PHYSICAL REVIEW D VOLUME 54, NUMBER 10 15 NOVEMBER 1996

Renormalization group and spontaneous compactification of a higher-dimensional scalar field theory in curved spacetime

E. Elizalde*
Center for Advanced Studies CEAB, Consejo Superior de Investigaciones Cientificas, Camí de Santa Bàrbara, 17300 Blanes and Department ECM and IFAE, Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Spain

R. Kantowski†
Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73019

S. D. Odintsov‡
Tomsk Pedagogical University, 634041 Tomsk, Russia and Department ECM, Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Spain

(Received 25 January 1996)

The renormalization group (RG) is used to study the asymptotically free ϕ_6^3 theory in curved spacetime. Several forms of the RG equations for the effective potential are formulated. By solving these equations we obtain the one-loop effective potential as well as its explicit forms in the case of strong gravitational fields and strong scalar fields. Using ζ-function techniques, the one-loop vacuum energy and corresponding RG-improved vacuum energy are found for the Kaluza-Klein backgrounds $R^4\times S^1\times S^1$ and $R^4\times S^2$. They are given in terms of exponentially convergent series, appropriate for numerical calculations. A study of these vacuum energies as a function of compactification lengths and other couplings shows that spontaneous compactification can be qualitatively different when the RG-improved energy is used. [S0556-2821(96)01420-8]

PACS number(s): 04.62.+v, 11.10.Gh

I. INTRODUCTION

The renormalization group (RG) has long been used to ''improve'' loop corrections in perturbative quantum field theory. Gell-Mann and Low [1] first used it to study the asymptotic behavior of Green's functions and in the classic work of Coleman and Weinberg [2] the RG equation was used to improve the effective potential and to study spontaneous symmetry breaking. This is just one of the many different applications that the RG has had in quantum field theory. Recently, it has been employed to put lower limits on the Higgs boson mass of the standard model [3]. In this paper we put it to use in a renormalizable Kaluza-Klein model, arguing that RG improvements are necessary if stability of the internal dimensions are to be correctly predicted. We will also develop the RG technique in order to study the scalar effective potential in this model. Note that, as it usually happens, in spite of the fact that the Kaluza-Klein model chosen here will be renormalizable, the resulting compactified four-dimensional (4D) model will not be such, due to the presence of the infinite tower of massive Kaluza-Klein modes. Our purpose here will be to study the consequences of the renormalizability of the higher-dimensional theory, for instance, concerning the spontaneous compactification pattern.

The model that will be studied here (denoted by ϕ_6^3) is a renormalizable higher-dimensional ϕ^3 scalar field theory defined on a six-dimensional curved spacetime. The interest of this theory is given by the fact that it provides a very useful toy model for the study of string field theory [4], where also higher-dimensional interactions of type ϕ^3 are known to appear.

From another side, such a model is the simplest example of a renormalizable Kaluza-Klein theory. In particular, being still in six dimensions one could also consider more complicated higher-derivative models of the following sort:

$$L\simeq G^3+G\Box G+RG^2, \tag{1}$$

where $G\equiv G^a_{\mu\nu}$ is the field strength corresponding to the six-dimensional gauge field. When trying to understand the question of whether renormalizable Kaluza-Klein theories can lead to consequences that are different to some extent from the ones coming from nonrenormalizable theories, it is natural to start from the simplest model of this kind.

We start in Sec. II using the RG equations to arrive at the one-loop effective potential starting from a classical ϕ^3 scalar field theory on a six-dimensional curved space. This is a renormalizable theory which is coupled to the curvature tensor and its square. We additionally use the RG equations to find the asymptotic behavior of the effective potential when either the gravitational field is strong or when the scalar field is intense. In Sec. III we give the one-loop vacuum energies for this scalar field on backgrounds $R^4\times S^1\times S^1$ and $R^4\times S^2$. We then compute RG improvements to these energies. We conclude that qualitative changes have occurred, i.e., minima have disappeared from the vacuum energy and that Kaluza-Klein stability will be correspondingly affected. In the conclusions we also mention other possible applications of the RG techniques in the context of Kaluza-Klein theories, as higher-derivative theories and renormalizable theories in the ''modern sense.''

*Electronic address: eli@ecm.ub.es
†Electronic address: kantowski@phyast.nhn.uoknor.edu
‡Electronic address: sergei@ecm.ub.es

II. A RENORMALIZABLE SELF-INTERACTING SCALAR THEORY IN $D=6$ CURVED SPACETIME

As an example of a renormalizable theory in higher-dimensional curved spacetime, we consider the action in $D=6$

$$L=L_m+L_{\rm ext},$$

$$L_m=-\frac{1}{2}\phi\Box\phi+\frac{1}{2}M^2\phi^2+\frac{1}{3!}g\phi^3+h\phi+\frac{1}{2}\xi R\phi^2+\eta_1 R\phi + \eta_3 R^2\phi+\eta_4 R_{\mu\nu}R^{\mu\nu}\phi+\eta_5 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\phi, \quad (2)$$

$$L_{\rm ext}=-(\Lambda+kR+\alpha_1 R^2_{\mu\nu\alpha\beta}+\alpha_2 R^2_{\mu\nu}+\alpha_3 R^2+\alpha_4 R^3 + \alpha_5 R R^2_{\mu\nu}+\alpha_6 R R^2_{\mu\nu\alpha\beta}+\alpha_7 R_{\mu\nu}R^\mu_\sigma R^{\nu\sigma} + \alpha_8 R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}+\alpha_9 R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^\nu_{\lambda\rho\sigma} + \alpha_{10}R_{\mu\nu\rho\sigma}R^{\mu\nu}_{\lambda\tau}R^{\rho\sigma\lambda\tau}).$$

Here, L_m and $L_{\rm ext}$ are the Lagrangians of matter and external fields, respectively, and ϕ is a scalar. The Lagrangian (2) represents the generalization to curved space of a renormalizable ϕ_6^3 theory [5]. Such a theory in curved spacetime was considered a few years ago in Refs. [6–8]. Here, the notation of Ref. [7] will be adopted. In that reference, a one-loop analysis was carried out. The form of $L_{\rm ext}$ in Eq. (2), as well as of the nonminimal gravitational terms in L_m, are such as to make the theory multiplicatively renormalizable in curved spacetime. We will consider only spacetimes of constant curvature, excluding terms of the form $\phi\Box R$, etc., from the Lagrangian (2). Finally, $\lambda_i=\{M^2,g,h,\xi,\dots,\alpha_{10}\}$ are all coupling constants whose dimensionality is clear from the form of the Lagrangian (2).

One-loop divergences of the model (2) are found in Ref. [7]. They yield the following running coupling constants (we give here their explicit expressions in the massless sector only):

$$g^2(t)=g^2B^{-1}(t),\quad B(t)=1+\frac{3g^2t}{2(4\pi)^3},$$

$$\xi(t)=\frac{1}{5}+\left(\xi-\frac{1}{5}\right)B^{-5/9}(t),$$

$$\eta_1(t)=\eta_1 B^{1/18}(t), \quad (3)$$

$$\eta_3(t)=B^{1/18}(t)\left[\eta_3-\frac{1}{1200g}[B^{4/9}(t)-1]+\frac{1}{5g}\left(\xi-\frac{1}{5}\right)\times[B^{-1/9}(t)-1]+\frac{1}{2g}\left(\xi-\frac{1}{5}\right)^2[B^{-2/3}(t)-1]\right],$$

$$\eta_{4,5}(t)=B^{1/18}(t)\left[\eta_{4,5}\pm\frac{1}{120g}[B^{4/9}(t)-1]\right],$$

$$h(t)=hB^{1/18}(t).$$

It is clear from expression (3) that the theory is asymptotically free at high energies $[g^2(t)\to 0]$, and that it is asymptotically conformal invariant in the matter sector (see [9] for a review). From the complete set of one-loop divergences, given explicitly in Ref. [7], there are no problems in writing down all running coupling constants, including $M\neq 0$. To save space we have listed only those needed in this section.

Working with the massless version of the theory (2) we use Eq. (3) first to find the effective potential at one loop and second to find RG-improved asymptotic forms of this potential. We start by writing the effective action of this theory as

$$\Gamma=\Gamma|_{\phi=0}+\int d^6x\sqrt{g}V+\cdots, \quad (4)$$

where the first term is the vacuum energy and the second is the effective potential. Terms that have not been explicitly included provide nonconstant ϕ contributions to Γ. The multiplicative renormalizability of the theory guarantees that the effective action as well as the effective potential satisfy the RG equations

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_{\lambda_i}\frac{\partial}{\partial\lambda_i}-\gamma_\phi\phi\frac{\partial}{\partial\phi}\right)V=0, \quad (5)$$

where γ_ϕ is the γ function of the scalar field (computed here from [7]):

$$\gamma_\phi=\frac{g^2}{12(4\pi)^3}. \quad (6)$$

In order to find the effective potential as an expansion over curvature invariants, we will write the classical potential as [its form is clear from Eq. (2)]

$$V^{(0)}=\sum_i V_i^{(0)},\quad V_i^{(0)}=a_i\lambda_i P_i\varphi^{k_i}, \quad (7)$$

where the a_i are numerical multipliers, $k_i\geqslant 1$ are integers, and the P_i are curvature invariants. Applying the method described in Ref. [10] (see also [8]), we can solve the RG equations (5) for a potential of the form (7). Restricting ourselves to one loop and using the tree-level potential (7) as boundary condition, we find (we skip technical details)

$$V=\frac{1}{6}g\phi^3-\frac{g^3\phi^3}{12(4\pi)^3}\left(\ln\frac{\phi^2}{\mu^2}-\frac{11}{3}\right)+h\phi+\frac{\xi}{2}R\phi^2 - \frac{g^2}{4(4\pi)^3}\left(\xi-\frac{1}{6}\right)R\phi^2\left(\ln\frac{\phi^2}{\mu^2}-3\right)+\eta_1R\phi+\eta_3R^2\phi + \eta_4R^2_{\mu\nu}\phi+\eta_5R^2_{\mu\nu\alpha\beta}\phi-\frac{g\phi}{(4\pi)^3}\left(\ln\frac{\phi^2}{\mu^2}-2\right) \times\left[\frac{1}{4}\left(\xi-\frac{1}{6}\right)^2R^2-\frac{1}{360}R^2_{\mu\nu}+\frac{1}{360}R^2_{\mu\nu\alpha\beta}\right]. \quad (8)$$

This is the one-loop effective potential up to terms quadratic in the curvature. It is clear that this potential is not bounded from below (this is the well-known instability of the φ_6^3 theory). This kind of potential is useful for studying six-dimensional cosmology coupled to a φ_6^3 quantum field.

Another application of the RG equations to the effective potential V is to study the asymptotics of the effective potential in curved spacetime [11,8]. The homogeneity condition of V has the form

$$V(e^{2t}\phi,e^{d_{\lambda_i}t}\lambda_i,e^{-2t}g_{\alpha\beta},e^t\mu)=e^{6t}V(\phi,\lambda_i,g_{\alpha\beta},\mu), \tag{9}$$

where $t=$ const and d_{λ_i} is the dimension of λ_i. Relation (9) leads to the equations

$$(\partial_t+\mu\partial_\mu+d_{\lambda_i}\lambda_i\partial_{\lambda_i}+2\phi\partial_\phi-6)V(\phi,e^{-2t}g_{\alpha\beta},\dots)=0, \tag{10}$$

$$\left(\partial_t+\mu\partial_\mu+d_{\lambda_i}\lambda_i\partial_{\lambda_i}-2g_{\alpha\beta}\frac{\delta}{\delta g_{\alpha\beta}}-6\right)\times V(e^{2t}\phi,g_{\alpha\beta},\dots)=0, \tag{11}$$

where the parameters of the potential that are not written explicitly are not scaled. Combining Eq. (5) with Eqs. (10) and (11), we obtain

$$[\partial_t-(\beta_{\lambda_i}-d_{\lambda_i}\lambda_i)\partial_{\lambda_i}+(\gamma_\phi+2)\phi\partial_\phi-6]V(\phi,e^{-2t}g_{\alpha\beta},\dots)=0, \tag{12}$$

$$\left\{\partial_t+(1+\gamma_\phi/2)^{-1}\left[-(\beta_{\lambda_i}-d_{\lambda_i}\lambda_i)\partial_{\lambda_i}-2g_{\alpha\beta}\frac{\delta}{\delta g_{\alpha\beta}}-6\right]\right\}\times V(e^{2t}\phi,g_{\alpha\beta},\dots)=0. \tag{13}$$

The RG equations (12) and (13) describe the asymptotics of the effective potential. In particular, when $g_{\alpha\beta}\to e^{-2t}g_{\alpha\beta}$, $R^2\to e^{4t}R^2$, Eq. (12) gives the asymptotic behavior of the effective potential in a strong gravitational field. Similarly, Eq. (13) gives the behavior of V in the case of a strong scalar field. Solving Eq. (12) we get (see also [11,8])

$$V(\phi,e^{-2t}g_{\alpha\beta},\lambda_i)=e^{6t}V(\phi(t),g_{\alpha\beta},\lambda_i(t)),$$

$$\dot\lambda_i(t)=\beta_{\lambda_i}(t)-d_{\lambda_i}\lambda_i(t),\quad \lambda_i(0)=\lambda_i, \tag{14}$$

$$\dot\phi(t)=-[2+\gamma_\phi(t)]\phi(t),\quad \phi(0)=\phi.$$

Selecting the leading coupling constants from Eq. (3) and using Eq. (6), we obtain

$$V(\phi,e^{-2t}g_{\alpha\beta},\lambda_i)\sim e^{6t}\phi(t)[\eta_3(t)R^2+\eta_4(t)R^2_{\mu\nu}+\eta_5(t)R^2_{\mu\nu\alpha\beta}], \tag{15}$$

where

$$\phi(t)=\phi e^{-2t}B^{-1/18}(t). \tag{16}$$

Thus, the asymptotics of the effective potential in a strong gravitational field are defined by the nonminimal interaction of the scalar with the quadratic curvature invariants. Such approximations can be useful in studying quantum effects in the early Universe (e.g., in the Kaluza-Klein framework).

In a similar way, we can solve Eq. (13), with the result

$$V(e^{2t}\phi,g_{\alpha\beta},\lambda_i)=\exp\left[6\int_0^t dt'A(t')\right]V(\phi,g_{\alpha\beta}(t),\tilde\lambda_i(t)), \tag{17}$$

where

$$A(t)=\left[1+\frac{\gamma_\phi(t)}{2}\right]^{-1},\quad \phi(t)=\phi,$$

$$\dot g_{\alpha\beta}(t,x)=2A(t)g_{\alpha\beta}(t,x),\quad g_{\alpha\beta}(0,x)=g_{\alpha\beta}(x),$$

$$\dot{\tilde\lambda}_i(t)=A(t)[\beta_{\tilde\lambda_i}(t)-d_{\tilde\lambda_i}\tilde\lambda_i(t)]. \tag{18}$$

As we see, contrary to what happens with Eqs. (14) for the effective couplings, the multiplier $A(t)$ appears on the right-hand side (RHS) of Eqs. (18). Using arguments similar to the ones given in Ref. [12] (where the procedure to study the asymptotics of the effective potential in flat spacetime was developed), one can show that the presence of $A(t)$ does not influence the asymptotics of the effective couplings. Again, due to the fact that the theory is asymptotically free, it is natural to expect that the asymptotic behavior of the effective potential is given by the lowest order of perturbation theory, with the parameters replaced by the corresponding effective couplings.

Now, since $\phi(t)=\phi$ and the effective curvature is always small, $R(t)\sim e^{-2t}$ [see Eq. (18)], we get

$$V(e^{2t}\phi,g_{\alpha\beta},\lambda_i)=\tfrac{1}{6}e^{6t}g(t)\phi^3. \tag{19}$$

The asymptotic value of the effective potential, in the limit of strong scalar curvature, is not bounded from below. This result can be useful for the study of six-dimensional quantum cosmology near the initial singularity. We conclude this discussion of the application of RG equations to the effective potential for the curved spacetime ϕ_6^3 theory and go on to an application of the RG equations to the vacuum energy.

III. THE VACUUM ENERGY IN THE ϕ_6^3 THEORY ON A KALUZA-KLEIN SPACETIME

Starting from the works [13,14], the vacuum energy of matter and gravitational fields on spherically compactified internal spaces was calculated and the process of quantum spontaneous compactification was studied. For a review and a list of references of papers on related questions concerning Kaluza-Klein theories, see [15,9]. In particular, in [16–21] and [22] vacuum energies were evaluated for scalar fields, etc. (including gravity) defined on even-dimensional compactified spaces. In most of these studies only the divergent parts (in dimensional regularization) of the vacuum energies were evaluated.

Our goal here is to obtain the RG-improved one-loop vacuum energies corresponding to the theory (2) on two Kaluza-Klein backgrounds, namely $R^4\times S^1\times S^1$ and $R^4\times S^2$, and to investigate the process of spontaneous compactification.

A. $R^4\times S^1\times S^1$ space

At the one-loop level, the vacuum energy is given by

$$\Gamma^{(1)}=\tfrac{1}{2}\mathrm{Tr}\ln(-\Box+M^2). \tag{20}$$

The calculation can be done with the help of ζ function regularization (for an introduction, see [23]). The spectrum has the form

$$\lambda = k_4^2 + \left(\frac{2\pi n_1}{L_1}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + X, \tag{21}$$

with $X = M^2$ here, and the corresponding ''Euclideanized'' ζ function is

$$\zeta_E(s) = \sum_\lambda \lambda^{-s} = \frac{1}{\Gamma(s)} \sum_\lambda \int_0^\infty dt\, t^{s-1} e^{-\lambda t}$$
$$= \frac{1}{\Gamma(s)} \sum_{n_1 n_2} \int \frac{d^4k}{(2\pi)^4} \int_0^\infty dt\, t^{s-1} \times \exp\left\{ -\left[k^2 + \left(\frac{2\pi n_1}{L_1}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + X \right] t \right\}$$
$$= \frac{1}{4(2\pi)^{2(s+1)}(s-1)(s-2)} \times \left[\left(\frac{X}{4\pi^2}\right)^{2-s} + E\left(s-2; L_1^{-2}, 0, L_2^{-2}; \frac{X}{4\pi^2}\right)\right]. \tag{22}$$

$E(s;a,b,c;q)$ is the ζ function introduced and studied in [24]:

$$E(s;a,b,c;q) \equiv \sum_{m,n \in \mathbf{Z}}{}' (am^2 + bmn + cn^2 + q)^{-s}, \quad \mathrm{Re}(s) > 1. \tag{23}$$

In the general theory [24], one requires that $a, c > 0$, that the discriminant

$$\Delta = 4ac - b^2 = \left(\frac{2}{L_1 L_2}\right)^2 > 0, \tag{24}$$

and that $am^2 + bmn + cn^2 + q \neq 0$, for all $m, n \in \mathbf{Z}$. These conditions are all satisfied in this case. The analytic continuation [25] of this ζ function is

$$E(s;a,b,c;q) = -q^{-s} + \frac{2\pi q^{1-s}}{(s-1)\sqrt{\Delta}} + \frac{4}{\Gamma(s)} \left[\left(\frac{q}{a}\right)^{1/4} \left(\frac{\pi}{\sqrt{qa}}\right)^s \sum_{k=1}^\infty k^{s-1/2} K_{s-1/2}\left(2\pi k \sqrt{\frac{q}{a}}\right) + \sqrt{\frac{q}{a}} \left(2\pi \sqrt{\frac{a}{q\Delta}}\right)^s \sum_{k=1}^\infty k^{s-1} \right.$$
$$\times K_{s-1}\left(4\pi k \sqrt{\frac{aq}{\Delta}}\right) + \sqrt{\frac{2}{a}} (2\pi)^s \sum_{k=1}^\infty k^{s-1/2} \cos(\pi k b/a) \sum_{d|k} d^{1-2s} \left(\Delta + \frac{4aq}{d^2}\right)^{1/4 - s/2}$$
$$\left. \times K_{s-1/2}\left(\frac{\pi k}{a} \sqrt{\Delta + \frac{4aq}{d^2}} \right) \right]. \tag{25}$$

This explicit form (25) and its derivative (given below) appeared for the first time in [25]. It is remarkable that the only simple pole $(s=1)$ is so explicit in Eq. (25). This expression also has excellent convergence properties, in fact, for large q the convergence behavior of the series of Bessel functions is at least exponential. Particular values for $s = -n$, $n = 0,1,2,3, \ldots$ are

$$E(-n;a,b,c;q) = -q^n - \frac{2\pi}{n+1} \frac{q^{n+1}}{\sqrt{\Delta}}, \tag{26}$$

and

$$E(-n;a,b,c;0) = 0. \tag{27}$$

For the corresponding derivative at zero we have

$$E'(0;a,b,c;q) = -\frac{2\pi q}{\sqrt{\Delta}} + \left(1 + \frac{2\pi q}{\sqrt{\Delta}}\right) \ln q - 2\ln(1 - e^{-2\pi\sqrt{q/a}}) + 4\sqrt{\frac{q}{a}} \sum_{n=1}^\infty n^{-1} K_1\left(4n\pi \sqrt{\frac{aq}{\Delta}}\right)$$
$$+ 4\sum_{n=1}^\infty n^{-1} \cos(n\pi b/a) \sum_{d|n} d \exp\left[-\frac{\pi n}{a} \left(\Delta + \frac{4aq}{d^2}\right)^{1/2}\right], \tag{28}$$

and, in general, for $s = -n$, $n = 0,1,2,3, \ldots,$

$$E'(-n;a,b,c;q) = -\frac{2\pi q^{n+1}}{(n+1)^2\sqrt{\Delta}} + q^n \left(1 + \frac{2\pi q}{(n+1)\sqrt{\Delta}}\right) \ln q + 4\frac{(-1)^n}{n!\,\pi^n} \left[q^{n/2+1/4} a^{n/2-1/4} \right.$$
$$\times \sum_{k=1}^\infty k^{-n-1/2} K_{n+1/2}\left(2\pi k \sqrt{\frac{q}{a}}\right) + 2^{-n} \left(\frac{q}{a}\right)^{(n+1)/2} \Delta^{n/2} \sum_{k=1}^\infty k^{-n-1} K_{n+1}\left(4\pi k \sqrt{\frac{aq}{\Delta}}\right)$$
$$\left. + \frac{2^{-n+1/2}}{\sqrt{a}} \sum_{k=1}^\infty k^{-n-1/2} \cos(\pi k b/a) \sum_{d|k} d^{2n+1} \left(\Delta + \frac{4aq}{d^2}\right)^{k/2+1/4} K_{n+1/2}\left(\frac{\pi k}{a}\sqrt{\Delta + \frac{4aq}{d^2}}\right)\right]. \tag{29}$$

These are the only expressions needed for what follows. We want to evaluate the effective action $\Gamma^{(1)}/V_4$, where

$$\Gamma^{(1)}=\tfrac{1}{2}[\zeta'_E(0)+\zeta_E(0)\ln\mu^2], \tag{30}$$

and V_4 is the four volume, $V_4\equiv\int d^4x$. The result is immediate from the expressions above:

$$\begin{aligned}\frac{\Gamma^{(1)}}{V_4}=&\frac{M^6L_1L_2}{128\pi^3}\left(-\frac{11}{36}+\frac{1}{6}\ln\frac{M^2}{\mu^2}\right)+2\sqrt{\frac{2}{\pi}}\frac{M^{5/2}}{L_1^{3/2}}\\&\times\sum_{n=1}^{\infty}n^{-5/2}K_{5/2}(nML_1)+\frac{M^3L_1}{4\pi L_2^2}\\&\times\sum_{n=1}^{\infty}n^{-3}K_3(nML_1)+\frac{2\sqrt{2}\pi^2}{L_1^{3/2}}\sum_{n=1}^{\infty}n^{-5/2}\\&\times\sum_{d|n}d^5\left(\frac{4}{L_2^2}+\frac{M^2}{\pi^2d^2}\right)^{5/4}K_{5/2}\left(\pi nL_1\sqrt{\frac{4}{L_2^2}+\frac{M^2}{\pi^2d^2}}\right).\end{aligned} \tag{31}$$

Notice that the result is given in terms of a rapidly convergent series, very well suited for numerical computation. In the massless case $(M^2=0)$, we are left with the last term

$$\frac{\Gamma^{(1)}}{V_4}\Big|_{M^2=0}=\frac{16\,\pi^2}{L_1^{3/2}L_2^{5/2}}\sum_{n=1}^{\infty}n^{-5/2}\sigma_5(n)K_{5/2}\left(2\pi n\frac{L_1}{L_2}\right). \tag{32}$$

B. $R^4\times S^2$ space

In this case, for simplicity, the vacuum energy will be calculated for the massless theory only

$$\Gamma^{(1)}=\tfrac{1}{2}\mathrm{Tr}\ln(-\Box+\xi R). \tag{33}$$

The spectrum is now

$$\lambda=k_4^2-\Lambda_l^2+X, \tag{34}$$

where $X=\xi R$. For the two-sphere $R=2/r^2$ when written in terms of the sphere's radius r. For scalar fields,

$$\Lambda_l^2=-\frac{l(l+1)}{r^2},\quad l=0,1,2,\ldots \tag{35}$$

with associated multiplicities

$$D_l=2l+1. \tag{36}$$

The corresponding ζ function is

$$\begin{aligned}\zeta_E(s)&=\frac{\Gamma(s-2)}{16\pi^2\Gamma(s)}\sum_l D_l(\Lambda_l^2+X)^{2-s}\\&=\frac{r^{2(s-2)}}{16\pi^2(s-1)(s-2)}\sum_{l=0}^{\infty}(2l+1)[(l+1/2)^2\\&\quad+(Xr^2-1/4)]^{2-s}\\&=-\frac{r^{2(s-2)}}{16\pi^2(s-1)(s-2)(s-3)}\\&\quad\times\frac{\partial}{\partial c}F(s-3;c;Xr^2-1/4)|_{c=1/2},\end{aligned} \tag{37}$$

where $F(s;c;q)$ is another typical ζ function studied in full detail in [24]:

$$F(s;c;q)\equiv\sum_{n=0}^{\infty}[(n+c)^2+q]^{-s}\equiv G(s;1,c;q). \tag{38}$$

From the general asymptotic expansion of $G(s;a,c;q)$ in powers of q^{-1} (see [24]),

$$\begin{aligned}G(s;a,c;q)\equiv&\sum_{n=0}^{\infty}[a(n+c)^2+q]^{-s}\\&\sim\frac{q^{-s}}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+s)}{m!}\left(\frac{q}{a}\right)^{-m}\zeta_H(-2m,c)\\&+\sqrt{\frac{\pi}{a}}\frac{\Gamma(s-1/2)}{2\Gamma(s)}q^{1/2-s}\\&+\frac{2\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}\\&\times\sum_{n=1}^{\infty}n^{s-1/2}\cos(2\pi nc)K_{s-1/2}(2\pi n\sqrt{q/a}),\end{aligned} \tag{39}$$

we easily obtain the asymptotic expansion

$$\begin{aligned}\zeta_E(s)\sim&-\frac{r^{2(s-2)}}{16\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n(1-2^{1-2n})B_{2n}\Gamma(s+n-3)}{n!\Gamma(s)}\\&\times(Xr^2-1/4)^{3-s-n},\end{aligned} \tag{40}$$

where the B_{2n} are Bernoulli numbers. This yields immediately

$$\begin{aligned}\zeta_E(0)=&\frac{1}{16\pi^2r^4}(Xr^2-1/4)^3\left[-\frac{1}{6}+\frac{1}{24}(Xr^2-1/4)^{-1}\right.\\&\left.-\frac{7}{480}(Xr^2-1/4)^{-2}+\frac{31}{8064}(Xr^2-1/4)^{-3}\right],\end{aligned} \tag{41}$$

and

$$\begin{aligned}\zeta_E'(0)=&-\zeta_E(0)\ln\left(X-\frac{1}{4r^2}\right)+\frac{1}{16\pi^2r^4}(Xr^2-1/4)^3\\&\times\left[-\frac{11}{36}+\frac{1}{16}(Xr^2-1/4)^{-1}-\frac{7}{480}(Xr^2-1/4)^{-2}\right.\\&\left.+\sum_{n=4}^{\infty}\frac{(-1)^{n+1}(1-2^{1-2n})B_{2n}}{n(n-1)(n-2)(n-3)}(Xr^2-1/4)^{-n}\right].\end{aligned} \tag{42}$$

Finally,

$$\frac{\Gamma^{(1)}}{V_4}=\frac{1}{32\pi^2 r^4}\left(2\xi-\frac{1}{4}\right)^3\left\{\left[-\frac{11}{36}+\frac{1}{16}(2\xi-1/4)^{-1}-\frac{7}{480}(2\xi-1/4)^{-2}+\sum_{n=4}^{\infty}\frac{(-1)^{n+1}(1-2^{1-2n})B_{2n}}{n(n-1)(n-2)(n-3)}(2\xi-1/4)^{-n}\right]+\ln\left(\frac{\mu^2 r^2}{2\xi-1/4}\right)\left[-\frac{1}{6}+\frac{1}{24}(2\xi-1/4)^{-1}-\frac{7}{480}(2\xi-1/4)^{-2}+\frac{31}{8064}(2\xi-1/4)^{-3}\right]\right\}, \quad (43)$$

which is an asymptotic expansion for large ξ, valid for $\xi>1/8$. The optimal truncation of this asymptotic expansion is obtained after the $n=4$ term. The point $\xi=1/8$ has nothing to do with the conformal coupling value but instead depends on how the expansion of $\zeta_E(s)$ was done. This result ceases to be valid when $\xi\leqslant 1/8$. For the particular value $\xi=1/8$, we can go back to the definition of the ζ function and show that F reduces to an ordinary Riemann ζ function $F(s;1/2;0)=\zeta_H(2s;1/2)=(2^{2s}-1)\zeta(2s)$, and $(\partial/\partial c)\zeta_H(s;c)=-s\zeta_H(s+1;c)$, resulting in

$$\zeta_E(s)=\frac{r^{2(s-2)}(2^{2s-5}-1)}{8\pi^2(s-1)(s-2)}\zeta(2s-5), \quad \xi=1/8. \quad (44)$$

Then,

$$\frac{\Gamma^{(1)}}{V_4}=\frac{1}{2^{11}\pi^3 r^4}\left\{\frac{31}{504}\left[\frac{3}{2}+\ln(\mu^2 r^2)\right]-\frac{\ln 2}{252}-31\zeta'(-5)\right\}, \quad \xi=1/8. \quad (45)$$

For $\xi<1/8$ the expansion (43) is replaced by a convergent series stemming from the binomial expansion of $F(s;,c;q)$ in powers of q, $0\leqslant q<1$ [see Eq. (38)]. This can be easily done, as described in detail in [24,25]. One would then have expressions which cover the whole range of values of ξ. However, in order to limit discussion we will illustrate the physical argument with the help of the $\xi>1/8$ case, i.e., Eq. (43). Similar considerations would also apply to the $\xi\leqslant 1/8$ cases.

C. Spontaneous compactification

Using the vacuum energies that we have calculated above, we can now study the process of quantum spontaneous compactification on Kaluza-Klein backgrounds (see, e.g., [15]). In particular, we want to investigate consequences of using RG improvements to these energies on spontaneous compactification.

Turning back to our first example, we will now consider a ϕ_6^3 theory on a $R^4\times S^1\times S^1$ background where, for simplicity, we set $L_1=L_2=L$. The effective action which takes into account the one-loop corrections (31) is given by

$$\frac{\Gamma}{V_4}=-\Lambda L^2+\frac{\Gamma^{(1)}}{V_4}. \quad (46)$$

The conditions of spontaneous compactification are:

$$\Gamma=0, \quad \frac{\partial}{\partial L}\left(\frac{\Gamma}{V_4}\right)=0. \quad (47)$$

Note that the topology of the external dimensions actually defines the gravitational metric. The first of Eqs. (47) gives the condition for the vanishing of the effective cosmological constant. At the same time, the second of Eqs. (47) is simply the gravitational equation (or Einstein equation), which results from the variation of the effective action over the metric. This has been explicitly shown in all detail in Ref. [13].

In the case under discussion, we have $\Gamma=\Gamma(\Lambda,M^2,\mu,L)$. Having two conditions and four parameters, the expectations of finding some solution of Eq. (47) are great. We will fix M^2 and μ and consider Γ as a function of the compactification length L only. In Fig. 1 we call this effective action (i.e., Γ divided by the four volume) simply $V(L)$, and show its form explicitly for some specified values of M^2,μ, and Λ that correspond to one of those situations in which spontaneous compactification takes place for a definite value of the compactification length L. Note that in this case we are beyond the range of validity of our approximation (which is analogous to that of the Coleman-Weinberg potential [2]), because of the large logarithmic contribution. For reasonably small values of the log (where the one-loop result can be trusted), that is up to $|\ln(M^2/\mu^2)|\simeq 1$, there is no minimum.

Let us now see how this picture changes, in general, when we take into account RG effects, e.g., when we enlarge the parameter space. As the theory is multiplicatively renormalizable, the effective action satisfies the RG equation (5). This equation can be solved using the method of the characteristics, yielding the so-called RG-improved effective action (or Wilsonian effective action [26]). The corresponding RG-improved effective potential [2] has been widely discussed in

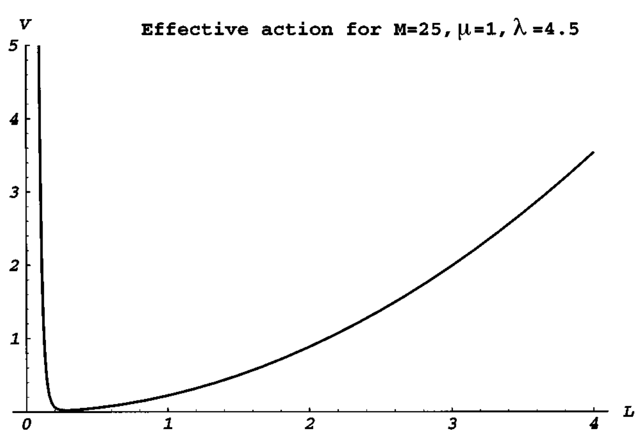

FIG. 1. The effective action $V\equiv\Gamma/V_4$, as a function of L, for the specified values of M^2, μ, and Λ (in units of 10^4). They correspond to a situation in which spontaneous compactification takes place. At the minimum, the value of the compactification length L is selected.

renormalizable theories with a Higgs boson sector, both in flat [3] and in curved spacetime [27].

The solution of the RG equation (5) (at $\phi=0$) for the effective action gives

$$\Gamma(\lambda_i, g_{\alpha\beta}, \mu) = \Gamma(\lambda_i(t), g_{\alpha\beta}, \mu e^t), \tag{48}$$

where the effective couplings are given in Eq. (3). As the boundary condition for Eq. (48), it is convenient to use the one-loop effective action (46). Then, the RG-improved effective action is given by the same expression (46) but with the following changes of variables:

$$M^2 \to M^2(t) = M^2\left(1+\frac{3g^2 t}{2(4\pi)^3}\right)^{-5/9},$$

$$\mu^2 \to \mu^2(t) = \mu^2 e^{2t},$$

$$\Lambda \to \Lambda(t) = \Lambda - \frac{M^2}{6g^2}\left[\left(1+\frac{3g^2 t}{2(4\pi)^3}\right)^{-2/3} - 1\right]. \tag{49}$$

In order to define t we may choose the standard and most natural condition of dropping out the logarithmic term (for more details and different ways of defining t, see the last reference in [3])

$$M^2(t) = \mu^2 e^{2t}. \tag{50}$$

The solution of Eq. (50) determines the value of t as a function of g, M^2, and μ. Fixing M^2 and μ, as before, we now obtain the corresponding picture for the RG-improved effective potential Γ as a function of L. This is depicted in Fig. 2 for specified values of M, μ, Λ, and g that correspond to those of Fig. 1. The value of t which is a solution of Eq. (50) is $t=3.2182$. Differences in the effective potential Γ as a function of L caused by the RG improvement can be seen by comparing Figs. 1 and 2. A virtue of the RG improvement is that the domain of validity of the approximation is greatly enlarged, i.e., it is now permissible to let the quotient M^2/μ^2 take on values as large as those shown in Fig. 1. As is now clearly observable in these figures, the RG improvement can dramatically modify the spontaneous compactification pattern. As actually happens in this case, the minimum completely disappears. At best, the minimum can turn into an inflection point and appear something like the inflection point in Fig. 3. Figure 3, however, is another non-RG-improved case to which we have applied the RG improvement (see Fig. 4) to find that not only is compactification not possible, the inflection point has disappeared. As can be easily seen these two examples are typical of what happens for the $R^4\times S^1\times S^1$ background, i.e., the RG improvement destroys any chance of spontaneous compactification.

The same type of compactification calculation can be repeated for the theory defined on the background $R^4\times S^2$. Using the same principles as above, we can write

$$\frac{\Gamma}{V_4} = \frac{1}{V_4}\int d^6x\sqrt{g}L_{\text{ext}} + \frac{\Gamma^{(1)}}{V_4}, \tag{51}$$

where $L_{\text{ext}} = L_{\text{ext}}(\Lambda, k, \alpha_1, \dots, \alpha_{10})$ [see Eq. (2)], and $\Gamma^{(1)}$ is given by Eq. (43). Expression (51) yields the one-loop effective action. The corresponding RG-improved effective action is given by the same formula (51) but with the substitutions:

$$\xi \to \xi(t) = \frac{1}{5} + \left(1-\frac{1}{5}\right)B^{-5/9}(t), \quad \alpha_i \to \alpha_i(t), \tag{52}$$

where $i=4,5,\dots,10$. The explicit form of $\alpha_i(t)$ can be easily obtained from Refs. [7,8]. One can establish the same comparison as before between the results of the spontaneous compactification process corresponding to the one loop and to the RG-improved effective actions on $R^4\times S^2$. In this case there are a total of 14 parameters to satisfy just two conditions. Setting aside exceptional situations, a great variety of possibilities occur. For this case the difference in the process of compactification introduced by the RG improvement of the effective action is found again. We omit plots similar to the previous example. Regretfully enough, in our analysis we have not been able to find a single example of a more ''constructive'' kind, that is, a model unstable at one loop, which is stabilized by the RG correction. For this second background we can only say that a most plausible conjecture is that such a stabilization cannot occur. Similar patterns of spontaneous compactification on other backgrounds have been considered elsewhere. In particular, we have also investigated backgrounds of the form $M_4\times H_2$, where H_2 is a

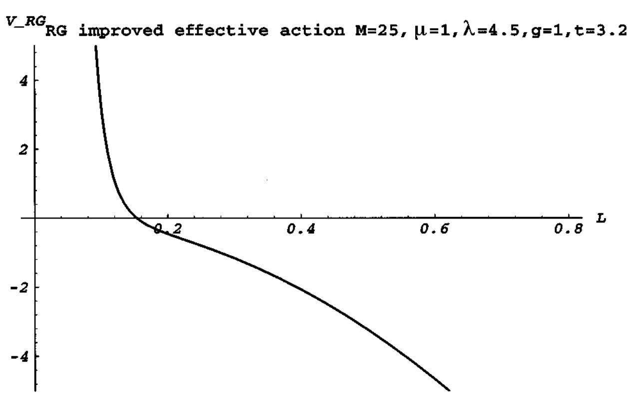

FIG. 2. Plot of the renormalization-group-improved effective action for values of M, μ, Λ, and g that correspond to those of Fig. 1. When comparing with Fig. 1, it is clearly observed that the RG improvement can modify the spontaneous compactification pattern.

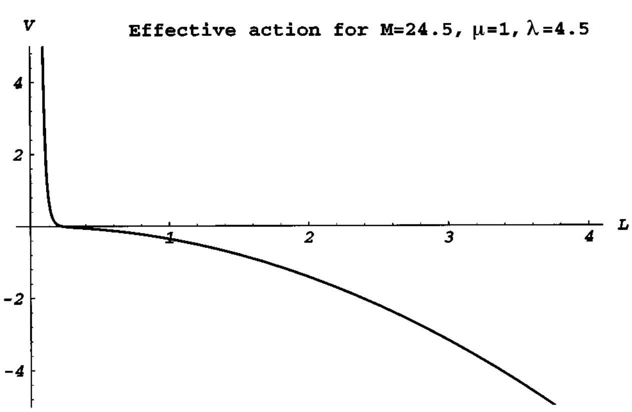

FIG. 3. Plot of the effective action for values of M, μ, Λ, and g that yield an inflection point, a situation that stays on the verge of spontaneous compactification.

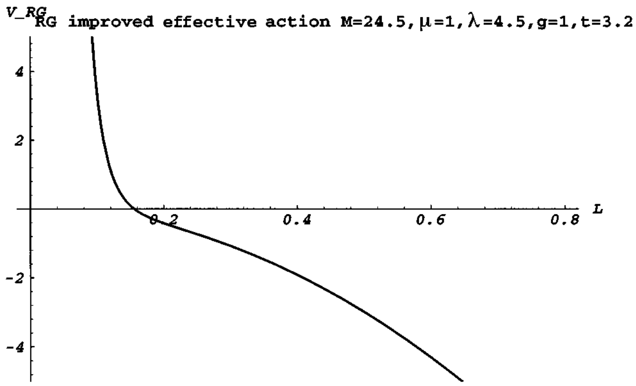

FIG. 4. Plot of the RG-improved effective action for values of M, μ, Λ, and g corresponding to Fig. 3. The inflection point has disappeared and we are driven far away from spontaneous compactification.

two-dimensional hyperbolic space (for an introduction and calculations of vacuum energies on hyperbolic spaces, see [23]). The numerical analysis of this case yields the same qualitative result, no possibility exists for a stable spontaneous compactification within the RG-improved effective potential.

IV. CONCLUSIONS

In this work we have investigated renormalization group effects in the ϕ_6^3 curved spacetime theory. Using this model as an example, the usefulness of the RG improvement in higher-dimensional theories has been demonstrated by calculating the one-loop effective potential and its asymptotics in strong fields. By using the one-loop vacuum energy and the RG improved vacuum energy on Kaluza-Klein backgrounds $R^4\times S^1\times S^1$ and $R^4\times S^2$, we have additionally shown that the RG improvement can lead to significant qualitative differences in spontaneous compactification. In both examples, the RG improvement destabilized a minimum or destroyed an inflection point that existed at one loop. Even though we could not find a point where an inflection could be stabilized by the RG improvement, we cannot exclude such a possibility in more elaborate backgrounds. However, we can, with certainty, conclude that one-loop predictions in Kaluza-Klein theories cannot be trusted.

There are not many theories in higher dimensions which are renormalizable in the standard way. But the number increases by the introduction of higher-derivative kinetic terms (probably at the expense of spoiling unitarity). For example, in $D=6$ one can consider a gauge theory with $F^3_{\mu\nu}$ terms as kinetic terms plus any other term not prohibited by dimensional arguments (they can have equal or lower dimensionality) or gauge invariance. Such a theory will be renormalizable in the same sense as R^2 gravity in $D=4$ is (see [9] for a review).

On the other hand, one can consider Kaluza-Klein theories to be renormalizable in the sense of the inclusion of an infinite number of additional terms and their corresponding counterterms (for a recent discussion, see [28]). Under these circumstances, the RG analysis can be applied again, but, of course, the RG equations will be infinite in number. Nevertheless, there are ways of truncating them in a systematic and consistent way, keeping just a finite number of terms. This can be done by considering, say just one-loop effects (in the even-dimensional case), or terms up to some particular order in the derivatives. As we have shown here, if the goal is studying spontaneous quantum compactification, the RG improved vacuum energy should be used.

ACKNOWLEDGMENTS

S.D.O. is grateful to the members of the Department ECM, Barcelona University, for warm hospitality. This work was supported by DGICYT (Spain) Project No. PB93-0035 and Grant No. SAB93-0024, by CIRIT (Generalitat de Catalunya) Grant No. GR94-8001, and by RFFR Project No. 94-020324. R.K. was supported by the U.S. Department of Energy.

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