced. The conventional heuristic derivation of the Hawking lifetime for black hole evaporation uses the energy-time uncertainty principle on the event horizon scale, R_g, to determine a temperature for the black hole which, under the assumption that the black hole is a black body, then allows one to use Stefan's law to calculate the lifetime of the black hole for complete evaporation of its mass to occur. The argument leading from Eqs. (1) and (2) to Eq. (6) is different: the application of the Wigner inequality to the event horizon scale predicts the Hawking lifetime directly without the assumption that the black hole is a black body radiator. The black body character could be inferred from the form of Eq. (6). The second Wigner inequality tells us that the minimum time interval that the black hole can be used to measure is just the light travel time across the black hole's horizon:

$$T_{\rm min} > \left(\frac{\hbar T_{\rm max}}{Mc^2} \right)^{1/2} = \frac{2GM}{c^3} = \frac{R_g}{c}. \qquad (7)$$

Thus we are led to view the quantum black hole as an information-processing system in which the number of computational steps is equal to $T_{\rm max}/T_{\rm min} \sim (M/m_P)^2$ where m_P is the Planck mass. This gives the number of bits required to specify the information content of the black hole as the event horizon area in Planck units, as expected from the identification of a black hole entropy (see, for example, [4]).

These results provoke the speculation that, at the quantum cosmological level, the conditions under time might be robustly measured [5] by a hypothetical ''observer'' may provide some constraints upon the nature of the Universe or on the conditions under which the concept of time remains coherent. Their simplicity reinforces the central importance of black holes as the simplest and most fundamental constructs of spacetime, linking together our concepts of information, gravity, and quantum uncertainty.

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[2] C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982); R. P. Feynman, in *Between Quantum and Cosmos* (Princeton University Press, Princeton, NJ, 1988), p. 523.

[3] S. W. Hawking, Nature (London) **248**, 30 (1974). The time $T_{\rm max}$ given in Eq. (6) is the lifetime for the emission of one species. If additional species are evaporated in the Hawking process the lifetime is reduced below $T_{\rm max}$ by a factor equal to the total number of spin states evaporated.

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[5] It is well known that the energy-time uncertainty principle is not a rigorous consequence of quantum mechanics because time is not an operator in the theory and the choice of time is not unique. By choosing a time that is defined external to the system being measured this heuristic form of the uncertainty principle can be evaded, see K. Kharev, Int. J. Theor. Phys. **21**, 311 (1982); J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Oxford University Press, Oxford, 1986), p. 671. The same strictures apply to Wigner's inequalities. For a rigorous discussion of energy-time uncertainty relations in quantum theory see P. Pfeifer and J. Fröhlich, Rev. Mod. Phys. **67**, 759 (1995).