QED in inhomogeneous magnetic fields

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A lower bound is placed on the fermionic determinant of Euclidean quantum electrodynamics in three dimensions in the presence of a smooth, finite-flux, static, unidirectional magnetic field $\mathbf{B}(\mathbf{r})=(0,0,B(\mathbf{r}))$, where $B(\mathbf{r})\geqslant 0$ or $B(\mathbf{r})\leqslant 0$ and $\mathbf{r}$ is a point in the xy plane. Bounds are also obtained for the induced spin for $(2+1)$-dimensional QED in the presence of $\mathbf{B}(\mathbf{r})$. An upper bound is placed on the fermionic determinant of Euclidean QED in four dimensions in the presence of a strong, static, directionally varying, square-integrable magnetic field $\mathbf{B}(\mathbf{r})$ on $\mathbb{R}^3$. [S0556-2821(96)04822-9]

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I. INTRODUCTION

In quantum electrodynamics and indeed in all gauge theories coupled to fermions the fermionic determinant is fundamental. Without substantially more knowledge of this determinant a nonperturbative analysis of QED in the continuum with dynamical fermions will remain impossible. The reader is reminded that the fermionic determinant results from the integration over the fermionic degrees of freedom in the presence of a potential A_μ. This determinant combines with the potential's gauge-fixed Gaussian measure $d\mu(A)$ to produce a one-loop effective action $S_{\text{eff}}\propto\ln\det$ that is exact and on which every physical process in QED depends, thereby justifying our opening statement.

In order to make this paper reasonably self-contained we will retrace some material previously covered in [1–4]. Schwinger's proper time definition of the fermionic determinant in Wick-rotated Euclidean quantum electrodynamics in four dimensions [5–7] is the most useful one for our purpose here:

$$\ln\det{}_{\text{ren}}(1-S\not{A})=\frac{1}{2}\int_0^\infty\frac{dt}{t}\left(\text{Tr}\left\{e^{-P^2t}-\exp\left[-\left(D^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)t\right]\right\}+\frac{\|F\|^2}{24\pi^2}\right)e^{-tm^2}, \tag{1.1}$$

where $D_\mu=P_\mu-A_\mu$; S denotes the free fermion Euclidean propagator; m is the unrenormalized fermion mass; $\sigma^{\mu\nu}=(1/2i)[\gamma^\mu,\gamma^\nu]$, $\gamma^{\mu\dagger}=-\gamma^\mu$, and $\|F\|^2=\int d^4xF^2_{\mu\nu}(x)$, $F_{\mu\nu}$ being the field strength tensor. The coupling e has been absorbed into the potential. For future reference note that $eF_{\mu\nu}$ has the invariant dimension of m^2 in any space-time dimension. Included in the definition is the second-order charge renormalization subtraction at zero momentum transfer that is required for the integral to converge for small t, as indicated by the determinant's subscript. The determinant is gauge invariant, depending only on invariant combinations of $F_{\mu\nu}$ and their derivatives. Definition (1.1) continues to hold for Euclidean three-dimensional QED (QED_3) and QED_2 except that the charge renormalization subtraction is omitted.

Now the determinant is part of a functional integral over A_μ, and if the gauge field is given an infrared cutoff—a mass term—then A_μ is concentrated on $\mathcal{S}'$, the Schwartz space of real-valued tempered distributions. As we have noted [1,3,4], there is a need to regulate in any dimension. One possibility is to replace A_μ in the determinant and anywhere else it appears in the functional integral, except in $d\mu(A)$, with the smoothed, polynomial bounded C^∞ potential $A^\Lambda_\mu(x)=(h_\Lambda*A_\mu)(x)$, where A_μ is convoluted with an ultraviolet cutoff function $h_\lambda\in\mathcal{S}$, the functions of rapid decrease [8]. This introduces a regulated photon propagator since

$$\int d\mu(A)A^\Lambda_\mu(x)A^\Lambda_\nu(y)=D^\Lambda_{\mu\nu}(x-y), \tag{1.2}$$

where $D^\Lambda_{\mu\nu}$'s Fourier transform is such that $\hat{D}^\Lambda_{\mu\nu}\propto|\hat{h}_\Lambda|^2$, $\hat{h}_\Lambda$ denoting the Fourier transform of h_Λ. For example, let $\hat{h}_\Lambda\in C^\infty_0$ with $\hat{h}_\Lambda(k)=1$, $k^2\leqslant\Lambda^2$ and $\hat{h}_\Lambda(k)=0$, $k^2\geqslant 2\Lambda^2$. The point of all this is that one might just as well assume that A_μ in Eq. (1.1) is C^∞ and polynomial bounded to begin with. If one succeeds in calculating a useful determinant one can then replace the potential in $F_{\mu\nu}$ with A^Λ_μ before the final functional integration over the gauge field. Or one may select some other regularization procedure.

Essentially we are instructed to integrate over all potentials, which requires knowledge of the determinant for all fields. What all fields means depends on the dimensionality of space-time. In Euclidean space we need the determinant for fields $\mathbf{B}$ and $\mathbf{E}$ in four dimensions, $\mathbf{B}$ in three dimensions, and a unidirectional magnetic field B in two dimensions. We have shown in [1] that an integral of the fermionic determinant in QED_2 over the fermion's mass gives the determinant in QED_4 for the field $\mathbf{B}=(0,0,B(x,y))$. It will be shown in Sec. II that the determinant in QED_3 may be calculated in the same way for this $\mathbf{B}$ field. And we will show in Sec. III that a mass integral of the fermionic determinant in QED_3 gives the determinant in QED_4 for a static, directionally varying magnetic field $\mathbf{B}(\mathbf{r})$.

The author has repeatedly encountered the assertion that the fermionic determinant of QED_2 is known explicitly. This is true for the case of massless fermions—the Schwinger

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model [9]—but not for the all-important case of massive fermions considered here. We note in passing that there is evidence that the massive fermionic determinant in QED_2 is discontinuous at $m=0$ for magnetic fields with nonvanishing flux [3]. This would imply that the Schwinger model's fermionic determinant cannot in general be obtained as the zero-mass limit of QED_2's.

As the representation (1.1) makes clear, the calculation of a fermionic determinant is really just a problem in quantum mechanics involving the calculation of the energy levels and their degeneracy of the Pauli operator

$$(\not{P}-\not{A})^\dagger(\not{P}-\not{A})=(P-A)^2+\frac{1}{2}\,\sigma^{\mu\nu}F_{\mu\nu}\geqslant 0. \quad (1.3)$$

Since the determinant is required for general fields, probably the best that can be done at present is to make estimates that place stringent bounds on the determinant. Inevitably it is the Zeeman term σF that complicates matters. If it is simply ignored then the zero modes of the Pauli operator are absent, thereby causing an unacceptable modification of the infrared behavior of QED.

It is by now a piece of folklore that the Pauli operator in two space dimensions in a unidirectional magnetic field $B\to 0$ at infinity has associated eigenvalues with finite degeneracy. This is necessary if one is to make sense out of the trace in Eq. (1.1) or any other definition of a determinant the author is aware of. This question has been discussed in [1,3,4]. We know in particular that polynomial, infinite flux, unidirectional magnetic fields are associated with infinite degeneracy [10]. Whether infinite flux in general implies an infinitely degenerate ground state is not known. Some results in this direction are given in [11]. Here we will consider only those unidirectional magnetic fields with finite flux, which is consistent with the introduction of a volume cutoff and which is required to define QED before taking the thermodynamic limit.

Before listing the known bounds on the determinants, including those obtained here, we mention two analytic calculations for finite flux fields: the determinant in QED_2 for the radially symmetric cylindrical field [3],

$$B(r)=\frac{\Phi}{2\pi}\,\frac{\delta(r-a)}{a}, \quad (1.4)$$

and the determinant in QED_3 for the field [12]

$$B(x,y)=\frac{B}{[\cosh(x/\lambda)]^2}, \quad (1.5)$$

localized in a strip of finite extent in the y direction.

Table I summarizes the known bounds on the fermionic determinants in QED. The lower bounds are for the fields $\mathbf{B}=(0,0,B(x,y))$, $B(x,y)\geqslant 0$ or $B(x,y)\leqslant 0$. The lower bound for QED_3 is new and will be dealt with in Sec. II. The upper bound on QED_4's determinant for a static, square integrable, directionally varying magnetic field $\mathbf{B}(\mathbf{r})$, where $\mathbf{r}$ is a point in $\mathbb{R}^3$, is also new and will be established in Sec. III. The other bounds have been previously derived. While the bounds for $QED_{2,3}$ indicate stability, the lower bound for QED_4, for the class of static magnetic fields considered, indicates that the contribution of the virtual fermion currents to the effective energy at the one-loop level is unbounded from below as the field's flux is increased. As noted above, it is the one-loop effective action, or energy in the special case of static fields in Euclidean space, that is relevant to the nonperturbative analysis of QED. Section III C is devoted to establishing bounds on the induced spin in planar QED with finite mass fermions in the presence of inhomogeneous background magnetic fields.

Finally, we would like to comment on the case of general static fields $\mathbf{B}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ in QED_4. It seems to be taken for granted that the effective Lagrangian for constant $\mathbf{B}$ and $\mathbf{E}$ [5,14] is an indication of the behavior of the fermionic determinant for general fields, provided one accepts the fudging of the thermodynamic limit involved. Now it is well known that $F_{\mu\nu}$ can be reduced to block diagonal form for constant fields by two rotations in $\mathbb{R}^4$ (corresponding to a Lorentz boost and a rotation in Minkowski space). As a result the constant field case reduces to the calculation of the spectrum of two uncoupled harmonic oscillators describing the planar motion of two independent charged particles in the normal magnetic and electric (in the Euclidean sense) fields $\frac{1}{2}(|\mathbf{B}+\mathbf{E}|\pm|\mathbf{B}-\mathbf{E}|)$. Therefore, constant fields are not generic in any sense, and the completely open problem of general static fields may yet prove to be of substantial interest.

TABLE I. Bounds on fermionic determinants. The lower bounds in QED_2 (Ref. [4]) and QED_3 (see Sec. II) are for the field $\mathbf{B}=(0,0,B(x,y))$, $B(x,y)\geqslant 0$. For $B(x,y)\leqslant 0$ replace B with $-B$. The upper bound for QED_2 has no restriction on the sign of $B(x,y)$. The upper bounds for QED_3 (Ref. [8]) and QED_4 (see Sec. III) are for a static, directionally varying field $\mathbf{B}(\mathbf{r})$, $\mathbf{r}\in R^3$. Z and T denote the size of the boxes in the z and t directions. The lower bounds for $QED_{2,3}$ are representative; better but more complicated bounds may be found in Sec. II and in Ref. [4].

QED_2	
	$-\frac{\|B\|^2}{4\pi m^2}\leqslant \ln\det\leqslant 0$
QED_3	
	$-\frac{Z}{6\pi}\int d^2r\,\lvert B\rvert^{3/2}\leqslant \ln\det\leqslant 0$
QED_4	
	$\frac{ZT\|B\|^2}{48\pi^2}\leqslant \lim_{\lambda\to\infty}\left(\frac{\ln\det_{ren}(\lambda B)}{\lambda^2\ln\lambda}\right)\leqslant\frac{T\|\mathbf{B}\|^2}{6\pi^2}$

II. THREE-DIMENSIONAL QED

A. Connection between the fermionic determinants in QED_3 and QED_2

We choose for the Dirac matrices in three dimensions the 2×2 matrices $\gamma^\mu=(i\sigma_1,i\sigma_2,i\sigma_3)$, where the σ_i's are the Pauli matrices. In this case definition (1.1) of the determinant in QED_3 reduces to

$$\ln\det{}_{QED_3}=\frac{1}{2}\int_0^\infty\frac{dt}{t}\,\mathrm{Tr}(e^{-P^2t}-\exp\{-[(\mathbf{P}-\mathbf{A})^2-\boldsymbol{\sigma}\cdot\mathbf{B}]t\})e^{-tm^2}. \quad (2.1)$$

This definition (and regularization) of the fermionic determinant is parity conserving and gives no Chern-Simons term, which is known to be regularization dependent [15]. Such a term may always be added. In order to relate $\det_{\mathrm{QED}_3}$ to Euclidean QED in two dimensions let $\mathbf{B}=(0,0,B(\mathbf{r}))$, $\mathbf{A}=(\mathbf{A}_\perp(\mathbf{r}),0)$, and $\mathbf{A}_\perp=(A_x,A_y)$, where $\mathbf{r}$ is a point in the xy plane. Enclosing the z axis (which may also be called the time axis) in a large box of length Z we get

$$\ln \det_{\mathrm{QED}_3}(m^2)=\frac{Z}{4\pi^{1/2}}\int_0^\infty \frac{dt}{t^{3/2}}\,\mathrm{Tr}(e^{-\mathbf{P}_\perp^2 t}-\exp\{-[(\mathbf{P}_\perp-\mathbf{A}_\perp)^2-\sigma_3 B]t\})e^{-tm^2}, \tag{2.2}$$

where we used

$$\mathrm{Tr}_{\mathrm{space}}e^{-P_3^2 t}=\frac{Z}{(4\pi t)^{1/2}}; \tag{2.3}$$

the remaining trace in Eq. (2.2) is over space and spin indices.

The fermionic determinant in Euclidean QED_2 (denoted by $\det_{\mathrm{Sch}}$ in Refs. [1–4]) is

$$\ln \det_{\mathrm{QED}_2}(m^2)=\frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}(e^{-\mathbf{P}_\perp^2 t}-\exp\{-[(\mathbf{P}_\perp-\mathbf{A}_\perp)^2-\sigma_3 B]\})e^{-tm^2}. \tag{2.4}$$

Using nothing more than $\int_0^\infty dE\,\exp(-tE^2)=(\pi/4t)^{1/2}$ we get the connection between the two determinants:

$$\ln \det_{\mathrm{QED}_3}(m^2)=\frac{Z}{\pi}\int_0^\infty dE\,\ln \det_{\mathrm{QED}_2}(E^2+m^2)=\frac{Z}{2\pi}\int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2-m^2}}\ln \det_{\mathrm{QED}_2}(M^2). \tag{2.5}$$

As for B, we are assuming that it is a smooth, polynomial bounded C^∞ function with finite flux; it will also be assumed to be square integrable.

As a check on Eq. (2.5) one may substitute the second-order contribution to QED_2's determinant obtained by expanding Eq. (2.4),

$$\ln \det_{\mathrm{QED}_2}=-\frac{1}{2\pi}\int \frac{d^2k_\perp}{(2\pi)^2}|\hat{B}(\mathbf{k}_\perp)|^2\times\int_0^1 dz\,\frac{z(1-z)}{z(1-z)k_\perp^2+m^2}+O(B^4), \tag{2.6}$$

and obtain the canonical result

$$\ln \det_{\mathrm{QED}_3}=-\frac{Z}{4\pi}\int \frac{d^2k_\perp}{(2\pi)^2}|\hat{B}(\mathbf{k}_\perp)|^2\times\int_0^1 dz\,\frac{z(1-z)}{[z(1-z)k_\perp^2+m^2]^{1/2}}+O(B^4), \tag{2.7}$$

for the unidirectional field $B(\mathbf{r})$.

An immediate consequence of Eq. (2.5) is that the ‘‘diamagnetic’’ bound in QED_2 [8,13],

$$\det_{\mathrm{QED}_2}\leqslant 1, \tag{2.8}$$

implies a ‘‘diamagnetic’’ bound in QED_3 for the static field $\mathbf{B}=(0,0,B(\mathbf{r}))$,

$$\det_{\mathrm{QED}_3}\leqslant 1. \tag{2.9}$$

The term ‘‘diamagnetic’’ is placed in quotation marks as it is really an expression of the paramagnetic property of fermions as definitions (2.1) and (2.4) make clear.

B. Lower bound on $\ln \det_{\mathrm{QED}_3}$

It is now possible to obtain a lower bound on $\ln \det_{\mathrm{QED}_3}$ with the aid of Eq. (2.5) for the field $\mathbf{B}=(0,0,B(\mathbf{r}))$, where $B(\mathbf{r})\geqslant 0$ or $B(\mathbf{r})\leqslant 0$, and $\mathbf{r}$ is a point in the xy plane. For $B(\mathbf{r})\geqslant 0$ we showed in [4] that

$$\ln \det_{\mathrm{QED}_2}\geqslant\frac{1}{4\pi}\int d^2r\left[B(\mathbf{r})-[B(\mathbf{r})+m^2]\ln\left(1+\frac{B(\mathbf{r})}{m^2}\right)\right]. \tag{2.10}$$

For $B\leqslant 0$, simply replace B with $-B$. Thus Eqs. (2.5) and (2.10) give

$$\begin{aligned}\ln \det_{\mathrm{QED}_3}&\geqslant\frac{Z}{8\pi^2}\int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2-m^2}}\int d^2r\left[B(\mathbf{r})-[B(\mathbf{r})+M^2]\ln\left(1+\frac{B(\mathbf{r})}{M^2}\right)\right]\\&=\frac{Z|m|^3}{6\pi}\int d^2r\left[1+\frac{3B(\mathbf{r})}{2m^2}-\left(1+\frac{B(\mathbf{r})}{m^2}\right)^{3/2}\right].\end{aligned} \tag{2.11}$$

This is our main bound. A simpler, less stringent bound can be obtained by noting that

$$1+\frac{3}{2}x-(1+x)^{3/2}\geqslant -x^{3/2},\quad x\geqslant 0, \tag{2.12}$$

in which case

$$\ln \det_{\mathrm{QED}_3}\geqslant-\frac{Z}{6\pi}\int d^2r|B(\mathbf{r})|^{3/2}. \tag{2.13}$$

The absolute value has been added to include the two possible signs of B. Note that this bound is uniform in the fermion's mass. The reader is again cautioned that our results (2.11) and (2.13) have only been established for magnetic fields that have the same sign everywhere.

As a formal check on Eq. (2.13) we can compare it with Redlich's [16] result for the zero-mass limit of $\ln \det_{\mathrm{QED}_3}$ for the case of a constant magnetic field. Combining his Eq. (4.25) with Eq. (2.13) requires

$$\lim_{m\to 0} \ln \det_{\mathrm{QED}_3} = -\frac{V\zeta(3/2)}{4\pi^2\sqrt{2}} B^{3/2} \geqslant -\frac{V}{6\pi} B^{3/2}, \tag{2.14}$$

or $\zeta(3/2) \leqslant 2^{3/2}\pi/3$, where $\zeta(3/2)$ is the Riemann ζ function

$$\zeta(3/2) = \sum_{n=1}^{\infty} n^{-3/2}, \tag{2.15}$$

and V is the volume of a large box in $\mathbb{R}^3$. Since $\zeta(3/2)=2.612$ to four significant figures, Eq. (2.14) implies $2.612 \leqslant 2.962$.

If the z axis is relabeled as the time axis then the effective one-loop energy $\mathcal{E}$ of QED_3 is bounded by

$$0 \leqslant \mathcal{E} \leqslant \frac{1}{6\pi} \int d^2r |B(\mathbf{r})|^{3/2}, \tag{2.16}$$

where the lower bound comes from the diamagnetic bound (2.9). Hence our results support stability for the class of static magnetic fields considered here.

As another check on our results consider the field of Ref. [12] given by Eq. (1.5). Equation (2.16) gives the bound

$$0 \leqslant \mathcal{E} \leqslant L\lambda (eB)^{3/2}/12, \tag{2.17}$$

where the coupling e has been restored, and L is the length of the strip in the y direction. The authors of Ref. [12] calculated $\mathcal{E}$ analytically. The zero-mass limit of $\mathcal{E}$, given by their Eq. (22), allows a direct check on Eq. (2.17):

$$\mathcal{E} = \frac{L\lambda (eB)^{3/2}}{8\sqrt{2}\pi} \left[\zeta(3/2) - \frac{15}{16\pi} \zeta(5/2) \frac{1}{eB\lambda^2} + \cdots \right]. \tag{2.18}$$

Thus Eqs. (2.17) and (2.18) give $0.073 - \cdots \leqslant 0.083$.

C. Induced spin

Using the above results we can obtain a lower bound on the spin induced in the vacuum by a static, unidirectional magnetic field for all finite values of the fermion mass. In 2+1 dimensions the normal ordered spin density in the field $\mathbf{B}(\mathbf{r}) = (0,0,B(\mathbf{r}))$ derived from the potential $\mathbf{A} = (\mathbf{A}_\perp(\mathbf{r}),0)$ is given by

$$\begin{aligned} S(\mathbf{r};B) &= \frac{1}{2} \left\langle \left[\psi^\dagger(\mathbf{r},t) \frac{1}{2} \sigma_3, \psi(\mathbf{r},t) \right]_- \right\rangle \\ &= -\frac{1}{4} \lim_{\epsilon\downarrow 0} \sum_n \int_C \frac{d\omega}{2\pi i} e^{-i\omega\epsilon} \psi_n^\dagger(\mathbf{r}) \sigma_3 \psi_n(\mathbf{r}) \\ &\quad \times [(E_n - \omega)^{-1} + (E_n + \omega)^{-1}], \end{aligned} \tag{2.19}$$

where the contour C runs below the negative real ω axis, passes through the origin, and continues running above the positive real ω axis. The ψ_n are energy eigenstates

$$H\psi_n = E_n \psi_n,$$

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p}_\perp - \mathbf{A}_\perp) + \beta m, \tag{2.20}$$

with $\gamma^1 = i\sigma_1$, $\gamma^2 = i\sigma_2$, $\beta = \sigma_3$, and $\boldsymbol{\alpha} = \beta\boldsymbol{\gamma}$. Then

$$\begin{aligned} S(\mathbf{r};B) &= -\frac{1}{4} \lim_{\epsilon\downarrow 0} \int_C \frac{d\omega}{2\pi i} e^{-i\omega\epsilon} \\ &\quad \times \mathrm{tr}\langle \mathbf{r} | (\not{P}_\perp - \not{A}_\perp + m - \omega\sigma_3)^{-1} \\ &\quad + (\not{P}_\perp - \not{A}_\perp + m + \omega\sigma_3)^{-1} | \mathbf{r} \rangle \\ &= -\frac{m}{2} \lim_{\epsilon\downarrow 0} \int_C \frac{d\omega}{2\pi i} e^{-i\omega\epsilon} \mathrm{tr}\langle \mathbf{r} | ((\mathbf{P}_\perp - \mathbf{A}_\perp)^2 \\ &\quad - \sigma_3 B + m^2 - \omega^2)^{-1} | \mathbf{r} \rangle. \end{aligned} \tag{2.21}$$

Now rotate the ω contour 90° counterclockwise while letting $\epsilon \to -i\epsilon$ to effect a Wick rotation. This gives

$$\begin{aligned} S(\mathbf{r},B) &= -m \int_0^\infty \frac{dE}{2\pi} \mathrm{tr}\langle \mathbf{r} | ((\mathbf{P}_\perp - \mathbf{A}_\perp)^2 \\ &\quad - \sigma_3 B + m^2 + E^2)^{-1} | \mathbf{r} \rangle. \end{aligned} \tag{2.22}$$

To make sense out of this the spin density at $B=0$ has to be subtracted out. Changing the integration variable to $M^2 = E^2 + m^2$ and integrating over the xy plane, we obtain the total induced spin:

$$\begin{aligned} S(B) - S(0) &= -\frac{m}{4\pi} \int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \mathrm{Tr}[((\mathbf{P}_\perp - \mathbf{A}_\perp)^2 \\ &\quad - \sigma_3 B + M^2)^{-1} - (P_\perp^2 + M^2)^{-1}]. \end{aligned} \tag{2.23}$$

We will now relate the induced spin to the determinants $\det_{\mathrm{QED}_3}$ and $\det_{\mathrm{QED}_2}$. From Eq. (2.4),

$$\begin{aligned} \frac{\partial}{\partial m^2} \ln \det_{\mathrm{QED}_2} &= \frac{1}{2} \mathrm{Tr}[((\mathbf{P}_\perp - \mathbf{A}_\perp)^2 - \sigma_3 B + M^2)^{-1} \\ &\quad - (P_\perp^2 + M^2)^{-1}], \end{aligned} \tag{2.24}$$

and hence

$$S(B) - S(0) = -\frac{m}{2\pi} \int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \frac{\partial}{\partial M^2} \ln \det_{\mathrm{QED}_2}(M^2). \tag{2.25}$$

Since $\det_{\mathrm{QED}_2}$ is even in B so is the induced spin.

From Eq. (2.2), after relabeling the z axis the time axis, we get

$$\begin{aligned} \frac{\partial}{\partial m} \ln \det_{\mathrm{QED}_3} &= \frac{mT}{2\pi^{1/2}} \int_0^\infty \frac{dt}{t^{1/2}} \mathrm{Tr}(\exp\{-[(\mathbf{P}_\perp - \mathbf{A}_\perp)^2 \\ &\quad - \sigma_3 B]t\} - e^{-P_\perp^2 t}) e^{-tm^2}. \end{aligned} \tag{2.26}$$

Again using $t^{-1/2} = (4/\pi)^{1/2} \int_0^\infty dE \exp(-tE^2)$ and then changing the integration variable to $M^2 = E^2 + m^2$ gives

$$\begin{aligned} \frac{\partial}{\partial m} \ln \det_{\mathrm{QED}_3} &= \frac{mT}{2\pi} \int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2 - m^2}} \mathrm{Tr}[((\mathbf{P}_\perp - \mathbf{A}_\perp)^2 - \sigma_3 B \\ &\quad + M^2)^{-1} - (P_\perp^2 + M^2)^{-1}]. \end{aligned} \tag{2.27}$$

Comparing Eq. (2.27) with (2.23) gives

$$\frac{\partial}{\partial m}\ln \det_{\mathrm{QED}_3} = -2T[S(B)-S(0)], \tag{2.28}$$

which is what one expects from formal manipulation of the fermionic functional integral for $\det_{\mathrm{QED}_3}$.

We are now in a position to obtain bounds on the induced spin. From Eqs. (5) and (6) in [4],

$$\frac{\partial}{\partial m^2}\ln \det_{\mathrm{QED}_2} \leqslant \frac{\Phi}{4\pi m^2} - \frac{1}{4\pi}\int d^2r\, \ln\left(1+\frac{B(\mathbf{r})}{m^2}\right), \tag{2.29}$$

where it is again assumed that $B(\mathbf{r})\geqslant 0$ or $B(\mathbf{r})\leqslant 0$ with $\mathbf{r}$ a point in the xy plane and $\Phi=\int d^2r\, B(\mathbf{r})$. Substituting Eq. (2.29) in Eq. (2.25) gives, for $m>0$,

$$\begin{aligned} S(B)-S(0) &\geqslant \frac{m}{8\pi^2}\int_{m^2}^{\infty}\frac{dM^2}{\sqrt{M^2-m^2}}\left[\int d^2r\, \ln\left(1+\frac{B(\mathbf{r})}{M^2}\right) - \frac{\Phi}{M^2}\right] \\ &= \frac{m}{4\pi}\int d^2r\left[[B(\mathbf{r})+m^2]^{1/2} - \frac{B(\mathbf{r})}{2m} - m\right], \end{aligned} \tag{2.30}$$

while, for $m<0$,

$$S(B)-S(0) \leqslant -\frac{|m|}{4\pi}\int d^2r\left[[B(\mathbf{r})+m^2]^{1/2} - \frac{B(\mathbf{r})}{2|m|} - |m|\right]. \tag{2.31}$$

For $B\leqslant 0$ simply change the sign of B in Eqs. (2.30) and (2.31). Elementary estimates indicate that the integrals in Eqs. (2.30) and (2.31) converge if $B\in L^2(\mathbb{R}^2)$.

Of particular interest is the $m=0$ limit of the induced spin since this is related to the vacuum condensate $\langle\psi\psi\rangle_{m=0}$ in the presence of an inhomogeneous background magnetic field. If the range of B is finite and independent of m, then the $m=0$ limit may be safely taken, giving

$$[S(B)-S(0)]_{m\downarrow 0} \geqslant -\Phi/8\pi \tag{2.32}$$

and

$$[S(B)-S(0)]_{m\uparrow 0} \leqslant \Phi/8\pi. \tag{2.33}$$

These limits are consistent with the results of Parwani [17]. Comparing Eq. (2.28) with Eqs. (2.32) and (2.33) it is evidently possible for $\ln\det_{\mathrm{QED}_3}$ to have a discontinuous mass derivative at $m=0$.

Finally, the vacuum condensate for the magnetic field

$$B(r) = B(1+r^2/R^2)^{-2}, \tag{2.34}$$

has been calculated by Dunne and Hall [18]. Assume $B>0$. Since the authors use 4×4 γ matrices, their result has to be divided by a factor of 2 to correct for this and by another factor of 2 to relate their condensate to the spin density in Eq. (2.19). Their Eq. (29) then gives

$$[S(\mathbf{r};B)-S(\mathbf{r};0)]_{m\to 0} = -\mathrm{sgn}(m)\left(1-\frac{2\pi}{\Phi}\right)\frac{B(r)}{8\pi}, \tag{2.35}$$

and hence

$$[S(B)-S(0)]_{m\to 0} = -\mathrm{sgn}(m)\left(1-\frac{2\pi}{\Phi}\right)\frac{\Phi}{8\pi}. \tag{2.36}$$

The result (2.36) is therefore consistent with our results (2.32) and (2.33).

III. FOUR-DIMENSIONAL QED

A. Connection between the fermionic determinants in QED_3 and QED_4

In order to make the connection we choose the static potential $A_\mu=(0,\mathbf{A}(\mathbf{r}))$. It is assumed that $\mathbf{A}$ is polynomial bounded C^∞ and that $\mathbf{A}\in L^3(\mathbb{R}^3)$. Why $\mathbf{A}$ is chosen to be in L^3 will be explained below (see also end of Sec. III C). We will require that the magnetic field $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ be square integrable. If $\mathbf{B}\in L^2(\mathbb{R}^3)$ and $\mathbf{A}$ is also assumed to be in the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, then, by Sobolev-Talenti-Aubin inequality [19],

$$\int d^3r\, \mathbf{B}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r}) \geqslant (27\pi^4/16)^{1/3}\sum_{i=1}^{3}\left(\int d^3r |A_i(\mathbf{r})|^6\right)^{1/3}, \tag{3.1}$$

that is, $\mathbf{A}\in L^6(\mathbb{R}^3)$ as well as $L^3(\mathbb{R}^3)$. As a simple consequence of this and Hölder's inequality [20],

$$\|fg\|_r \leqslant \|f\|_p\|g\|_q,$$

$$p^{-1}+q^{-1}=r^{-1}, \quad 1\leqslant p,\ q,\ r\leqslant\infty, \tag{3.2}$$

$\mathbf{A}\in \bigcap_{3\leqslant p\leqslant 6} L^p(\mathbb{R}^3)$. No assumption has to be made about finite flux as it is always zero. Finally, we choose the chiral representation of the γ matrices so that $\sigma_{ij}=\begin{pmatrix}-\sigma_k & 0\\ 0 & -\sigma_k\end{pmatrix}$ with $i,j,k=1,2,3$ in cyclic order.

Following these preliminaries, Eq. (1.1) for QED_4's fermionic determinant reduces to

$$\begin{aligned} \ln\det_{\mathrm{ren}} = \frac{T}{2}\int_0^\infty \frac{dt}{t}\Bigg[\frac{2}{(4\pi t)^{1/2}}\,\mathrm{Tr}\big(e^{-\mathbf{P}^2 t} \\ - \exp\{-[(\mathbf{P}-\mathbf{A})^2 - \boldsymbol{\sigma}\cdot\mathbf{B}]t\}\big) + \frac{\|\mathbf{B}\|^2}{12\pi^2}\Bigg]e^{-tm^2}, \end{aligned} \tag{3.3}$$

where T is the dimension of the time box and $\|\mathbf{B}\|^2=\int d^3r\,\mathbf{B}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r})$. We have used Eq. (2.3) again for $\mathrm{Tr}_{\mathrm{space}}e^{-P_0^2 t}$, exchanging Z for T. The factor 2 in Eq. (3.3) comes from a partial spin sum; the remaining spin trace is over a two-dimensional space. The determinant in QED_3 in the presence of $\mathbf{B}(\mathbf{r})$ is, from Eq. (1.1),

$$\ln \det_{\mathrm{QED}_3}(m^2) = \frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}\big(e^{-\mathbf{P}^2 t} - \exp\{-[(\mathbf{P}-\mathbf{A})^2 - \boldsymbol{\sigma}\cdot\mathbf{B}]t\}\big)e^{-tm^2}. \tag{3.4}$$

Thus we get the connection between QED_3 and QED_4 for static magnetic fields:

$$\begin{aligned}\ln \det_{\mathrm{ren}} &= \frac{2T}{\pi}\int_0^\infty dE\left(\ln \det_{\mathrm{QED}_3}(E^2+m^2) + \frac{\|\mathbf{B}\|^2}{24\pi^{3/2}}\int_0^\infty \frac{dt}{t^{1/2}}\, e^{-(E^2+m^2)t}\right)\\ &= \frac{T}{\pi}\int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2-m^2}}\left(\ln \det_{\mathrm{QED}_3}(M^2) + \frac{\|\mathbf{B}\|^2}{24\pi\sqrt{M^2}}\right).\end{aligned} \tag{3.5}$$

In order to get the upper bound on $\ln \det_{\mathrm{ren}}$ in Table I it is useful to isolate the second-order contribution to $\ln \det_{\mathrm{QED}_3}$. Denoting the remainder by $\ln \det_4$ we get

$$\ln \det_{\mathrm{QED}_3}(1 - S\mathcal{A}) = -\frac{1}{4\pi}\int \frac{d^3k}{(2\pi)^3}\,|\hat{\mathbf{B}}(\mathbf{k})|^2 \times \int_0^1 dz\, \frac{z(1-z)}{[z(1-z)k^2+m^2]^{1/2}} + \ln \det_4(1-S\mathcal{A}), \tag{3.6}$$

where the subscript 4 in Eq. (3.6) is explained below; it does not refer to four dimensions. We have changed the arguments of $\det_{\mathrm{QED}_3}$ in preparation for making contact with the formal identity $\ln \det(1+x) = \mathrm{Tr}\ln(1+x)$; see Eq. (3.7) below. The first term on the right-hand side of Eq. (3.6) was obtained by expanding Eq. (3.4) to second order. Graphically, $\ln \det_4$ is the sum of all even order one-loop fermion graphs in three dimensions, beginning with the fourth-order box graph since definition (3.4) respects Furry's theorem or C invariance. Thus, restoring e,

$$\ln \det_4(1 - eS\mathcal{A}) = -\sum_{n=4}^\infty \frac{e^n}{n}\,\mathrm{Tr}(S\mathcal{A})^n. \tag{3.7}$$

The operator $S\mathcal{A}$ is a bounded operator on $L^2(\mathbb{R}^3, \sqrt{k^2+m^2}\,d^3k; \mathbb{C}^2)$ for $\mathbf{A}\in L^p(\mathbb{R}^3)$ for some $p>3$, which is the case here. In addition, $S\mathcal{A} = (\not{P}+m)^{-1}\mathcal{A}(X)$ belongs to the trace ideal $\mathcal{C}_p$ for $p>3$ $[\mathcal{C}_n = \{A \,|\, \|A\|_n^n \equiv \mathrm{Tr}((A^\dagger A)^{n/2}) < \infty\}]$ [6,21–23]. As a result, the eigenvalues $1/e_n$ of the compact operator $S\mathcal{A}$ (none of which are real for $m\neq 0$; see Sec. III C) are such that $\sum_{n=1}^\infty |e_n|^{-p} < \infty$ [24]. Therefore, the series in Eq. (3.7) has a finite radius of convergence, although our analysis will not rely on this. More will be said about $\det_4$ for general e in Sec. III C. For the present, note that it is defined for all real e by Eqs. (3.4) and (3.6) [see Eq. (3.14) below]. But already one begins to see the usefulness of $\mathbf{A}\in L^3(\mathbb{R}^3)$.

Inserting Eq. (3.6) in Eq. (3.5) gives

$$\ln \det_{\mathrm{ren}} = \frac{T}{4\pi^2}\int \frac{d^3k}{(2\pi)^3}\,|\hat{\mathbf{B}}(\mathbf{k})|^2 \int_0^1 dz\, z(1-z)\ln\left[\frac{z(1-z)k^2+m^2}{m^2}\right] + \frac{T}{\pi}\int_{m^2}^\infty \frac{dM^2}{\sqrt{M^2-m^2}}\ln \det_4(M^2). \tag{3.8}$$

The first term on the right-hand side is the second-order vacuum polarization contribution to $\det_{\mathrm{ren}}$ in a static magnetic field, renormalized at zero momentum transfer.

B. Upper bound on $\ln \det_{\mathrm{ren}}$

An upper bound can be put on $\ln \det_{\mathrm{ren}}$ in a general static magnetic field $\mathbf{B}(\mathbf{r})$ with the help of Eq. (3.8) and the diamagnetic inequality for QED_3 [8]:

$$|\det_{\mathrm{QED}_3}(1 - eS\mathcal{A})| \le 1, \tag{3.9}$$

where $\mathbf{A}$ is the smooth potential we introduced in Sec. III A. For $m\neq 0$ $\det_{\mathrm{QED}_3}$ has no zeros for real e (see Sec. III C) and if $\det_{\mathrm{QED}_3}|_{e=0} = 1$, then we can write, instead of Eq. (3.9),

$$0 < \det_{\mathrm{QED}_3}(1 - eS\mathcal{A}) \le 1. \tag{3.10}$$

A few comments on Eqs. (3.9) and (3.10) are in order. The diamagnetic inequality is general and follows for any determinant that is obtained as the continuum limit of a lattice theory obeying reflection positivity. On the lattice Wilson fermions may be used, and since they are CP invariant, there is no Chern-Simons term [25]. The fact that the continuum limit of $\det_{\mathrm{QED}_3}$ coincides with definition (3.4) follows from Seiler's Statement 5.4 and his Eq. (7.20) (without the counterterm which is not needed in QED_3) in Ref. [6].

Now Eqs. (3.10) and (3.6) imply that

$$\ln \det_4(1 - S\mathcal{A}) \le \frac{1}{4\pi}\int \frac{d^3k}{(2\pi)^3}\,|\hat{\mathbf{B}}(\mathbf{k})|^2 \times \int_0^1 dz\, \frac{z(1-z)}{[z(1-z)k^2+m^2]^{1/2}}. \tag{3.11}$$

This remarkable consequence of the paramagnetism of charged spin-$\frac{1}{2}$ fermions implies that all the nonlinearities of $\ln \det_4$ are bounded by a quadratic in the magnetic field. Inserting Eq. (3.11) into Eq. (3.8) gives, for $\|\mathbf{B}\|^2 \ge |m|$ (restoring e, recall that $e^2\|\mathbf{B}\|^2 = e^2\int d^3r\, \mathbf{B}\cdot\mathbf{B}$ has the dimension of mass in both three and four dimensions)

$$\ln \det_{\text{ren}} \leqslant \frac{T}{4\pi^2}\int \frac{d^3k}{(2\pi)^3}|\hat{\mathbf{B}}(\mathbf{k})|^2\int_0^1 dz\, z(1-z)\ln\left[\frac{z(1-z)k^2+m^2}{m^2}\right]$$

$$+\frac{T}{4\pi^2}\int_{m^2}^{\|\mathbf{B}\|^4}\frac{dM^2}{\sqrt{M^2-m^2}}\int \frac{d^3k}{(2\pi)^3}|\hat{\mathbf{B}}(\mathbf{k})|^2\int_0^1 dz\,\frac{z(1-z)}{[z(1-z)k^2+M^2]^{1/2}}+\frac{T}{\pi}\int_{\|\mathbf{B}\|^4}^{\infty}\frac{dM^2}{\sqrt{M^2-m^2}}\ln \det_4(M^2)$$

$$\leqslant \frac{T}{4\pi^2}\int \frac{d^3k}{(2\pi)^3}|\hat{\mathbf{B}}(\mathbf{k})|^2\int_0^1 dz\, z(1-z)\ln\left[\frac{4\|\mathbf{B}\|^4+2z(1-z)k^2-2m^2}{m^2}\right]+\frac{T}{\pi}\int_{\|\mathbf{B}\|^4}^{\infty}\frac{dM^2}{\sqrt{M^2-m^2}}\ln \det_4(M^2). \quad (3.12)$$

The argument of the logarithm in the last line of Eq. (3.12) has been simplified somewhat using $2\sqrt{xy}\leqslant x+y$ for $x,y\geqslant 0$.

The last term in Eq. (3.12) can be estimated for strong fields. Thus let $\mathbf{B}\to\lambda\mathbf{B}$, $\lambda>0$. Then

$$\ln \det_{\text{ren}} \underset{\lambda\gg 1}{\leqslant} \frac{\lambda^2 T\|\mathbf{B}\|^2}{24\pi^2}\ln\left(\frac{4\lambda^4\|\mathbf{B}\|^4}{m^2}\right)+\frac{T}{\pi}\int_{\lambda^4\|\mathbf{B}\|^4}^{\infty}\frac{dM^2}{\sqrt{M^2-m^2}}\ln \det_4(\lambda\mathbf{B},M^2)+O(\lambda^{-2}). \quad (3.13)$$

Evidently the large mass behavior of $\ln \det_4$ is needed in Eq. (3.13). Equations (3.4) and (3.6) can be combined to give

$$\ln \det_4(m^2)=\frac{1}{2}\int_0^\infty \frac{dt}{t}\left[\mathrm{Tr}(e^{-P^2t}-\exp\{-[(\mathbf{P}-\mathbf{A})^2-\boldsymbol{\sigma}\cdot\mathbf{B}]t\})+\frac{t^{1/2}}{2\pi^{3/2}}\int_0^1 dz\, z(1-z)\int \frac{d^3k}{(2\pi)^3}|\hat{\mathbf{B}}(\mathbf{k})|^2 e^{-k^2 z(1-z)t}\right]e^{-tm^2}, \quad (3.14)$$

so that the large mass limit will come from the small-t region of $\det_4$'s proper time representation. Carrying out the small-t expansion we find

$$\mathrm{Tr}(\exp\{-[(\mathbf{P}-\mathbf{A})^2-\boldsymbol{\sigma}\cdot\mathbf{B}]t\}-e^{-P^2t})$$
$$=(4\pi t)^{-3/2}\int d^3r\left[\frac{2}{3}t^2\mathbf{B}\cdot\mathbf{B}+\frac{2}{15}t^3\mathbf{B}\cdot\boldsymbol{\nabla}^2\mathbf{B}-\frac{2}{45}t^4(\mathbf{B}\cdot\mathbf{B})^2+\frac{1}{70}t^4\mathbf{B}\cdot\boldsymbol{\nabla}^4\mathbf{B}+O(t^5\mathbf{B}\cdot\mathbf{B}\mathbf{B}\cdot\boldsymbol{\nabla}^2\mathbf{B},t^5\mathbf{B}\cdot\boldsymbol{\nabla}^6\mathbf{B})\right], \quad (3.15)$$

which, together with Eq. (3.14), gives the large-mass expansion of $\ln \det_4$:

$$\ln \det_4=\frac{1}{2}\int_0^\infty\frac{dt}{t}\int d^3r\left[\frac{t^{5/2}}{180\pi^{3/2}}(\mathbf{B}\cdot\mathbf{B})^2+O(t^{7/2}\mathbf{B}\cdot\mathbf{B}\mathbf{B}\cdot\boldsymbol{\nabla}^2\mathbf{B})\right]e^{-tm^2}$$
$$=\frac{\int(\mathbf{B}\cdot\mathbf{B})^2}{480\pi|m|^5}+O\left(\frac{\int\mathbf{B}\cdot\mathbf{B}\mathbf{B}\cdot\boldsymbol{\nabla}^2\mathbf{B}}{|m|^7}\right). \quad (3.16)$$

Then

$$\int_{\lambda^4\|\mathbf{B}\|^4}^{\infty}\frac{dM^2}{\sqrt{M^2-m^2}}\ln \det_4(\lambda\mathbf{B},M^2)$$
$$=\frac{\int(\mathbf{B}\cdot\mathbf{B})^2}{960\pi\|\mathbf{B}\|^8\lambda^4}+O(\lambda^{-8}), \quad (3.17)$$

and hence Eqs. (3.13) and (3.17) give the bound in Table I:

$$\lim_{\lambda\to\infty}\frac{\ln \det_{\text{ren}}(\lambda\mathbf{B})}{\lambda^2\ln\lambda}\leqslant\frac{T\|\mathbf{B}\|^2}{6\pi^2}. \quad (3.18)$$

We note that this upper bound for a general static field **B** is greater by a factor of 2 than the case when **B** is unidirectional [1].

The bound (3.18) relies on the proper time definition of the fermionic determinant in Eq. (1.1). Recall that it incorporates mass-shell renormalization. Therefore the question arises as to whether the bound (3.18) is renormalization scale dependent. It is not.

It is easy to demonstrate this because we are dealing with QED_4 in external fields. The only divergent graph is the second-order photon self-energy term. Hence any finite renormalization scale change will only introduce an ambiguity quadratic in the field and so cannot change the bound (3.18).

To show this explicitly refer back to Eq. (3.8): the logarithm in the photon self-energy term vanishes at $k^2=0$, corresponding to on-shell renormalization. Suppose we subtracted at $k^2=\mu^2$. Then referring to Eq. (3.8) again we get

$$\ln \det_{\text{ren}}(m^2,\mu^2)=\ln \det_{\text{ren}}(m^2,\mu^2=0)+\frac{T}{4\pi^2}\|\mathbf{B}\|^2\times\int_0^1 dz\, z(1-z)\ln\left[\frac{m^2}{z(1-z)\mu^2+m^2}\right]. \quad (3.19)$$

The two determinants differ only by a quadratic in the field strength, as promised. Therefore the bound (3.18), which indicates a growth faster than quadratic, is not altered by a renormalization scale change.

Let us conclude this section with a comment on the physics of Eq. (3.18). The main input was the ‘‘diamagnetic’’ bound given by the upper bound in Eq. (3.10). It is a reflection of the paramagnetic tendency of charged fermions in an external magnetic field as is evident from Eq. (3.4): the eigenvalues of the Pauli Hamiltonian are, on average, reduced in the presence of **B** relative to the **B**=0 case. The bound in Eq. (3.18) is saying that because of this there is a limit on how fast the one-loop effective action—due to the vacuum fermion current density induced by **B**—can grow. It is also interesting that the diamagnetic bound has come to us by a long chain of reasoning starting with QED_3 on a lattice, that it had lain dormant for about seventeen years, and then emerged again to tell us something nontrivial about QED_4.

C. Zeros of $\det_4$

In order to write Eq. (3.9) in the form (3.10) it is necessary to show that $\det_{\mathrm{QED}_3}$ or, equivalently $\det_4(1-eS\not{A})$, has no zeros for real e when $m\neq 0$. Instead of working in the Hilbert space $L^2(\mathbb{R}^3,\sqrt{k^2+m^2}d^3k;\mathbb{C}^2)$ introduced in Sec. III B we will make a similarity transformation on $S\not{A}$, which does not change its eigenvalues, and deal with the integral operator

$$K=(P^2+m^2)^{1/4}S\not{A}(P^2+m^2)^{-1/4}, \tag{3.20}$$

on $L^2(\mathbb{R}^3,d^3r;\mathbb{C}^2)$ [21,26]. Let $\psi_n\in L^2$ be an eigenvector of K,

$$K\psi_n=\frac{1}{e_n}\,\psi_n\,. \tag{3.21}$$

Taking the Fourier transform of Eq. (3.21) gives

$$\int\frac{d^3k}{(2\pi)^3}\,\hat{\not{A}}(\mathbf{p}-\mathbf{k})(k^2+m^2)^{-1/4}\hat{\psi}_n(\mathbf{k})=\frac{1}{e_n}\,\frac{\not{p}+m}{(p^2+m^2)^{1/4}}\,\hat{\psi}_n(\mathbf{p}). \tag{3.22}$$

Its complex conjugate is

$$-\int\frac{d^3k}{(2\pi)^3}\,\hat{\psi}_n^\dagger(\mathbf{k})(k^2+m^2)^{-1/4}\hat{\not{A}}(\mathbf{k}-\mathbf{p})=\frac{1}{e_n^\star}\,\frac{\hat{\psi}_n^\dagger(\mathbf{p})(m-\not{p})}{(p^2+m^2)^{1/4}}. \tag{3.23}$$

Multiply Eq. (3.22) from the left by $\hat{\psi}^\dagger(\mathbf{p})(p^2+m^2)^{-1/4}$ and Eq. (3.23) from the right by $(p^2+m^2)^{-1/4}\hat{\psi}_n(\mathbf{p})$; add the two equations and integrate both sides over p to get

$$i\ \mathrm{Im}(e_n)\int d^3p(p^2+m^2)^{-1/2}\hat{\psi}_n^\dagger(\mathbf{p})\not{p}\hat{\psi}_n(\mathbf{p})=m\ \mathrm{Re}(e_n)\int d^3p(p^2+m^2)^{-1/2}|\hat{\psi}_n(\mathbf{p})|^2. \tag{3.24}$$

Since $\psi_n\in L^2$ so does $\hat{\psi}_n$. Therefore both integrals in Eq. (3.24) converge by inspection, and the integral on the right-hand side is not zero. Since $\Sigma_{n=1}^{\infty}|e_n|^{-p}<\infty$ for $p>3$, there are no zeros at the origin. Hence Eq. (3.24) implies $\mathrm{Im}(e_n)\neq 0$ if $m\neq 0$. A similar conclusion was reached in QED_4 by Adler [27] and in the pseudoscalar Yukawa model in 1+1 dimensions in a finite space-time volume by Seiler [26].

In Sec. III B we saw that, for the potentials we are considering, $S\not{A}\in\mathcal{C}_p$, $p>3$. Then by theorem 6.2 of Simon in Ref. [24] we can express $\det_4$ in terms of the eigenvalues of $S\not{A}$:

$$\det_4(1-eS\not{A})=\prod_{n=1}^{\infty}\left[\left(1-\frac{e}{e_n}\right)\exp\left(\sum_{k=1}^{3}(e/e_n)^k/k\right)\right]. \tag{3.25}$$

Since all of the eigenvalues are off the real axis for $m\neq 0$, $\det_4$ cannot vanish for real values of e, and therefore Eq. (3.10) follows from this, definition (3.6), and $\det_{\mathrm{QED}_3}|_{e=0}=1$.

Since there is a nonsingular matrix C such that $C^{-1}\gamma_\mu C=-\gamma_\mu^T$, namely $C=\gamma_2$ in the representation $\gamma_\mu=(i\sigma_1,i\sigma_2,i\sigma_3)$, C invariance is maintained and hence $\det_4$ is an even function of e. This and the reality of $\det_4$ for real e imply that the eigenvalues appear in quartets $\pm e_n,\pm e_n^\star$. The same conclusion in QED_4 was reached by the authors in Ref. [28].

It is not essential that $\mathbf{A}\in L^3(\mathbb{R}^3)$. Instead, one may assume that $\mathbf{A}\in L^6(\mathbb{R}^3)$ in the Coulomb gauge as required if **B** is square integrable. In this case $S\not{A}\in\mathcal{C}_6$ so that one must deal with $\det_6$, whose expansion begins in sixth order. The analysis above is trivially modified with the end result that Eq. (3.18) is unchanged. It should be mentioned that restricting **A** to $L^6(\mathbb{R}^3)$ is not a weaker assumption than requiring **A** to be in $L^3(\mathbb{R}^3)$ since the former condition restricts local singularities more severely.

Finally, the analysis we have used to show that $\det_{\mathrm{QED}_3}$ has no zeros for real e and $m\neq 0$ may be applied to $\det_{\mathrm{QED}_2}$. Here it must be kept in mind that A_μ behaves as a "winding" field in the gauge $\partial_\mu A^\mu=0$ with a $1/r$ fall off if the magnetic flux is nonvanishing. Stated differently, $A_\mu\in L^2(\mathbb{R}^2)$ only if the total flux is zero. By assuming $A_\mu\in L^3(\mathbb{R}^2)$ one can show that $S\not{A}\in\mathcal{C}_3$ and conclude that $\det_{\mathrm{QED}_2}$ is never negative if $\det_{\mathrm{QED}_2}|_{e=0}=1$.

[1] M. P. Fry, Phys. Rev. D **45**, 682 (1992).
[2] M. P. Fry, Phys. Rev. D **47**, 2629 (1993).
[3] M. P. Fry, Phys. Rev. D **51**, 810 (1995).
[4] M. P. Fry, Phys. Rev. D **53**, 980 (1996).
[5] J. Schwinger, Phys. Rev. D **82**, 664 (1951).
[6] E. Seiler, in *Gauge Theories: Fundamental Interactions and Rigorous Results*, Proceedings of the International School of Theoretical Physics, Poiana Brasov, Romania, 1981, edited by P. Dita, V. Georgescu, and P. Purice, Progress in Physics Vol. 5 (Birkhäuser, Boston, 1982), p. 263.

[7] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, Lecture Notes in Physics Vol. 220 (Springer, Berlin, 1985).
[8] D. Brydges, J. Fröhlich, and E. Seiler, Ann. Phys. (N.Y.) **121**, 227 (1979).
[9] J. Schwinger, Phys. Rev. **128**, 2425 (1962).
[10] J. E. Avron and R. Seiler, Phys. Rev. Lett. **42**, 931 (1979).
[11] K. Miller, Ph.D. thesis, Princeton University, 1982.
[12] D. Cangemi, E. D'Hoker, and G. Dunne, Phys. Rev. D **52**, R3163 (1995).
[13] D. H. Weingarten, Ann. Phys. (N.Y.) **126**, 154 (1980).
[14] H. Euler and B. Kockel, Naturwissenschaften **23**, 246 (1935); W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936); V. Weisskopf, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **14**, 6 (1936).
[15] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); J. Fröhlich and T. Kerler, Nucl. Phys. **B354**, 369 (1991).
[16] A. N. Redlich, Phys. Rev. D **29**, 2366 (1984).
[17] R. R. Parwani, Phys. Lett. B **358**, 101 (1995).
[18] G. Dunne and T. Hall, Phys. Rev. D **53**, 2220 (1996).
[19] E. H. Lieb and W. E. Thirring, in *Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann*, edited by E. H. Lieb, B. Simon, and A. S. Wightman (Princeton University Press, Princeton, NJ, 1976), p. 269.
[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness* (Academic, New York, 1975).
[21] E. Seiler and B. Simon, Commun. Math. Phys. **45**, 99 (1975).
[22] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics Vol. 159 (Springer, Berlin/Heidelberg/New York, 1982).
[23] B. Simon, *Trace Ideals and their Applications*, London Mathematical Society Lecture Note Series 35 (Cambridge University Press, Cambridge, England, 1979).
[24] B. Simon, Adv. Math. **24**, 244 (1977).
[25] E. Seiler (private communication).
[26] E. Seiler, Commun. Math. Phys. **42**, 163 (1975).
[27] S. L. Adler, Phys. Rev. D **16**, 2943 (1977).
[28] C. Itzykson, G. Parisi, and J.-B. Zuber, Phys. Rev. D **16**, 996 (1977); R. Balian, C. Itzykson, J.-B. Zuber, and G. Parisi, *ibid.* **17**, 1041 (1978).