

Massive and massless monopoles with non-Abelian magnetic charges

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We use the multimonopole moduli space as a tool for studying the properties of BPS monopoles carrying non-Abelian magnetic charges. For configurations whose total magnetic charge is purely Abelian, the moduli space for non-Abelian breaking can be obtained as a smooth limit of that for a purely Abelian breaking. As the asymptotic Higgs field is varied toward one of the special values for which the unbroken symmetry is enlarged to a non-Abelian group, some of the fundamental monopoles of unit topological charge remain massive but acquire non-Abelian magnetic charges. The BPS mass formula indicates that others should become massless in this limit. We find that these do not correspond to distinct solitons but instead manifest themselves as “non-Abelian clouds” surrounding the massive monopoles. The moduli space coordinates describing the position and U(1) phase of these massless monopoles are transformed into an equal number of non-Abelian global gauge orientation and gauge-invariant structure parameters characterizing the non-Abelian cloud. We illustrate this explicitly in a class of $Sp(2N)$ examples for which the full family of monopole solutions is known. We show in detail how the unbroken symmetries of the theory are manifested as isometries of the moduli space metric. We discuss the connection of these results to the Montonen-Olive duality conjecture, arguing in particular that the massless monopoles should be understood as the duals to the massless gauge bosons that appear as the mediators of the non-Abelian forces in the perturbative sector. [S0556-2821(96)00122-1]

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I. INTRODUCTION

Magnetic monopoles have been the object of intense interest ever since it was shown that they can arise as classical solutions in spontaneously broken gauge theories [1]. This interest is due in part to their role as predicted, although as yet undiscovered, particles that occur in all grand unified theories. Beyond their specific phenomenological implications, however, monopoles are of interest as examples of classical solitons. Like all solitons, they give rise after quantization to a type of particle that can be seen as complementary to those that arise as quanta of the elementary fields. The complementary nature of solitons and elementary quanta is particularly striking in theories with unbroken U(1) gauge symmetry, since Maxwell’s equations are invariant under a duality that interchanges magnetic and electric charges. This idea is made more concrete in the conjecture of Montonen and Olive [2] that in certain theories there might be an exact electromagnetic duality that exchanges solitons and elementary quanta, and weak and strong coupling.

In this paper we will be concerned with monopoles whose magnetic charge has a non-Abelian component, i.e., those whose long-range magnetic field transforms nontrivially under an unbroken non-Abelian subgroup of the gauge symmetry of the theory. Just as elementary quanta carrying non-Abelian electric-type charges display a much richer range of phenomena than those with purely Abelian charges, there are some curious new properties that arise with non-Abelian magnetic charges. Some of these are associated with the long-distance behavior of these monopoles. Attempts to ap-

ply a time-dependent global non-Abelian gauge rotation to obtain a dyonic object carrying both electric and magnetic non-Abelian charges are frustrated by the nonnormalizability of certain zero modes [3] and, at a deeper level, by the inability to define global non-Abelian charge in the presence of a monopole [4,5]. Also, Brandt and Neri, and Coleman [6] have shown that, regardless of the physics that governs the structure of their core, monopoles carrying more than a minimal non-Abelian magnetic charge are unstable against decay into minimally charged objects. (This result does not apply in the BPS limit.) There are other new phenomena suggested by the possibility of electromagnetic duality. In particular, one would expect the massless electrically charged gauge bosons to have magnetic counterparts. Although duality would predict that these should be massless, it is not obvious how to obtain a zero energy soliton; one of our goals will be to gain more insight into the properties of these objects.

We work with an adjoint representation Higgs field Φ in the Bogomol’nyi-Prasad-Sommerfield (BPS) limit [7] in which the scalar field potential is ignored and a nonzero Higgs expectation value is imposed as a boundary condition at spatial infinity. In this limit static monopole solutions obey the first order equations¹

$$B_i = D_i \Phi . \quad (1.1)$$

Because the Higgs field is massless in the BPS limit, it mediates a long-range force. For static monopoles, this force exactly balances their magnetic force.

We also use the moduli space approximation [8], in which the dynamics of the many degrees of freedom of the soliton

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¹Our conventions are such that $F_{ij} = \partial_i A_j - \partial_j A_i + ie[A_i, A_j] = \epsilon_{ijk} B_k$ and $D_i \Phi = \partial_i \Phi + ie[A_i, \Phi]$.

solution is effectively reduced to that of a small number of collective coordinates z_i . For BPS monopoles, the absence of static interactions implies that the collective coordinate Lagrangian consists only of a kinetic energy term, which can be written in the form

$$L = \frac{1}{2} g_{ij}(z) \dot{z}_i \dot{z}_j, \quad (1.2)$$

where g_{ij} may be interpreted as a metric on the moduli space spanned by the z_i . If the monopole solutions are known for arbitrary values of the collective coordinates, then the moduli space metric can be obtained in a straightforward manner from the zero modes about these solutions. Even if the general solution is not known, as is usually the case, it is sometimes possible to determine the moduli space metric. This was first done by Atiyah and Hitchin [9], who found the two-monopole moduli space metric for the case of $SU(2)$ broken to $U(1)$. Recently, the metric for two monopoles in a theory with an arbitrary group broken to a purely Abelian subgroup was found [10–12]. Finally, in Ref. [13] we proposed a family of metrics for the moduli spaces of a somewhat larger class of multimonopole solutions in higher rank gauge groups.²

These last results are the starting point for our present investigation. We begin in Sec. II by reviewing some of the properties of BPS monopoles. An adjoint Higgs field can break a rank r gauge group G to either $U(1)^r$ or to $K \times U(1)^{r-k}$, where K is a semisimple group of rank $k < r$. The former case, which we will refer to as maximal symmetry breaking (MSB), occurs for generic values of Φ . There are r topologically conserved charges, one for each $U(1)$ factor. Associated with these are r fundamental monopoles, each carrying a single unit of one of these topological charges; all other BPS solutions can be understood as multimonopole solutions containing appropriate numbers of the various fundamental monopoles.

The latter case, with a non-Abelian unbroken symmetry (NUS), occurs for special values of Φ . For these values some of the fundamental monopoles of the MSB case survive as massive solitons but acquire non-Abelian magnetic charge in the sense that their long-range magnetic field has nonvanishing components in K . Taken at face value, the BPS mass formulas indicate that certain other fundamental MSB monopoles (also with non-Abelian magnetic charge) become massless in the NUS limit; these are just the duals to the massless gauge bosons that were mentioned above. Their interpretation is complicated by the fact that as the massless limit is approached the core radii of the corresponding classical monopole solutions tend to infinity while at the same time the fields all tend toward their vacuum values.

In this paper we investigate the properties of these non-Abelian monopoles by following the behavior of MSB solutions as the asymptotic Higgs field is varied toward the NUS

value. For configurations whose total magnetic charge acquires a non-Abelian component when one passes from the MSB to the NUS case, one encounters various pathologies, of which the behavior of the massless monopoles described above is just one example. To avoid these difficulties, we use the approach of Refs. [5,17] and consider only combinations of monopoles whose non-Abelian charges cancel. As we shall see, for such “magnetically color-neutral” combinations the approach to the NUS case is quite smooth.

Each such combination of NUS magnetic charges is the limit of a unique set of MSB magnetic charges. Index theory methods reveal that the moduli spaces for the two cases have the same dimension. It therefore seems quite plausible that the moduli space metric for the NUS case should be simply the appropriate limit of the metric for the corresponding set of MSB charges.

In Sec. III we test this explicitly for an example with gauge group $SO(5)$, with MSB and NUS symmetry breakings to $U(1) \times U(1)$ and $SU(2) \times U(1)$, respectively. In the former case there are two fundamental monopole solutions. Because the sum of the magnetic charges of these two remains purely Abelian as one passes to the NUS case, the solutions containing two distinct fundamental monopoles are precisely the sort of color-neutral combinations that we want. For the MSB case, the metric for the corresponding eight-dimensional two-monopole moduli space is known from the results of Ref. [10]. For the NUS case, the full eight-parameter family of solutions was found some time ago [18]. We use these to calculate the NUS moduli space metric directly and verify that it is indeed the expected limit of the MSB metric.

Despite this smooth behavior of the metric, the interpretation of the coordinates on the moduli space is quite different for the cases of Abelian and non-Abelian symmetry breaking. In the MSB case the generic solution has a natural interpretation in terms of two widely separated monopoles, each of which is specified by the three spatial coordinates of its center and a single $U(1)$ phase angle. As the NUS limit is approached, one of the fundamental monopole solutions retains its nonzero mass and finite core radius. The mass of the other fundamental monopole approaches zero while, as noted above, its radius, *in the absence of any other monopoles*, tends to infinity. However, the behavior of this massless monopole is modified dramatically by the presence of a massive monopole.

This can be seen by considering an MSB solution containing two such monopoles separated by a distance r_0 that is much larger than either of their core radii. As the NUS limit is approached, the core of the would-be massless monopole expands until its radius becomes comparable to r_0 . It then begins to lose its identity as an isolated soliton and instead is manifested as a “non-Abelian cloud” of radius $\sim r_0$ surrounding the massive monopole. Within this cloud there is a Coulomb magnetic field corresponding to a combination of Abelian and non-Abelian magnetic charge, but the non-Abelian component disappears for $r \gg r_0$. In the NUS non-Abelian limit, one of the position coordinates of the massless monopole is transformed into a parameter specifying the radius of the non-Abelian cloud, while its other two position coordinates combine with its $U(1)$ phase angle to specify the global $SU(2)$ orientation of the solution.

²In a recent paper, Murray [14] has shown that these metrics coincide with those on the space of Nahm data [15] for the unitary gauge groups. More recently, Chalmers [16] has given a proof that they are the only smooth hyper-Kähler metrics that possess the right symmetry properties as well as the correct asymptotic behavior, and thus are in fact the exact moduli space metrics.

In the last part of Sec. III, we consider the semiclassical quantization of the moduli space coordinates describing this non-Abelian cloud. We find that there is a tower of states carrying both spin and electric-type $SU(2)$ charge (“isospin”), with the magnitudes of the isospin and spin being equal.

In Sec. IV we consider some more complex cases. The first of these involves a color-neutral combination of $(N+1)$ monopoles in a theory with $Sp(2N+2)$ broken to $Sp(2N) \times U(1)$. [For $N=1$ this reduces to the $SO(5)=Sp(4)$ example of Sec. III.] The $N+1$ monopoles become distinct fundamental monopoles upon maximal symmetry breaking, and so the MSB moduli space metric given in Ref. [13] is applicable to this case. With $Sp(2N) \times U(1)$ as the unbroken group, N of these monopoles become massless and coalesce in a cloud about the single massive monopole. In fact, the full family of solutions for this case can be obtained from embedding of the $SO(5)$ solutions of Sec. III. As with the $SO(5)$ case, one can verify that the moduli space metric obtained from such exact monopole solutions is identical to the NUS limit of the MSB moduli space metric. This example also illustrates very nicely how monopole coordinates are transformed into parameters describing the structure and gauge orientation of the cloud. As we will show, what used to be the relative position and $U(1)$ coordinates of the $N+1$ monopoles can be assembled into $2N$ complex (or N quaternionic) variables on which the unbroken group $Sp(2N)$ acts triholomorphically, defining a set of Killing vector fields of the algebra of $Sp(2N)$. These leave invariant a single combination of the monopole coordinates that becomes the radius of the non-Abelian cloud.

The next step is to examine solutions with two massive monopoles in the NUS limit. For either $Sp(2N+2)$ or $SU(N+2)$ broken to $SU(N) \times U(1)^2$ there are magnetically color-neutral configurations with $(N-1)$ massless and two massive monopoles, each of which individually carries a nonzero non-Abelian magnetic charge. Again they belong to the class of multimono- poles for which we have a MSB moduli space metric. We are unable to compare its NUS limit to the exact metric in this case, because the complete family of such multimono- pole solutions is unknown. Instead, we examine its symmetry properties in the NUS limit, which must include a $U(N)$ triholomorphic isometry coming from the unbroken gauge group, and find the right set of Killing vectors. As in the previous case, we can construct a single invariant from the massless monopoles coordinates that fixes the size of the non-Abelian cloud surrounding the two massive monopoles.

We cannot carry out the analysis at this level for other cases, since we know neither the general solutions nor the moduli space metric. However, as we describe in Sec. V, it is still possible to make some progress in understanding non-Abelian monopoles in other groups. From the root structure of the group, we can determine the transformation properties of the massive fundamental monopoles under the unbroken gauge group and see how they can be combined to yield configurations with no net non-Abelian magnetic charge. Each such combination requires a fixed number of massless monopoles, whose coordinates combine to give the various global gauge and cloud structure parameters. Using group theory arguments, we can in most cases determine (and in

the remaining ones bound) the number of structure parameters. In general there are more than one, suggesting that the non-Abelian cloud can have considerable structure.

One of motivations for this work was the possibility of an exact electromagnetic duality. In particular, it has been conjectured that in $N=4$ supersymmetric Yang-Mills theories there is a correspondence between electrically and magnetically charged states. While some of the magnetic states required by this duality are based straightforwardly on the fundamental monopole solutions, others must be obtained as threshold bound states; the latter can be related to normalizable harmonic forms on the moduli space. In Sec. VI we note some of the implications of our results for this conjectured duality and discuss some of the issues related to the threshold bound states.

Finally, in Sec. VII we summarize our results and make some concluding remarks. Some detailed calculations relating to the isometries of the moduli spaces studied in Sec. IV C are contained in the Appendix.

II. REVIEW OF BPS MONOPOLES

We begin by recalling the main features of the BPS monopoles in an $SU(2)$ gauge theory [7]. We fix the normalization of the magnetic charge in the unbroken $U(1)$ by writing the asymptotic magnetic field as

$$B_i^a = \frac{g \hat{r}_i}{4 \pi r^2} \frac{\Phi^a}{|\Phi|}. \quad (2.1)$$

Topological arguments then show that g must be quantized in integer multiples of $4 \pi/e$. The monopole solution carrying one unit of magnetic charge may be written as

$$\Phi^a = \hat{r}^a H(r), \quad A_i^a = \epsilon_{aim} \hat{r}^m A(r), \quad (2.2)$$

where v is the asymptotic magnitude of the Higgs field and

$$A(r) = \frac{v}{\sin e v r} - \frac{1}{e r}, \quad H(r) = v \coth e v r - \frac{1}{e r}. \quad (2.3)$$

The solutions carrying $n > 1$ units of magnetic charge can all be understood as multimono- pole solutions. The dimension of the moduli space for a given n can be determined by studying the zero modes about an arbitrary solution, i.e., the perturbations that preserve Eq. (1.1) to first order. By requiring that these perturbations satisfy the background gauge condition

$$0 = D_i \delta A_i + i e [\Phi, \delta \Phi] \equiv D_\mu \delta A_\mu, \quad (2.4)$$

we ensure that the zero mode is orthogonal to all modes obtained by gauge transformation of the original solution with gauge functions that vanish at spatial infinity.³ This

³In the second equality we have adopted a notation in which Φ is treated as the fourth component A_4 of a vector potential A_μ , with ∂_4 acting on any quantity being identically zero. We will always use Greek indices to indicate that this four-dimensional notation is being used; Roman indices should always be understood to run from 1 to 3.

leaves only a single normalizable gauge mode, corresponding to the single generator of the unbroken $U(1)$. Index theory methods show [19] that there are $4n$ linearly independent normalizable zero modes; when the monopoles are separated far away from each other, the corresponding coordinates on the moduli space having natural interpretations as the positions and $U(1)$ phase angles of n unit monopoles.

Now consider an arbitrary gauge group G of rank r . Its generators can be chosen to be k commuting operators H_i , normalized by $\text{tr } H^i H^j = \delta^{ij}$, which span the Cartan subalgebra, together with ladder operators, associated with the roots α , which obey

$$[\mathbf{H}, E_\alpha] = \alpha E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha \cdot \mathbf{H}. \quad (2.5)$$

One can choose a basis of r simple roots with the property that all other roots are linear combinations of these with integer coefficients all of the same sign. A particularly convenient basis may be chosen as follows. Let Φ_0 be the asymptotic value of Φ in some fixed direction. We choose this to lie in the Cartan subalgebra and then define a vector \mathbf{h} by

$$\Phi_0 = \mathbf{h} \cdot \mathbf{H}. \quad (2.6)$$

We then require that the simple roots all have non-negative inner products with \mathbf{h} . If the symmetry breaking is maximal, there are no roots orthogonal to \mathbf{h} and there is unique set of simple roots β_a obeying this condition. If instead there are roots orthogonal to \mathbf{h} , then the sublattice formed by such roots is the root lattice for some semisimple group K of rank $k < r$, and the unbroken gauge group is $U(1)^{r-k} \times K$. In this case we denote by γ_j the simple roots orthogonal to \mathbf{h} and write the remainder as β_a . Here the choice of simple roots is not unique, with the various possibilities being related by elements of the Weyl group of K .

We can also require that, in the direction chosen to define Φ_0 , the asymptotic magnetic field lie in the Cartan subalgebra and be of the form

$$B_i = \frac{\hat{r}_i}{4\pi r^2} \mathbf{g} \cdot \mathbf{H}. \quad (2.7)$$

Topological arguments lead to the quantization condition [20]

$$\mathbf{g} = \frac{4\pi}{e} \left[\sum_a n_a \beta_a^* + \sum_j q_j \gamma_j^* \right], \quad (2.8)$$

where

$$\alpha^* = \frac{\alpha}{\alpha^2} \quad (2.9)$$

is the dual of the root α and the n_a and q_j are non-negative integers. The n_a are the topologically conserved charges. For a given solution they are uniquely determined and gauge invariant, even though the corresponding β_a may not be. The q_j are neither gauge invariant nor conserved.

For maximal symmetry breaking there is a unique fundamental monopole solution associated with each of the r topological charges. To obtain these, we first note that any root α defines an $SU(2)$ subgroup generated by

$$t^1(\alpha) = \frac{1}{\sqrt{2\alpha^2}} (E_\alpha + E_{-\alpha}),$$

$$t^2(\alpha) = -\frac{i}{\sqrt{2\alpha^2}} (E_\alpha - E_{-\alpha}),$$

$$t^3(\alpha) = \alpha^* \cdot \mathbf{H}. \quad (2.10)$$

If $A_i^s(\mathbf{r}; v)$ and $\Phi^s(\mathbf{r}; v)$ give the $SU(2)$ solution corresponding to a Higgs expectation value v , then the fundamental monopole corresponding to the root β_a is given by [19]

$$A_i(\mathbf{r}) = \sum_{s=1}^3 A_i^s(\mathbf{r}; \mathbf{h} \cdot \beta_a) t^s(\beta_a),$$

$$\Phi(\mathbf{r}) = \sum_{s=1}^3 \Phi^s(\mathbf{r}; \mathbf{h} \cdot \beta_a) t^s(\beta_a) + (\mathbf{h} - \mathbf{h} \cdot \beta_a^* \beta_a) \cdot \mathbf{H}. \quad (2.11)$$

It carries topological charges

$$n_b = \delta_{ab} \quad (2.12)$$

and has mass

$$m_a = \frac{4\pi}{e} \mathbf{h} \cdot \beta_a^*. \quad (2.13)$$

All other BPS solutions can be understood as multimonopole solutions containing $N = \sum n_a$ fundamental monopoles. These include both solutions, containing many widely separated fundamental monopoles, that are obviously composite and spherically symmetric solutions whose compositeness is revealed only by analysis of their zero modes. The latter solutions are obtained by replacing β_a in Eq. (2.11) by any composite root α ; their topological charges are equal to the coefficients in the expansion

$$\alpha^* = \sum_a n_a \beta_a^*. \quad (2.14)$$

The moduli space for these multimonopole solutions has $4N$ dimensions, corresponding to three position variables and a single $U(1)$ phase for each of the component fundamental monopoles. The full moduli space and its metric are known for $N=2$. For $N>2$ the metric for the case where all the component fundamental monopoles are all distinct was given in Ref. [13]; for all other cases, the explicit form of the metric is known only for the region of moduli space corresponding to widely separated fundamental monopoles.

Matters are somewhat more complicated when the unbroken gauge group is non-Abelian [21]. If $\beta_a \cdot \mathbf{H}$ commutes with the generators of K (i.e., if β_a is not linked in the Dynkin diagram to one of the γ_j), the construction described above yields a unique fundamental monopole carrying a single unit of topological charge. The identification of the fundamental solutions for the remaining β_a is less straightforward. The Weyl group of K takes each of these β_a to one or more other roots, any of which could have been chosen as a simple root instead of β_a . Using any of these in the embedding construction leads to a solution, carrying a single

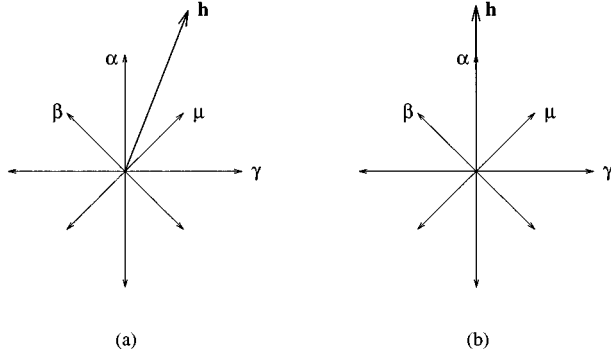


FIG. 1. The unitary gauge Higgs expectation value \mathbf{h} in the root space of $SO(5)$. The symmetry is maximally broken for (a) and only partially for (b).

unit of topological charge, that is simply a global gauge rotation of the original solution. In addition, it is sometimes possible to have a root α that is not related to β_a by a Weyl reflection but that nevertheless gives an expansion

$$\alpha^* = \beta_a^* + \sum_j q_j \gamma_j^*. \quad (2.15)$$

Insertion of such a root into Eq. (2.11) yields a solution that is gauge inequivalent to the solution based on β_a , yet still carries unit topological charge.⁴ As we will see illustrated in the next section, there is a continuous family of gauge-inequivalent solutions with unit topological charge that interpolate between the α - and β_a -embedding solutions.

If the long-range magnetic field has a non-Abelian component (i.e., if $\mathbf{g} \cdot \boldsymbol{\gamma}_j \neq 0$), the index theory methods used to count zero modes in Refs. [19] and [21] fail for technical reasons related to the slow falloff of the non-Abelian field at large distance. These difficulties do not arise if $\mathbf{g} \cdot \boldsymbol{\gamma}_j = 0$, in which case the number of normalizable zero modes is

$$p = 4 \left[\sum_a n_a + \sum_j q_j \right]. \quad (2.16)$$

(It is possible to write p in the form $\sum c_a n_a$, but this is somewhat misleading because, as we will see, there are some zero modes that cannot be associated with any single fundamental monopole.)

III. $SO(5)$ EXAMPLE

A. Monopoles in $SO(5)$ gauge theory

Many of the issues we want to address are illustrated in a particularly simple fashion if the gauge group G is $SO(5)$, whose root lattice is shown in Fig. 1. If \mathbf{h} is oriented as shown in Fig. 1(a), there is maximal symmetry breaking, to the subgroup $U(1) \times U(1)$, while if \mathbf{h} is as in Fig. 1(b), the unbroken gauge group is $SU(2) \times U(1)$ with the $SU(2)$ de-

finied by the long root⁵ $\boldsymbol{\gamma}$. In this section we will examine the behavior as \mathbf{h} is rotated toward α (i.e., as the mass of the $\boldsymbol{\gamma}$ vector meson is taken to zero) and see to what extent the properties of the monopoles with $SU(2) \times U(1)$ symmetry breaking can be obtained as limits of the MSB case.

In the maximally broken case, with \mathbf{h} oriented as in Fig. 1(a), the simple roots are the two labeled β and $\boldsymbol{\gamma}$. The corresponding fundamental monopoles, with masses

$$m_\beta = \frac{4\pi}{e} \mathbf{h} \cdot \boldsymbol{\beta}^*, \quad m_\boldsymbol{\gamma} = \frac{4\pi}{e} \mathbf{h} \cdot \boldsymbol{\gamma}^*, \quad (3.1)$$

are obtained by $SU(2)$ embeddings as in Eq. (2.11). Their central cores have radii

$$R_\beta \sim (e \mathbf{h} \cdot \boldsymbol{\beta})^{-1}, \quad R_\boldsymbol{\gamma} \sim (e \mathbf{h} \cdot \boldsymbol{\gamma})^{-1}, \quad (3.2)$$

which are set by the masses of the corresponding electrically charged vector bosons.

The $SU(2)$ embeddings defined by α and μ give two other spherically symmetric solutions but, as discussed in Sec. II, these are actually multimonomole solutions. Because

$$\alpha^* = \beta^* + \boldsymbol{\gamma}^*, \quad \mu^* = \beta^* + 2\boldsymbol{\gamma}^*, \quad (3.3)$$

the former is a two-monopole solution that maps to a single point of an eight-dimensional moduli space, while the latter is a three-monopole configuration, with the corresponding moduli space having 12 dimensions. Note that, even though these last two solutions are composite, their cores are actually smaller than those of either of the fundamental monopoles. Essentially, this is because the vector boson mass that sets the core size depends on the local, rather than the asymptotic, value of the Higgs field.

With the non-Abelian symmetry breaking that results when \mathbf{h} is orthogonal to $\boldsymbol{\gamma}$, as in Fig. 1(b), β and $\boldsymbol{\gamma}$ can still be chosen as the simple roots. However, there is no longer a solution with \mathbf{g} parallel to $\boldsymbol{\gamma}$. Furthermore, the solutions with $e\mathbf{g}/4\pi$ equal to β^* , $\beta^* + \boldsymbol{\gamma}^*$, and $\beta^* + 2\boldsymbol{\gamma}^*$ that corresponded to one, two, and three monopoles in the MSB case are now all degenerate, with any solution with $e\mathbf{g}/4\pi = \beta^* + 2\boldsymbol{\gamma}^*$ being gauge equivalent to one with $e\mathbf{g}/4\pi = \beta^*$.

The way in which this behavior emerges from the MSB case as \mathbf{h} is rotated toward α can be rather subtle. Consider, for example, the $\boldsymbol{\gamma}$ monopole solution. This exists for all nonzero values of $m_\boldsymbol{\gamma}$, but not if $m_\boldsymbol{\gamma} = 0$. As $m_\boldsymbol{\gamma}$ decreases, the core of this monopole spreads out to increasingly large distances, while the magnitudes of the gauge fields at any fixed point in the core become ever smaller. Thus, to an observer who measures fields only within a fixed region of space, the monopole becomes effectively undetectable when $m_\boldsymbol{\gamma}$ is sufficiently small. From a more global point of view, on the other hand, the limit is not smooth. Similarly, since the moduli spaces for solutions with $e\mathbf{g}/4\pi = \beta^*$ and $e\mathbf{g}/4\pi = \beta^* + 2\boldsymbol{\gamma}^*$ have 4 and 12 dimensions, respectively, they cannot have a common limit, even though an observer

⁴Such solutions were referred to as degenerate fundamental monopoles in Ref. [21].

⁵There is an inequivalent breaking to $SU(2) \times U(1)$ where the unbroken $SU(2)$ is the subgroup defined by a short root; this case is not of interest to us here.

confined to a finite volume would not be able to distinguish between the case where m_γ was precisely zero and that where it very small, but nonzero.

Since these difficulties are associated, at least in part, with the appearance of a non-Abelian magnetic charge with its associated Coulomb field, one hope that the $m_\gamma \rightarrow 0$ limit would be smoother for the solutions with $e\mathbf{g}/4\pi = \boldsymbol{\beta}^* + \boldsymbol{\gamma}^*$, whose magnetic charge remains purely Abelian. Index theory methods can be applied to these solutions for either maximal or nonmaximal symmetry breaking, and in both cases show that the moduli space is eight dimensional. It thus seems quite plausible that the moduli space metric for the latter case might be the $m_\gamma \rightarrow 0$ limit of the moduli space for the former. To test this conjecture, we will obtain the moduli space metric for the NUS case directly from the explicit solutions that were found in Ref. [18], and then compare this with the $m_\gamma \rightarrow 0$ limit of the MSB metric that was obtained in Ref. [13].

B. Eight-parameter family of solutions

We begin by describing the solutions of Ref. [18]. We start with some notation. For any Hermitian element P of the Lie algebra we define two real vectors $\mathbf{P}_{(1)}$ and $\mathbf{P}_{(2)}$ and 2×2 matrix $P_{(3)}$ obeying $P_{(3)}^* = -\tau_2 P_{(3)} \tau_2$ by

$$P = \mathbf{P}_{(1)} \cdot \mathbf{t}(\boldsymbol{\alpha}) + \mathbf{P}_{(2)} \cdot \mathbf{t}(\boldsymbol{\gamma}) + \text{tr} P_{(3)} M, \quad (3.4)$$

where $\mathbf{t}(\boldsymbol{\alpha})$ and $\mathbf{t}(\boldsymbol{\gamma})$ are defined as in Eq. (2.10) and

$$M = \frac{i}{\sqrt{\boldsymbol{\beta}^2}} \begin{pmatrix} E_\beta & -E_{-\mu} \\ E_\mu & E_{-\beta} \end{pmatrix}. \quad (3.5)$$

Note that a 2π gauge rotation generated by any of the $t^a(\boldsymbol{\gamma})$ changes the sign of $P_{(3)}$ but leaves the other components of P invariant. The commutation relations of the generators imply that the components of $R = [P, Q]$ are

$$\begin{aligned} \mathbf{R}_{(1)} &= i\mathbf{P}_{(1)} \times \mathbf{Q}_{(1)} - \text{tr} P_{(3)}^\dagger \boldsymbol{\tau} \mathbf{Q}_{(3)}, \\ \mathbf{R}_{(2)} &= i\mathbf{P}_{(2)} \times \mathbf{Q}_{(2)} - \text{tr} P_{(3)} \boldsymbol{\tau} \mathbf{Q}_{(3)}^\dagger, \\ R_{(3)} &= \frac{1}{2} [\mathbf{P}_{(1)} \cdot \boldsymbol{\tau} \mathbf{Q}_{(3)} - \mathbf{Q}_{(3)} \boldsymbol{\tau} \cdot \mathbf{P}_{(2)} \\ &\quad - \mathbf{Q}_{(1)} \cdot \boldsymbol{\tau} P_{(3)} + P_{(3)} \boldsymbol{\tau} \cdot \mathbf{Q}_{(2)}]. \end{aligned} \quad (3.6)$$

The family of spherically symmetric solutions found in [18] can be written as

$$\begin{aligned} A_{i(1)}^a &= \epsilon_{aim} \hat{r}_m A(r), & \phi_{(1)}^a &= \hat{r}_a H(r), \\ A_{i(2)}^a &= \epsilon_{aim} \hat{r}_m G(r), & \phi_{(2)}^a &= \hat{r}_a G(r), \\ A_{i(3)} &= \tau_i F(r), & \phi_{(3)} &= -iIF(r). \end{aligned} \quad (3.7)$$

Here $A(r)$ and $H(r)$ are the SU(2) monopole functions given in Eq. (2.3), while the other two coefficient functions obey

$$0 = G' + \left(eG + \frac{2}{r} \right) G + 4eF^2, \quad (3.8)$$

$$0 = F' + \frac{e}{2} (H - 2A + G) F, \quad (3.9)$$

together with the boundary conditions $G(0) = F(\infty) = G(\infty) = 0$. There is no constraint on $F(0)$, although the gauge freedom noted below Eq. (3.5) can be used to make it positive. These equations have a one-parameter family of solutions

$$F(r) = \frac{v}{\sqrt{8} \cosh(evr/2)} L(r, a)^{1/2}, \quad G(r) = A(r) L(r, a), \quad (3.10)$$

where

$$L(r, a) = [1 + (r/a) \coth(evr/2)]^{-1} \quad (3.11)$$

and the parameter⁶ a has the dimension of length and ranges from 0 to ∞ . In these formulas $v = \mathbf{h} \cdot \boldsymbol{\alpha}$.

When $a = 0$ the monopole is invariant under the unbroken SU(2), since the doublet and triplet components of the fields, proportional to $F(r)$ and $G(r)$, vanish identically. If $a \neq 0$, these components are nonvanishing and can be thought of as constituting a ‘‘non-Abelian cloud’’ about the monopole. The effect of a on the long-range tail of $G(r)$ is particularly striking. For $1/ev \lesssim r \lesssim a$, this falls as $1/r$, thus yielding the Coulomb magnetic field appropriate to a non-Abelian magnetic charge. At larger distances, however, the falloff increases to $1/r^2$, showing that the magnetic charge is actually purely Abelian. Not surprisingly, the limit $a \rightarrow \infty$ gives a solution that is a gauge transformation of the $\boldsymbol{\beta}$ embedding of the SU(2) monopole, for which \mathbf{g} actually does have a non-Abelian component.

With the MSB case in mind, one might think of these solutions as being superpositions of a $\boldsymbol{\beta}$ monopole and a $\boldsymbol{\gamma}$ monopole. The fact that it has a finite core radius, even though Eq. (3.2) gives $R_\gamma = \infty$ in the NUS limit, can be seen as analogous to the contraction of the cores in the analogous MSB superposition that was noted below Eq. (3.3).

This one-parameter family of solutions can be extended to an eight-parameter family by the action of the symmetries of the theory. Three of the additional parameters correspond to spatial translations of the solution, while the remaining four are obtained by applying global SU(2) \times U(1) transformations generated by $\mathbf{t}(\boldsymbol{\gamma})$ and $t^3(\boldsymbol{\beta})$.

C. Zero modes

The moduli space metric can be obtained directly from the zero modes about these solutions, provided that these modes satisfy the background gauge condition $D_\mu \delta A_\mu = 0$. In order to satisfy this condition, it may be necessary to add an infinitesimal gauge transformation to the zero modes ob-

⁶This is related to the parameter $b = F(0)$ used in Ref. [18] by $eva = 16b^2/(1 - 8b^2)$.

tained by varying the parameters in the solution, so that the zero mode corresponding to a collective coordinate z will in general take the form

$$\delta_z A_\mu = \partial_z A_\mu + D_\mu \epsilon_z. \quad (3.12)$$

Once these zero modes have been found, the moduli space metric is given by

$$g_{ij} = \int d^3x \operatorname{tr}(\delta_i A_\mu \delta_j A_\mu). \quad (3.13)$$

The determination of the zero modes is simplified considerably by the fact that one zero mode can be used to generate three others. If we define

$$\psi(x) = I \delta \phi(x) + i \sigma_j \delta A_j(x), \quad (3.14)$$

then the three self-duality equations plus the background gauge condition for δA_μ are equivalent to the Dirac equation

$$\sigma_\mu D_\mu \psi = 0, \quad (3.15)$$

where $\sigma_4 \equiv i$ [22]. Right multiplication of a solution ψ by any unitary 2×2 matrix yields another solution ψ' , which can be transformed back to give a new zero mode $\delta' A_\mu$. In particular, if we have a zero mode δA_μ , then multiplication of the corresponding ψ on the right by $i \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}$ (where $\hat{\mathbf{n}}$ is a unit three-vector) yields a new Dirac solution that can be decomposed to give

$$\delta' \phi = -\hat{n}_i \delta A_i, \quad \delta' A_i = \hat{n}_i \delta \phi + \epsilon_{ijk} \hat{n}_j \delta A_k. \quad (3.16)$$

By making three orthogonal choices for $\hat{\mathbf{n}}$, we can obtain three zero modes that are orthogonal to each other and to the original mode; the four modes clearly have the same norm.

We consider first the mode corresponding to an infinitesimal change in the parameter a . Because a enters only through the function L ,

$$\partial_a A_{\mu(1)} = 0, \quad \partial_a A_{\mu(2)} = \frac{\partial_a L}{L} A_{\mu(2)}, \quad \partial_a A_{\mu(3)} = \frac{\partial_a L}{2L} A_{\mu(3)}. \quad (3.17)$$

To see whether this is already in background gauge, we must calculate

$$D_\mu \partial_a A_\mu = \partial_\mu \delta A_\mu + i e [A_\mu, \delta A_\mu]. \quad (3.18)$$

It is trivial to verify the vanishing of the singlet and triplet components of this quantity. The remaining component is

$$\begin{aligned} D_\mu \partial_a A_{\mu(3)} &= \partial_j \delta A_{j(3)} \\ &+ \frac{i e \partial_a L}{4L} [\mathbf{A}_{\mu(1)} \cdot \boldsymbol{\tau} \mathbf{A}_{\mu(3)} + A_{\mu(3)} \boldsymbol{\tau} \cdot \mathbf{A}_{\mu(2)}]. \end{aligned} \quad (3.19)$$

In the first term on the right we can interchange the spatial differentiation and the variation of a . To evaluate the second term we make use of the fact that $\partial_a L/L = 2 \partial_a F/F$. Using Eq. (3.7) we then find that

$$D_\mu \partial_a A_{\mu(3)} = \hat{\mathbf{r}} \cdot \boldsymbol{\tau} \left[\partial_a F' + \frac{e}{2} \partial_a F (H - 2A + 3G) \right] = 0, \quad (3.20)$$

where the last equality follows from the variation of Eq. (3.9) together with the relation $\partial_a G/G = 2 \partial_a F/F$. Thus, this mode satisfies the background gauge condition without the need for any additional gauge transformation, and so

$$\delta_a A_\mu = \partial_a A_\mu. \quad (3.21)$$

We can now use Eq. (3.16) to generate three other zero modes from this mode. Substitution of the expression (3.17) for $\delta_a A_\mu$ into this equation gives a mode that can be written in the form⁷

$$\delta' A_\mu = D_\mu \Lambda = \partial_\mu \Lambda + i e [A_\mu, \Lambda], \quad (3.22)$$

where the only nonzero components of Λ are

$$\Lambda_{(2)}(r) = -\hat{\mathbf{n}} \frac{\partial_a L}{eL} = -\frac{\hat{\mathbf{n}}}{e} \left[\frac{1}{a} - \frac{1}{r} + O(r^{-2}) \right]. \quad (3.23)$$

This new mode is just a global SU(2) zero mode, already in background gauge. Its relation to the gauge rotation angle is given by $e \Lambda(\infty)$; from Eq. (3.23), we see that the mode corresponding to a shift δa maps to one corresponding to an SU(2) rotation by an angle $\delta \psi = \delta a/a$.

The three translation zero modes are given by spatial derivatives of the solution combined with appropriate gauge transformations. Once these are found, Eq. (3.16) can be used to obtain the eighth, U(1), mode. We do not actually need the form of these four modes, but we will make use of the fact that they are orthogonal to each other and to the other four zero modes. This orthogonality is clearly expected on physical grounds. To verify it, we first note that the translation modes transform under spatial rotations as the components of a vector, and so must be orthogonal to the other five modes, which are rotational scalars. It then follows that the Dirac mode from which these arise is orthogonal to the Dirac mode obtained from the SU(2) and δa modes; since the U(1) mode arises from the former Dirac mode, it must be orthogonal to the latter four modes.

D. Moduli space metric

We can now proceed to determine the moduli space metric. Symmetry considerations and the properties of the BPS mass formula constrain its form considerably. The subspace corresponding to the translation modes is clearly R^3 , with a natural set of coordinates given by the location of the center of the monopole. The metric on this subspace relates the kinetic energy to the spatial velocity, and so is proportional to the monopole mass, which depends only on the magnetic charge. Hence, it must be independent not only of the position coordinates and SU(2) and U(1) parameters, but also of

⁷Showing that $\delta' A_{\mu(1)}$ and $\delta' A_{\mu(3)}$ are of this form is trivial. To verify the result for $\delta' A_{\mu(2)}$, one must make use of the identity $(\partial_a L/L)' = 2(\partial_a F/F)' = -eG(\partial_a L/L)$ which is obtained by differentiating Eq. (3.9) with respect to a .

the parameter a . Similarly, since the metric component $g_{\chi\chi}$ in the subspace spanned by the $U(1)$ phase angle contributes to the leading corrections to the dyon mass through a term of the form $Q_\chi^2/2g_{\chi\chi}$, it too must be independent of all eight parameters. The subspace spanned by the $SU(2)$ parameters must be simply the standard mapping of $SU(2)$ onto a three-sphere, with a radius that might depend on a but not on the position or $U(1)$ phase angle. Finally, the metric in the one-dimensional δa subspaces can depend at most on a .

Thus the metric on the eight-dimensional moduli space must be of the form

$$ds^2 = B d\mathbf{x}^2 + C d\chi^2 + I_1(a) da^2 + I_2(a) (\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad (3.24)$$

where B and C are constants, and the one-forms σ_j are defined by

$$\begin{aligned} \sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, \\ \sigma_3 &= d\psi + \cos\theta d\phi, \end{aligned} \quad (3.25)$$

with the $SU(2)$ Euler angles θ , ϕ , and ψ having periodicities π , 2π , and 4π , respectively.

From Eq. (3.13), we see that $I_1(a)$ is simply the norm of the δa mode; from its construction, it is obvious that the $SU(2)$ mode of Eq. (3.22) has the same norm. Hence,

$$\begin{aligned} I_1(a) &= \int d^3x \operatorname{tr} (\delta' A_\mu \delta' A_\mu) = \int d^3x \operatorname{tr} (D_\mu \Lambda)^2 \\ &= \int d^3x \partial_j [\operatorname{tr} (\Lambda D_j \Lambda)] = \frac{4\pi\kappa}{e^2 a}, \end{aligned} \quad (3.26)$$

with $\kappa \equiv \operatorname{tr} t^3(\boldsymbol{\gamma}) t^3(\boldsymbol{\gamma}) = 1/\boldsymbol{\gamma}^2$. In the second equality we have used the fact that $\delta' A_\mu$ obeys the background gauge condition, while in the last we have used Eq. (3.23). To obtain $I_2(a)$ we need only multiply this by the square of the factor $\delta a/\delta\psi = a$ that followed from $\Lambda(\infty)$. Finally, B and C can be related to the monopole mass $M \equiv m_\beta$ with the aid of the BPS dyon mass formula. We thus find that the moduli space metric is

$$ds^2 = M d\mathbf{x}^2 + \frac{16\pi^2}{Me^4} d\chi^2 + \frac{4\pi\kappa}{e^2} \left[\frac{da^2}{a} + a(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right]. \quad (3.27)$$

To put this in a more standard form, we define $\rho = 2\sqrt{a}$ and obtain

$$ds^2 = M d\mathbf{x}^2 + \frac{16\pi^2}{Me^4} d\chi^2 + \frac{4\pi\kappa}{e^2} \left[d\rho^2 + \frac{\rho^2}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right]. \quad (3.28)$$

The quantity in square brackets is just the metric for R^4 written in polar coordinates, with the unfamiliar factor of $1/4$ arising from the normalization of the σ_j , and so the moduli space is

$$\mathcal{M} = R^3 \times S^1 \times R^4, \quad (3.29)$$

with the natural flat metric. (The second factor is S^1 , rather than R^1 , because of the periodicity of χ .)

We want to compare this with the metric for the moduli space of solutions containing one $\boldsymbol{\beta}$ and one $\boldsymbol{\gamma}$ monopole in the MSB case. In Ref. [13], it was shown that this space is of the form

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{Z}. \quad (3.30)$$

Here \mathcal{M}_0 is the Taub-NUT (Newman-Unti-Tamburino) space with metric

$$\begin{aligned} \mathcal{G}_{\mathcal{M}_0} &= \left(\mu + \frac{g^2 \lambda}{8\pi r} \right) [dr^2 + r^2 \sigma_1^2 + r^2 \sigma_2^2] \\ &+ \left(\frac{g^2 \lambda}{8\pi} \right)^2 \left(\mu + \frac{g^2 \lambda}{8\pi r} \right)^{-1} \sigma_3^2, \end{aligned} \quad (3.31)$$

with the reduced mass $\mu = m_\beta m_\gamma / (m_\beta + m_\gamma)$ and the magnetic coupling $g = 4\pi/e$. The constant λ encodes the strength of coupling between the two monopoles,

$$\lambda = -2 \boldsymbol{\gamma}^* \cdot \boldsymbol{\beta}^* = 2\kappa, \quad (3.32)$$

where the second equality follows from the fact that $\boldsymbol{\gamma}$ is a long simple root of the non-simply-laced $SO(5)$ algebra. The division by Z denotes the fact that there is an identification of points

$$(\chi, \psi) = \left(\chi + 2\pi, \psi + \frac{4m_\gamma}{m_\beta + m_\gamma} \pi \right). \quad (3.33)$$

Using Eq. (3.32) and the relation between g and e , we see that, as μ and m_γ tend to zero, $\mathcal{G}_{\mathcal{M}_0}$ approaches the metric for the relative moduli space that we found for the $SU(2) \times U(1)$ breaking, provided that we identify the radial coordinate r with the cloud size parameter a . Furthermore, in this limit the identification (3.33) reduces to $(\chi, \psi) = (\chi + 2\pi, \psi)$, and so the division by Z acts only on the R^1 factor, allowing us to make the replacement $R^1/Z = S^1$. Thus the moduli space metric for the NUS case is indeed the expected limit of that for the MSB case.

Although the metric varies smoothly as one case goes over into the other, there is a curious transformation in the meaning of the moduli space coordinates, specifically those on the four-dimensional subspace that remains after the center-of-mass coordinates and overall $U(1)$ phase have been factored out. With maximal symmetry breaking these coordinates are the distance r between the $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ monopoles, the angles θ and ϕ that specify the direction from one monopole to the other, and the relative $U(1)$ phase angle ψ . As μ tends toward 0 and the $\boldsymbol{\gamma}$ monopole ceases to be a distinct soliton, the monopole separation r becomes instead a measure of the size of the non-Abelian cloud, while the directional angles θ and ϕ combine with ψ to give the coordinates in the internal symmetry space.

E. Quantum mechanics of the moduli space coordinates

In the moduli space approximation, one assumes that at sufficiently low energy the classical dynamics of the monopoles is mimicked by the free motion of a point particle on the moduli space. Quantizing this motion should then give the low energy quantum mechanics of the monopoles. This reduces the quantum mechanics of interacting monopoles to a nonlinear σ model with the moduli space as the target manifold. (When there are fermionic zero modes present, one must modify the σ model to include fermionic coordinates, but here we want to confine our attention to the purely bosonic part.)

When the symmetry breaking is maximal, all bosonic coordinates on the moduli space have a clear physical interpretation as either positions or U(1) phase angles of individual monopoles. The periodicity of the latter leads to the quantization of the dyonic charges. On the NUS moduli space of the SO(5) solution found above, the center-of-mass variables still have this interpretation. Since the corresponding portion of the moduli space is a flat $R^3 \times S^1$, a natural basis of energy eigenstates is given by plane waves on R^3 with a periodic dependence on the ‘‘internal’’ S^1 ; these describe a freely propagating dyon with quantized electric U(1) charge.

The relative part of this moduli space is a flat R^4 , whose coordinates may be taken as the cloud size parameter a together with SU(2) gauge collective coordinates that span the transverse three-spheres. R^4 admits an SO(4) = SU(2) \times SU(2) isometry. Let us call the respective SU(2) generators $iL^{(a)}$ and $iK^{(a)}$, $a=1,2,3$. The wave function is then decomposed as

$$\Psi_{\mathcal{M}_0} = \sum A_{jkl}^E f_E^{(j)}(a) \mathcal{D}_{kl}^{(j)}(\theta, \phi, \psi), \quad (3.34)$$

where the $\mathcal{D}_{kl}^{(j)}$ are the three-dimensional spherical harmonics that satisfy

$$\begin{aligned} -L^{(a)}L^{(a)} \mathcal{D}_{kl}^{(j)} &= -K^{(a)}K^{(a)} \mathcal{D}_{kl}^{(j)} = j(j+1) \mathcal{D}_{kl}^{(j)}, \\ iL^{(3)}\mathcal{D}_{kl}^{(j)} &= l\mathcal{D}_{kl}^{(j)}, \\ iK^{(3)}\mathcal{D}_{kl}^{(j)} &= k\mathcal{D}_{kl}^{(j)}, \end{aligned} \quad (3.35)$$

and $f_E^{(j)}(a)$ solves the eigenvalue equation

$$-\frac{1}{a} \frac{d}{da} a^2 \frac{d}{da} f_E^{(j)} + \frac{j(j+1)}{a} f_E^{(j)} = E f_E^{(j)}. \quad (3.36)$$

As usual with representations of an SU(2) group, the eigenvalues l and k are either integers or half-integers and are bounded by $-j$ and j .

We will see in the next section that one triplet of generators, $K^{(a)}$, induces SU(2) global gauge transformations, and so the eigenvalue $j(j+1)$ encodes the electric SU(2) (isospin) charge of the resulting state. The other triplet $L^{(a)}$ is nothing but the angular momentum in the center-of-mass frame. Hence there is a tower of (non-BPS) states carrying both spin and isospin; the fact that the eigenvalues of $L^{(a)}L^{(a)}$ are identical to those of $K^{(a)}K^{(a)}$ implies that the *spin of the chromodynamic state is identical to its isospin*.

This identity can be understood by considering the MSB case first. Both monopoles are then massive and the angular momentum of the system is the sum of the orbital angular momentum and an anomalous contribution, proportional to the relative electric U(1) charge q , of the form $q\hat{\mathbf{r}}$. Because these two contributions are orthogonal, $|q|$ gives a lower bound on the magnitude of the total angular momentum that is saturated when the orbital part vanishes. As the NUS limit is approached, the relative U(1) is promoted to an SU(2), and so $|q|$ becomes the isospin. At the same time, one of the monopoles becomes massless and is manifested as a spherically symmetric cloud about the other, and so the ‘‘orbital’’ angular momentum disappears. The equality of the isospin $|q|$ and the spin j then follows.

It is worth noting that this identity should hold beyond the BPS limit. Introducing a mass term for the Higgs scalar would lift the degeneracy along the a direction, and so we would expect to find a family of SO(5) solutions similar to the above BPS solution but with a definite size for the non-Abelian cloud. Because of the unbroken SU(2), the non-Abelian gauge zero modes would still span a three-sphere in the appropriate moduli space and so should lead after quantization to a tower of chromodyons with the same eigenvalues for spin and isospin as before.

The quantization of the last collective coordinate a is less transparent. Solving Eq. (3.36) for the ground state ($E=0$) radial wave function $f_0^{(0)}$, for instance, we find a unique solution that is regular at the origin,

$$\Psi_{\mathcal{M}_0}(a) = \text{const}, \quad (3.37)$$

which is just the non-normalizable, zero-momentum plane wave on the R^4 with radial distance ρ . In terms of the three-dimensional monopole separation or cloud size parameter a , however, we have a nontrivial probability distribution

$$|\Psi_{\mathcal{M}_0}|^2 \rho^3 d\rho \sim \frac{1}{a} (a^2 da). \quad (3.38)$$

The proper physical interpretation of this result is just one of the puzzles related to these states that we hope to investigate in the future.

IV. SYMMETRY OF THE MODULI SPACE

In the previous section we showed in an SO(5) example that the NUS moduli space for a family of configurations carrying no net non-Abelian magnetic charge could be obtained as a limit of the known two-monopole MSB moduli space. More generally, the metric for the MSB moduli space was given in Ref. [10] for all cases in which the monopoles are all fundamental and distinct.

Before presenting this metric, we need some notation. When a simple gauge group is maximally broken to its Cartan subgroup, the fundamental monopoles are in one-to-one correspondence with the r simple roots β_a ($a=1, \dots, r$) of the original gauge group. A pair of such monopoles interact with each other if and only if $\beta_a \cdot \beta_b \neq 0$. In the Dynkin diagram (see Fig. 2) such pairs are indicated by linked circles. In any simple gauge group of rank r , there are precisely $r-1$ such links. We will label these links by an index A , and denote by \mathbf{r}_A the relative position vector between the pair of fundamental monopoles connected by the A th link. Likewise, ψ_A is the linear combination of internal U(1) angles

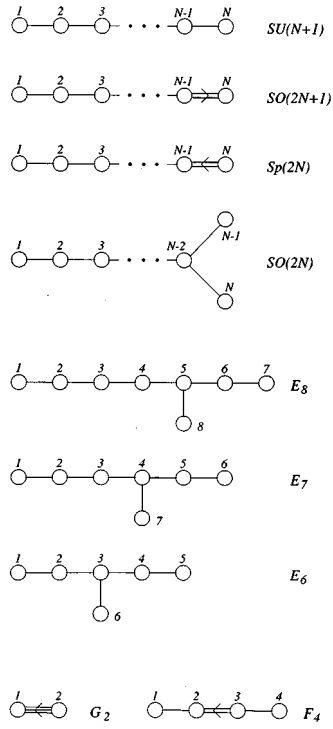


FIG. 2. Dynkin diagrams of all simple groups. For the non-simply-laced cases, the arrow points toward the short roots. We have also labeled the simple roots for later reference.

that is conjugate to the relative $U(1)$ electric charge between the two monopoles. Finally, we generalize Eq. (3.32) by defining λ_A to be -2 times the inner product of the duals of the roots joined by the A th link.

The relative part of the moduli space metric can then be written as

$$\mathcal{G}_{\text{rel}} = \sum_{A,B} C_{AB} d\mathbf{r}_A \cdot d\mathbf{r}_B + \sum_{A,B} \left(\frac{g^2}{8\pi} \right)^2 \lambda_A \lambda_B (C^{-1})_{AB} \times (d\psi_A + \mathbf{w}_A \cdot d\mathbf{r}_A) (d\psi_B + \mathbf{w}_B \cdot d\mathbf{r}_B). \quad (4.1)$$

Here the matrix C is

$$C_{AB} = \mu_{AB} + \delta_{AB} \frac{g^2 \lambda_A}{8\pi r_A}, \quad (4.2)$$

where μ_{AB} may be interpreted as a reduced-mass matrix, and $\mathbf{w}_A(\mathbf{r}_A)$ is the vector potential due to a negative unit charged Dirac monopole at $\mathbf{r}_A = 0$.

In this section, we consider two types of configurations. The first is a direct generalization of the $SO(5)$ case, and consists of one massive and N massless monopoles in a theory with $Sp(2N+2)$ broken to $U(1) \times Sp(2N)$. Again the moduli space metric can be found by direct calculation and then compared to the result obtained by the limiting procedure. The second involves two massive and $N-1$ massless monopoles in a theory with either $Sp(2N+2)$ or $SU(N+2)$ broken to $U(1)^2 \times SU(N)$. In this case, a direct calculation of the metric is not possible, since the full family of monopole solutions is not known. Although we cannot verify with certainty that the limiting procedure yields the

correct metric, we can check its consistency by examining the symmetry of the moduli space metric. The unbroken gauge symmetry must be realized as an isometry of the moduli space that preserves the hyper-Kähler structure, and the correct metric must exhibit such properties. For both classes of theories, we give explicit forms of the corresponding triholomorphic Killing vector fields.

A. Unbroken $Sp(2N)$

The simplest class of examples arises when the gauge group $Sp(2N+2)$ is broken to $U(1) \times Sp(2N)$; the $SO(5) = Sp(4) \rightarrow U(1) \times SU(2)$ example of the last section is a special case of this. We write the simple roots as β_1 and γ_j ($j=2, \dots, N+1$), with the indices corresponding to the numbering of roots in Fig. 2. The sum

$$\frac{e\mathbf{g}}{4\pi} = \beta_1^* + \sum_{j=2}^{N+1} \gamma_j^* \quad (4.3)$$

is orthogonal to the γ_j 's that span the root lattice of the unbroken $Sp(2N)$, and gives the magnetic charge of a configuration containing a single massive β_1 fundamental monopole surrounded by a cloud of massless monopoles that cancel the long-range non-Abelian field.

As mentioned above, this can be regarded as a generalization of the $SO(5)$ example of the previous section. In fact, we can identify an $SO(5)$ subgroup of $Sp(2N+2)$, generated by the pair $\boldsymbol{\gamma} = \gamma_{N+1}$ and $\boldsymbol{\beta} = \beta_1 + \sum_{j=2}^N \gamma_j$, in which the $SO(5)$ solutions of the previous section can be embedded. This embedding makes the Higgs expectation value \mathbf{h} proportional to $\boldsymbol{\beta}^* + \boldsymbol{\gamma}^* = \beta_1^* + \sum_{j=2}^{N+1} \gamma_j^*$ which is just what is needed to ensure that the unbroken group is $Sp(2N) \times U(1)$. Note that, even though the form of the solution remains intact, the number of massless monopoles associated with this embedded solution is now N rather than 1.

Further solutions can be obtained by gauge transforming such an embedded solution by elements of the unbroken $Sp(2N)$, but not all generators of $Sp(2N)$ transform it non-trivially. A generic embedded solution is left invariant by $Sp(2N-2)$, and this tells us that there must be at least $\dim[Sp(2N)/Sp(2N-2)] = 4N-1$ global gauge zero modes. Since we already know that the $SO(5)$ solution contains one parameter that fixes the size of the non-Abelian cloud, the general $Sp(2N+2)$ solution must admit at least one such parameter. Together, these account for all $4N$ coordinates of the relative moduli space. Let us now proceed to determine the metric of this space.

Consider a point on the moduli space corresponding to a generic $SO(5)$ embedded solution. Since the geometry of the gauge orbit can depend only on the parameter a , evaluating the metric at such a point determines the metric everywhere. Of the $4N-1$ gauge generators that act on this point non-trivially, three arise from the simple embedding and form an $SU(2)$, generated by $\mathbf{t}(\gamma_{N+1})$, which keep the solution within the $SO(5)$ subgroup. The other $4N-4$ gauge zero modes about such a point are generated by the ladder operators associated with the $2N-2$ positive roots $\boldsymbol{\nu}_j = \gamma_j + \gamma_{j+1} + \dots + \gamma_N$ and $\boldsymbol{\mu}_j = \gamma_j + \gamma_{j+1} + \dots + \gamma_{N+1}$ with $j=2, \dots, N$. The associated zero modes $\delta A_\mu = D_\mu \Lambda$ satisfy

$$D_\mu D^\mu \Lambda = 0. \quad (4.4)$$

The solutions of this equation are found to be of the form

$$\Lambda = \epsilon(r)T, \quad (4.5)$$

where T is any linear combination of the $4N-4$ ladder operators above, appropriately normalized, and the radial function ϵ satisfies

$$\frac{d\epsilon}{dr} + \frac{1}{2}G\epsilon = 0, \quad (4.6)$$

where $G(r)$ is given by Eq. (3.10) and $\epsilon(\infty) = 1/e$.

For generic values of a , the total gauge orbit must be topologically given by $\text{Sp}(2N)/\text{Sp}(2N-2) = S^{4N-1}$, possibly up to a division by a discrete group. Together with the fact that the last $4N-4$ gauge zero modes do not involve any of the $\text{SU}(2)$ generators $t^a(\gamma_{N+1})$, this allows us to decompose the metric in the form

$$\mathcal{G}_{\text{rel}} = \left[I_1(a)da^2 + I_2(a)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \sum_{s,t=4}^{4N-1} \tilde{I}_3^{st}(a)\sigma_s\sigma_t \right], \quad (4.7)$$

where

$$I_1(a) = \frac{4\pi\kappa}{e^2} \frac{1}{a}, \quad I_2(a) = \frac{4\pi\kappa}{e^2} a \quad (4.8)$$

were obtained in the previous section and $\{\sigma_s/2, s=1, \dots, 4N-1\}$ is an orthonormal frame on a unit sphere S^{4N-1} .

Further, the functional form of the gauge zero modes in Eq. (4.5) is independent of the generator T , and so we may choose $4N-4$ orthogonal T 's, each of whose zero modes is given by Eq. (4.5) with one and the same function $\epsilon(r)$. Then,

$$\tilde{I}_3^{st} = I_3 \delta_{st}, \quad l \quad 4 \leq s, t \leq 4N-1, \quad (4.9)$$

with

$$I_3(a) = \int d^3x \text{tr} D_\mu \Lambda D^\mu \Lambda = \oint_\infty dS_\mu \text{tr} \Lambda D^\mu \Lambda = \frac{2\pi\kappa'}{e^2} a. \quad (4.10)$$

Recall that $\kappa = \text{tr} t^3(\gamma_{N+1}) t^3(\gamma_{N+1})$, where the $\text{SU}(2)$ generators $t^a(\gamma_{N+1})$ induce unit shifts along the σ_a 's with $a=1,2,3$. We must fix $\kappa' = \text{tr} T^2$ so that T will induce a unit shift along a σ_s with $4 \leq s \leq 4N-1$. The action of $\text{Sp}(2N)$ on an S^{4N-1} is found by embedding $\text{Sp}(2N)$ into $\text{SO}(4N)$; after normalizing all generators with respect to the invariant bilinear form of $\text{Sp}(2N)$, we find that the generators associated with the short roots μ_j and ν_j and those associated with the long root γ_{N+1} shift a point corresponding to a generic $\text{SO}(5)$ embedding by unequal distances. The ratio turns out to be $1/\sqrt{2}$, so that the appropriate normalization for T is such that

$$\kappa' = 2\kappa. \quad (4.11)$$

The relative moduli space metric is then

$$\mathcal{G}_{\text{rel}} = \frac{4\pi\kappa}{e^2} \left[\frac{1}{a} da^2 + a \sum_{s=1}^{4N-1} \sigma_s \sigma_s \right]. \quad (4.12)$$

After a coordinate redefinition $\rho = 2\sqrt{a}$, we obtain

$$\begin{aligned} \mathcal{G}_{\text{rel}} &= \frac{4\pi\kappa}{e^2} \left[d\rho^2 + \rho^2 \frac{\sum_{s=1}^{4N-1} \sigma_s \sigma_s}{4} \right] = \frac{4\pi\kappa}{e^2} [d\rho^2 + \rho^2 d\Omega_{4N-1}^2] \\ &= \frac{4\pi\kappa}{e^2} \mathcal{G}_{\text{flat}}, \end{aligned} \quad (4.13)$$

showing that the moduli space metric is that of a flat Euclidean space R^{4N} . The smoothness of the metric at origin then requires the gauge orbit to be S^{4N-1} globally, and so the relative moduli space is strictly R^{4N} .

To compare this to the NUS limit of the MSB metric in Eq. (4.1), let us first note that $\lambda_A = \lambda = -2\beta^* \cdot \gamma^*$ for all A . We also need the fact that the μ_{AB} all vanish if there is only a single massive monopole. The NUS limit of Eq. (4.1) is then

$$\mathcal{G}_{\text{rel}} = \frac{g^2\lambda}{8\pi} \sum_A \left[\frac{1}{r_A} d\mathbf{r}_A^2 + r_A (d\psi_A + \cos\theta_A d\phi_A)^2 \right], \quad (4.14)$$

where we have rewritten the vector potential $\mathbf{w}_A \cdot d\mathbf{r}_A$ in polar coordinates. After a coordinate redefinition $\rho_A = 2\sqrt{r_A}$, we see that this MSB metric is a sum of N copies of the flat R^4 metric,

$$\begin{aligned} \mathcal{G}_{\text{rel}} &= \frac{g^2\lambda}{8\pi} \sum_A \left[d\rho^2 + \rho^2 \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{4} \right]_A \\ &= \frac{g^2\lambda}{8\pi} \sum_A [d\rho^2 + \rho^2 d\Omega_3^2]_A = \frac{g^2\lambda}{8\pi} \mathcal{G}_{\text{flat}}. \end{aligned} \quad (4.15)$$

Since $\lambda/2 = \kappa$ and $eg = 4\pi$, this is the same as the metric of Eq. (4.13), thus verifying that the two approaches produce the same result.

It is curious that the single $\text{Sp}(2N)$ -invariant a can be written as the sum of all distances between adjacent (as defined by the Dynkin diagram) monopoles, massive or massless alike; i.e.,

$$a = \frac{1}{4}\rho^2 = \frac{1}{4} \sum_A \rho_A^2 = \sum_A r_A. \quad (4.16)$$

However, the fact that there is only a single invariant parameter implies that the individual r_A 's are not invariant (the results of the next section will make this more explicit). Thus, the positions of the massless monopoles do not have a gauge-invariant meaning, emphasizing again that the massless monopoles should not be regarded as localized objects.

As for the $\text{SO}(5)$ example of Sec. III, a tower of non-BPS states carrying non-Abelian electric charge can be constructed by semiclassical quantization of the moduli space coordinates. The degeneracy of these states will be greater than in that example, reflecting the greater symmetry of the higher-dimensional moduli space.

B. $\text{Sp}(2N)$ triholomorphic isometry of R^{4N}

Because $\text{Sp}(2N)$ is an unbroken symmetry of the field equations and of the boundary conditions, its action on the solutions will manifest itself as metric-preserving diffeomorphisms of the relative moduli space R^{4N} . In fact, since the relative moduli space is a flat R^{4N} , it possesses a larger isometry group, $\text{SO}(4N)$. However, the $\text{Sp}(2N)$ subgroup acquires a special significance because it is the maximal subgroup of $\text{SO}(4N)$ that preserves the hyper-Kähler structure of the manifold. This triholomorphicity is a generic feature of isometries associated with the gauge rotation.

Let us introduce a pair of complex coordinates $\xi_A = x_A^1 + ix_A^2$ and $\zeta_A = x_A^3 + ix_A^4$ in the R^4 spanned by \mathbf{r}_A and ψ_A . These are related to the relative Euler angles and the monopole separation r_A by

$$\begin{aligned}\xi_A &= 2\sqrt{r_A}\cos(\theta_A/2)e^{-i(\phi_A+\psi_A)/2}, \\ \zeta_A &= 2\sqrt{r_A}\sin(\theta_A/2)e^{-i(\phi_A-\psi_A)/2}.\end{aligned}\quad (4.17)$$

In effect, we have chosen a particular complex structure on R^{4N} . Given this complex structure, the Kähler form is

$$\begin{aligned}w^{(3)} &= \frac{i}{2}\sum_A (d\xi_A \wedge d\xi_A^* + d\zeta_A \wedge d\zeta_A^*) \\ &= -\sum_A \left(\frac{1}{r_A} dr_A^1 \wedge dr_A^2 + dr_A^3 \wedge (d\psi_A + \cos\theta_A d\phi_A) \right),\end{aligned}\quad (4.18)$$

where in the second line we have used

$$r_A^1 - ir_A^2 = \frac{\zeta_A \xi_A}{2}, \quad r_A^3 = \frac{\xi_A \xi_A^* - \zeta_A^* \zeta_A}{4}.\quad (4.19)$$

Each factor of R^4 admits an $\text{SO}(4) = \text{SU}(2) \times \text{SU}(2)$ isometry. The first $\text{SU}(2)$ is generated by

$$\begin{aligned}L_A^{(3)} &= \frac{i}{2} \left(\xi_A \frac{\partial}{\partial \xi_A} + \zeta_A \frac{\partial}{\partial \zeta_A} - \xi_A^* \frac{\partial}{\partial \xi_A^*} - \zeta_A^* \frac{\partial}{\partial \zeta_A^*} \right) = -\frac{\partial}{\partial \phi_A}, \\ L_A^{(+)} &= i \left(\zeta_A^* \frac{\partial}{\partial \xi_A} - \xi_A^* \frac{\partial}{\partial \zeta_A} \right), \\ L_A^{(-)} &= i \left(\xi_A \frac{\partial}{\partial \zeta_A^*} - \zeta_A \frac{\partial}{\partial \xi_A^*} \right).\end{aligned}\quad (4.20)$$

After reexpressing these in terms of \mathbf{r}_A and ψ_A , and writing $L_A^{(\pm)} = L_A^{(1)} \pm iL_A^{(2)}$ we can add these to obtain

$$\mathbf{L} = \sum_A \mathbf{L}_A = \sum_A \left[-\mathbf{r}_A \times \left(\nabla_A - \mathbf{w}_A \frac{\partial}{\partial \psi_A} \right) - \hat{\mathbf{r}}_A \frac{\partial}{\partial \psi_A} \right],\quad (4.21)$$

which is the standard form for the generators of three-dimensional rotations in the presence of the vector potentials \mathbf{w}_A . Under appropriate rotations induced by \mathbf{L} , the Kähler two-form $w^{(3)}$ is transformed into the other two Kähler forms

$$\begin{aligned}w^{(a)} = w_{\text{flat}}^{(a)} &\equiv -\sum_A \left(\frac{1}{2r_A} \epsilon^{abc} dr_A^b \wedge dr_A^c \right. \\ &\quad \left. + dr_A^a \wedge (d\psi_A + \cos\theta_A d\phi_A) \right),\end{aligned}\quad (4.22)$$

which are needed to complete the hyper-Kähler structure of the moduli space.

In contrast, the second $\text{SU}(2)$ is holomorphic and thus cannot rotate the complex structure. Its generators are given by

$$\begin{aligned}K_A^{(3)} &= \frac{i}{2} \left(-\xi_A \frac{\partial}{\partial \xi_A} + \zeta_A \frac{\partial}{\partial \zeta_A} + \xi_A^* \frac{\partial}{\partial \xi_A^*} - \zeta_A^* \frac{\partial}{\partial \zeta_A^*} \right) = \frac{\partial}{\partial \psi_A}, \\ K_A^{(+)} &= i \left(\xi_A \frac{\partial}{\partial \zeta_A} - \zeta_A^* \frac{\partial}{\partial \xi_A^*} \right), \\ K_A^{(-)} &= i \left(\zeta_A \frac{\partial}{\partial \xi_A} - \xi_A^* \frac{\partial}{\partial \zeta_A^*} \right).\end{aligned}\quad (4.23)$$

Since these $K_A^{(a)}$'s commute with the $L_A^{(a)}$'s and since \mathbf{L} induces rotations among the $w^{(a)}$'s, the $K_A^{(a)}$'s are in fact triholomorphic; i.e., they preserve the hyper-Kähler structure of the moduli space:

$$\mathcal{L}_{K_A^{(a)}} w_{\text{flat}}^{(a')} = 0.\quad (4.24)$$

These $\text{SU}(2)$'s are clearly part of the $\text{Sp}(2N)$ isometry, with the $K_A^{(3)}$'s forming a set of N commuting generators that can be taken to be the generators of the Cartan subalgebra.

To complete the $\text{Sp}(2N)$, we recall that $\text{Sp}(2N)$ contains an $\text{SU}(N) \times \text{SU}(2)$ subgroup. Let the $\text{SU}(N)$ be generated by the simple roots $\{\gamma_2, \dots, \gamma_N\}$ and the $\text{SU}(2)$ by $\{\gamma_{N+1}\}$. Since this $\text{SU}(2)$ maps a given $\text{SO}(5)$ embedded solution to another embedded solution in the same $\text{SO}(5)$ subgroup, it must be realized on the moduli space by the $K_N^{(a)}$'s, which form the unique triholomorphic $\text{SU}(2)$ that preserves the four-plane $r_1 = r_2 = \dots = r_{N-1} = 0$. The $\text{SU}(N)$, on the other hand, rotates one R^4 to another; its Killing vectors are

$$T_A = \frac{1}{2}(K_A^{(3)} - K_{A+1}^{(3)}),$$

$$E_{AB} = \frac{i}{\sqrt{2}} \left(\xi_A \frac{\partial}{\partial \xi_B} + \zeta_A^* \frac{\partial}{\partial \zeta_B^*} - \xi_B^* \frac{\partial}{\partial \xi_A^*} - \zeta_B \frac{\partial}{\partial \zeta_A} \right), \quad A \neq B.\quad (4.25)$$

Commuting the E_{AB} 's with the $K_B^{(\pm)}$'s results in another set of ladder operators,

$$\begin{aligned}\tilde{E}_{AB}^{(+)} &= \frac{i}{\sqrt{2}} \left(\xi_A \frac{\partial}{\partial \zeta_B} + \xi_B \frac{\partial}{\partial \zeta_A} - \zeta_A^* \frac{\partial}{\partial \xi_B^*} - \zeta_B^* \frac{\partial}{\partial \xi_A^*} \right), \quad A < B, \\ \tilde{E}_{AB}^{(-)} &= \frac{i}{\sqrt{2}} \left(\zeta_A \frac{\partial}{\partial \xi_B} + \zeta_B \frac{\partial}{\partial \xi_A} - \xi_A^* \frac{\partial}{\partial \zeta_B^*} - \xi_B^* \frac{\partial}{\partial \zeta_A^*} \right), \quad A < B,\end{aligned}\quad (4.26)$$

which give the remaining generators of $\text{Sp}(2N)$. These all commute with \mathbf{L} and are therefore triholomorphic. We thus have the required triholomorphic isometry

$$\{K_A^{(3)}, E_{AB}, \tilde{E}_{AB}^{(\pm)}, K_A^{(\pm)}, 1 \leq A \neq B \leq N\} \rightarrow \text{Sp}(2N). \quad (4.27)$$

These generators have particularly simple interpretations in terms of an orthonormal basis \mathbf{e}_j ($j=1, \dots, N$) in the root space of $\text{Sp}(2N)$. [In this basis, $\boldsymbol{\gamma}_{j+1} = (\mathbf{e}_j - \mathbf{e}_{j+1})/2$ for $j < N$, while $\boldsymbol{\gamma}_{N+1} = \mathbf{e}_N$.] The correspondence between roots and ladder operators is then simply

$$E_{AB} \rightarrow \frac{1}{2}(\mathbf{e}_A - \mathbf{e}_B), \quad \tilde{E}_{AB}^{(\pm)} \rightarrow \pm \frac{1}{2}(\mathbf{e}_A + \mathbf{e}_B), \quad K_A^{(\pm)} \rightarrow \pm \mathbf{e}_A. \quad (4.28)$$

Finally, we note that in terms of the complex coordinates, the invariant a takes the simple form

$$a = \sum_A r_A = \frac{1}{4} \sum_A (\xi_A \xi_A^* + \zeta_A^* \zeta_A), \quad (4.29)$$

which is manifestly invariant under the transformations generated by the Killing vectors.

C. Unbroken $\text{SU}(N)$

A slightly more involved example arises when either $\text{Sp}(2N+2)$ or $\text{SU}(N+2)$ is broken to $\text{SU}(N) \times \text{U}(1)^2$. One finds that the magnetic charge

$$\frac{e\mathbf{g}}{4\pi} = \boldsymbol{\beta}_1^* + \sum_{j=2}^N \boldsymbol{\gamma}_j^* + \boldsymbol{\beta}_{N+1}^* \quad (4.30)$$

(where the roots are again numbered in accordance with the Dynkin diagram of Fig. 2) is orthogonal to the $\boldsymbol{\gamma}_j$'s that span the unbroken $\text{SU}(N)$. This corresponds to a combination of two massive monopoles, associated with $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_{N+1}^*$ and having masses m_1 and m_{N+1} , and $N-1$ massless monopoles.

The relative moduli space again has the topology of R^{4N} and can be covered by the coordinate system $\{\xi_A, \zeta_A\}$ defined in the previous subsection. However, this moduli space is no longer flat. Referring to the results of Ref. [13], one finds that the reduced mass matrix μ_{AB} no longer vanishes. Instead, the N^2 elements of this matrix are all equal to the reduced mass of the two massive monopoles: i.e.,

$$\mu_{AB} = \bar{\mu} \equiv \frac{m_1 m_{N+1}}{m_1 + m_{N+1}} \quad \text{all } A \text{ and } B. \quad (4.31)$$

Using this and the fact that $\lambda_A = \lambda$ is again independent of the link index A , we find that the NUS limit of the MSB metric is

$$\begin{aligned} \mathcal{G}_{\text{rel}} &= \frac{g^2 \lambda}{8\pi} \mathcal{G}_{\text{flat}} + \bar{\mu} \left(\sum_A d\mathbf{r}_A \right)^2 \\ &\quad - \frac{g^2 \lambda \bar{\mu}}{g^2 \lambda + 8\pi \bar{\mu} \sum_B r_B} \left(\sum_A r_A (d\psi_A + \cos\theta_A d\phi_A) \right)^2. \end{aligned} \quad (4.32)$$

The metric is still hyper-Kähler, as it must be, but the three independent Kähler forms are now given by [23,24]

$$\begin{aligned} w_{\text{SU}(N)}^{(a)} &= -\frac{1}{2} \sum_{A,B} C_{AB} \epsilon^{abc} dr_A^b \wedge dr_B^c \\ &\quad - \frac{g^2 \lambda}{8\pi} \sum_A dr_A^a \wedge (d\psi_A + \cos\theta_A d\phi_A) \\ &= \frac{g^2 \lambda}{8\pi} w_{\text{flat}}^{(a)} - \frac{\bar{\mu}}{2} \epsilon^{abc} \left(\sum_A dr_A^b \right) \wedge \left(\sum_B dr_B^c \right). \end{aligned} \quad (4.33)$$

As in the previous example, this moduli space must reflect the symmetries of the theory. There must be three Killing vector fields that generate three-dimensional rotations of the multimono- pole solution, while the unbroken gauge symmetry must be realized as a triholomorphic $\text{U}(N)$ isometry with appropriate Killing vectors. Now note that if the original gauge group is $\text{Sp}(2N+2)$, this example reduces to our previous one in the limit $\bar{\mu} \rightarrow 0$. Hence, in that limit the rotational Killing vectors for the present case must reduce to the \mathbf{L} of Eq. (4.21), while the triholomorphic Killing vectors must reduce to those that generate the $\text{U}(N)$ subgroup of $\text{Sp}(2N)$ in Eq. (4.25).

In fact, the vector fields \mathbf{L} in Eq. (4.21) generate the rotational $\text{SU}(2)$ isometry of the general MSB metric in Eq. (4.1) [13], and thus are Killing vectors on the NUS moduli space as well. Further, it turns out, as we show in the Appendix, that the $\text{SU}(N)$ generators in Eq. (4.25) are Killing vectors on this curved moduli space and also preserve the hyper-Kähler structure. Together with a simultaneous rotation of all ψ_A , this completes the $\text{U}(N)$ triholomorphic isometry induced by the action of the unbroken gauge group on the multimono- pole solutions,

$$\{T_A, E_{AB}, 1 \leq A \neq B \leq N\} \rightarrow \text{SU}(N),$$

$$K \equiv \sum_A K_A^{(3)} \rightarrow \text{U}(1). \quad (4.34)$$

Physically K corresponds to the relative electric $\text{U}(1)$ charge of the two massive monopoles. This $\text{U}(N)$ contains the N $\text{U}(1)$ generators $K_A^{(3)} = \partial/\partial\psi_A$ that clearly preserve both the metric and the hyper-Kähler structures of the general MSB metric. This is not true in general for the E_{AB} 's, but the detailed calculation in the Appendix shows that they are all triholomorphic and metric preserving in the present NUS limit. An important consistency check is to see if these vector fields preserve such gauge-invariant quantities as the relative position vector and the relative $\text{U}(1)$ charge of the two

massive monopoles.⁸ The latter is clearly invariant since its charge operator $-iK$ commutes with all generators of $SU(N)$, while the former is also invariant if

$$E_{AB} \left[\sum_A \mathbf{r}_A \right] = 0. \quad (4.36)$$

One can show this explicitly using Eq. (4.19).

The $4N$ coordinates of the moduli space can be related to the physical parameters of the multimonomole solution. An embedding argument similar to that of Sec. IV A shows that the gauge orbit of a generic point in the moduli space is of the form $U(N)/U(N-2)$. This implies that the number of gauge modes is $4N-4$, including one that corresponds to the relative $U(1)$ phase of the two massive monopoles. Three more parameters must correspond to the relative position vector between the two massive monopoles. This leaves only one gauge-invariant coordinate to characterize the non-Abelian cloud. A natural choice for this coordinate is just $a = \sum_A r_A$, which we know from our previous results to be $U(N)$ invariant. In the simplest case, with $N=2$, the gauge orbits are ellipsoids, $a = r_1 + r_2 = \text{const}$, with focal points at the two massive monopoles; these become three-spheres if the two massive monopoles coincide.

As with our previous examples, the semiclassical quantization of the moduli space coordinates will lead to a tower of chromodynamic states. A new feature here is that the relevant ‘‘moment of inertia’’ will increase with the separation between the two massive monopoles, in a fashion similar to that found in Ref. [5].

V. CHARACTERISTICS OF GENERAL MODULI SPACES

In the examples of the previous two sections we were able to analyze monopoles in theories with unbroken non-Abelian symmetries by taking the appropriate limit of the MSB case. For combinations of monopoles such that the long-range magnetic field was invariant under the action of the unbroken non-Abelian symmetry, the NUS limit of the MSB moduli space was shown to possess an isometry corresponding to the unbroken gauge symmetry. We found that color-magnetically neutral combinations were composed of a number of massive monopoles surrounded by a non-Abelian cloud that could be viewed as arising from the coalescence of a number of massless monopoles carrying purely non-

Abelian magnetic charges. In this section we will consider the general case of a simple group G of rank r broken to $K \times U(1)^{r-k}$ with a simple group K of rank k . (The extension to semisimple K is straightforward.) As before, we denote the simple roots of K by $\boldsymbol{\gamma}_i$ and the remaining roots of G by $\boldsymbol{\beta}_a$, with numbering corresponding to the Dynkin diagrams of Fig. 2.

Although we do not know the moduli space metric for most cases, we can still learn a good deal about how the massive and massless monopoles combine to form neutral configurations. In such configurations the magnetic charge vector \mathbf{g} must be orthogonal to every root of K ; i.e.,

$$0 = \frac{e\mathbf{g}}{4\pi} \cdot \boldsymbol{\gamma}_j = \sum_{a=1}^{r-k} n_a \boldsymbol{\beta}_a^* \cdot \boldsymbol{\gamma}_j + \sum_{i=1}^k q_i \boldsymbol{\gamma}_i^* \cdot \boldsymbol{\gamma}_j, \quad (5.1)$$

for all j . The sum of any two or more such \mathbf{g} 's will also satisfy this condition; we will concentrate here on the ‘‘minimal’’ cases, for which \mathbf{g} cannot be decomposed as such a sum.

The number of normalizable zero modes about such solutions, i.e., the dimension of the moduli space \mathcal{M} , is equal to $4(n+q)$, where $n = \sum n_a$ and $q = \sum q_i$ are the number of massive and massless fundamental monopoles, respectively [21]. The examples described above suggest that $4n$ of these describe the position coordinates and $U(1)$ phases of the massive monopoles, while the remaining $4q$ describe the non-Abelian cloud. Of the latter, some describe the size and, possibly, other gauge-invariant characteristics of the non-Abelian cloud and the rest correspond to global non-Abelian gauge rotations of the configuration. The number of such gauge modes can be as large as the dimension of K , but is less if the generic solution is invariant under some group $K' \subset K$. (Note that K' need not be semisimple.) The number of parameters describing the gauge-invariant structure of the cloud is then

$$\mathcal{N}_{\text{structure}} = 4q - \dim[K/K']. \quad (5.2)$$

The problem of finding the minimal \mathbf{g} 's can be phrased in terms of group representations. Each of the massive monopoles transforms according to a representation of the dual group K_{dual} spanned by the $\boldsymbol{\gamma}_j^*$. (The dual group enters here because the magnetic charge $\boldsymbol{\beta}_a^*$ is a weight vector with respect to the dual root system spanned by the $\boldsymbol{\gamma}_j^*$ [20].) The desired \mathbf{g} 's correspond to collections of massive monopoles that can be combined with a number of adjoint representation massless monopoles to form a group singlet.⁹

⁸In fact this is sufficient to show that E_{AB} generates a symmetry of the monopole dynamics. Consider the first order form of the Lagrangian

$$\begin{aligned} L(\bar{\boldsymbol{\mu}}) &= \frac{1}{2} \sum_{A,B} C_{AB} [\dot{\mathbf{r}}_A \cdot \dot{\mathbf{r}}_B - q_A q_B] + \sum_A q_A (\dot{\psi}_A + \cos \theta_A \dot{\phi}_A) \\ &= L(\bar{\boldsymbol{\mu}}=0) + \frac{\bar{\boldsymbol{\mu}}}{2} \left[\left(\sum_A \dot{\mathbf{r}}_A \right)^2 - \left(\sum_A q_A \right)^2 \right]. \end{aligned} \quad (4.35)$$

The metric is recovered by integrating out the conserved charges q_A (which is conjugate to ψ_A) and replacing velocities by line elements. $L(\bar{\boldsymbol{\mu}}=0)$ is by itself invariant since it describes free motion on R^{4N} . The invariance of the relative position $\sum_A \mathbf{r}_A$ and the relative $U(1)$ charge $\sum q_A$ then implies the invariance of the whole Lagrangian.

⁹An equivalent approach starts from the observation that any fundamental weight is a linear combination of the simple roots with non-negative rational coefficients. By definition, a fundamental weight is orthogonal to all but one simple root, and so any linear combination of the fundamental weights associated with the broken simple roots $\boldsymbol{\beta}_a^*$'s of G_{dual} is automatically orthogonal to the $\boldsymbol{\gamma}_j^*$'s and hence to the $\boldsymbol{\gamma}_j$'s. To obtain a minimal \mathbf{g} , one simply adds a number of such fundamental weights in such a way that the coefficients of the $\boldsymbol{\beta}_a^*$'s and of the $\boldsymbol{\gamma}_j^*$'s in the final expression are all non-negative integers; these coefficients then give the number of massive and massless monopoles required for the configuration.

The representations of the massive fundamental monopoles can be identified with the aid of the Dynkin diagram. Consider the monopole corresponding to the root β_a and let γ_j be the root of K to which β_a is linked in the Dynkin diagram. (If β_a is not linked to a root of K , then the monopole transforms as a singlet.) If $\lambda \equiv -2\beta_a^* \cdot \gamma_j^* = 1$, then the monopole transforms according to the complex conjugate of the basic representation of K_{dual} corresponding to γ_j^* ; if $\lambda > 1$, then the monopole transforms as a symmetric product of λ such representations.

With these ideas in mind, let us recall the case of $\text{Sp}(2N+2) \rightarrow \text{Sp}(2N) \times \text{U}(1)$ that was considered in Sec. IV A. The single massive monopole is linked to the first root of $\text{Sp}(2N)$, with $\lambda = 1$. It therefore transforms according to the vector representation of the dual group, $K_{\text{dual}} = \text{SO}(2N+1)$. Since the adjoint representation of an orthogonal group is the antisymmetric product of two vectors, and the antisymmetric product of $2N+1$ vectors is a singlet, a color-neutral combination can be obtained by combining the massive monopole with N massless monopoles, in agreement with our previous results.

In the other case considered in Sec. IV, with $K = K_{\text{dual}} = \text{SU}(N)$ and G being either $\text{SU}(N+2)$ or $\text{Sp}(2N+2)$, the two massive monopoles were linked to the first and last simple roots of the unbroken $\text{SU}(N)$, both with $\lambda = 1$. These therefore transform under the defining representations F and \bar{F} . The neutral combination of Eq. (4.30) corresponds to the fact that a group singlet can be formed by combining an F and an \bar{F} with a number of adjoints. However, this is not the only possibility. A singlet can also be constructed from N F 's (or N \bar{F} 's) together with some adjoint representation objects. The corresponding color-neutral magnetic charge is

$$\frac{e\mathbf{g}}{4\pi} = N\beta_1^* + \sum_{j=1}^{j=N-1} (N-j)\gamma_{j+1}^*. \quad (5.3)$$

This describes a family of solutions composed of N massive and $N(N-1)/2$ massless monopoles, with a moduli space of dimension $2N^2 + 2N$. The positions and $\text{U}(1)$ phases of the N massive monopoles account for $4N$ of these. There appears to be no invariance subgroup, and so there are $N^2 - 1$ gauge modes from the global $\text{SU}(N)$ rotations. This leaves $(N-1)^2$ structure parameters that encode the gauge-invariant characteristics of the non-Abelian cloud; this shows that the cloud can have much more structure than it did in our $\text{SO}(5)$ example.

With other choices for G , additional representations of $\text{SU}(N)$ can arise. If $G = \text{SO}(2N+1)$, one can have a massive monopole linked to the last simple root of $\text{SU}(N)$ with $\lambda = 2$, corresponding to the symmetric rank-2 tensor representation S , while with $G = \text{SO}(2N)$ a massive monopole can be linked to the next to last root of $\text{SU}(N)$, with $\lambda = 1$, giving an antisymmetric rank two tensor Λ . In addition, the even orthogonal groups allow the possibility of two different monopoles transforming as fundamentals; this can happen if the last two simple roots of $\text{SO}(2N)$ are broken but γ_{N-2} is not. A few more possibilities arise for low values of N by taking G to be an exceptional group. An antisymmetric rank-3 tensor representation Δ can be obtained if $G = E_6$,

E_7 , or E_8 , while there is a breaking of G_2 to $\text{SU}(2) \times \text{U}(1)$ that gives monopoles transforming under the spin-3/2 representation of $\text{SU}(2)$.

For $K = \text{SO}(2N+1)$, $\text{SO}(2N)$, or $\text{Sp}(2N)$, there is one type of massive monopole, transforming under the defining or vector representation V , if G is a classical group. If G is exceptional, there can also be massive monopoles transforming under the spinor representations corresponding to the last root of $\text{SO}(2N+1)$ and the last two roots of $\text{SO}(2N)$ or under the 14-dimensional representation corresponding to the last root of $\text{Sp}(6)$.

In Table I we list, for the case where the original gauge group is a classical group, the various ways in which these representations can be combined to give minimal configurations with vanishing non-Abelian magnetic charge. The overall group G that is shown is the smallest one that allows the neutral combination shown; in most cases, a larger G is also possible. In the table we also give the decomposition of \mathfrak{g} into simple roots, with the coefficients corresponding to massive monopoles indicated by boldface type. The remaining coefficients give the number of massless monopoles, from which in turn the total number of gauge and cloud structure zero modes can be obtained.

For the neutral combinations in which the number of component massive monopoles is independent of the rank of the group, the number of massless monopoles grows linearly with N . Since the dimension of K grows quadratically with the rank, there must be a nontrivial invariance subgroup K' . In all such cases, the generic solution for sufficiently high rank can be obtained by an embedding of a lower rank solution. Thus, the solutions studied in Sec. IV A for $\text{Sp}(2N+2)$ broken to $\text{Sp}(2N)$ could all be obtained by embedding the $\text{SO}(5)$ solution, and had $K' = \text{Sp}(2N-2)$, while the solutions with two massive monopoles considered in Sec. IV C were all equivalent to embeddings either of $\text{SU}(4)$ solutions [if $G = \text{SU}(N+2)$] or of $\text{Sp}(6)$ solutions [if $G = \text{Sp}(2N+2)$], and had $K' = \text{U}(N-2)$.

The other entries in Table I with nontrivial K' can all be determined by studying appropriate embeddings. As an example, for $G = \text{SO}(k+2)$ broken to $\text{SO}(k)$ ($k \geq 4$) there are color-magnetically neutral solutions containing two massive fundamental monopoles. To deal with these, we first consider the case of $\text{SO}(6) \rightarrow \text{SO}(4)$. Viewing this as $\text{SU}(4) \rightarrow \text{SU}(2) \times \text{SU}(2)$, it is not hard to construct an approximate solution with the two massive monopoles widely separated that clearly has no invariance group. The corresponding moduli space has $\dim \text{SO}(4) = 6$ gauge parameters and $8 - 6 = 2$ cloud structure parameters. By embedding these solutions in the larger orthogonal groups, we see that for $k \geq 4$, $\mathcal{N}_{\text{structure}} \geq 2$ and $K' \subseteq \text{SO}(k-3)$. In order that the number of parameters be consistent with the decompositions of \mathfrak{g} shown in the table, these inequalities must be saturated, indicating that the embeddings give the generic solution.

Finally, for the combinations where the number of massless monopoles grows with N the generic solution cannot be obtained by embedding from a smaller group, and we expect K' to be trivial.

The results for when the initial gauge group G is exceptional are summarized in Table II. We have used a notation $4\mathbf{n} + 4q$ for the dimension of the moduli space, with the

TABLE I. Minimal singlet combinations of massive fundamental monopoles when the original gauge group is classical. The symbols for the representations of the dual group K_{dual} are as follows: F is the defining representation of $SU(N)$, while Λ and S are the antisymmetric and the symmetric products of two F 's respectively; V is either the defining representation of a symplectic group or the vector representation of an orthogonal group; finally, an overbar denotes complex conjugation. The total magnetic charge of the singlet combination is written as a row vector of the integer coefficients appearing in Eq. (2.8), ordered according to the Dynkin diagram of the original gauge group G in Fig. 1, with the n_j indicated by boldface type.

K	Singlet	G	$e\mathbf{g}/4\pi$	K'	$\mathcal{N}_{\text{structure}}$
SU(N)	\bar{F}^N	SU($N+1$)	($\mathbf{N}, N-1, \dots, 2, 1$)	.	$(N-1)^2$
	F^N	Sp($2N$)	($1, 2, \dots, N-1, \mathbf{N}$)	.	$(N-1)^2$
	$F\bar{F}$	SU($N+2$)	($\mathbf{1}, 1, \dots, 1, \mathbf{1}$)	U($N-2$)	1
		Sp($2N+2$)			
	$S^{N/2}$	SO($2N+1$)	($1, 2, \dots, N-1, \mathbf{N}/2$)	.	$(N-1)^2$
		(even N)			
	S^N	SO($2N+1$)	($2, 4, \dots, 2N-2, \mathbf{N}$)	.	$3N^2-4N+1$
		(odd N)			
	$\bar{F}^2 S$	SO($2N+3$)	($\mathbf{2}, 2, \dots, 2, \mathbf{1}$)	.	5 ($N=2$)
				.	8 ($N=3$)
				U($N-4$)	9 ($N\geq 4$)
	$\Lambda^{N/2}$	SO($2N$)	($1, 2, \dots, N-2, N/2-1, \mathbf{N}/2$)	.	N^2-4N+1
		(even N)			($N>2$)
	Λ^N	SO($2N$)	($2, 4, \dots, 2N-4, N-2, \mathbf{N}$)	.	$3N^2-8N+1$
	(odd N)				
$F^n F'^{N-n}$	SO($2N+2$)	($1, 2, \dots, N-1, \mathbf{n}, \mathbf{N-n}$)	.	$(N-1)^2$	
$\bar{F}^2 \Lambda$	SO($2N+2$)	($\mathbf{2}, 2, \dots, 2, \mathbf{1}, \mathbf{1}$)	.	4 ($N=3$)	
			U($N-4$)	5 ($N\geq 4$)	
$\bar{F}^2 F F'$	SO($2N+4$)	($\mathbf{2}, 2, \dots, 2, \mathbf{1}, \mathbf{1}$)	.	5 ($N=2$)	
			.	8 ($N=3$)	
			U($N-4$)	9 ($N\geq 4$)	
SO($2N+1$)	V^2	SO($2N+3$)	($\mathbf{2}, 2, \dots, 2, \mathbf{1}$)	SO($2N-3$)	2 ($N\geq 2$)
Sp($2N$)	V	Sp($2N+2$)	($\mathbf{1}, 1, \dots, 1$)	Sp($2N-2$)	1
SO($2N$)	V^2	SO($2N+2$)	($\mathbf{2}, 2, \dots, 2, \mathbf{1}, 1$)	SO($2N-4$)	2 ($N\geq 3$)

boldface numeral indicating the degrees of freedom associated with the massive monopoles. Except for the trivial case $G=G_2$, we do not know the invariant subgroup K' , and so the number in the final column is in general a lower bound obtained from Eq. (5.2) by assuming that K' is trivial. When this yields a nonpositive number, we have written “1?” in the last column.

VI. DUALITY AND THRESHOLD BOUND STATES

The classical BPS multimonopole solutions that we have been studying can be naturally embedded in an $N=4$ supersymmetric Yang-Mills theory. It has been conjectured [2] that such theories possess an exact electromagnetic duality under which the spectrum of electrically charged elementary particles is mirrored by that of the magnetically charged particles.¹⁰ More precisely, the magnetically charged objects in a theory with gauge group G should be in one-to-one correspondence with the electrically charged objects in a theory with the dual group G_{dual} . In the simplest cases, this

correspondence is between the states based on the elementary quanta and those based on simple soliton solutions. Thus, in the $N=4$ supersymmetric theory with SU(2) broken to U(1), the states dual to the electrically charged vector mesons are obtained from the unit charged monopole and antimonopole solutions.¹¹ However, more complex situations can arise, even when the unbroken group is purely Abelian. Consider, for example, the case of SU(3) broken to U(1)². There are three electrically charged vector bosons, whose charges in the two unbroken U(1) factors are (1,0), (0,1), and (1,1); in the BPS limit, the mass of the third of these is the sum of the masses of the first two. The duals to the first two objects are the fundamental monopoles of the theory, but the dual of the third is a threshold bound state of the two fundamental monopoles. This state can be constructed semiclassically by considering the supersymmetric quantum mechanics of two-monopole systems or, equivalently, by studying a supersymmetric σ model on the corresponding moduli space [26,27]. In the latter approach, the bound states are in one-

¹⁰Duality also makes predictions concerning the dyonic states carrying both electric and magnetic charges; we do not discuss these here.

¹¹If the SU(2) theory has only $N=2$ supersymmetry, the unit charged monopoles are actually dual to quarks in SU(2) doublets. See Ref. [25] for detailed studies of a conformally invariant model with four families of quarks.

TABLE II. Minimal singlet combinations of massive fundamental monopoles when the original gauge group is exceptional. The notation is similar to that in Table I. Δ is the antisymmetric product of three F 's, while Ψ and $\Psi^{(\pm)}$ are spinor representations of odd- and even-dimensional orthogonal groups. In some cases, an irreducible representation is denoted by its dimension inside the square brackets.

K	Singlet	G	$e\mathbf{g}/4\pi$	$\dim[\mathcal{M}]$	$\mathcal{N}_{\text{structure}} \geq$
SU(2)	$[4]^2$	G_2	(2,3)	$8+12=20$	9
	F^2	G_2	(1,2)	$8+4=12$	1
SU(4)	$F\bar{F}\Lambda^2$	E_6	(1,2,3,2,1,2)	$16+28=44$	13
SU(5)	$\bar{F}\Lambda^2$	E_6	(1,2,3,2,1,2)	$12+32=44$	8
	$\bar{F}^3\bar{\Lambda}$	E_6	(3,3,3,2,1,1)	$16+36=52$	12
	$F^2\bar{F}\Lambda$	E_7	(1,2,3,4,3,2,2)	$20+48=68$	24
SU(6)	Δ^2	E_6	(1,2,3,2,1,2)	$8+36=44$	1
	$F^2\Lambda^2$	E_7	(1,2,3,4,3,2,2)	$16+52=68$	17
	\bar{F}^3A	E_7	(3,3,3,3,2,1,1)	$16+48=64$	13
	$F\bar{F}\Lambda^3$	E_8	(1,2,3,4,5,3,1,3)	$20+68=88$	33
SU(7)	Δ^7	E_7	(3,6,9,12,8,4,7)	$28+168=196$	120
	$F\Lambda^3$	E_8	(1,2,3,4,5,3,1,3)	$16+72=88$	24
	$\Delta^5\bar{F}$	E_8	(1,3,5,7,9,6,3,5)	$24+132=156$	84
SU(8)	Δ^8	E_8	(3,6,9,12,15,10,5,8)	$32+240=272$	177
SO(5)	$V\Psi^2$	F_4	(1,2,3,2)	$12+20=32$	10
SO(7)	$[14]^2$	F_4	(2,3,4,2)	$8+36=44$	15
Sp(6)	Ψ^2	F_4	(1,2,3,2)	$8+24=32$	3
SO(8)	$[\Psi^{(+)}]^4[\Psi^{(-)}]^2$	E_6	(4,5,6,4,2,3)	$24+72=96$	44
SO(10)	$[\Psi^{(+)}]^4$	E_6	(4,5,6,4,2,3)	$16+80=96$	35
	$V[\Psi^{(+)}]^2$	E_7	(1,2,3,4,3,2,2)	$12+56=68$	11
SO(12)	$[\Psi^{(-)}]^2$	E_7	(1,2,3,4,3,2,2)	$8+60=68$	1?
	$V^2[\Psi^{(-)}]^2$	E_8	(2,3,4,5,6,4,2,3)	$16+100=116$	34
SO(14)	$[\Psi^{(-)}]^4$	E_8	(2,4,6,8,10,7,4,5)	$16+168=184$	77
E_6	$[27]^3$	E_7	(3,4,5,6,4,2,3)	$12+96=108$	18
E_7	$[56]^2$	E_8	(2,3,4,5,6,4,2,3)	$8+108=116$	1?

to-one correspondence with the harmonic forms on the moduli space that satisfy an appropriate normalizability condition [28,29]. Such a normalizable harmonic form was found recently in Refs. [10] and [11].

Now let us consider the extension of these ideas to theories with unbroken non-Abelian subgroups. A new feature that arises here is the presence of massless elementary excitations in the electrically charged sector. The duals to these should also be massless, and so cannot be solitons of the ordinary sort; they are presumably the massless monopoles that form the non-Abelian clouds that we have found. For the massive particles, on the other hand, the duality picture should be much closer to that of the MSB case, except that some of the particles transform under nontrivial representations of the unbroken gauge group.

As an example, take the case of $SU(N)$ broken to $SU(N-1)\times U(1)$. In the electrically charged sector, the N^2-1 gauge bosons of the original group can be decomposed into $N(N-2)$ massless $SU(N-1)$ gauge bosons, a neutral massless $U(1)$ gauge boson, $(N-1)$ massive bosons with positive $U(1)$ charge belonging to the fundamental representation of $SU(N-1)$, and $(N-1)$ massive bosons with negative $U(1)$ charge belonging to the antifundamental representation of $SU(N-1)$. As noted above, the duals of the massless gauge bosons are the massless monopoles and antimonopoles (except for the case of the neutral bosons, which

are self-dual). The dual to the positively (negatively) charged massive multiplet is the fundamental monopole (antimonopole), which, according to the arguments of the previous section, corresponds to a fundamental (antifundamental) representation multiplet.

If this group is broken further, to $SU(N-2)\times U(1)^2$, the elementary particle sector contains two nondegenerate massive fundamental $SU(N-2)$ multiplets with $U(1)$ charges (1,0) and (0,1); these are dual to the two kinds of massive fundamental monopoles. There is also a massive $SU(N-2)$ singlet that carries one unit of each of the $U(1)$ charges. Its dual must be a threshold bound state containing one of each of the fundamental monopoles and $N-1$ massless monopoles. Such a state would correspond to a normalizable harmonic form on the moduli space we discussed in Sec. IV C; in order to be unique, this form must be either a self-dual or anti-self-dual $2N$ -form [28]. With further breaking [e.g., to $SU(N-3)\times U(1)^3$], additional bound states, containing some monopoles with purely Abelian charges, would also be required.

Other symmetry-breaking patterns can be studied in a similar fashion. In Table III we list the breakings of simple groups such that the unbroken group is a product of a simple group times a product of $U(1)$ factors and the fundamental monopoles all carry non-Abelian charges. (The latter requirement implies that every broken simple root is linked to an

TABLE III. Representations of massive fundamental monopoles and their threshold bound states. The notation for representations follows that of Tables I and II.

G	G_{dual}	Unbroken dual group	Massive monopoles	Bound states
$SU(N+1)$	$SU(N+1)$	$SU(N) \times U(1)$	F	None
		$U(1) \times SU(N-1) \times U(1)$	\bar{F}, F	$F\bar{F} \Rightarrow [1]$
$SO(2N+1)$	$Sp(2N)$	$U(1) \times Sp(2N-2)$	V	$VV \Rightarrow [1]$
		$SU(N) \times U(1)$	S	None
		$U(1) \times SU(N-1) \times U(1)$	\bar{F}, S	$\bar{F}S \Rightarrow F$
				$\bar{F}\bar{F}S \Rightarrow [1]$
$Sp(2N)$	$SO(2N+1)$	$U(1) \times SO(2N-1)$	V	None
		$SU(N) \times U(1)$	F	$FF \Rightarrow \Lambda$
		$U(1) \times SU(N-1) \times U(1)$	F, \bar{F}	$F\bar{F} \Rightarrow [1]$
				$FF \Rightarrow \Lambda$ $FF\bar{F} \Rightarrow F$
$SO(2N)$	$SO(2N)$	$U(1) \times SO(2N-2)$	V	None
		$SU(N) \times U(1)$	Λ	None
		$SU(N-1) \times U(1)^2$	F, F'	$FF' \Rightarrow \Lambda$
		$U(1) \times SU(N-1) \times U(1)$	\bar{F}, Λ	$\bar{F}\Lambda \Rightarrow F$
		$U(1) \times SU(N-2) \times U(1)^2$	F, F', \bar{F}	$F\bar{F} \Rightarrow [1]$
				$F'\bar{F} \Rightarrow [1]$ $FF' \Rightarrow \Lambda$ $FF'\bar{F} \Rightarrow F$
G_2 (dim=14)	G_2	$U(1) \times SU(2)$	[4]	$[4] \times [4] \Rightarrow [1]$
		$SU(2) \times U(1)$	[2]	$[2] \times [2] \Rightarrow [1]$ $[2] \times [2] \times [2] \Rightarrow [2]$
F_4 (dim=52)	F_4	$Sp(6) \times U(1)$	[14]	$[14] \times [14] \Rightarrow [1]$
		$U(1) \times SO(7)$	Ψ	$\Psi\Psi \Rightarrow V$
		$U(1) \times SO(5) \times U(1)$	V, Ψ	$\Psi\Psi \Rightarrow [1]$
				$V\Psi \Rightarrow \Psi$ $V\Psi\Psi \Rightarrow V$ $VV\Psi\Psi \Rightarrow [1]$
E_6 (dim=78)	E_6	$U(1) \times SU(6)$	Δ	$\Delta\Delta \Rightarrow [1]$
		$SO(10) \times U(1)$	$\Psi^{(+)}$	None
		$U(1) \times SO(8) \times U(1)$	$V, \Psi^{(-)}$	$V\Psi^{(-)} \Rightarrow \Psi^{(+)}$
		$U(1) \times SU(4) \times U(1)^2$	\bar{F}, F, Λ	$\bar{F}F \Rightarrow [1]$
				$\bar{F}\Lambda \Rightarrow F$ $F\Lambda \Rightarrow \bar{F}$ $\bar{F}F\Lambda \Rightarrow \Lambda$
				$\bar{F}F\Lambda\Lambda \Rightarrow [1]$
E_7 (dim=133)	E_7	$U(1) \times E_6$	[27]	None
		$U(1) \times SU(7)$	Δ	$\Delta\Delta \Rightarrow \bar{F}$
		$SO(12) \times U(1)$	$\Psi^{(-)}$	$\Psi^{(-)}\Psi^{(-)} \Rightarrow [1]$
		$U(1) \times SO(10) \times U(1)$	$V, \Psi^{(+)}$	$V\Psi^{(+)} \Rightarrow \Psi^{(-)}$
		$U(1) \times SU(6) \times U(1)$	\bar{F}, Δ	$V\Psi^{(+)}\Psi^{(+)} \Rightarrow [1]$
				$\bar{F}\Delta \Rightarrow \Lambda$
				$\Delta\Delta \Rightarrow [1]$
				$\bar{F}\Delta\Delta \Rightarrow \bar{F}$
		$\bar{F}F \Rightarrow [1]$ $\bar{F}\Lambda \Rightarrow F$ $F\Lambda \Rightarrow \bar{\Lambda}$ $F\Lambda\Lambda \Rightarrow [1]$ $\bar{F}F\Lambda \Rightarrow \Lambda$ $\bar{F}F\Lambda\Lambda \Rightarrow \bar{F}$ $\bar{F}F\Lambda\Lambda \Rightarrow [1]$		

TABLE III. (Continued.)

G	G_{dual}	Unbroken dual group	Massive monopoles	Bound states
E_8 (dim=248)	E_8	$U(1) \times E_7$	$[56]$	$[56] \times [56] \Rightarrow [1]$
		$U(1) \times SU(8)$	Δ	$\Delta \Delta \Rightarrow \bar{\Lambda}$ $\Delta \Delta \Delta \Rightarrow F$
		$SO(14) \times U(1)$	$\Psi^{(-)}$	$\Psi^{(-)} \Psi^{(-)} \Rightarrow V$
		$U(1) \times SO(12) \times U(1)$	$\Psi^{(-)}, V$	$\Psi^{(-)} \Psi^{(-)} \Rightarrow [1]$ $\Psi^{(-)} V \Rightarrow \Psi^{(+)}$ $\Psi^{(-)} \Psi^{(-)} V \Rightarrow V$ $\Psi^{(-)} \Psi^{(-)} V V \Rightarrow [1]$
		$U(1) \times SU(7) \times U(1)$	\bar{F}, Δ	$\bar{F} \Delta \Rightarrow \Lambda$ $\Delta \Delta \Rightarrow \bar{F}$ $\bar{F} \Delta \Delta \Rightarrow \bar{\Lambda}$ $\bar{F} \Delta \Delta \Delta \Rightarrow F$ $\bar{F} \bar{F} \Delta \Delta \Delta \Rightarrow [1]$
		$U(1) \times SU(6) \times U(1)^2$	\bar{F}, F, Λ	$\bar{F} F \Rightarrow [1]$ $\bar{F} \Lambda \Rightarrow F$ $F \Lambda \Rightarrow \Delta$ $F \Lambda \Lambda \Rightarrow \bar{F}$ $\bar{F} F \Lambda \Rightarrow \Lambda$ $\bar{F} F \Lambda \Lambda \Rightarrow \bar{\Lambda}$ $F F \Lambda \Lambda \Rightarrow [1]$ $\bar{F} F \Lambda \Lambda \Lambda \Rightarrow [1]$ $\bar{F} F F \Lambda \Lambda \Rightarrow \bar{F}$ $\bar{F} F F \Lambda \Lambda \Lambda \Rightarrow \bar{F}$ $\bar{F} F F F \Lambda \Lambda \Rightarrow [1]$

unbroken root in the Dynkin diagram.) For each of these we have listed the representations of the fundamental monopoles and indicated the bound states containing such monopoles that are required by duality. These examples can in most cases be embedded in larger groups, in which case there will also be purely Abelian fundamental monopoles and additional bound states containing these.

In principle, all the bound states listed in Table III must be realized as harmonic forms on appropriate moduli spaces. However, actually finding these forms is a rather nontrivial problem. For the case of distinct fundamental monopoles with the moduli space metric given in Ref. [13], Gibbons [30] recently gave an answer for the threshold bound state in the MSB case: a middle form constructed as a wedge product of a number of harmonic two-forms that are associated with the Killing vectors $\partial/\partial\psi_A$. One might have hoped that this construction would carry over to the present NUS limit and produce the expected harmonic $2N$ -form on the moduli space for two massive and $N-1$ massless monopoles of $Sp(2N+2)$ or $SU(N+2)$ broken to $SU(N) \times U(1)^2$. Unfortunately, this is not the case. Although the harmonicity of the middle form is likely to be preserved, the normalizability is not. Further, this middle form is invariant only under the Cartan subgroup of the unbroken gauge group $SU(N)$, implying that the corresponding state is electrically charged and cannot be the purely magnetic threshold bound state. This difficulty is compounded by the fact that even the MSB moduli space metric is unknown for most cases.

Finally, we want to emphasize the fact that some of the required bound states transform nontrivially under the unbro-

ken gauge group. As we have noted, there are pathologies associated with configurations that have nonzero long-range non-Abelian magnetic fields, and a meaningful moduli space emerges only if we insist that the total magnetic charge be purely Abelian. Since the nonsinglet bound states necessarily involve only some of the monopoles described by the moduli space, the corresponding harmonic forms cannot be normalizable in the usual sense.

VII. CONCLUSION

In this paper we have used the multimono- pole moduli space as a tool for investigating the properties of monopoles carrying non-Abelian magnetic charges. If the net magnetic charge is purely Abelian, the moduli space for the case with an unbroken non-Abelian subgroup can be obtained as a smooth limit of that for the MSB case. In this limit the moduli space describes multimono- pole solutions that are composed of one or more color-magnetically neutral combinations of monopoles. In each of the latter there are a number of massive fundamental monopoles, corresponding to embeddings of the $SU(2)$ monopole, that carry both Abelian and non-Abelian magnetic charge. These are surrounded by a cloud within which there is a nonzero non-Abelian magnetic field.

By studying the approach to the NUS limit, we are led to interpret this cloud as being composed of massless mono- poles carrying purely non-Abelian magnetic charges. These can be understood as limits of the fundamental monopoles of the MSB case that correspond to simple roots of the unbro-

ken non-Abelian subgroup. However, they differ from the other fundamental monopoles in that there is no classical soliton corresponding to an isolated massless monopole. When they coalesce to form a non-Abelian cloud, they lose their identity as individual objects. Thus, although the number of parameters remains unchanged as one goes from the MSB case to the NUS case, the position and $U(1)$ orientations of these monopoles are transformed into gauge orientation and structure parameters describing the cloud as a whole.

There are a number of outstanding issues to be addressed. We have worked entirely within the context of the BPS limit. To what extent do our results apply to models (such as realistic grand unified theories) that have nonvanishing Higgs potentials? Such models will still have a number of massive fundamental monopoles belonging to representations of the dual of the unbroken gauge group. At least for Higgs boson masses small compared to the vector meson masses, the leading effects of the departure from the BPS limit could be incorporated by adding to the moduli space Lagrangian a potential energy depending on the monopole separations and the cloud structure parameters. Presumably at least some of the color-magnetically neutral combinations of monopoles are stably bound (both classically and quantum mechanically) by this potential energy [31], since the Brandt-Neri-Coleman analysis [6] shows that stable configurations with large non-Abelian magnetic charge are impossible.

Another important question is that of how our largely classical analysis must be modified to take into account quantum effects. We discussed briefly in Sec. III the quantization of the moduli space coordinates and the nature of the low energy eigenstates of the moduli space Hamiltonian. However, we have not addressed at all the question of how the moduli space itself might be modified by quantum corrections. (Note that the BPS limit can be maintained under quantum corrections in theories with extended supersymmetry.) For example, at the classical level the energy does not depend on the values of the cloud structure parameters, but the corresponding degeneracy does not seem to be required by the BPS conditions at the quantum level. Does this mean that one-loop effects modify the low energy moduli space Lagrangian? It would be clearly desirable to go beyond the semiclassical approximation and make a connection with the work of Seiberg and Witten [32]. One would also like to understand what the effects of confinement on non-Abelian magnetic charges are and how they should be incorporated.

Perhaps most interesting are the questions connected with the duality hypothesis. Particularly intriguing is the role of the massless monopoles, which are naturally recognized as being the objects that are dual to the massless gauge bosons carrying electric-type color charges. In fact, if the electric and magnetic sectors are to be on an equal setting, the full multiplet of gauge bosons should have a counterpart comprising not only the massless monopoles and antimonopoles, but also neutral gauge particles corresponding to the Cartan subalgebra. In one sense, the latter should be seen as being their own dual, just as the photon is in the $SU(2) \rightarrow U(1)$ case. However, the fact that the choice of the Cartan subalgebra for the unbroken group is not gauge invariant shows that the particular separation into monopoles, antimonopoles, and self-dual objects is to some extent arbitrary. Clearly,

there is much to be learned about these objects. Indeed, one might hope that a fuller understanding of these massless monopoles could form the basis for a dual approach to non-Abelian interactions that would prove complementary to that based on the perturbative gauge bosons.

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APPENDIX A: TRIHOLOMORPHIC $SU(N)$ ISOMETRY

We start with the observation that the E_{AB} 's preserve $\sum_A r_A$, which is the relative position vector between the two massive monopoles. This can be seen by rewriting the vector field in three-dimensional coordinates:

$$E_{AB} = e^{-i(\psi_A - \psi_B)/2} \left[f_{AB}^{(a)} \left(\frac{\partial}{\partial r_A^a} - \frac{\partial}{\partial r_B^a} \right) + g_{AB} \frac{\partial}{\partial \psi_A} + g_{BA}^* \frac{\partial}{\partial \psi_B} \right]. \quad (\text{A1})$$

(The details of the $N \times N$ matrices $f^{(a)}$ and g will not matter here.) Recalling that the scalar quantity $\sum_A r_A$ is also invariant, one can easily see that the metric in Eq. (4.32) is invariant if and only if the Lie derivative of the one-form

$$\begin{aligned} \Omega &\equiv \sum_A r_A (d\psi_A + \cos\theta_A d\psi_A) \\ &= \frac{i}{4} \sum_A (\xi_A^* d\xi_A - \xi_A d\xi_A^* - \zeta_A^* d\zeta_A + \zeta_A d\zeta_A^*) \end{aligned} \quad (\text{A2})$$

vanishes. The Lie derivative of the differential form can be succinctly written as

$$\mathcal{L}_{E_{AB}} \Omega = d\langle E_{AB}, \Omega \rangle + \langle E_{AB}, d\Omega \rangle. \quad (\text{A3})$$

The two terms cancel each other with Ω given as in Eq. (A2) (this is easiest to see in complex coordinates), and so the E_{AB} 's are indeed Killing vector fields.

To show that E_{AB} is triholomorphic, we compute the Lie derivative of the Kähler form $w_{SU(N)}^{(a)}$:

$$\mathcal{L}_{E_{AB}} w_{SU(N)}^{(a)} = d\langle E_{AB}, w_{SU(N)}^{(a)} \rangle + \langle E_{AB}, dw_{SU(N)}^{(a)} \rangle. \quad (\text{A4})$$

The Kähler forms are closed, so that the second term is null, while the first term is

$$\begin{aligned} d\langle E_{AB}, w_{SU(N)}^{(a)} \rangle &= \frac{g^2 \lambda}{8\pi} d\langle E_{AB}, w_{\text{flat}}^{(a)} \rangle \\ &= -\frac{\bar{\mu}}{2} d \left\langle E_{AB}, \epsilon^{abc} \left(\sum_A dr_A^b \right) \wedge \left(\sum_B dr_B^c \right) \right\rangle. \end{aligned} \quad (\text{A5})$$

Because E_{AB} is orthogonal to $\Sigma_A dr_A^a$ the $\bar{\mu}$ -dependent term vanishes identically. Then,

$$\mathcal{L}_{E_{AB}} w_{\text{SU}(N)}^{(a)} = \frac{g^2 \lambda}{8\pi} d \langle E_{AB}, w_{\text{flat}}^{(a)} \rangle = \frac{g^2 \lambda}{8\pi} \mathcal{L}_{E_{AB}} w_{\text{flat}}^{(a)} = 0. \quad (\text{A6})$$

This concludes the proof that the E_{AB} 's are triholomorphic Killing vector fields. It follows that the NUS metric in Eq. (4.32) admits a $U(N)$ isometry that preserves its hyper-Kähler structure.

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