Generalized Raychaudhuri equations for strings in the presence of an antisymmetric tensor field

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The generalized Raychaudhuri equations derived by Capovilla and Guven are exclusively for extremal, timelike Nambu-Goto membranes. In this article, we construct the corresponding equations for string world sheets in the presence of a background Kalb-Ramond field. We analyze the full set of equations by concentrating on special cases in which the generalized shear or the generalized rotation or both are set to zero. If only the generalized shear is set to zero then it is possible to identify the components of the generalized rotation with the projections of the field strength of the Kalb-Ramond potential. [S0556-2821(96)03318-8]

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I. INTRODUCTION

The importance of the consequences resulting out of the analysis of the Raychaudhuri equations for timelike or null geodesic congruences [1–3] in the proof of the singularity theorems of general relativity (GR) [3,2] is a well-known fact today. At the same time, string and membrane theories have been extremely popular among theoretical physicists of different specializations. Apart from it being a candidate for quantum gravity and unification of forces there are a multitude of situations ranging from particle physics to biology where membranes or strings play a pivotal role in describing a system or analyzing a phenomenon. Given the diverse areas in which strings and membranes are applied it is perhaps worthwhile to know the generalizations of the basic equations for curves (such as the geodesic equation, the Jacobi equation and the Raychaudhuri equation) to the case of surfaces. Earlier papers by Guven [4], Larsen and Frolov [5] and Carter [6] have succeeded in generalizing the equation of geodesic deviation to the case of timelike surfaces. More recently, Capovilla and Guven [7] have generalized the Raychaudhuri equations for timelike geodesic congruences to the case of families of D-dimensional, timelike, extremal, Nambu-Goto surfaces in an N-dimensional background. Subsequently, several illustrative examples of these equations were constructed by this author in [8,9]. The generalization of the notion of geodesic focusing to families of surfaces was also introduced and discussed in some detail.

However, it is necessary to realize that there are several other actions that arise in string and membrane theories which are different from the Nambu-Goto or Polyakov actions. Differences arise in various ways—the presence of background fields other than gravity [10], supersymmetrizations [12], rigidity corrections [11], and so on. What are the generalizations of the Raychaudhuri equations for these actions? In this paper we attempt to analyze one such action—the one in the presence of a background antisymmetric tensor field (the Kalb-Ramond field).

The generalized Raychaudhuri equations now contain many extra new terms which embody several nontrivialities. We discuss special cases in which either the shear or the rotation or both are set to zero and thereby obtain simplified sets of equations. For the case in which only the shear is set to zero it is possible to identify the projections of the antisymmetric tensor field with the components of the rotation in a special way.

The paper is organized as follows. Section II provides some background material based on the paper by Capovilla and Guven. In Sec. III we write down and analyze the equations for the case of strings in the presence of an antisymmetric tensor field. Finally, in Sec. IV we conclude with remarks on future directions.

II. BACKGROUND

In this section we briefly review the work of Capovilla and Guven [7] on the generalization of the Raychaudhuri equations.

The geometric objects under consideration are D-dimensional, timelike surfaces embedded in an N-dimensional background. We denote x^μ ($\mu=0,1,2,\ldots,N-1$) as the background spacetime coordinates and ξ^a ($a=0,1,2,\ldots,D-1$) as the surface coordinates. Tangents and normals are defined in the usual way and we can construct an orthonormal, spacetime basis $\{E_a^\mu, n_i^\mu\}$ at each point on the surface. The projections of the spacetime covariant derivatives of E_a^μ and n_i^μ along the surface tangents define the extrinsic curvatures, Ricci rotation coefficients, and the twist potentials through the standard Gauss-Weingarten formulas given as

$$D_a E_b = \gamma_{ab}^c E_c - K_{ab}^i n_i, \tag{1}$$

$$D_a n_i = K_{ab}^i E^b + \omega_a^{ij} n_j, \tag{2}$$

where $D_a \equiv E_a^\mu D_\mu$ (D_μ being the usual spacetime covariant derivative). The quantities K_{ab}^i (extrinsic curvature), ω_a^{ij} and γ_{ab}^c are defined as

$$K_{ab}^i = -g(D_a E_b, n^i) = K_{ba}^i, \tag{3}$$

$$\omega_a^{ij} = g(D_a n^i, n^j), \tag{4}$$

$$\gamma_{abc} = g(D_a E_b, E_c) = -\gamma_{acb}. \tag{5}$$

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In order to analyze deformations normal to the world sheet we need to consider the normal gradients of the spacetime basis set. The corresponding analogues of the Gauss-Weingarten equations are

$$D_i E_a = J_{aij} n^j + S_{abi} E^b, \tag{6}$$

$$D_i n_j = -J_{aij} E^a + \gamma_{ij}^k n_k, \tag{7}$$

where $D_i \equiv n_i^\mu D_\mu$. The quantities J_a^{ij}, S_{abi}, and γ_{ij}^k are defined as

$$S_{ab}^i = g(D^i E_a, E_b) = -S_{ba}^i, \tag{8}$$

$$\gamma_{ijk} = g(D_i n_j, n_k) = -\gamma_{ikj}, \tag{9}$$

$$J_a^{ij} = g(D^i E_a, n^j). \tag{10}$$

The quantity J_{aij} is the most crucial object in our discussion because it is, in a sense, a measure of the allowed orthogonal deformations of the world surface which is necessary in obtaining information of the behavior of families of surfaces. One can think of J_{aij} as a quantity which essentially is a surface analogue of the gradient of the tangent vector field in the case of geodesics. The major point of difference with the case of geodesics is the fact that one has one quantity along each of the tangential directions on the surface.

The evolution equations for the J_{aij} are the generalized Raychaudhuri equations.

Instead of taking the proper time derivative of J_{aij} we look at the covariant world-sheet derivative of this quantity. This turns out to be (for details see Appendix of [7]) given as

$$\tilde{\nabla}_b J_a^{ij} = -\tilde{\nabla}^i K_{ab}^j - J_{bk}^i J_a^{kj} - K_{bc}^i K_a^{cj} - g(R(E_b, n^i) E_a, n^j), \tag{11}$$

where the extrinsic curvature tensor components are $K_{ab}^i = -g_{\mu\nu} E_a^\alpha (D_\alpha E_b^\mu) n^{\nu i}$.

On tracing over world-sheet indices we get

$$\tilde{\nabla}_a J^{aij} = -J_{ak}^i J^{akj} - K_{ac}^i K^{acj} - g(R(E_a, n^i) E^a, n^j), \tag{12}$$

where we have used the equation for extremal, timelike, Nambu-Goto membranes (i.e., $K^i = 0$). As we shall see in the next section, it is this condition which will change and thereby introduce all the nontrivialities when we introduce a background antisymmetric tensor field.

The antisymmetric part of Eq. (12) is given as

$$\tilde{\nabla}_b J_a^{ij} - \tilde{\nabla}_a J_b^{ij} = G_{ab}^{ij}, \tag{13}$$

where $g(R(Y_1, Y_2) Y_3, Y_4) = R_{\alpha\beta\mu\nu} Y_1^\alpha Y_2^\beta Y_3^\mu Y_4^\nu$ and

$$G_{ab}^{ij} = -J_{bk}^i J_a^{kj} - K_{bc}^i K_a^{cj} - g(R(E_b, n^i) E_a, n^j) - (a \to b). \tag{14}$$

One can further split J_{aij} into its symmetric traceless, trace, and antisymmetric parts $(J_a^{ij} = \Sigma_a^{ij} + \Lambda_a^{ij} + (1/N - D)\delta^{ij}\theta_a)$ and obtain the evolution equations for each of these quantities. The one we shall be concerned with mostly is given as

$$\Delta\gamma + \frac{1}{2}\partial_a \gamma \partial^a \gamma + (M^2)_i^i = 0 \tag{15}$$

with the quantity $(M^2)^{ij}$ given as

$$(M^2)^{ij} = K_{ab}^i K^{abj} + R_{\mu\nu\rho\sigma} E_a^\mu n^{\nu i} E^{\rho a} n^{\sigma j}. \tag{16}$$

∇_a is the world-sheet covariant derivative $(\Delta = \nabla^a \nabla_a)$ and $\partial_a \gamma = \theta_a$. Notice that we have set Σ_a^{ij} and Λ_a^{ij} equal to zero. This is possible only if the symmetric traceless part of $(M^2)^{ij}$ is zero. One can check this by looking at the full set of generalized Raychaudhuri equations involving Σ_a^{ij}, Λ_a^{ij}, and θ_a [4]. For geodesic curves the usual Raychaudhuri equations can be obtained by noting that $K_{00}^i = 0$, the J_{aij} are related to their spacetime counterparts $J_{\mu\nu a}$ through the relation $J_{\mu\nu a} = n_\mu^i n_\nu^j J_{aij}$, and the θ is defined by contracting with the projection tensor $h_{\mu\nu}$.

The θ_a or γ basically tell us how the spacetime basis vectors change along the normal directions as we move along the surface. If θ_a diverges somewhere, it induces a divergence in J_{aij}, which, in turn means that the gradients of the spacetime basis along the normals have a discontinuity. Thus the family of world sheets meet along a curve and a cusp/kink is formed. This, we claim, is a focusing effect for extremal surfaces analogous to geodesic focusing in GR where families of geodesics focus at a point if certain specific conditions on the matter stress energy are obeyed.

III. STRINGS IN THE PRESENCE OF AN ANTISYMMETRIC TENSOR FIELD

In the presence of a background antisymmetric tensor field (the Kalb-Ramond field) the usual Nambu-Goto–Polyakov action gets generalized by the addition of an extra term. We, therefore, have

$$S = S_{\rm NG} + \int \epsilon^{ca} B_{\mu\nu} \partial_c x^\mu \partial_a x^\nu d^2\xi. \tag{17}$$

The field equation resulting from the variation of the action with respect to x^μ is given as

$$K^i = \frac{1}{2} n_\lambda^i g^{\alpha\lambda} \epsilon^{ca} \partial_c x^\beta \partial_a x^\nu H_{\beta\alpha\nu}, \tag{18}$$

where

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu}. \tag{19}$$

We can also write the field equation in the following alternative form by choosing the conformal gauge. In that case we have

$$\partial_a \partial^a x^\lambda + \Gamma_{\rho\sigma}^\lambda \partial_a x^\rho \partial^a x^\sigma = -\frac{1}{2} \epsilon^{ca} \partial_c x^\beta \partial_a x^\nu g^{\alpha\lambda} H_{\alpha\nu\beta}. \tag{20}$$

The constraints, of course, remain the same as for the Nambu-Goto action for extremal surfaces.

Given the above field equation we can ask—When does the world sheet retain its minimal character even though background fields or potentials may be present? This implies investigating the conditions under which the right-hand side (RHS) of the equation becomes zero.

First, if $H_{\mu\nu\rho}=0$, i.e., $B_{\mu\nu}$ is pure gauge the RHS disappears and we are left with an equation which is the same as the original Nambu-Goto equation.

Additionally, if we choose stationary strings then it is not necessary that all components of $H_{\mu\nu\rho}$ be zero. Given the stationary string ansatz $[t=\tau, x^i=x^i(\sigma)]$ we find that

$$K^i = n^{qi} x^{p\prime} H_{0qp}. \tag{21}$$

Thus, if $H_{0qp}=0 \forall p,q$ then $K_i=0$ and the world sheet remains a minimal surface.

On the other hand, for circular strings in a spherically symmetric background $[t=t(\tau), l=l(\tau), \theta=\pi/2, \phi=\sigma]$ we have

$$K^i = -n^{\alpha i}\dot{t} H_{\alpha 03} - n^{\alpha i}\dot{l} H_{\alpha 13}. \tag{22}$$

Therefore, with

$$H_{103}=H_{203}=H_{213}=0, \tag{23}$$

we end up with $K_i=0$ and the minimality of the surface is retained.

We now need to evaluate the quantity $\tilde{\nabla}^i K^j$. It is this term which will add up to the generalized Raychaudhuri equation for the usual Nambu-Goto action and result in the major differences.

After some amount of algebra one finds the following expression for $\tilde{\nabla}^i K^j$:

$$\tilde{\nabla}^i K^j = \frac{1}{2}\epsilon^{ca} J^{ij}_m (E^m_\lambda E^\beta_c E^\nu_a H^\lambda_{\beta\nu}) - \epsilon^{ca} S^i_{cd} (E^{d\beta} E^\nu_a n^j_\lambda H^\lambda_{\beta\nu}) - \epsilon^{ca} J^{ik}_c (n^j_\lambda n^\beta_k E^\nu_a H^\lambda_{\beta\nu}). \tag{24}$$

The quantities in brackets in the above expression can be thought of as projections of $H_{\mu\nu\rho}$ along normal and tangential directions. Thus, we end up with

$$\tilde{\nabla}^i K^j = \epsilon^{ca}\left(\frac{1}{2} J^{ij}_m H^m_{ca} - S^i_{cd} H^{jd}_a - J^{ik}_c H^j_{ka}\right), \tag{25}$$

where we have assumed

$$D^i H_{\mu\nu\rho}=0. \tag{26}$$

Note that the first term can be very easily shown to be equal to zero since the world-sheet indices take only two values (σ,τ). The second term for the case of the string can be written as

$$S^i_{cd}\epsilon^{ca} H^{jd}_a = \epsilon^{\tau\sigma} S^i_{\tau\sigma} H^{j\sigma}_\sigma + \epsilon^{\sigma\tau} S^i_{\sigma\tau} H^{j\tau}_\tau, \tag{27}$$

and it is obvious that it is equal to zero by the antisymmetry of H^{ia}_b with respect to a,b indices. Hence only the last term in Eq. (25) survives.

We are now in a position to input this into the term containing $\tilde{\nabla}^i K^j$ in the generalized Raychaudhuri equation for string theory in the presence of a background antisymmetric tensor field.

The equation obtained by tracing the expression for $\tilde{\nabla}_b J^{ij}_a$ with respect to the world-sheet indices is given as

$$\tilde{\nabla}_a J^{aij} = -J^i_{ak} J^{akj} - K^i_{ac} K^{acj} + \epsilon_{ca} J^{ik}_c H^j_{ka} - g(R(E_a, n^i)E^a, n^j). \tag{28}$$

We now use the splitting of J^{ij}_a into its symmetric traceless, antisymmetric, and trace parts:

$$J^{ij}_a = \Sigma^{ij}_a + \Lambda^{ij}_a + \frac{1}{N-D}\delta^{ij}\theta_a. \tag{29}$$

The equations for each of these quantities—θ_a, Σ_{aij}, and Λ_{aij} turn out to be

$$\tilde{\nabla}_a \theta^a + \Sigma_{ak}{}^i \Sigma^{aki} + \Lambda_{ak}{}^i \Lambda^{aki} + \frac{1}{N-D}\theta_a\theta^a + (M^2)^i_i - \epsilon^{ca}\Lambda^{ik}_c H_{ika} = 0, \tag{30}$$

$$\tilde{\nabla}_a \Lambda^{aij} - \Sigma^{a[i}_k \Sigma^{j]k}_a + \Lambda^{ak[i}\Lambda^{j]}_{a\ k} - 2\Lambda^{ak[i}\Sigma^{j]}_{a\ k} - \epsilon^{ca}\Sigma^{k[i}_c H^{j]}_{ka} + \epsilon^{ca}\Lambda^{k[i}_c H^{j]}_{ka} + \frac{2}{N-D}\epsilon^{ca}\theta_c H^{ij}_a + \frac{2}{N-D}\theta^a \Lambda^{ij}_a = 0, \tag{31}$$

$$\tilde{\nabla}_a \Sigma^{aij} + (\Lambda^{aik}\Lambda^j_{ak} + \Sigma^{aik}\Sigma^j_{ak})^{\rm str} + \frac{2}{N-D}\Sigma^{ij}_a\theta^a + [(M^2)^{ij}]^{\rm str} - \epsilon^{ca}\Sigma^{k(i}_c H^{j)}_{ka} + \epsilon^{ca}\Lambda^{k(i}_c H^{j)}_{ka} = 0, \tag{32}$$

where str denotes symmetric traceless part of a matrix.

The first fact to note in the above equations is that if we set Λ and Σ both equal to zero the second equation leads to a relation between the θ_a and the H^{ij}_a which is given as

$$\epsilon^{ca}\theta_c H^{ij}_a = 0. \tag{33}$$

This constrains the choice of θ_a and H^{ij}_a If $H^{ij}_\sigma = H^{ij}_\tau$ then, as a consequence we have $\theta_\sigma = \theta_\tau$. Therefore, the $\theta_a\theta^a$ term in the first equation vanishes and we end up with the equation

$$\tilde{\nabla}_a \theta^a + (M^2)^i_i = 0. \tag{34}$$

Furthermore, if we assume only $\Lambda^{ij}_a = 0$ and

$$\Sigma^{ij}_\sigma = \Sigma^{ij}_\tau, \quad H^{ij}_\sigma = H^{ij}_\tau, \tag{35}$$

we end up with Eq. (33) and a similar equation for Σ_{aij}:

$$\tilde{\nabla}_a \Sigma^{aij} - [(M^2)^{ij}]^{\rm str} = 0. \tag{36}$$

Finally, we set only Σ equal to zero. Therefore we end up with the following algebraic relation resulting from the third equation:

$$(\Lambda^{aik}\Lambda^j_{ak})^{\rm str} + \epsilon^{ca}\Lambda^{k(i}_c H^{j)}_{ka} = 0, \tag{37}$$

assuming that the symmetric traceless part of the object $(M^2)^{ij}$ is equal to zero.

This above algebraic relation can be satisfied (for the string case) if we assume the following to hold true:

$$\Lambda_{\tau}{}^{j}{}_{k}=H^{j}_{k\sigma}, \quad \Lambda^{j}_{\sigma k}=H^{j}_{k\tau}. \tag{38}$$

Thus, a geometric quantity Λ^{ij}_{a} has been related to a physical object: the field strength of the Kalb-Ramond potential. It would be worthwhile trying to understand how the generalized rotation has its physical meaning in the projections of the $H_{\mu\nu\rho}$.

With this identification of the Λ with the projections of H we find that the second equation reduces to the simple equation

$$\tilde{\nabla}_{a}\Lambda^{a}_{ij}=0, \tag{39}$$

and the equation for the generalized expansion θ turns out to be

$$\tilde{\nabla}_{a}\theta^{a}+\frac{1}{N-D}\theta_{a}\theta^{a}+(M^{2})^{i}_{i}=0. \tag{40}$$

Let us now turn to the antisymmetric equations. Recall that the $\tilde{\nabla}^{i}K^{j}$ term does not contribute to these equations. They are given by

$$2\tilde{\nabla}_{[a}\Lambda^{ij}_{b]}=-2\Lambda^{k[i}_{[a}\Lambda^{j]}_{b]k}-2\Sigma^{k[i}_{[a}\Sigma^{j]}_{b]k}-\Omega^{ij}_{ab}, \tag{41}$$

$$2\partial_{[a}\theta_{b]}=0, \tag{42}$$

$$2\tilde{\nabla}_{[a}\Sigma^{ij}_{b]}=-2(\Lambda^{ik}_{[a}\Lambda^{j}_{b]k}+\Sigma^{ik}_{[a}\Sigma^{j}_{b]k}{}^{\rm str}+4\Lambda^{k(i}_{[a}\Lambda^{j)}_{b]k}. \tag{43}$$

First, let us look at the case with $\Sigma^{ij}_{\sigma}=\Sigma^{ij}_{\tau}$, $H^{ij}_{\sigma}=H^{ij}_{\tau}$ and $\Lambda^{ij}_{a}=0$. With these assumptions, we can easily check that the antisymmetric equations essentially reduce to one equation given by

$$\tilde{\nabla}_{a}\Sigma_{aij}=0. \tag{44}$$

However, one of the traced equations (36) matches with the above equation only if $[(M^{2})^{ij}]^{\rm str}=0$.

For the case in which $\Sigma^{ij}_{a}=0$, the identification of the H with the Λ results in some constraints on the H or the Λ which have to be satisfied in order to have a consistent solution of the full set of equations. These constraints turn out to be as follows.

The second of these equations results in the relation $\theta_{a}=\partial_{a}\gamma$. The third is satisfied identically with the assumption of the relation between H and Λ. The first equation (with the choice $\Omega^{ij}_{a}=0$) yields an extra constraint on the H which reads

$$\tilde{\nabla}_{[a}\Lambda^{ij}_{b]}=2H^{kj}_{[b}H^{i}_{a]}k. \tag{45}$$

For the string world sheet in a background four-dimensional spacetime one ends up with the following constraints on H:

$$\tilde{\nabla}_{\sigma}H^{ij}_{\tau}=\tilde{\nabla}_{\tau}H^{ij}_{\sigma}, \tag{46}$$

$$\tilde{\nabla}_{\sigma}H^{ij}_{\sigma}=\tilde{\nabla}_{\tau}H^{ij}_{\tau}. \tag{47}$$

This is because there are only two normals to the world sheet and the i,j indices in H^{ij}_{a} are antisymmetric. The first of these two equations follows from Eq. (39) while the second one is a descendant of Eq. (44).

If we further assume that the string world sheet is flat then the covariantized world-sheet derivatives are reduced to ordinary derivatives on the world sheet. The pair of first-order partial differential equations can therefore be thought of as a single second-order wave equation for either H^{ij}_{σ} or H^{ij}_{τ} given as

$$\frac{\partial^{2}H^{ij}_{\tau,\sigma}}{\partial\tau^{2}}-\frac{\partial^{2}H^{ij}_{\tau,\sigma}}{\partial\sigma^{2}}=0. \tag{48}$$

Therefore, with the identification of the projections of the $H_{\mu\nu\rho}$ field with the generalized rotation Λ_{aij} we end up with an equation for the γ and two constraints on the H field. This is all we need to solve in order to analyze focusing effects for world sheets which are extremal solutions of the Nambu-Goto action with an antisymmetric tensor field added to it. In fact, apart from the extrinsic curvature which will now contain some objects related to the H field and the constraints on the H we have nothing more to analyze except the usual generalized Raychaudhuri equation for the θ. The crucial result of this paper is the demonstration of the fact that the generalized rotation can indeed be related to the projections of the $H_{\mu\nu\rho}$ field. This in turn leads us to a system of equations which are tractable.

One might ask—What are the generalized Raychaudhuri equations for strings in a background three-dimensional spacetime? Note that in this case the string world sheet is a hypersurface and therefore considerable simplifications occur. One can very easily show that $\tilde{\nabla}^{i}K^{j}$ is identically zero (the term containing a product of normals vanishes because of the fact that there is only one normal now and H^{ij}_{a} is antisymmetric in its i,j indices. Therefore the generalized Raychaudhuri equations are the same as for the case without an antisymmetric tensor field.

IV. CONCLUDING REMARKS

The aim of this paper has been to derive the generalized Raychaudhuri equations for strings in the presence of a background antisymmetric tensor field. We have analyzed several special cases by choosing the shear, the rotation, or both to zero. It turns out that if the shear is set to zero then it is possible to identify the projections of the field strength of the Kalb-Ramond potential with the generalized rotation. This, in fact demonstrates that a geometric object can be related to a physical quantity. Recall, that in the geodesic case one could give a physical meaning to the rotation. In the paper of Capovilla and Guven such a physical meaning for the generalized shear or rotation was lacking. We have, in this paper, been able to make some progress along this direction for at least one of these quantities.

A multitude of open issues remains in this area. As an extension to this paper one can work out the generalized Raychaudhuri equations for other actions which were mentioned in the Introduction. Apart from this, one has to understand the issue of focusing of surfaces in a better and more general way without referring to specific examples. Thereafter, one can address the question of spacetime singularities and their relation to string focusing effects.

In drawing parallels with the basic equations of GR-geodesic equation, deviation equation, Raychaudhuri equations, and the Einstein equation one now notices that in the context of strings we actually have the first three. The fourth one is, however, not there. However, recall that GR as a theory has the unique feature: the geodesic equation (trajectory of test particles) can be derived from the Einstein equations. Therefore, one can frame the question—What is the ''Einstein equation'' which will lead to the string equations of motion? An answer to this question will perhaps help us to understand the relation between strings, gravity, and spacetime geometry in a novel way.

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