Note on restoring manifest rotational symmetry in hyperfine and fine structure in light-front QED

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We study the part of the renormalized, cutoff QED light-front Hamiltonian that does not change particle number. The Hamiltonian contains interactions that must be treated in second-order bound-state perturbation theory to obtain hyperfine structure. We show that a simple unitary transformation leads directly to the familiar Breit-Fermi spin-spin and tensor interactions, which can be treated in degenerate first-order bound-state perturbation theory, thus simplifying analytic light-front QED calculations. To the order in momenta we need to consider, this transformation is equivalent to a Melosh rotation. We also study how the similarity transformation affects spin-orbit interactions. [S0556-2821(96)05222-8]

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I. INTRODUCTION

Light-front Hamiltonian field theory is being developed as a tool for solving bound-state problems in QCD [1,2]. Cutoffs are introduced that can be lowered using a similarity renormalization group [3], and renormalization can be completed either using coupling coherence [4] or by fixing counterterms to repair symmetries violated by the cutoffs. In the simplest procedure the renormalized, cutoff Hamiltonians are computed perturbatively and may then be diagonalized nonperturbatively to obtain low-lying bound states. Each stage has an approximation scheme associated with it: in the first step, the effective Hamiltonian is calculated to a given order in perturbation theory; and in the second step the effective Hamiltonian is divided into a dominant part, which defines the starting bound-state wave functions, and a perturbation, which is treated to a given order using bound-state perturbation theory. Interactions that change particle number are treated perturbatively. As a consequence, different Fock states decouple and one is left with few-body problems in the leading order. It is therefore important to know to what extent bound states are accurately described by the effective interactions that do not change particle number. A principal signature of the truncation errors in these schemes is a violation of rotational invariance, which is a dynamical symmetry in light-front field theory.

There are two reasons why rotational symmetry is complicated in this approach. The first is that we have formulated the theory on the light front. Rotations are dynamical, and light-front spinors depend on the choice of the z axis. In the weak-coupling limit, however, one expects that rotations should become simple because both boost and rotational symmetries are kinematic in the nonrelativistic limit. The second source of complexity is the regularization and renormalization scheme that we use [1,2]. This has profound effects in QCD [5], but in QED, to obtain the leading interactions cutoffs can effectively be removed. Therefore, to disentangle the two problems it is useful to study QED in the nonrelativistic limit.

Jones *et al.* [6] have studied positronium in this approach, detailing renormalization and explicitly completing the calculation of hyperfine splitting in the ground state. To second order in the coupling there are new interactions between the electron and positron that arise from eliminating matrix elements of the Hamiltonian involving high-energy photon emission and absorption. These new interactions have spin-independent as well as spin-dependent parts. The spin-independent interaction, combined with the instantaneous exchange interaction, leads to the Coulomb interaction, and the leading-order problem reduces to the familiar equal-time Schrödinger equation [2].

The light-front spin-dependent interactions appear to be different from the spin-dependent interactions found in an equal-time formulation, even in the nonrelativistic limit. For example, if one calculates the hyperfine splitting in the positronium ground state, first-order bound-state perturbation theory gives incorrect results. The triplet state is not degenerate, and the energy of the singlet state is incorrect. This is because the effective Hamiltonian contains a term that does not give a contribution in first-order bound-state perturbation theory, but its contribution in second-order bound-state perturbation theory is of the same order in α as the terms contributing in the first order, $O(\alpha^4)$. Jones *et al.* [6] summed the second-order bound-state perturbation theory analytically, and showed that it leads to the correct hyperfine splitting in the ground state of positronium. Kaluza and Pirner encountered the same problem [7], and they completed the sum numerically.

We propose an alternative approach. We find a simple unitary transformation of the Hamiltonian that alters the problematic term so that it enters at first-order in bound-state perturbation theory. We find the transformation order-by-order in powers of momenta. In order to restore the hyperfine splitting, we need to find the unitary transformation only to the next-to-leading order. A unitary transformation does not change the eigenvalues, and the transformation we obtain makes the calculation much easier. It turns out that the unitary transformation to this order is an expansion of the so called Melosh transformation [8] to next-to-leading order in powers of momenta. The simple formalism enables us to study how the similarity transformation affects the spin-dependent structure of the effective Hamiltonian. This issue is addressed in the third section. The last section contains our conclusions.

II. SPIN-SPIN INTERACTION IN QED

The first step in the calculation is to generate a Hamiltonian renormalized to $O(\alpha)$. We start with a canonical light-front Hamiltonian regulated by a cutoff on the change in free energy at each interaction vertex. This cutoff is lowered by a similarity transformation:

$$H_{\Lambda_{n-1}}=U^{-1}(\Lambda_{n-1})H_{\Lambda_n}U(\Lambda_{n-1}). \tag{1}$$

U is a unitary operator designed to bring the Hamiltonian toward band-diagonal form in light-front energy. As the cutoff is lowered by this transformation, one explicitly encounters dependence on the original cutoff and renormalization is required to allow the initial cutoff to be taken to infinity. One method to accomplish this is to find a Hamiltonian that reproduces itself in form after the canonical coupling and masses are allowed to explicitly run with the cutoff. This is coupling coherence [4], and it is relatively straightforward to find the coupling coherent QED Hamiltonian to $O(\alpha)$ [2,6]. For more details on the similarity renormalization scheme we refer the reader to Refs. [1–3], coupling coherence is discussed in [2,4], and for applications see Refs. [5,6].

The effective Hamiltonian for a fermion and an antifermion generated by the similarity transformation [3] using coupling coherence [4] is

$$H_{\rm eff}=H_{\rm free}+V_1+V_2+V_{2\rm eff}, \tag{2}$$

where $H_{\rm free}$ is the kinetic energy, V_1 is $O(e)$ emission and absorption, V_2 is the $O(e^2)$ instantaneous exchange interaction, and $V_{2\rm eff}$ includes the $O(e^2)$ effective interactions that reproduce the effects of photon exchange above the cutoff. For simplicity, we consider the case where the fermion and antifermion have equal mass.

Kinetic energy is diagonal in momentum space, and matrix elements of the interactions are nonzero only between states with energy difference smaller than $\Lambda^2/\mathcal{P}^+$, which is reflected by an overall cutoff function in the equations below. If the cutoff is chosen within certain limits, the cutoff functions can be approximated by 1 to leading order in α [2].

Let us consider matrix elements of the Hamiltonian between states containing a fermion and antifermion pair. The kinetic energy is diagonal in momentum space:

$$\frac{p^{\perp 2}+m^2}{p^+}+\frac{k^{\perp 2}+m^2}{k^+}.$$

The emission and absorption of a photon enters at second order.

Let p_i, k_i be the light-front three-momenta carried by the fermion and antifermion; σ_i, λ_i are their light-front helicities; $u(p,\sigma)$, $v(k,\lambda)$ are their spinors; index $i=1,2$ refers to the initial and final states, respectively. The instantaneous exchange interaction mixes states of different momenta,

$$-e^2\overline{u(}p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\overline{v(}k_1,\lambda_1)\gamma^\nu v(k_2,\lambda_2)\frac{1}{q^{+2}}\eta_\mu\eta_\nu\theta\left(\frac{\Lambda^2}{\mathcal{P}^+}-\left|(p_1^-+k_1^-)-(p_2^-+k_2^-)\right|\right), \tag{3}$$

and so do the effective interactions generated by the similarity transformation:

$$\begin{aligned}&-e^2\overline{u(}p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\overline{v(}k_1,\lambda_1)\gamma^\nu v(k_2,\lambda_2)\frac{1}{q^+}D_{\mu\nu}(q)\\&\times\left(\frac{\theta(|D_1|-\Lambda^2/\mathcal{P}^+)\,\theta(|D_1|-|D_2|)}{D_1}+\frac{\theta(|D_2|-\Lambda^2/\mathcal{P}^+)\,\theta(|D_2|-|D_1|)}{D_2}\right)\theta\left(\frac{\Lambda^2}{\mathcal{P}^+}-\left|(p_1^-+k_1^-)-(p_2^-+k_2^-)\right|\right),\end{aligned} \tag{4}$$

where $D_{\mu\nu}(q)=(q^{\perp 2}/q^{+2})\eta_\mu\eta_\nu+(1/q^+)(\eta_\mu q^\perp{}_\nu+\eta_\nu q^\perp{}_\mu)-g^\perp_{\mu\nu}$ is the photon propagator in light-front gauge, $\eta_\mu=(0,\eta_+=1,0,0)$; $q=p_1-p_2$ is the exchanged momentum, with $q^-=q^{\perp 2}/q^+$; D_1, D_2 are energy denominators: $D_1=p_1^- - p_2^- - q^-$ and $D_2=k_2^- - k_1^- - q^-$. It is convenient to add Eqs. (3) and (4) together, leading to

$$\begin{aligned}&e^2\overline{u(}p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\overline{v(}k_2,\lambda_2)\gamma^\nu v(k_1,\lambda_1)\Bigg[g_{\mu\nu}\left(\frac{\theta(|D_1|-\Lambda^2/\mathcal{P}^+)\,\theta(|D_1|-|D_2|)}{q^+D_1}+\frac{\theta(|D_2|-\Lambda^2/\mathcal{P}^+)\,\theta(|D_2|-|D_1|)}{q^+D_2}\right)\\&-\frac{\eta_\mu\eta_\nu}{2q^{+2}}\left(1-\frac{\theta(|D_1|-\Lambda^2/\mathcal{P}^+)\,\theta(|D_1|-|D_2|)D_2}{D_1}-\frac{\theta\left(\left|D_2\right|-\frac{\Lambda^2}{\mathcal{P}^+}\right)\theta(|D_2|-|D_1|)D_1}{D_2}\right)\Bigg]\theta\left(\frac{\Lambda^2}{\mathcal{P}^+}-\left|(p_1^-+k_1^-)-(p_2^-+k_2^-)\right|\right).\end{aligned} \tag{5}$$

In what follows, we approximate the cutoff functions by 1, which is allowed to leading order for the range of cutoffs $e^2m^2 \ll \Lambda^2 \ll em^2$ [2].

The ''$\eta_\mu \eta_\nu$'' term is spin-independent, it vanishes on-shell, and it is at least one power of momentum higher than the leading spin-independent piece of the ''$g_{\mu\nu}$'' term. As we explain later, it does not affect spin-spin and tensor interactions, but it may influence the spin-orbit interaction.

In what follows we use Jacobi momenta in the center-of-mass frame:

$$p_i^+ = x_i P^+, \quad p_i^\perp = \kappa_i^\perp, \quad k_i^+ = y_i P^+, \quad k_i^\perp = -\kappa_i^\perp,$$

where $y_i = 1 - x_i$; and we replace four-component spinors $u(p,\sigma)$, $v(k,\lambda)$ with two-component spinors by substituting

$$u(p,\sigma) = \sqrt{2/p^+}(p^+ + \beta m - \vec{\alpha}^\perp \cdot \vec{p}_\perp)\Lambda_+ \begin{pmatrix} \xi_\sigma \\ 0 \end{pmatrix}, \tag{6}$$

and similarly for $v(k,\lambda)$ with $m \to -m$ and $\sigma \to \overline{\sigma}$. Here $\beta = \gamma^0$, $\vec{\alpha} = \gamma^0 \vec{\gamma}$ are Dirac matrices, $\Lambda_+ = \frac{1}{4}\gamma^- \gamma^+$ is a projection operator, and ξ_σ is a two-component spinor,

$$\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From now on we will write the Hamiltonian as an operator which acts in the cross product space of these two-component spinors.

After the following change of variables, which defines p_z,

$$x_i = \frac{\sqrt{\vec{p_i}^2 + m^2} + p_{iz}}{2\sqrt{\vec{p_i}^2 + m^2}}, \quad y_i = \frac{\sqrt{\vec{k_i}^2 + m^2} + k_{iz}}{2\sqrt{\vec{k_i}^2 + m^2}}, \tag{7}$$

where $\vec{k_i} = -\vec{p_i}$, and the three-momentum in the center of mass frame is then $\vec{p} \equiv (\kappa^\perp, p_z)$, we take the nonrelativistic limit of the Hamiltonian.

The energy denominators become

$$\begin{aligned} q^+ D_1 &= -\vec{q}^2 + \left(\frac{p_{1z}}{m\sqrt{1+\vec{p_1}^2/m^2}} - \frac{p_{2z}}{m\sqrt{1+\vec{p_2}^2/m^2}}\right)(\vec{p_1}^2 - \vec{p_2}^2) + \left[\frac{2 + \vec{p_1}^2/m^2 + \vec{p_2}^2/m^2}{\sqrt{1+\vec{p_1}^2/m^2}\sqrt{1+\vec{p_2}^2/m^2}} - 2\right] p_{1z}p_{2z} \simeq -\vec{q}^2, q^+ D_2 = -\vec{q}^2 \\ &\quad - \left(\frac{p_{1z}}{m\sqrt{1+\vec{p_1}^2/m^2}} - \frac{p_{2z}}{m\sqrt{1+\vec{p_2}^2/m^2}}\right)(\vec{p_1}^2 - \vec{p_2}^2) + \left[\frac{2 + \vec{p_1}^2/m^2 + \vec{p_2}^2/m^2}{\sqrt{1+\vec{p_1}^2/m^2}\sqrt{1+\vec{p_2}^2/m^2}} - 2\right] p_{1z}p_{2z} \simeq -\vec{q}^2, \end{aligned} \tag{8}$$

and the interaction Hamiltonian reduces to

$$4e^2(2m)^2\left[\frac{1}{-\vec{q}^2}(1+d)\right](v_0 + v_{\text{spin}}) + v_{\eta_\mu\eta_\nu}, \tag{9}$$

where v_0 and v_{spin} come from the $g_{\mu\nu}$ term in Eq. (5). v_0 is spin-independent and v_{spin} depends on spins. d denotes corrections from energy denominators that we discuss in the next section, together with the spin independent $v_{\eta_\mu\eta_\nu}$ that arises from the $\eta_\mu\eta_\nu$ term in Eq. (5). We have dropped an overall factor of $\sqrt{x_1x_2(1-x_1)(1-x_2)}$ in the Hamiltonian which is absorbed by a similar factor in the definition of the two-body wave function.

The corrections from energy denominators do not influence the discussion of spin-dependent structure, because they enter as an overall factor multiplying the entire $g_{\mu\nu}$ term. This will become clear later. To second order in powers of momenta,

$$\begin{aligned} v_0 = 1 &+ \frac{1}{4m^2}(\vec{p}_1 + \vec{p}_2)^2 + \frac{1}{2m^2}\vec{p}_1 \cdot \vec{p}_2 + \frac{1}{2m^2}\vec{p}_1^\perp \cdot \vec{p}_2^\perp \\ &+ \frac{3}{4m^2}((\vec{p_1})_z^2 + (\vec{p_2})_z^2), \end{aligned} \tag{10}$$

and

$$\begin{aligned} v_{\text{spin}} = &-\frac{i}{2m}((\vec{k}_1 - \vec{k}_2) \times \vec{\sigma}_a)_z - \frac{i}{2m}((\vec{p}_1 - \vec{p}_2) \times \vec{\sigma}_b)_z \\ &+ \frac{1}{4m^2}((\vec{k}_1 - \vec{k}_2) \times \vec{\sigma}_a)_\perp \cdot ((\vec{p}_1 - \vec{p}_2) \times \vec{\sigma}_b)_\perp \\ &+ 3\frac{i}{4m^2}(\vec{k}_2 \times \vec{k}_1) \cdot \vec{\sigma}_a + 3\frac{i}{4m^2}(\vec{p}_2 \times \vec{p}_1) \cdot \vec{\sigma}_b \\ &+ \frac{i}{4m^2}(\vec{k}_2 \times \vec{k}_1)_z \cdot (\vec{\sigma}_a)_z + \frac{i}{4m^2}(\vec{p}_2 \times \vec{p}_1)_z \cdot (\vec{\sigma}_b)_z \\ &+ \frac{i}{4m^2}(\vec{k}_1)_z \cdot (\vec{k}_1 \times \vec{\sigma}_a)_z - \frac{i}{4m^2}(\vec{k}_2)_z \cdot (\vec{k}_2 \times \vec{\sigma}_a)_z \\ &+ \frac{i}{4m^2}(\vec{p}_1)_z \cdot (\vec{p}_1 \times \vec{\sigma}_b)_z - \frac{i}{4m^2}(\vec{p}_2)_z \cdot (\vec{p}_2 \times \vec{\sigma}_b)_z. \end{aligned} \tag{11}$$

We can immediately see that the first two terms in v_{spin}, which are linear in momentum, will lead to difficulties in bound-state perturbation theory. In first-order bound-state perturbation theory they integrate to zero, but they enter at the second-order of bound-state perturbation theory, bringing the same power of momentum as the familiar term $(\vec{q}\times\vec{\sigma}_a)_\perp\cdot(\vec{q}\times\vec{\sigma}_b)_\perp$. So in order to obtain correct splitting of the ground-state triplet and singlet states using this Hamiltonian, one has to sum second-order bound-state perturbation theory using all bound and scattering electron-positron states [6]. We note that the remaining terms in Eq. (11) give rise to part of the spin-orbit interaction.

The key to resolving this nuisance is to recognize that the spin-independent v_0 and the spin-dependent $v_{\rm spin}$ are both multiplied by the same energy denominators. We seek a unitary transformation which, applied to the spin-independent term, generates terms that cancel the unwanted linear terms, and restore rotational invariance in the $(\vec{q}\times\vec{\sigma}_a)\cdot(\vec{q}\times\vec{\sigma}_b)$ term.

Consider the transformation

$$U_\alpha=1+\frac{i}{2m}(\vec{\mathcal{P}}_\alpha\times\vec{\sigma}_\alpha)_z-\frac{1}{2}\frac{\mathcal{P}_\alpha^{\perp 2}}{4m^2} \tag{12}$$

for each particle α. This transformation is clearly unitary to second order in momentum, which is all we require here. For two particles a and b in the initial and final states,

$$U^\dagger_{\rm initial}=\left[1-\frac{i}{2m}(\vec{k}_1\times\vec{\sigma}_a)_z-\frac{1}{2}\frac{k_1^{\perp 2}}{4m^2}\right]\times\left[1-\frac{i}{2m}(\vec{p}_1\times\vec{\sigma}_b)_z-\frac{1}{2}\frac{p_1^{\perp 2}}{4m^2}\right], \tag{13}$$

and

$$U_{\rm final}=\left[1+\frac{i}{2m}(\vec{k}_2\times\vec{\sigma}_a)_z-\frac{1}{2}\frac{k_2^{\perp 2}}{4m^2}\right]\times\left[1+\frac{i}{2m}(\vec{p}_2\times\vec{\sigma}_b)_z-\frac{1}{2}\frac{p_2^{\perp 2}}{4m^2}\right]. \tag{14}$$

The Hamiltonian transforms as

$$H\rightarrow U_f H U_i^\dagger, \tag{15}$$

leading to new operators $\tilde{v}_0$ and $\tilde{v}_{\rm spin}$,

$$\tilde{v}_0=1+\frac{1}{2m^2}(\vec{p}_1+\vec{p}_2)^2+\frac{1}{2m^2}(\vec{p}_1^{\,2}+\vec{p}_2^{\,2}) \tag{16}$$

to the leading order, and

$$\begin{aligned}\tilde{v}_{\rm spin}=&-\frac{1}{4m^2}(\vec{q}\times\vec{\sigma}_a)\cdot(\vec{q}\times\vec{\sigma}_b)\\&+\frac{3i}{4m^2}(\vec{k}_2\times\vec{k}_1)\cdot\vec{\sigma}_a+\frac{3i}{4m^2}(\vec{p}_2\times\vec{p}_1)\cdot\vec{\sigma}_b\\&-\frac{i}{m^2}(\vec{p}_2\times\vec{p}_1)\cdot\vec{\sigma}_b-\frac{i}{m^2}(\vec{k}_2\times\vec{k}_1)\cdot\vec{\sigma}_a\\&+\frac{i}{4m^2}(\vec{p}_1)_z\cdot(\vec{p}_1\times\vec{\sigma}_b)_z-\frac{i}{4m^2}(\vec{p}_2)_z\cdot(\vec{p}_2\times\vec{\sigma}_b)_z\\&+\frac{i}{4m^2}(\vec{k}_1)_z\cdot(\vec{k}_1\times\vec{\sigma}_a)_z-\frac{i}{4m^2}(\vec{k}_2)_z\cdot(\vec{k}_2\times\vec{\sigma}_a)_z.\end{aligned} \tag{17}$$

The corrections from energy denominators [i.e., d in Eq. (8)] do not affect spin-dependent interactions to this order. The term in the unitary transformation designed to remove $(\vec{q}\times\vec{\sigma})_z$ also removes $d(\vec{q}\times\vec{\sigma})_z$, because d is an overall factor multiplying both v_0 and $v_{\rm spin}$.

The rotationally noninvariant terms that do not mix initial and final state momenta [e.g., $(i/4m^2)(\vec{p}_1)_z\cdot(\vec{p}_1\times\vec{\sigma}_b)_z$] can be removed by adding terms of that form to the unitary transformation at second order in momentum:

$$U_\alpha\rightarrow U_\alpha+\frac{i}{4m^2}(\mathcal{P}_\alpha)_z\cdot(\vec{\mathcal{P}}_\alpha\times\vec{\sigma}_\alpha)_z. \tag{18}$$

The resultant spin-dependent interactions are

$$\begin{aligned}\tilde{v}_{\rm spin}=&-\frac{1}{4m^2}(\vec{q}\times\vec{\sigma}_a)\cdot(\vec{q}\times\vec{\sigma}_b)\\&+\frac{3i}{4m^2}(\vec{k}_2\times\vec{k}_1)\cdot\vec{\sigma}_a+\frac{3i}{4m^2}(\vec{p}_2\times\vec{p}_1)\cdot\vec{\sigma}_b\\&-\frac{i}{m^2}(\vec{p}_2\times\vec{p}_1)\cdot\vec{\sigma}_b-\frac{i}{m^2}(\vec{k}_2\times\vec{k}_1)\cdot\vec{\sigma}_a,\end{aligned} \tag{19}$$

which is the familiar Breit-Fermi interaction.

The $\eta_\mu\eta_\nu$ term in Eq. (5), which we ignored so far, is spin-independent and already one power of momentum higher than the leading spin-independent term in v_0. Therefore, to order two powers of momentum higher than the leading spin-independent term, it does not affect the spin-spin and tensor interactions. It may affect the spin-orbit interactions. But at least as far as the spin-spin structure, we can now diagonalize the new Hamiltonian using states that are related to the original states as

$$|\tilde{\psi}\rangle=U|\psi\rangle. \tag{20}$$

It should be mentioned that the unitary transformation presented here is a next-to-leading order expansion of the Melosh transformation [8]:

$$\frac{m+x_\alpha M_0-i(\mathcal{P}_\alpha^\perp\times\vec{\sigma}_\alpha)_z}{\sqrt{(m+x_\alpha M_0)^2+\mathcal{P}_\alpha^{\perp 2}}}. \tag{21}$$

III. SIMILARITY TRANSFORMATION AND FINE STRUCTURE

In this section we consider corrections that arise due to the similarity transformation, i.e., the $\eta_\mu\eta_\nu$ term and the corrections due to energy denominators in the $g_{\mu\nu}$ term in Eq. (5). For completeness, we mention that the finite cutoffs also introduce corrections, the size of which depends on the specific choice of Λ [2].

In the previous considerations we omitted corrections due to energy denominators in the $g_{\mu\nu}$ term in Eq. (5), or d in Eq. (9). These corrections do not affect the spin-dependent terms, but they do produce a spin-independent correction:

$$4e^2(2m)^2\left[\frac{d}{\vec{q}^2}\right]=4e^2(2m)^2\frac{1}{\vec{q}^2}\left[\frac{|q_z||\vec{q}\cdot(\vec{p}_1+\vec{p}_2)|}{m\vec{q}^2}\right], \tag{22}$$

where we have dropped the omnipresent $\sqrt{x_1x_2(1-x_1)(1-x_2)}$ as before. Similarly, any corrections of this term due to the finite cutoff do not affect spin-dependent interactions.

We now address the $v_{\eta_\mu\eta_\nu}$ term, and its effect on the spin-orbit interaction. Dropping the omnipresent $\sqrt{x_1x_2(1-x_1)(1-x_2)}$, the $\eta_\mu\eta_\nu$ term gives

$$4e^2\frac{1}{2(x_1-x_2)^2}\left[\frac{\theta(|D_1|-|D_2|)(q^+D_1-q^+D_2)}{q^+D_1}+\frac{\theta(|D_2|-|D_1|)(q^+D_2-q^+D_1)}{q^+D_2}\right]. \tag{23}$$

To the lowest order in momentum this equals (for details see Appendix)

$$v_{\eta_\mu\eta_\nu}=4e^2(2m)^2\frac{1}{\vec{q}^2}\left[\frac{|q_z||\vec{q}\cdot(\vec{p_1}+\vec{p_2})|}{mq_z^2}\right]. \tag{24}$$

The unitary transformation (15) applied to this term produces[1]

$$4e^2(2m)^2\frac{1}{\vec{q}^2}\left[\frac{|q_z||\vec{q}\cdot(\vec{p_1}/m+\vec{p_2}/m)|}{q_z^2}\right]\times\left[1-\frac{i}{2m}(\vec{q}\times\vec{\sigma}_a)_z+\frac{i}{2m}(\vec{q}\times\vec{\sigma}_b)_z\right]. \tag{25}$$

All of these corrections are nonanalytic. This is a consequence of using a nonanalytic cutoff function in the similarity transformation. If the nonanalytic spin-dependent corrections do not vanish, a simple angular momentum operator does not emerge even in the nonrelativistic limit. Fortunately, these spin-dependent terms integrate to zero in first-order bound state perturbation theory, since they are odd under parity. Terms that appear at higher orders must be paired with other terms from second-order bound state perturbation theory and with higher-order terms from similarity transformation. Corrections that arise due to the finite value of the cutoff Λ do not influence the lowest order spin-dependent interactions for the same reasons.

IV. CONCLUSIONS

We have studied the part of the effective QED Hamiltonian that does not change particle number. We have shown that the light-front spin-dependent interactions reduce to the familiar Breit-Fermi interactions. This can be achieved by a simple unitary transformation corresponding to a change of spinor basis. As a consequence of sharp cutoff functions in the similarity transformation, there are nonanalytic corrections. These nonanalytic corrections produce spin-independent corrections at $O(\alpha^3)$, but they do not affect the spin-dependent splittings at order $O(\alpha^4)$. Photon exchange below the cutoff is needed to remove the corrections at order $O(\alpha^3)$.

Our primary motivation for restricting this study to the part of the Hamiltonian that does not change the particle number is QCD. The approach suggested by Wilson *et al.* [1] builds on suppressing the exchange of low energy gluons by introducing a gluon mass, or by using confinement [2]. A mass gap causes higher Fock states to decouple when the cutoff is lowered below the gap, producing a constituent approximation for QCD.

As far as aspects discussed here, in QCD questions about rotational symmetry become more complicated, because the finite size of the cutoff comes into play. It is straightforward to show that the procedure we outline here does not completely restore manifest rotational symmetry in the spin-spin interaction in QCD. It is still useful, however, because it helps to separate violations of manifest rotational symmetry caused by using light-front spinors from the violations due to the cutoff [5].

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APPENDIX: $\eta_\mu\eta_\nu$ TERM

Let us concentrate on the expression in the square brackets in Eq. (23). From Eq. (8) one can see that the energy denominators have the form

$$q^+D_1=a+X,$$
$$q^+D_2=a-X, \tag{A1}$$

where

$$X=\left(\frac{p_{1z}}{m\sqrt{1+\vec{p_1}^2/m^2}}-\frac{p_{2z}}{m\sqrt{1+\vec{p_2}^2/m^2}}\right)(\vec{p_1}^2-\vec{p_2}^2) \tag{A2}$$

and

$$a=-\vec{q}^2+O(p^6). \tag{A3}$$

The difference between the energy denominators is

$$q^+D_1-q^+D_2=2X. \tag{A4}$$

Step functions can be expressed as

$$\theta(|D_1|-|D_2|)=\theta(|a+X|-|a-X|)=\theta(aX),$$
$$\theta(|D_2|-|D_1|)=\theta(|a-X|-|a+X|)=\theta(-aX). \tag{A5}$$

Using these expressions, to the lowest nonvanishing order,

[1]This term can affect spin-orbit splittings even if one does not use the unitary transformation to rotate the spins. In that case, there would be corrections in second-order bound state perturbation theory, arising from the product of Eq. (24) with the first two terms in v_{spin} [see Eq. (11)].

$$\left[\frac{\theta(|D_1|-|D_2|)(q^+D_1-q^+D_2)}{q^+D_1} + \frac{\theta(|D_2|-|D_1|)(q^+D_2-q^+D_1)}{q^+D_2}\right]$$

$$=2\left[\frac{X\theta(aX)}{a+X}-\frac{X\theta(-aX)}{a-X}\right]\simeq 2\left|\frac{X}{a}\right|+O\left(\frac{X^2}{a^2}\right). \quad \text{(A6)}$$

It is easy to show that even if the cutoff is kept in place, the nonanalytic corrections are still present.

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