Quark mass correction to the string potential

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A consistent method for calculating the interquark potential generated by the relativistic string with massive ends is proposed. In this approach, the interquark potential in the model of the Nambu-Goto string with pointlike masses at its ends is calculated. At first the calculation is done in the one-loop approximation. For obtaining a finite result under summation over eigenfrequencies of the Nambu-Goto string with massive ends, an appropriate renormalization procedure is suggested. It is shown that in this case the Lüscher term in the string potential acquires a dependence on the quark mass which results in the reduction of the absolute value of this term. Then the interquark potential in the string model with new boundary conditions is calculated by making use of the variational estimation of the corresponding functional integral. In this case the quark mass correction results in decreasing the critical distance (deconfinement radius) in the string potential. In the framework of the developed approach, the one-loop interquark potential in the model of the relativistic string with rigidity is also calculated. [S0556-2821(96)01520-2]

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I. INTRODUCTION

The investigation of the quark interaction at large distances is outside the QCD perturbation theory. Usually, in this field the lattice simulations and string models are used.

The calculation of the quark interaction in the framework of string models has a rather long history (see, for example, papers [1–9] and references therein). In all these investigations, without exception, only the static interquark potential has been considered. It implies that the quarks are assumed to be infinitely heavy. Obviously, this potential, by definition, does not depend on the quark masses. The assumption about infinitely heavy quarks is rather crude, at least for u and d quarks with (constituent) masses about 200–300 MeV that is significantly less than the characteristic hadronic mass scale ~ 1 GeV. It is clear that in a general case the interquark potential should depend on quark masses. Both the general approach to this problem in the framework of QCD [10] and the numerical calculations of the light and heavy meson spectra in potential models [11–13] testify to this. Certainly, in this case one should talk not about the static potential generated, for example, by a relativistic string connecting quarks but simply about the interaction potential between quarks having a finite mass rather than an infinite one.

The aim of the present paper is an attempt to extend the standard approach to the calculation of the interquark potential in the framework of the string models [1,5–9] to the case of the finite quark masses. It turns out that this program can be realized. To this end the boundary conditions in the string model in question should be modified and a new renormalization procedure for summation over string eigenfrequencies should be developed.

In a proposed approach, a correction to the string potential due to the finite quark masses is calculated both in the Nambu-Goto string model and in the Polyakov-Kleinert rigid string model.

In the Nambu-Goto string with massive ends the quark potential is calculated first in the one-loop approximation of perturbation theory for arbitrary dimension of space-time D and then via a variational estimation of the corresponding functional integral in the limit $(D-2)\to\infty$. As is known, the static quark potential generated by the Nambu-Goto string in the one-loop approximation is compiled by two terms: the linearly rising confinement potential (classical part) and the first quantum correction usually referred to as the universal Lüscher term [14,15]. It is worthwhile to remember that this term is nothing other than the Casimir energy of the string. When the ends of the string are loaded by pointlike masses (quarks) then the Lüscher term proves to be dependent on the quark masses. It is not unnatural because the Casimir energy, as is known [16,17] is essentially determined by the boundary conditions imposed on the field variables (in the case under consideration, on the string coordinates). For calculating the Casimir energy in the model of the string with massive ends, a subtraction procedure is suggested that includes the renormalization of the string tension and quark mass. With the help of it, a finite value of the Casimir energy in this string model is derived in a unique fashion. It proves that the allowance for the finite quark masses results in reducing the absolute value of the Lüscher term. At certain values of the model parameters (string tension, its length, and quark mass), the ratio of the Lüscher term calculated with an allowance for the finite quark mass to the value of this term in the Nambu-Goto string with fixed ends can be reduced to ≈ 0.2. Having defined the Lüscher term in the string potential as the Casimir energy of the string, we cannot certainly expect that its dependence on the string length R will be $1/R$ at any boundary conditions. Really, in the Nambu-Goto string with massive ends, this dependence turns out to be rather complicated [see Eq. (3.25)] and only in the limiting cases when $m\to\infty$ or $m\to 0$ we obtain the universal behavior $1/R$ [see Eq. (3.26)].

A variational calculation of the potential generated by the

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Nambu-Goto string with massive ends in the limit $(D-2)\to\infty$ results in a radical expression [Eq. (4.22)]. As is known, the string potential calculated in this approach is not determined at all the distances R but only at $R>R_c$, where the critical radius R_c in the case of the Nambu-Goto string with fixed ends is given by [1] $R_c^2=\pi(D-2)/(12\,M_0^2)$. Taking into account the finite quark mass results in reducing the value of the critical radius R_c (see Fig. 3). The potential curves, being displaced to lower distances, preserve their form (Fig. 2).

In the rigid string model with massive ends the interquark potential is calculated in the one-loop approximation. When confined to the quadratic approximation in the Polyakov-Kleinert action, the dynamical variables (string position vector) can be presented as a sum of two terms, $\mathbf{u}(t,r)=\mathbf{u}_1(t,r)+\mathbf{u}_2(t,r)$, where $\mathbf{u}_1(t,r)$ is a solution to the Nambu-Goto string with massive ends and $\mathbf{u}_2(t,r)$ is an additional variable caused by the extrinsic curvature in the Polyakov-Kleinert action. It is remarkable that the quark masses only affect $\mathbf{u}_1(t,r)$. This essentially simplifies the problem under consideration and enables us to use directly the results for the potential derived in the Nambu-Goto string with massive ends. In the one-loop approximation, the variables $\mathbf{u}_1(t,r)$ and $\mathbf{u}_2(t,r)$ give additive contributions to the interquark potential generated by rigid string. It is true both in the case of the fixed string ends and for the rigid string with massive ends. As a result, the quark mass correction to the one-loop potential generated by rigid string is reduced to the modification of the contribution from the variable $\mathbf{u}_1(t,r)$: the one-loop potential in the Nambu-Goto string with massive ends calculated before should be used here. The contribution to the potential of the string oscillations due to its rigidity does not exceed, in absolute value, the universal Lüscher term (Fig. 4).

The layout of the paper is as follows. In Sec. II the quadratic approximation for the Nambu-Goto string model with massive ends is developed. Upon linearization of the equations of motion and boundary conditions, the general solution to them is obtained. The eigenfrequencies of the string oscillations are determined by a transcendental equation. Then the canonical quantization of this model is outlined in short. In Sec. III, the interquark potential generated by the Nambu-Goto string with massive ends is calculated in the one-loop approximation of the perturbation theory. In order to remove the divergence, a new subtraction procedure is proposed. In Sec. IV, the interquark potential generated by the Nambu-Goto string with massive ends is calculated by making use of a variational estimation of the functional integral in the limit when $(D-2)\to\infty$. In Sec. V, the rigid string model with massive ends is treated. By making use of a quadratic approximation for the Polyakov-Kleinert action, the linear equations of motion and boundary conditions are derived. Then canonical quantization of this model is developed. And finally, the interquark potential generated in this string model is calculated in the one-loop approximation. In the Conclusions (Sec. VI) the obtained results are discussed in short and possible extensions of them are proposed. Some mathematical details of calculation are presented in Appendices A and B.

II. NAMBU-GOTO STRING WITH MASSIVE ENDS

The action of the Nambu-Goto string with pointlike masses attached to its ends is written as [18]

$$S=-M_0^2\int\int_\Sigma d\Sigma-\sum_{a=1}^{2}m_a\int_{C_a}ds_a\,, \tag{2.1}$$

where $d\Sigma$ is infinitesimal area of the string world surface, C_a $(a=1,2)$ are the world trajectories of the string massive ends, and M_0^2 is the string tension with the dimension of the mass squared $(\hbar=c=1)$.

For our calculations, it will be convenient to use the Gauss parametrization of the string world surface:

$$x^\mu(\xi)=(t,r;x^1(t,r),\dots,x^{D-2}(t,r))=(\xi^i;\mathbf{u}(\xi^i)),\quad i=0,1. \tag{2.2}$$

The vector field $u^j(t,r)$, $j=1,\dots,D-2$ corresponds to $D-2$ transverse components of x^μ, while $t=\xi^0$, $r=\xi^1$ are the coordinates on the string world sheet. The infinitesimal area $d\Sigma$ is given by $d\Sigma=\sqrt{-g}\,dt\,dr$, where g is the determinant of the induced metric on the world surface of the string, $g_{ij}=\partial_i x^\mu\,\partial_j x_\mu$, $i,j=0,1$. The metric of the D-dimensional space-time has the signature $(+,-,\dots,-)$.

In this parametrization, the induced metric g_{ij} has the components

$$g_{ij}=\begin{pmatrix}1-\dot{\mathbf{u}}^2 & -\dot{\mathbf{u}}\mathbf{u}'\\ -\dot{\mathbf{u}}\mathbf{u}' & -1-\mathbf{u}'^2\end{pmatrix},\quad i,j=0,1\,, \tag{2.3}$$

where $\mathbf{u}\mathbf{u}=\sum_{j=1}^{D-2}u^ju^j$, $\mathbf{u}_0=\partial\mathbf{u}/\partial t=\dot{\mathbf{u}}$, and $\mathbf{u}_1=\partial\mathbf{u}/\partial r=\mathbf{u}'$. Further, we will use the quadratic approximation for the action (2.1). This approximation is the basis of the perturbative calculations. In addition, when developing the $1/D$ expansion (see Sec. IV) string coordinates will actually be determined by the quadratic string action (in the general case, with parameters that should be varied).

In quadratic approximation we obtain, from Eq. (2.3),

$$-g=\det(g_{ij})\simeq1-\dot{\mathbf{u}}^2+\mathbf{u}'^2\,. \tag{2.4}$$

The line elements ds_a, $a=1,2$, take the form

$$ds_a\simeq\left[1-\frac{1}{2}\dot{\mathbf{u}}^2(t,\,r_a)\right]dt\,. \tag{2.5}$$

As a result, action (2.1) acquires the form

$$\begin{aligned}S\simeq&-M_0^2\,(t_2-t_1)R+\frac{M_0^2}{2}\int_{t_1}^{t_2}dt\int_0^R dr[\dot{\mathbf{u}}^2(t,r)-\mathbf{u}'^2(t,r)]\\&-(t_2-t_1)\sum_{a=1}^{2}m_a+\sum_{a=1}^{2}\frac{m_a}{2}\int_{t_1}^{t_2}dt\,\dot{\mathbf{u}}^2(t,r_a),\\&r_1=0,\quad r_2=R\,.\end{aligned} \tag{2.6}$$

We kept here the constant terms proportional to string tension M_0^2 and quark masses m_a. Obviously, they do not give contribution to the dynamical equations but they will be significant when calculating the string potential.

Variation of the action (2.6) leads to the equations of motion

$$\Box\ \mathbf{u}=0 \tag{2.7}$$

and boundary conditions

$$m\ \ddot{\mathbf{u}}=M_0^2\mathbf{u}', \quad r=0, \tag{2.8}$$

$$m\ \ddot{\mathbf{u}}=-M_0^2\mathbf{u}', \quad r=R\ , \tag{2.9}$$

where $\Box=\partial^2/\partial t^2-\partial^2/\partial r^2$ and we assume for simplicity that $m_1=m_2=m$. The case of arbitrary quark masses will be considered in our publication [19].

General solution to the boundary value problem [Eqs. (2.7)–(2.9)] is given by

$$u^j(t,r)=\frac{1}{\sqrt{2}M_0}\sum_{n\neq 0}\exp[-i\omega_n t]\frac{\alpha_n^j}{\omega_n}\,u_n(r),$$
$$j=1,2,\ldots,D-2\ . \tag{2.10}$$

Amplitudes α_n satisfy the usual rule of complex conjugation $\alpha_n^*=\alpha_{-n}$. The eigenfunctions $u_n(r)$ in Eq. (2.10) are defined by

$$u_n(r)=N_n\left[\cos(\omega_n r)-\frac{m\omega_n}{M_0^2}\sin(\omega_n r)\right], \tag{2.11}$$

where N_n's are the normalization constants

$$N_n^2=\left[\frac{m}{M_0^2}+\frac{R}{2}\left(1+\frac{m^2\omega_n^2}{M_0^4}\right)\right]^{-1}, \quad n=\pm 1,\pm 2,\ldots\ . \tag{2.12}$$

The eigenfrequencies ω_n are the roots of the trigonometric equation

$$\tan(\omega\ R)=\frac{2mM_0^2\omega}{m^2\omega^2-M_0^4}\ . \tag{2.13}$$

On the ω axis these roots are placed symmetrically around zero. Hence, they can be numbered in the following way: $\omega_{-n}=-\omega_n$, $n=1,2,\ldots$. Therefore, it will be sufficient to consider only the positive roots. The eigenfunction $u_n(r)$ obeys the orthogonality conditions

$$\int_0^R dr\ u_n(r)\ u_m(r)\ \varepsilon(r)=\delta_{nm}, \tag{2.14}$$

where the weight function $\varepsilon(r)$ is given by

$$\varepsilon(r)=1+\frac{m}{M_0^2}\left[\delta(r)+\delta(R-r)\right]\ . \tag{2.15}$$

It is interesting to note that the functions $u_n'(r)$ satisfy the usual orthogonality conditions

$$\int_0^R dr\ u'_n(r)\ u'_m(r)=\omega_n^2\ \delta_{nm}, \tag{2.16}$$

where the eigenfrequencies ω_n are solutions of Eq. (2.13).

The density of the canonical momentum $p^j(t,r)$ is defined in a standard way

$$p^j(t,r)\ =\ \frac{\partial L}{\partial \dot{u}^j}=M_0^2\dot{u}^j(t,r)\ \varepsilon(r)\ , \tag{2.17}$$

in which L is the Lagrangian density in action (2.6). Obviously, in the problem under consideration, we can assume that the total momentum of the string with massive ends vanishes:

$$P^j(t)=\int_0^R dr\ p^j(t,r)=0\ . \tag{2.18}$$

The canonical Hamiltonian is defined by

$$H=\int_0^R dr\ [\mathbf{p}(t,r)\ \dot{\mathbf{u}}(t,r)-L]$$
$$=\frac{M_0^2}{2}\int_0^R dr\ [\dot{\mathbf{u}}^2(t,r)\varepsilon(r)+\mathbf{u}'^2(t,r)]\ . \tag{2.19}$$

In terms of the amplitudes α_n^j, it reads

$$H=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{D-2}(\alpha_n^j\ \alpha_n^{j+}+\alpha_n^{j+}\ \alpha_n^j)\ . \tag{2.20}$$

In quantum theory, $u^j(t,r)$ and its conjugate momentum $p^j(t,r)$ become operators with canonical commutation relations

$$[u^i(t,r),\ p^j(t,r')]\ =\ i\ \delta^{ij}\ \delta(r-r')\ . \tag{2.21}$$

This implies that the Fourier coefficients become operators and satisfy the relations

$$[\alpha_n^i,\alpha_m^j]=\omega_n\delta^{ij}\ \delta_{n+m,0}, \tag{2.22}$$

$$i,j\ =\ 1,\ldots,D-2\ , \quad n,m\ =\ \pm 1,\pm 2,\ldots\ .$$

The creation and annihilation operators in Fock space a_n^+ and a_n are introduced in the usual way

$$\alpha_n^j=\sqrt{\omega_n}\ a_n^j, \quad \alpha_n^{j\ +}=\sqrt{\omega_n}\ a_n^{j\ +}, \tag{2.23}$$

$$[a_n^i,a_m^{j\ +}]=\delta^{ij}\ \delta_{nm}, \quad n,m\ =\ 1,2,\ldots\ . \tag{2.24}$$

In terms of them, the Hamiltonian (2.19) takes the form

$$H=\sum_{n=1}^{\infty}\sum_{j=1}^{D-2}\omega_n\ a_n^{j+}\ a_n^j+\frac{D-2}{2}\sum_{n=1}^{\infty}\omega_n\,. \tag{2.25}$$

The last term in Eq. (2.25) is the usual Casimir energy [16,17]. When calculating the interquark potential generated by string this term gives the Lüscher correction [14,15] (the first quantum correction to the classical, linearly rising potential).

III. ONE-LOOP POTENTIAL GENERATED BY NAMBU-GOTO STRING WITH MASSIVE ENDS

In this section we shall investigate the interquark potential generated by a string with massive ends via perturbation calculations. We define the string potential $V(R)$ in terms of the functional integral in a standard way [1,7–9]:

$$\exp[-\beta\, V(R)]=\int [D\mathbf{u}]\exp\{-S^{\beta}[\mathbf{u}]\}, \quad \beta\to\infty\ , \tag{3.1}$$

where β is inverse temperature and $S^{\beta}[\mathbf{u}]$ is an Euclidean version of the action (2.1) calculated for finite ''time'' interval $0\leqslant t<\beta$. As usual, the dynamical variables $\mathbf{u}(t,r)$ should satisfy periodic conditions in the time variable t:

$$\mathbf{u}(t,r)=\mathbf{u}(t+\beta,r)\ . \tag{3.2}$$

In this section we confine ourselves to the one-loop approximation for interquark potential. Therefore, S^{β} in Eq. (3.1) should be substituted by its quadratic part S_0^{β}. By analogy with Eq. (2.6), we obtain

$$S_0^{\beta}=M_0^2\beta R+\frac{M_0^2}{2}\int_0^{\beta}dt\int_0^R dr[\dot{\mathbf{u}}^2(t,r)+\mathbf{u}'^2(t,r)]+\beta\sum_{a=1}^{2} m_a+\sum_{a=1}^{2}\frac{m_a}{2}\int_0^{\beta}dt\ \dot{\mathbf{u}}^2(t,r_a),$$
$$r_1=0,\quad r_2=R\ . \tag{3.3}$$

Variation of Eq. (3.3) results in the equations of motion

$$\Delta\mathbf{u}=0\ , \tag{3.4}$$

and boundary conditions

$$m_1\ddot{\mathbf{u}}=-M_0^2\mathbf{u}', \tag{3.5}$$

$$m_2\ddot{\mathbf{u}}=M_0^2\mathbf{u}', \tag{3.6}$$

where $\Delta=\partial^2/\partial t^2+\partial^2/\partial r^2$ is two-dimensional Laplacian. As one would expect, Eqs. (3.4)–(3.6) are deduced from Eq. (2.7)–(2.9) through formal substitution of t by it.

Functional integration should be done over the functions $\mathbf{u}(t,r)$ obeying periodicity condition in t [Eq. (3.2)] and boundary conditions (3.5) and (3.6). In this case, after integrating by parts, action (3.3) can be written in the form

$$S_0^{\beta}=M_0^2\beta R+\beta\sum_{a=1}^{2} m_a+\frac{M_0^2}{2}\int_0^{\beta}dt\int_0^R dr\ \mathbf{u}(t,r)(-\Delta)\mathbf{u}(t,r)\ . \tag{3.7}$$

Thus, by imposing the boundary conditions (3.5) and (3.6) on functions $\mathbf{u}(t,r)$ we remove the contributions, proportional to the δ functions, to S_0^{β} of the pointlike masses at the string ends. Now, effects of these masses are taken into account through the boundary conditions (3.5) and (3.6) and ultimately through the string eigenfrequencies.

Substituting Eq. (3.7) in Eq. (3.1) and carrying out the functional integration, we arrive at the result

$$V(R)=M_0^2R+\sum_{a=1}^{2} m_a+\frac{D-2}{2\beta}\ \mathrm{Tr}\ \ln(-\Delta)\ ,\quad \beta\to\infty\ . \tag{3.8}$$

Here, we have exactly taken into account the number of the field variables $u^j(t,r)$, $j=1,2,\dots,D-2$. Therefore, the operator $(-\Delta)$ in Eq. (3.8) should be assumed now to act on the scalar function obeying conditions (3.2) and (3.5), (3.6). In addition, the known property of the functional determinants $\mathrm{Tr}\ \ln(-a\,\Delta)=\mathrm{Tr}\ \ln(-\Delta)$, where a being an arbitrary constant [20], has been used.

For calculating the functional trace in Eq. (3.8) the eigenvalues of the operator $(-\Delta)$ are needed

$$-\Delta\varphi_{nm}=\lambda_{nm}\varphi_{nm}\,. \tag{3.9}$$

Eigenfunction $\varphi_{nm}(t,r)$ must satisfy the periodicity condition (3.2) and boundary conditions (3.5) and (3.6). However, when determining the eigenvalues λ_{nm} in Eq. (3.9) these boundary conditions prove to be awkward. With allowance for the equations of motion (3.4) we transform Eqs. (3.5) and (3.6) to the form

$$m_1\mathbf{u}''=M_0^2\mathbf{u}',\ r=0;\quad m_2\mathbf{u}''=-M_0^2\mathbf{u}',\ r=R\ . \tag{3.10}$$

New boundary conditions (3.10) evidently give rise to the same Eq. (2.13) for string eigenfrequencies.

Eigenfunctions of the operator $(-\Delta)$ satisfying periodicity condition (3.2) and boundary conditions (3.10) have the form

$$\varphi_{nm}(t,r)=e^{i\Omega_n t}u_m(r)\ , \tag{3.11}$$

where $\Omega_n=2\pi n/\beta$, $n=0,\pm1,\pm2,\dots$ are the Matsubara frequencies following from the periodicity condition (3.2), and $u_m(r)$, $m=1,2,\dots$ are defined in Eqs. (2.11)–(2.13). Substituting Eq. (3.11) in Eq. (3.9), we obtain

$$\lambda_{nm}=\Omega_n^2+\omega_m^2,\quad n=0,\pm1,\pm2,\dots,m=1,2,\dots\ . \tag{3.12}$$

Now, we calculate the functional trace in Eq. (3.8):

$$\mathrm{Tr}\ \ln(-\Delta)=\sum_{n=-\infty}^{+\infty}\sum_{m=1}^{\infty}\ln\left[\left(\frac{2\pi n}{\beta}\right)^2+\omega_m^2\right]=\sum_{n=-\infty}^{+\infty}\sum_{m=1}^{\infty}\ln[(2\pi n)^2+\beta^2\omega_m^2]\ . \tag{3.13}$$

Summation over the Matsubara frequencies in Eq. (3.13) can be accomplished by making use of the known methods [21]. One of them is presented in Appendix A. As a result, we obtain

$$\mathrm{Tr}\ \ln(-\Delta)=2\sum_{m=1}^{\infty}\left[\frac{\beta\omega_m}{2}+\ln(1-e^{-\beta\omega_m})\right]\ . \tag{3.14}$$

Substituting Eq. (3.14) in Eq. (3.8) and taking the limit $\beta\to\infty$, the final formula for the potential $V(R)$ assumes the form

$$V(R)=M_0^2R+\sum_{a=1}^{2} m_a+\frac{D-2}{2}\sum_{m=1}^{\infty} \omega_m . \tag{3.15}$$

Thus, the first quantum correction to the string potential is the Casimir energy for the string with given boundary conditions. Certainly, we could, at the very beginning, obtain Eq. (3.15) proceeding from the energy of the string found in the preceding section [see Eq. (2.25)]. We gave here detailed derivation of Eq. (3.15) by making use of the functional integration technique because many points of this consideration will be used while investigating the string potential in the limit $D\to\infty$ (see next section).

Let us turn to calculation of the Casimir energy of the string entering Eq. (3.15):

$$E_C=\frac{1}{2}\sum_{k=1}^{\infty} \omega_k . \tag{3.16}$$

Obviously, this sum diverges and to obtain finite physical value of this energy, regularization and following renormalization are needed. For simplicity, the case of equal quark masses $m_1=m_2=m$ will be treated further. The general case $m_1\neq m_2$ will be investigated in our forthcoming publication [19].

The Casimir energy (3.16) is a function of quark mass $E_C(m)$ and for two limiting values $m=\infty$ (immobile quarks) and $m=0$ (free ends of the string) it can be calculated easily by making use of the Riemann zeta-function renormalization. When $m=\infty$ and $m=0$, the frequency equation (2.13) gives $\omega_n=n\pi/R,\ \ n=1,2,\ldots$. The corresponding Casimir energy is

$$E_C(m=\infty)=E_C(m=0)=\frac{\pi}{2R}\sum_{n=1}^{\infty} n=\frac{\pi}{2R}\zeta(-1)=-\frac{\pi}{24R} . \tag{3.17}$$

The renormalized Casimir energy at finite value of m, $E_C(m)$, should satisfy these limiting conditions. This requirement, as will be shown below, determines subtraction procedure uniquely.

To sum over the string eigenfrequencies in Eq. (3.16) we use the following integral formula from the complex analysis [22]. Let us consider an analytical function $f(\omega)$ with zeros of order n_k at points $\omega=\omega_k$ and with poles of order p_l at points $\omega=\tilde{\omega}_l$ in a region bounded by a contour C. From Cauchy's theorem it follows that

$$\frac{1}{2\pi i}\oint_C d\omega\ \omega\ \frac{f'(\omega)}{f(\omega)}=\frac{1}{2\pi i}\oint_C d\omega\ \omega\ [\ln f(\omega)]'$$
$$=\sum_k n_k\omega_k-\sum_l p_l\tilde{\omega}_l . \tag{3.18}$$

In order to get rid of the poles in Eq. (3.18) we rewrite the frequency equation (2.13) in the form

$$f(\omega)=2mM_0^2\omega\cos(\omega R)-(m^2\omega^2-M_0^4)\sin(\omega R)=0 . \tag{3.19}$$

Substituting Eq. (3.19) in Eq. (3.18), we deduce

$$E_C(m)=\frac{1}{4\pi i}\oint_C \omega\ d\omega\frac{d}{d\omega}[\ln f(\omega)] , \tag{3.20}$$

where counter C encloses the real positive semiaxis where the roots of Eq. (3.19) are placed. As the function $f(\omega)$ in Eq. (3.19) has no singularities on the right half plane, the counter C can be transformed into a semicircle with radius Λ and the segment of the imaginary axis $(-i\Lambda,\ i\Lambda)$. At any finite Λ, the counter integral in Eq. (3.20) is finite. Therefore, this integral can be treated as a regularized value of $E_C^{\mathrm{reg}}(m,R)$. We have noted here exact dependence of E_C on the length of the string R (distance between quarks).

To carry out the renormalization of the Casimir energy we must, as usual [16,17,23], subtract from $E_C^{\mathrm{reg}}(m,R)$ the value of this energy at $R\to\infty$:

$$E_C^{\mathrm{ren}}(m,R)=E_C^{\mathrm{reg}}(m,R)-E_C^{\mathrm{reg}}(m,R\to\infty)|_{\Lambda\to\infty} . \tag{3.21}$$

Contribution to the finite (renormalized) value of $E_C^{\mathrm{ren}}(m,R)$ gives only the integral along the imaginary axis in Eq. (3.20). This will be seen from the final result. Therefore, we confine ourselves to the integral along the imaginary axes in complex plane ω:

$$E_C^{\mathrm{reg}}(m,R)=-\frac{1}{4\pi}\int_{-\Lambda}^{\Lambda}y\ dy\{\ln[2mM_0^2y\cosh(yR)$$
$$+(m^2y^2+M_0^4)\sinh(yR)]\}' . \tag{3.22}$$

Integrating by parts and dropping the terms outside the integral (formally they vanish), we arrive at the formula

$$E_C^{\mathrm{reg}}(m,R)=\frac{1}{2\pi}\int_0^{\Lambda}dy\ \ln[2mM_0^2y\cosh(yR)$$
$$+(m^2y^2+M_0^4)\sinh(yR)] . \tag{3.23}$$

In order to renormalize E_C^{reg} we must subtract from Eq. (3.23) the asymptotic of E_C^{reg} when $R\to\infty$:

$$E_C^{\mathrm{reg}}(m,R\to\infty)=\frac{1}{2\pi}\int_0^{\Lambda}dy\ \ln[e^{yR}(my+M_0^2)^2/2] . \tag{3.24}$$

Substituting Eqs. (3.23) and (3.24) into Eq. (3.21), we obtain renormalized Casimir energy[1]

[1] In Ref. [24] other subtraction procedures have been considered which satisfy condition (3.17) only at $m=\infty$.

$$E_C^{\rm ren}(m,R)=\frac{1}{2\pi}\int_0^\infty dy\ \ln\left\{1-e^{-2Ry}\left(\frac{my-M_0^2}{my+M_0^2}\right)^2\right\}$$
$$=\frac{1}{2\pi R}\int_0^\infty dx\ \ln\left\{1-e^{-2x}\left(\frac{x-q}{x+q}\right)^2\right\}\ , \tag{3.25}$$

where q is a dimensionless parameter, $q=M_0^2R/m$. Analysis of integrand in Eq. (3.25) shows that integral along the semicircle of radius Λ, that has been discarded above, actually vanishes when $\Lambda\to\infty$. One can easily convince that $E_C^{\rm ren}(m,R)$ obeys limiting conditions (3.17):

$$E_C^{\rm ren}(m=0,R)=E_C^{\rm ren}(m=\infty,R)$$
$$=\frac{1}{2\pi}\int_0^\infty dy\ \ln(1-e^{-2Ry})$$
$$=-\frac{\pi}{24R}\ . \tag{3.26}$$

Subtraction of divergent at $\Lambda\to\infty$ contribution (3.24) is interpreted as a transition to physical values of string tension M_0^2 and quark masses m_a in Eq. (3.15). For simplicity, we will not introduce any new notation for renormalized parameters of the model. The final formula for the interquark potential derived in the one-loop approximation reads

$$V(R)=M_0^2R+(D-2)E_C^{\rm ren}(m,R)\ , \tag{3.27}$$

where $E_C^{\rm ren}(m,R)$ is determined in Eq. (3.25). The constant $2m$, giving the rest energy of quarks, has been dropped in this formula.

It is natural to compare Eq. (3.27) with the one-loop interquark potential generated by string with fixed ends [1]:

$$V^{\rm fixed}(R)=M_0^2R-\frac{(D-2)\pi}{24R}\ . \tag{3.28}$$

The last term here is the universal Lüscher term independent of the concrete form of the string action but calculated for fixed ends of the string [14,15]. A clear picture of the quark mass influence on the string potential in any given approximation provides the ratio of $E_C^{\rm ren}(m,R)$ to the Lüscher term in Eq. (3.28), i.e., to $E_C^{\rm ren}(m=\infty,R)$. This ratio depends only on the dimensionless parameter $q=M_0^2R/m$ and is given by the formula[2]

$$\eta(q)=\frac{E_C^{\rm ren}(m,R)}{E_C^{\rm ren}(m=\infty,R)}=-\frac{12}{\pi^2}\int_0^\infty dx\ \ln\left[1-e^{-2x}\left(\frac{q-x}{q+x}\right)^2\right]\ . \tag{3.29}$$

Deviation of $\eta(q)$ from 1 characterizes the contribution of the quark masses to the first quantum correction to the string potential. The function $\eta(q)$ is plotted in Fig. 1. In the region $q\approx0.2$, it is dropped to ≈0.2. If one assumes that the string length is of the order of the Compton wavelength of quark $R\sim m^{-1}$, then the maximal alteration of the Lüscher term happens at the quark mass $m\sim2.2M_0$, i.e., for sufficiently heavy quarks (in string models $M_0\sim0.4$ GeV).

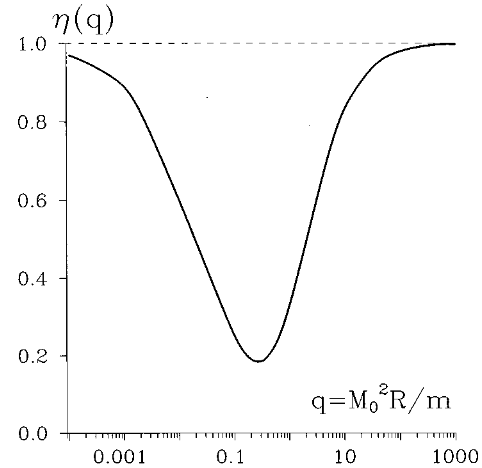

FIG. 1. Function $\eta(q)$ [see definition (3.29)] describes the effect of the finite quark mass on the value of the Lüscher term in string potential.

IV. VARIATIONAL ESTIMATION OF THE STRING POTENTIAL

In the preceding section the string potential has been calculated in the one-loop approximation by using the perturbative theory for an arbitrary dimension of space-time D. Otherwise, this potential can be investigated in the limit $D\to\infty$ by making use of the variational estimation of the functional integral [1,4–9].

Let us turn to the initial Eq. (3.1) determining the string potential

$$e^{-\beta\ V(R)}=\int[D\mathbf{u}]\ \exp\{-S^\beta[\mathbf{u}]\},\quad \beta\to\infty\ , \tag{4.1}$$

where S^β is the Euclidean action

$$S^\beta[\mathbf{u}]=M_0^2\int_0^\beta dt\int_0^R dr\sqrt{\det(\delta_{ij}+\partial_i\mathbf{u}\ \partial_j\mathbf{u})}$$
$$+\sum_{a=1}^2 m_a\int_0^\beta dt\ \sqrt{1+\dot{\mathbf{u}}^2(t,r_a)}, \tag{4.2}$$
$$i,j=0,1,\quad r_1=0,\quad r_2=R\ .$$

The $1/(D-2)$ expansion is carried out in a standard way [1]. Let us introduce the composite field σ_{ij} for $\partial_i\mathbf{u}\ \partial_j\mathbf{u}$ and constrain $\sigma_{ij}=\partial_i\mathbf{u}\ \partial_j\mathbf{u}$ through the Lagrange multiplier α^{ij}. By using the exponential parametrization of the δ function, with the understanding that the α^{ij} functional integrals run from $-i\infty$ to $+i\infty$, Eqs. (4.1) and (4.2) become

[2]In Ref. [25] other subtraction procedure has been used in the model under consideration and as a result the formula for $\eta(q)$ obtained there does not coincide with Eq. (3.29).

$$e^{-\beta V(R)}=\int [D\mathbf{u}][D\alpha][D\sigma]\exp\{-S^{\beta}[\mathbf{u},\alpha,\sigma]\}, \quad \beta\to\infty , \tag{4.3}$$

where

$$S^{\beta}[\mathbf{u},\alpha,\sigma]=\frac{M_0^2}{2}\int_0^{\beta}dt\int_0^R dr[2\sqrt{\det(\delta_{ij}+\sigma_{ij})}+\alpha^{ij}(\partial_i\mathbf{u}\;\partial_j\mathbf{u}-\sigma_{ij})]+\sum_{a=1}^{2}m_a\int_0^{\beta}dt\sqrt{1+\dot{\mathbf{u}}^2(t,r_a)},$$
$$i,j=0,1, \quad r_1=0, \quad r_2=R . \tag{4.4}$$

Further, the $1/(D-2)$ expansion is constructed in the following way [1]. Functional integral over string coordinates $\mathbf{u}$ in Eq. (4.3) can be done exactly as action (4.4) is quadratic in $\mathbf{u}$. Functional integrals over α^{ij} and σ_{ij} are estimated by making use of the variational method, the stationary values of these functional variables being given by constant (independent on t and r) diagonal matrices $\alpha^{ij}=\delta^{ij}\alpha^j$, $\sigma_{ij}=\delta_{ij}\sigma_j$ (no summation over j).

When integrating over $\mathbf{u}$ in Eq. (4.3), we have to take into account the quark contribution to action (4.4). As in the preceding section, we do this by imposing on $\mathbf{u}$ corresponding boundary conditions. Variation of action (4.4) with respect to string coordinates results in the equations of motion

$$\Delta_{\alpha}\mathbf{u}\equiv\alpha^0\ddot{\mathbf{u}}+\alpha^1\mathbf{u}''=0, \tag{4.5}$$

and boundary conditions

$$\frac{m_1}{\sqrt{1+\sigma_0}}\,\ddot{\mathbf{u}}=-M_0^2\alpha^1\mathbf{u}', \quad r=0, \tag{4.6}$$

$$\frac{m_2}{\sqrt{1+\sigma_0}}\,\ddot{\mathbf{u}}=M_0^2\alpha^1\mathbf{u}', \quad r=R . \tag{4.7}$$

We require that the functional variables $\mathbf{u(t,r)}$ in Eqs. (4.3) and (4.4) satisfy the boundary conditions (4.6) and (4.7). Then, integrating by parts in the second term in Eq. (4.4), we can present action $S^{\beta}[\mathbf{u},\alpha,\sigma]$ in the form

$$S^{\beta}[\mathbf{u},\alpha,\sigma]=\frac{M_0^2}{2}\int_0^{\beta}dt\int_0^R dr\;\mathbf{u}(-\Delta_{\alpha})\mathbf{u}+M_0^2\beta\,R\left[\sqrt{(1+\sigma_0)(1+\sigma_1)}-\frac{1}{2}(\alpha^0\sigma_0+\alpha^1\sigma_1)\right], \tag{4.8}$$

where operator Δ_{α} is defined in Eq. (4.5). Thus, we arrive at the action exactly quadratic in the transverse string coordinates $\mathbf{u(t,r)}$. Functional integration over $\mathbf{u}$ in Eqs. (4.3) and (4.8) gives standard contribution to effective action $[(D-2)/2]$ Tr $\ln(-\Delta_{\alpha})$. In order to calculate this functional trace the eigenvalues of the operator $(-\Delta_{\alpha})$ are needed. To this end, we again transform boundary conditions (4.6) and (4.7) taking here into account Eq. (4.5):

$$\frac{m_1}{\sqrt{1+\sigma_0}}\,\mathbf{u}''=M_0^2\alpha^0\mathbf{u}', \quad r=0;$$

$$\frac{m_2}{\sqrt{1+\sigma_0}}\,\mathbf{u}''=-M_0^2\alpha^0\mathbf{u}', \quad r=R . \tag{4.9}$$

Equations of motion (4.5) with boundary conditions (4.6), (4.7), or (4.9) result in the same frequency equation

$$\tan(\omega R)=\frac{2m\overline{M_0^2}\omega}{m^2\omega^2-M_0^2} , \tag{4.10}$$

where $\overline{M_0^2}=\alpha^0\sqrt{1+\sigma_0}\,M_0^2$. One can be easily convinced that the eigenvalues λ_{nk} of the operator $(-\Delta_{\alpha})$ with boundary conditions (4.9) are given by

$$\lambda_{nk}=\alpha^0\left(\frac{2\pi n}{\beta}\right)^2+\alpha^1\omega_k^2 ,$$
$$n=0,\pm1,\pm2,\ldots, \quad k=1,2,\ldots , \tag{4.11}$$

where ω_k are the positive roots of Eq. (4.10). Using Eqs. (3.13) and (3.14), we obtain

$$\lim_{\beta\to\infty}\frac{1}{2\beta}\,\mathrm{Tr}\,\ln(-\Delta_{\alpha})=\frac{1}{2}\sqrt{\frac{\alpha^1}{\alpha^0}}\sum_{k=1}^{\infty}\omega_k=\sqrt{\frac{\alpha^1}{\alpha^0}}E_C , \tag{4.12}$$

where $E_C=(1/2)\Sigma_{k=1}^{\infty}\omega_k$ is the Casimir energy for the frequency equation (4.10). Now, E_C obviously depends on the variational parameters α^0 and σ_0. Finally, effective action that should be varied with respect to α^i, σ_i, $i=0,1$ assumes the form

$$S^{\beta}=M_0^2\beta R\left\{\sqrt{(1+\sigma_0)\;(1+\sigma_1)}-\frac{1}{2}\,(\alpha^0\sigma_0+\alpha^1\sigma_1)\right\}+\beta(D-2)\sqrt{\frac{\alpha^1}{\alpha^0}}\,E_C . \tag{4.13}$$

For calculating E_C, formula (3.25) should be used with substitution of M_0^2 by $\overline{M_0^2}$. Variation of Eq. (4.13) results in the equations

$$\alpha^0=\sqrt{\frac{1+\sigma_1}{1+\sigma_0}}+\frac{D-2}{M_0^2R}\sqrt{\frac{\alpha^1}{\alpha^0}}\,\frac{\partial E_C}{\partial\sigma_0} , \tag{4.14}$$

$$\alpha^1=\sqrt{\frac{1+\sigma_0}{1+\sigma_1}} , \tag{4.15}$$

$$\sigma_0=-\frac{D-2}{M_0^2R\alpha^0}\sqrt{\frac{\alpha^1}{\alpha^0}}\,E_C+2\frac{D-2}{M_0^2R}\sqrt{\frac{\alpha^1}{\alpha^0}}\,\frac{\partial E_C}{\partial\alpha^0} , \tag{4.16}$$

$$\sigma_1=\frac{D-2}{M_0^2R}\,\frac{E_C}{\sqrt{\alpha^0\alpha^1}} . \tag{4.17}$$

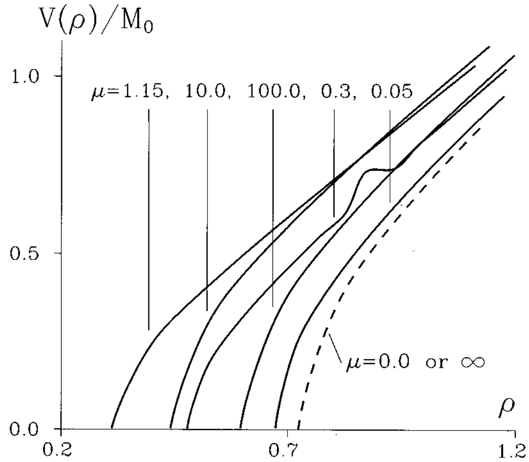

FIG. 2. The dimensionless interquark potential $V(\rho)/M_0$ as a function of $\rho=M_0R$ is shown at different values of the ratio $\mu=m/M_0$. All the curves start at the points on x axis presenting the value of ρ_c at given μ. The left- most curve correspond to the minimal value of ρ_c (see Fig. 3). When $\rho\to\infty$ the curves tend towards the linearly rising potential $V(\rho)\sim M_0^2\rho$. Dashed curve presents the interquark potential generated by string with fixed (or free) ends. When μ tends to infinity or to zero, the potential curves calculated by the formula (4.22) approach the dashed one.

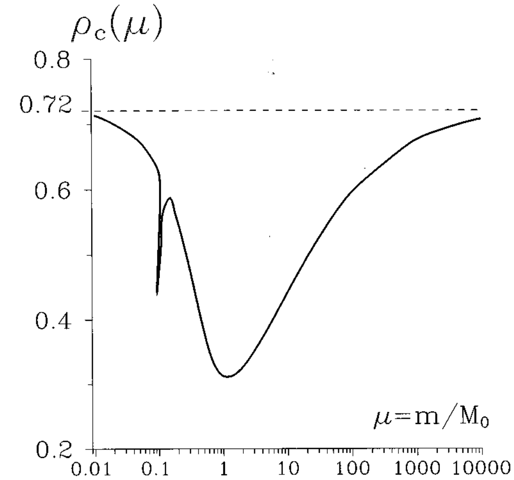

FIG. 3. The critical radius, $\rho_c=M_0R$, of the string potential is presented as a function of the dimensionless quark mass $\mu=m/M_0$. When μ tends to infinity or to zero, $\rho_c(\mu)$ approaches the critical radius of the potential generated by string with fixed (or free) ends $\rho_c^{\text{fixed}}\simeq 0.72$. The plot indicates that even for heavy quarks with $\mu=10^2$ (region of the top quark mass), the critical radius of the potential generated by a string with massive ends significantly differs from the limiting value ρ_c^{fixed}.

Now, we use simplifying assumption that enables us to write the solutions to Eqs. (4.14)–(4.17) in an analytical form. When calculating the Casimir energy E_C, we put $\alpha^0=1$ and $\sigma_0=0$. The limits of applicability of this approximation are discussed in Appendix B. Omitting terms with $\partial E_C/\partial\sigma_0$ and $\partial E_C/\partial\alpha^0$ in Eqs. (4.14) and (4.16), we arrive at the equations [1] solution of which is given by

$$\alpha^0=\sqrt{1-2\lambda}\ , \tag{4.18}$$

$$\alpha^1=\frac{1}{\sqrt{1-2\lambda}}\ , \tag{4.19}$$

$$\sigma_0=\frac{\lambda}{1-2\lambda}\ , \tag{4.20}$$

$$\sigma_1=-\lambda\ , \tag{4.21}$$

where $\lambda=-(D-2)\ E_C/(M_0^2R)$, the Casimir energy E_C being defined in Eq. (3.25).

Calculating the action (4.13) on the solutions (4.18)–(4.21), we arrive at the final expression for the string potential:

$$V(R)=M_0^2R\sqrt{1+\frac{2(D-2)}{M_0^2R}\ E_C(m,R)}\ . \tag{4.22}$$

In Fig. 2 the dimensionless string potential $V(R)/M_0$ is plotted as a function of the dimensionless distance $\rho=M_0R$ for different values of ratio $\mu=m/M_0$. For the chosen symmetrical quark configuration $m_1=m_2=m$, the Casimir energy $E_C(m,R)$ remains negative (see the preceding section). Therefore, Eq. (4.22) has sense only at such values of R when

$$1\geqslant-\frac{2(D-2)}{M_0^2R}\ E_C(m,R)\ . \tag{4.23}$$

Equality to zero of the radicand in Eq. (4.22) determines the critical radius R_c of the string potential. In the case of the infinitely heavy quarks, it is given by $M_0R_c=\sqrt{\pi/6}\simeq 0.72$ (for $D=4$). Figure 3 presents the dependence of the dimensionless critical radius $\rho_c=R_c/M_0$ on the ratio $\mu=m/M_0$. This plot shows that account of the finite quark masses results in reducing the critical radius. For a given quark mass configuration, $m_1=m_2=m$, the minimal value ($\simeq 0.31$) of ρ_c is reached at $\mu\simeq 1.15$. If we take as usual $M_0\simeq 0.4$ GeV then $m\simeq 0.5$ GeV.

V. QUARK MASS CORRECTIONS TO THE RIGID STRING POTENTIAL

In this section we calculate the quark mass corrections to the one-loop interquark potential in the framework of the rigid string model [5,26]. As is known, this model can be treated as an effective one, taking into account the finite thickness of gluonic tube [27,28]. The basic aim of this calculation is to show the principal applicability of the proposed method to the rigid string model with massive ends. Variational estimation of the interquark potential in the framework of this model will be published elsewhere.

The Polyakov-Kleinert action for a rigid string with massive ends has the form [5,26]

$$S=-M_0^2\int\int_\Sigma d\Sigma\left[1-\frac{\alpha}{2}\,r_s^2\ (\Delta_{L-B}x^\mu)\ (\Delta_{L-B}x_\mu)\right]$$
$$-\sum_{a=1}^{2}\ m_a\int_{C_a}ds_a\,, \tag{5.1}$$

where the new parameters r_s and α are, respectively, the radius of gluonic tube and a dimensionless constant; Δ_{L-B} is the Laplace-Beltrami operator for the induced metric g_{ij} on the string world sheet

$$\Delta_{L-B}=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\xi^i}\left(\sqrt{-g}\,g^{ij}\,\frac{\partial}{\partial\xi^j}\right). \tag{5.2}$$

In the Gauss parametrization (2.2), the operator (5.2), up to the second order in $\mathbf{u}$, can be written as

$$\Delta_{L-B}\simeq\Box+O(\mathbf{u}^2)\ , \tag{5.3}$$

where $\Box=\partial^2/\partial t^2-\partial^2/\partial r^2$. Now action (5.1) reads

$$S\simeq-M_0^2\int_{t_1}^{t_2}dt\int_0^R dr\left[1-\frac{1}{2}\,\dot{\mathbf{u}}^2+\frac{1}{2}\,\mathbf{u}'^2+\frac{\alpha}{2}\,r_s^2\,(\Box\mathbf{u})^2\right.$$
$$-\sum_{a=1}^{2}\frac{m_a}{2}\int_{t_1}^{t_2}dt\ \dot{\mathbf{u}}^2(t,r_a),\quad r_1=0,\quad r_2=R\ . \tag{5.4}$$

The equations of motion and boundary conditions for the action (5.4) are

$$(1+\alpha\ r_s^2\ \Box)\ \Box\ \mathbf{u}=0\ , \tag{5.5}$$

$$(1+\alpha\ r_s^2\ \Box)\ \mathbf{u}'=\frac{m}{M_0^2}\ \ddot{\mathbf{u}}\ ,\quad r\ =0\ , \tag{5.6}$$

$$(1+\alpha\ r_s^2\ \Box)\ \mathbf{u}'=-\frac{m}{M_0^2}\ \ddot{\mathbf{u}}\ ,\quad r=R\ , \tag{5.7}$$

$$\Box\ \mathbf{u}=0\ ,\quad r=0,R, \tag{5.8}$$

$(m_1=m_2=m)$. The Lagrangian in the action (5.4) depends on the first and the second derivatives of the string coordinates, therefore the number of derived boundary conditions is twice compared with those of the Nambu-Goto string.

The boundary value problem [Eqs. (5.5)–(5.8)] reduces to two independent ones. Indeed, equations of motion are given by the product of commuting differential operators $(1+\alpha\ r_s^2\ \Box)$ and $\Box$. Hence, the general solution to this equation can be represented as a sum of two terms

$$\mathbf{u}(t,r)=\mathbf{u}_1(t,r)+\mathbf{u}_2(t,r)\ , \tag{5.9}$$

where

$$\Box\ \mathbf{u}_1=0\ , \tag{5.10}$$

$$\mathbf{u}_1'=-\frac{m}{M_0^2}\ \ddot{\mathbf{u}}_1\ ,\quad r=0\ , \tag{5.11}$$

$$\mathbf{u}_1'=\frac{m}{M_0^2}\ \ddot{\mathbf{u}}_1\ ,\quad r=R\ , \tag{5.12}$$

and

$$(1+\alpha\ r_s^2\ \Box)\ \mathbf{u}_2=0\ , \tag{5.13}$$

$$\mathbf{u}_2(t,0)=\mathbf{u}_2(t,R)=0\ . \tag{5.14}$$

In this case, $\mathbf{u}_1(t,r)$ is the solution to the Nambu-Goto string with massive ends that we have analyzed in Sec. II. The string rigidity is taken into account by function $\mathbf{u}_2(t,r)$. The general solution to Eq. (5.13) obeying Eq. (5.14) can be presented as

$$u_2^j(t,r)=\frac{-1}{\sqrt{2}\ M_0}\sum_{n\neq0}\ \exp[i\,\nu_n\ t]\ \frac{\beta_n^j}{\nu_n}\ v_n(r)\ ,$$
$$j=1,2,\ \ldots\ ,D-2\ . \tag{5.15}$$

The eigenfunctions $v_n(r)$ are given by

$$v_n(r)=-\ v_{-n}=\sqrt{\frac{2}{R}}\,\sin\!\left(n\pi\ \frac{r}{R}\right),\quad n=1,2,\ \ldots\ . \tag{5.16}$$

For the natural frequencies ν_n in Eq. (5.15), we have

$$\nu_n=-\ \nu_{-n}=\sqrt{\left(\frac{n\pi}{R}\right)^2+\frac{1}{\alpha r_s^2}},\quad n=1,2,\ \ldots\ . \tag{5.17}$$

The amplitudes β_n^j satisfy the usual relations of complex conjugation $\beta_n^*=\beta_{-n}\,,\ \ n=1,2,\ \ldots\ .$

The Hamiltonian formulation of the model under consideration is developed in the following way. According to Ostrogradsky [29,30], the canonical variables are defined by

$$q_1^j=u^j\ ,\quad q_2^j=\dot{u}^j\ , \tag{5.18}$$

$$p_1^j=\frac{\partial L}{\partial\dot{u}^j}-\ p_2^j\,,\quad p_2^j=\frac{\partial L}{\partial\ddot{u}^j}\ , \tag{5.19}$$

$$j=1,2,\ \ldots\ ,D-2\ ,$$

where L is the Lagrangian density in action (5.4):

$$L=\frac{M_0^2}{2}\left[\varepsilon(r)\ \dot{\mathbf{u}}^2-\mathbf{u}'^2-\alpha\ r_s^2\ (\Box\mathbf{u})^2\right]\ . \tag{5.20}$$

Weight function $\varepsilon(r)$ was determined in Eq. (2.15). Putting L and Eq. (5.9) into Eqs. (5.18) and (5.19) and taking into account Eqs. (5.10) and (5.13), one obtains

$$\mathbf{q}_1=\mathbf{u}_1\ +\ \mathbf{u}_2\,,\quad \mathbf{q}_2=\dot{\mathbf{u}}_1+\dot{\mathbf{u}}_2\,, \tag{5.21}$$

$$\mathbf{p}_1=M_0^2\ [\varepsilon(r)+\alpha\ r_s^2\ \Box]\ \dot{\mathbf{u}}\ ,$$

$$\mathbf{p}_2=-\ \alpha\ r_s^2\ M_0^2\ \Box\ \mathbf{u}=M_0^2\ \mathbf{u}_2\,. \tag{5.22}$$

The canonical Hamiltonian is defined by

$$H=\int_0^R dr\ [\mathbf{p}_1\dot{\mathbf{q}}_1+\mathbf{p}_2\dot{\mathbf{q}}_2-L]\ . \qquad (5.23)$$

In terms of Fourier amplitudes it becomes

$$H=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{D-2}\ (\alpha_{nj}\alpha_{nj}^{+}+\alpha_{nj}^{+}\alpha_{nj})$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{D-2}\ (\beta_{nj}\beta_{nj}^{+}+\beta_{nj}^{+}\beta_{nj})\ . \qquad (5.24)$$

The quantum theory is based on the canonical commutation relations

$$[u_a^i(t,r),\ p_b^j(t,r')]=i\ \delta_{ab}\ \delta^{ij}\delta(r-r')\ ,$$

$$a=1,2\ ,\quad i,j=1,2,\ldots,D-2\ , \qquad (5.25)$$

or in terms of the Fourier amplitudes

$$[\alpha_n^i,\alpha_m^j]=\omega_n\ \delta^{ij}\ \delta_{n+m,0}\,,$$

$$[\beta_n^i,\beta_m^j]=\nu_n\ \delta^{ij}\ \delta_{n+m,0}\,,\quad n,m=\pm 1,\pm 2,\ldots\ . \qquad (5.26)$$

By introducing, in standard way, the annihilation and creation operators

$$a_n^i=(\omega_n)^{-1/2}\ \alpha_n^i\,,\qquad a_n^{i+}=(\omega_n)^{-1/2}\ \alpha_n^{i+}\,,$$

$$b_n^j=(\nu_n)^{-1/2}\ \beta_n^j\,,\qquad b_n^{j+}=(\nu_n)^{-1/2}\ \beta_n^{j+}\,, \qquad (5.27)$$

$$n=1,2,\ldots\ ,\qquad i,j=1,2,\ldots,D-2\ ,$$

the Hamiltonian operator (5.24) acquires the form

$$H=\sum_{n=1}^{\infty}\ \omega_n\sum_{j=1}^{D-2}\ a_n^{j+}\ a_n^j-\sum_{n=1}^{\infty}\ \nu_n\sum_{j=1}^{D-2}\ b_n^{j+}\ b_n^j$$

$$+\frac{D-2}{2}\left(\sum_{n=1}^{\infty}\ \omega_n-\sum_{n=1}^{\infty}\ \nu_n\right)\ . \qquad (5.28)$$

The last two terms in Eq. (5.28) define the Casimir energy in the model under consideration [31]. It is important to note that the second oscillation mode with frequencies ν_n, responsible for the string rigidity, gives a negative contribution to the energy as compared with the oscillation of the first mode with frequencies ω_n. This is also true for the Casimir energy [see the last two terms in Eq. (5.28)]. It is a direct consequence of the classical expression for the total energy in the rigid string model (5.24). We point out that this defect is typical in all the theories with higher derivatives. To remove it, certainly in the formal way only, the quantum states with negative norm are sometimes used [32–34].

Now, we calculate the rigid string potential in the one-loop approximation. Again, we shall treat the Euclidean version of the model under consideration.

The interquark potential is given by Eq. (3.1) with the Euclidean action ($t_1=0$, $t_2=\beta$)

$$S^{\beta}=M_0^2\int_0^{\beta}dt\int_0^R dr\left[1+\frac{1}{2}\ \mathbf{u}\ (1-\alpha\ r_s^2\ \Delta)\ (-\Delta)\mathbf{u}\ \right]. \qquad (5.29)$$

We have integrated here by parts and taken into account the boundary conditions. As in Sec. IV, the boundary terms in action (5.4) will be taken into account by finding the proper eigenvalues of the corresponding differential operators.

After functional integration, the potential takes the form

$$V(R)=M_0^2R+\frac{D-2}{2\beta}\ \mathrm{Tr}\ \ln[(1-\alpha r_s^2\Delta)(-\Delta)],\quad \beta\to\infty. \qquad (5.30)$$

Now, we turn to the calculation of the functional trace in Eq. (5.30):

$$\mathrm{Tr}\ \ln[(1-\alpha r_s^2\Delta)(-\Delta)]=\mathrm{Tr}\ \ln(1-\alpha r_s^2\Delta)+\mathrm{Tr}\ \ln(-\Delta)\ , \qquad (5.31)$$

provided that the operator $(1-\alpha r_s^2\Delta)$ should be supplemented by the boundary conditions (5.14) and the operator $(-\Delta)$ by Eq. (3.10). Analogously, with the calculations in Sec. III, we obtain

$$V(R)=M_0^2R+\frac{D-2}{2}\left(\sum_{k=1}^{\infty}\ \omega_k+\sum_{k=1}^{\infty}\ \nu_k\right), \qquad (5.32)$$

where ω_k are the eigenfrequencies of the Nambu-Goto string with massive ends (2.13) and ν_k are the string frequencies responsible for its rigidity [see Eq. (5.17)]. The last term in Eq. (5.32) has, in contrast to Eq. (5.28), a positive sign. As was mentioned above, it means that the formalism applied here effectively uses quantum states with negative norm to describe the string excitations with frequencies ν_k. The renormalized value of the first sum in Eq. (5.32), the Casimir energy of the Nambu-Goto string with massive ends, was derived in Sec. III [see Eq. (3.25)]. Now, we have to calculate only the second sum in Eq. (5.32) which is the Casimir energy due to the string oscillations with ''rigidity'' frequencies

$$E_C^{\mathrm{rigid}}=\frac{1}{2}\sum_{n=1}^{\infty}\ \nu_n=\frac{\pi}{2R}\sum_{n=1}^{\infty}\ \sqrt{n^2+\epsilon^2}, \qquad (5.33)$$

where the dimensionless parameter ϵ is given by $\epsilon^2=R^2/(\alpha_s\pi^2r_s^2)$. This sum has been considered in many problems (see, for example, [31,35,36]). Therefore, we directly use its finite value which is obtained by subtracting the analogous contribution of the infinite string

$$E_C^{\mathrm{rigid}}=-\frac{\epsilon}{2R}\sum_{n=1}^{\infty}\frac{K_1(2\pi n\epsilon)}{n}, \qquad (5.34)$$

where $K_1(z)$ is the modified Bessel function [37].

Finally, the one-loop interquark potential generated by Polyakov-Kleinert rigid string with massive ends has the form

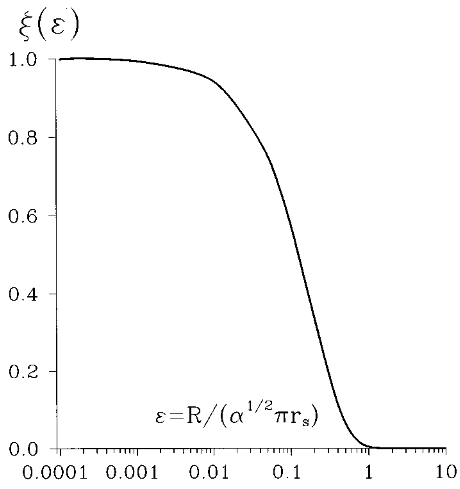

FIG. 4. Function $\xi(\epsilon)$ [see definition (5.36)] presents the contribution of the string rigidity to the quark potential in the one-loop approximation.

$$V(R)=M_0^2R+(D-2)[E_C(m,R)+E_C^{\rm rigid}(\alpha_s,R)], \tag{5.35}$$

where $E_C(m,R)$ is the Casimir energy of the Nambu-Goto string with massive ends [Eq. (3.25)] and $E_C^{\rm rigid}(\alpha_s,R)$ is the Casimir energy [Eq. (5.34)] due to the string rigidity.

In Sec. III the contribution of $E_C(m,R)$ to string potential has been analyzed by comparing it with the universal Lüscher term [see Eq. (3.29) and Fig. 1]. It is worthwhile to do the same with $E_C^{\rm rigid}$ by considering the ratio

$$\xi(\epsilon)=\frac{E_C^{\rm rigid}(\alpha_s,R)}{E_C(m=\infty)}=\frac{12\epsilon}{\pi}\sum_{n=1}^{\infty}\frac{K_1(2\pi n\epsilon)}{n} \tag{5.36}$$

that depends only on the dimensionless parameter ϵ. The plot of this function is presented in Fig. 4. It shows that, for any value of ϵ the contribution of the string rigidity to interquark potential does not exceed, in absolute value, the Lüscher term.

VI. CONCLUSION

In this paper we have developed a consistent method for calculating the interquark potential generated by relativistic string with pointlike masses (spinless quarks) at its ends. The obtained results indicate that the correction to the potential due to the finite quark masses turns out to be considerable both in the one-loop approximation and through the variational estimation of the string functional integral. The extension of the proposed method for investigating the model of the relativistic string with massive ends at finite temperature is of undoubted interest.

Finally, the following should be noted. In our approach only transverse vibrations of the string and its massive ends have been considered at fixed string length. Therefore, we have calculated really the contribution of the finite quark masses to the statical part of the interquark potential. Obviously, the longitudinal motions of the string and quarks at its ends also will give contribution to the string potential. But this problem is beyond the scope of our paper.

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APPENDIX A: SUMMATION OVER THE MATSUBARA FREQUENCIES

Here, we show how to cast Eq. (3.13) into Eq. (3.14) summing over the Matsubara frequencies. We shall proceed from the known representation of the entire function sinh$z$ in terms of the infinite product [37]

$$\sinh\frac{z}{2}=\frac{z}{2}\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4\pi^2n^2}\right). \tag{A1}$$

Taking the logarithm of both sides of this equation, we obtain

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{z^2}{4\pi^2n^2}\right)=\ln(e^z-1)-\ln z-\frac{z}{2}\ . \tag{A2}$$

Now, the sum over the Matsubara frequencies in Eq. (3.13), in view of Eq. (A2), can be rewritten as

$$\sum_{n=-\infty}^{\infty}\ln[(2\pi n)^2+\beta^2\omega_m^2]=2\sum_{n=1}^{\infty}\ln\left(1+\frac{\beta^2\omega_m^2}{4\pi^2n^2}\right)+4\sum_{n=1}^{\infty}\ln(2\pi n)+2\ln(\beta\omega_m)$$
$$=2\ln(e^{\beta\omega_m}-1)-\beta\omega_m+4\sum_{n=1}^{\infty}\ln(2\pi n)=2\ln(1-e^{-\beta\omega_m})+\beta\omega_m+4\sum_{n=1}^{\infty}\ln(2\pi n)\ . \tag{A3}$$

The last divergent term in Eq. (A3) should be dropped because after dividing by β [see Eq. (3.8)] and taking the limit $\beta\to\infty$, it vanishes. In view of this, Eq. (3.13) transforms to Eq. (3.14).

APPENDIX B: ANALYSIS OF APPROXIMATION USED FOR CONSTRUCTING THE SOLUTION TO VARIATIONAL EQUATIONS

Here, we consider the approximations which have been done for obtaining the solutions (4.18)–(4.21) to variational equations (4.14)–(4.17).

From boundary conditions (4.6) and (4.7), it follows that the dependence of E_C on σ_0 takes actually into account relativistic corrections in dynamical equations describing the motion of quarks. Therefore, putting in Eq. (4.6) and (4.7) $\sigma_0=0$, we restrict ourselves to nonrelativistic description of the quark dynamics. This is admissible for sufficiently heavy quarks.

The next assumption consists in setting, under calculation of E_C, $\alpha^0 \simeq 1$ and neglecting the last term in Eq. (4.16). Having done this approximation, we directly obtain solutions (4.18)–(4.21). Let us show that this solution leads to estimation of α^0 that agrees fairly well with the initial assumption $\alpha^0 \simeq 1$. In fact, proceeding from the definition (3.29), we obtain

$$\sigma_1 = -\eta(q)\frac{\pi(D-2)}{24\, M_0^2 R^2}\,. \tag{B1}$$

String model claims the description of the interquark potential at distances when $M_0 R \sim 1$. Analysis of the function $\eta(q)$ in Sec. III shows that $|\eta(q)|<1$. Hence, for $D=4$ we infer from Eq. (B1)

$$\sigma_1 \sim -\eta(q)/4 \sim -0.25\,. \tag{B2}$$

In view of this and Eqs. (4.18) and (4.21), we obtain

$$\alpha^0 = \sqrt{1+\sigma_1} \sim 0.9\,, \tag{B3}$$

which is in a fairly good agreement with our initial assumption $\alpha^0 \simeq 1$.

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