Exact CTP renormalization group equation for the coarse-grained effective action

Diego A. R. Dalvit*

Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires-Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

Francisco D. Mazzitelli†

Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires-Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina and Instituto de Astronomía y Física del Espacio, Casilla de Correo 67-Sucursal 28, 1428 Buenos Aires, Argentina

(Received 30 April 1996)

We consider a scalar field theory in Minkowski spacetime and define a coarse-grained closed time path (CTP) effective action by integrating quantum fluctuations of wavelengths shorter than a critical value. We derive an exact CTP renormalization group equation for the dependence of the effective action on the coarse-graining scale. We solve this equation using a derivative expansion approach. Explicit calculation is performed for the $\lambda\phi^4$ theory. We discuss the relevance of the CTP average action in the study of nonequilibrium aspects of phase transitions in quantum field theory. [S0556-2821(96)00722-9]

PACS number(s): 11.10.Hi

I. INTRODUCTION

The study of phase transitions in quantum field theory is of great interest in cosmology and particle physics. Cosmological inflationary models [1–3], the electroweak phase transition [4,5], and the formation of chiral condensates [6–8] are clear examples of the very interesting problems related to phase transitions in this context.

The first analyses of phase transitions in quantum field theory were based on the use of the finite temperature effective potential. The effective potential is useful only in quasistatic situations and, therefore, one must use the complete effective action in order to address the nonequilibrium aspects of the transition.

The usual effective action is not adequate to study initial value problems since it gives the evolution equations for ''in-out'' matrix elements of the background fields. The equations for these matrix elements are neither real nor causal. The solution to this problem is to use the so-called ''in-in'' or closed time path (CTP) effective action, introduced by Schwinger and Keldysh many years ago [9]. This action gives real and causal evolution equations for the ''in-in'' mean value of the background fields. The CTP effective action has been used to analyze inflationary models, anisotropy dissipation in the early Universe, the backreaction problem in semiclassical and stochastic gravity, the quantum to classical transition in quantum Brownian motion and Quantum Field Theory, etc. [10,11].

There is, however, one aspect that, to our knowledge, has not been fully investigated. Phase transitions occur via the formation and growth of *spatial* domains. Inside these domains, the order parameter of the transition evolves dynamically, and one is usually interested in computing its temporal evolution. The order parameter is, therefore, *the average of the quantum mean value of the field in a spatial volume of the size of the domain*.

In previous works, this problem has been addressed using different approaches. On the one hand, phase transitions have been analyzed using the CTP effective action, assuming that the order parameter depends only on time. Presumably, this time-dependent function describes the dynamics of the order parameter inside one typical domain [12]. It has been shown that domain growth is an effect characterized by the rapid evolution of (exponentially unstable) long wavelength modes. Such a dynamics can be nonperturbatively described by a Hartree approximation to the two-point correlation function [13]. On the other hand, phenomenological Langevin-like equations which account for dissipation and noise have been proposed and numerically solved (see Ref. [14] for one such type of calculation, where the order parameter is coupled to a thermal bath and a Markovian Langevin equation is put forward in order to mimic thermal phase mixing during a first-order phase transition).

In this paper, as a step towards a first principle calculation, we will proceed using an analogy with what is done in the context of condensed matter [15]. We will coarse grain our theory up to a length scale Λ^{-1} comparable to the initial size of a typical domain. In this way, we will define a ''coarse-grained effective action'' (CGEA), which will be basically the usual CTP action in which only those modes of the scalar field with $|\vec{q}|>\Lambda$ are integrated out. As a result of tracing the short wavelength modes (the so-called ''environmental'' degrees of freedom), the CTP average action, which depends on the long wavelength modes (the so-called ''system'' degrees of freedom), develops dissipation and noise terms.

We will derive an exact evolution equation for the dependence of the CGEA on the coarse-graining scale. In principle, it is possible to derive a Langevin equation for the long wavelength order parameter $\phi(\vec{x},t)$ starting from the CGEA. However, in order to make the analysis more tractable, here

*Electronic address: dalvit@df.uba.ar

†Electronic address: fmazzi@df.uba.ar

we will compute the CTP average action using some simple approximations. The major approximation we will make is to ignore spatial correlations between different domains. This allows the study of the dynamics for coarse-grained, time-dependent configurations $\phi(t)$ inside a given domain.

The paper is organized as follows: in Sec. II we define the CGEA and compute it in the one-loop approximation, making an adiabatic expansion. We should mention that by using such approximations we are not aiming at studying domain growth since, as we have already stated, they fail to describe such a dynamical process. Rather, our intention is to illustrate a simple calculation of the CTP effective action. In Sec. III we derive the exact renormalization group (RG) equation for the evolution of the CGEA, which is a generalization to CTP of the Euclidean Wegner and Houghton's RG equation [16]. Using a derivative expansion, in Sec. IV we reduce this functional equation to a system of coupled differential equations, which are then numerically solved. The interest of the results is not only restricted to nonzero coarse-graining scales, but also to provide nonperturbative approximations to standard $(\Lambda=0)$ quantum field theory. In Sec. V we make the conclusions.

In order to make contact with related works, we would like to mention that the CGEA defined here was originally introduced in Ref. [17] in order to study inflationary cosmology, and it was perturbatively evaluated in Ref. [18] in order to analyze the decoherence of the long wavelength sector of the $\lambda\phi^4$ field theory. It is also similar in spirit to the *Euclidean* average effective action proposed earlier [19–24]. The main difference between both actions is that the Euclidean action averages the field over a spacetime volume, while our CTP action averages the field over a spatial volume, and it is, therefore, more useful to study nonequilibrium situations.

In the above-mentioned works, and also in this paper, the coarse-grained or average effective action interpolates between the bare theory at $\Lambda=\Lambda_0$ (the ultraviolet cutoff) and the physical theory at the coarse-graining scale. Another possibility has been recently analyzed in Ref. [25], where an effective action is defined that interpolates between the physical theory at $T=0$ and the physical theory at $T\neq 0$.

II. THE CTP COARSE-GRAINED EFFECTIVE ACTION

Let us consider $\lambda\phi^4$ field theory in Minkowski spacetime. In order to deal with nonequilibrium scenarios we follow Schwinger-Keldysh formulation, doubling the number of fields and imposing CTP boundary conditions. The coarse-grained CTP effective action $S_\Lambda(\phi_+,\phi_-)$ is defined as

$$e^{iS_\Lambda(\phi_+,\phi_-)}\equiv\int\prod_{\Lambda_0>|\vec{q}|>\Lambda}\mathcal{D}[\phi_+(\vec{q},t)]\times\mathcal{D}[\phi_-(\vec{q},t)]e^{iS_{\rm cl}(\phi_+,\phi_-)},\qquad(1)$$

where $S_{\rm cl}(\phi_+,\phi_-)=S_{\rm cl}(\phi_+)-S_{\rm cl}(\phi_-)$. Note that a sharp cutoff Λ and an ultraviolet cutoff Λ_0 have been used. The functional integrals over the short wavelength modes are to be computed using standard CTP boundary conditions: the fields ϕ_+ and ϕ_- must have only negative and positive frequency modes, respectively, in the past $-T$, and match in the future T. In general, the CTP coarse-grained effective action has an imaginary part, related to noise, and a real part, related to dissipation. The equations of motion derived from it are, however, real and causal.

An exact calculation of the above-defined action is quite difficult, and it is, therefore, necessary to use approximation methods. One possible approach is to make perturbations in the coupling constant λ [18]. Another possibility that we will explore here is to use a loop expansion. In the one-loop approximation our effective action can be written as $S_\Lambda(\phi_+,\phi_-)=S_{\rm cl}(\phi_+)-S_{\rm cl}(\phi_-)+\Delta S_\Lambda(\phi_+,\phi_-)$, where the last term is linear in $\hbar$.

As mentioned in Introduction, we shall be concerned with background configurations that depend only on time $\phi_\pm(t)$. We split the complete field $\phi_\pm\to\phi_\pm(t)+\varphi_\pm$, where the fluctuations $\varphi_\pm$ contain spatial modes with $|\vec{q}|>\Lambda$. Note that we are assuming that the only mode with $|\vec{q}|<\Lambda$ is the spatial homogeneous one $(\vec{q}=0)$. The one-loop correction is

$$e^{i\Delta S_\Lambda(\phi_+(t),\phi_-(t))}=\int\prod_{\Lambda_0>|\vec{q}|>\Lambda}\mathcal{D}[\varphi_+]\mathcal{D}[\varphi_-]\times\exp\left[\frac{i}{2}\int dt\int\frac{d^3q}{(2\pi)^3}\times\left(\varphi_+\frac{\delta^2S_{\rm cl}}{\delta\phi_+\delta\phi_+}\varphi_+-\varphi_-\frac{\delta^2S_{\rm cl}}{\delta\phi_-\delta\phi_-}\varphi_-\right)\right]\times\exp\left(\frac{i}{2}\int\frac{d^3q}{(2\pi)^3}\int dt\times\frac{d}{dt}[\varphi_+\dot{\varphi}_+-\varphi_-\dot{\varphi}_-]\right),\qquad(2)$$

where the functional derivatives are evaluated at $\varphi_\pm=0$. The action for the quantum fluctuations is that of a free field with mass $M_\pm^2=V''(\phi_\pm)$, where V is the potential in the (bare) classical action. Their spatial Fourier modes are, therefore, harmonic oscillators with time-dependent frequency, namely, $w_{q,\pm}^2(t)=q^2+V''(\phi_\pm(t))$ with $q=|\vec{q}|$. The functional integral is quadratic and can be done straightforwardly. The result is

$$\Delta S_\Lambda(\phi_+(t),\phi_-(t))=\frac{i}{2}\int_{\Lambda_0>|\vec{q}|>\Lambda}\frac{d^3q}{(2\pi)^3}\times\ln[g_-(\vec{q},T)\dot{g}_+(\vec{q},T)-g_+(\vec{q},T)\dot{g}_-(\vec{q},T)],\qquad(3)$$

where the modes $g_\pm$ are solutions to $\ddot{g}_\pm(\vec{q},t)+w_{q,\pm}^2(t)g_\pm(\vec{q},t)=0$ satisfying CTP conditions in the past and having an arbitrary normalization in the future [we will present an explicit proof of Eq. (3) in Sec. IV].

Even in the one-loop approximation, the effective action is a very complicated object and additional approximations are needed in order to get analytic results. The simplest approximation one can think of is the adiabatic approximation [26], in which one neglects the excitations of the quantum fluctuation field due to the time dependence of the background field $\phi_\pm(t)$. This approximation misses the very important stochastic aspects of the theory. However, it will be useful as a warm-up and also to make contact with the Euclidean works.

We can write the mode functions as

$$g_{\pm}(\vec{q},t)=\frac{1}{\sqrt{2W_q(\phi_{\pm}(t))}}e^{\pm i\int_{-T}^{t}dt' W_q(\phi_{\pm}(t'))}, \tag{4}$$

where the functions $W_q(\phi_{\pm})$ satisfy

$$W_q^2+\frac{1}{2}\left[\frac{\ddot{W}_q}{W_q}-\frac{3}{2}\left(\frac{\dot{W}_q}{W_q}\right)^2\right]=w_q^2. \tag{5}$$

The adiabatic approximation consists in solving this equation using an expansion in derivatives of the background field. The result is

$$W_q^2=q^2+V''+\frac{5}{16}\left[\left(\frac{V'''}{q^2+V''}\right)^2-\frac{V''''}{4(q^2+V'')}\right]\dot{\phi}^2 - \frac{V'''}{4(q^2+V'')}\ddot{\phi}+\cdots, \tag{6}$$

where the ellipsis denotes higher derivative terms.

In the one-loop and adiabatic approximations, the average CTP effective action is, therefore, up to a surface term evaluated at $t=T$, given by $\Delta S_{\Lambda}(\phi_+,\phi_-)=\Delta S_{\Lambda}(\phi_+) - \Delta S_{\Lambda}(\phi_-)$, where

$$\Delta S_{\Lambda}(\phi)=\frac{1}{2}\int dt\int_{\Lambda}^{\Lambda_0}\frac{d^3q}{(2\pi)^3} \times\left(-\sqrt{q^2+V''}+\frac{\dot{\phi}^2}{32}\frac{V'''^2}{(q^2+V'')^{5/2}}\right). \tag{7}$$

Including the classical part, we can write the total effective action as

$$S_{\Lambda}(\phi)=\int dt\left(-V_{\Lambda}(\phi)+\frac{1}{2}[1+Z_{\Lambda}(\phi)]\dot{\phi}^2+\cdots\right), \tag{8}$$

where

$$V_{\Lambda}=V+\frac{1}{2}\int_{\Lambda}^{\Lambda_0}\frac{d^3q}{(2\pi)^3}\sqrt{q^2+V''} =V+\frac{1}{4\pi^2}\left[\frac{\Lambda_0}{4}\sqrt{\Lambda_0^2+V''}\left(\frac{V''}{2}+\Lambda_0^2\right)-\frac{V''^2}{8} \times\ln(\Lambda_0+\sqrt{\Lambda_0^2+V''})-\frac{\Lambda}{4}\sqrt{\Lambda^2+V''}\left(\frac{V''}{2}+\Lambda^2\right) +\frac{V''^2}{8}\ln(\Lambda+\sqrt{\Lambda^2+V''})\right] \tag{9}$$

and

$$Z_{\Lambda}=\frac{1}{32}\int_{\Lambda}^{\Lambda_0}\frac{d^3q}{(2\pi)^3}\frac{V'''^2}{(q^2+V'')^{5/2}} =\frac{1}{192\pi^2}\frac{V'''^2}{V''}\left[\frac{\Lambda_0^3}{(\Lambda_0^2+V'')^{3/2}}-\frac{\Lambda^3}{(\Lambda^2+V'')^{3/2}}\right]. \tag{10}$$

While the function $Z_{\Lambda}(\phi)$ is finite when $\Lambda_0\to\infty$, the potential $V_{\Lambda}(\phi)$ diverges in the UV. In order to make contact with the conventional renormalization schemes, we write the one-loop bare potential as $V(\phi)=\frac{1}{2}(m_R^2+\delta m^2)\phi^2+1/4!(\lambda_R+\delta\lambda)\phi^4$, where the renormalized mass and coupling constant are defined as

$$m_R^2\equiv\left.\frac{\partial^2 V_{\Lambda}}{\partial\phi^2}\right|_{\Lambda=\phi=0}, \quad \lambda_R\equiv\left.\frac{\partial^4 V_{\Lambda}}{\partial\phi^4}\right|_{\Lambda=\phi=0} \tag{11}$$

and the counterterms are

$$\delta m^2=-\frac{\lambda_R}{32\pi^2}\left[\frac{m_R^2}{2}+2\Lambda_0^2+m_R^2\ln\left(\frac{m_R^2}{4\Lambda_0^2}\right)\right],$$

$$\delta\lambda=-\frac{3\lambda_R^2}{32\pi^2}\left[2+\lambda_R^2\ln\left(\frac{m_R^2}{4\Lambda_0^2}\right)\right]. \tag{12}$$

Therefore, the renormalized potential is

$$V_{\Lambda}^{\rm ren}(\phi)=\frac{1}{2}m_R^2\phi^2\left(1-\frac{\lambda_R}{64\pi^2}\right)+\frac{1}{4!}\lambda_R\phi^4\left(1-\frac{3\lambda_R}{16\pi^2}\right) +\frac{1}{32\pi^2}\left[-\Lambda(2\Lambda^2+m_R^2+\tfrac{1}{2}\lambda_R\phi^2) \times\sqrt{\Lambda^2+m_R^2+\tfrac{1}{2}\lambda_R\phi^2}+(m_R^2+\tfrac{1}{2}\lambda_R\phi^2)^2 \times\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m_R^2+\frac{1}{2}\lambda_R\phi^2}}{m_R}\right)\right] \tag{13}$$

and the renormalized wave function renormalization is

$$Z_{\Lambda}^{\rm ren}(\phi)=\frac{1}{192\pi^2}\frac{\lambda_R^2\phi^2}{m_R^2+\frac{1}{2}\lambda_R\phi^2} \times\left[1-\frac{\Lambda^3}{(\Lambda^2+m_R^2+\frac{1}{2}\lambda_R\phi^2)^{3/2}}\right]. \tag{14}$$

The flow with the coarse-graining scale of the effective potential V_{Λ} and of the wave function factor Z_{Λ} in the one-loop approximation follows immediately from Eqs. (9) and (10):

$$\Lambda\frac{dV_{\Lambda}}{d\Lambda}=-\frac{\Lambda^3}{4\pi^2}\sqrt{\Lambda^2+V''},$$

$$\Lambda\frac{dZ_{\Lambda}}{d\Lambda}=-\frac{\Lambda^3}{64\pi^2}\frac{V'''^2}{(\Lambda^2+V'')^{5/2}}. \tag{15}$$

The equation for the effective potential has been previously obtained in Ref. [27] using a blocking procedure.

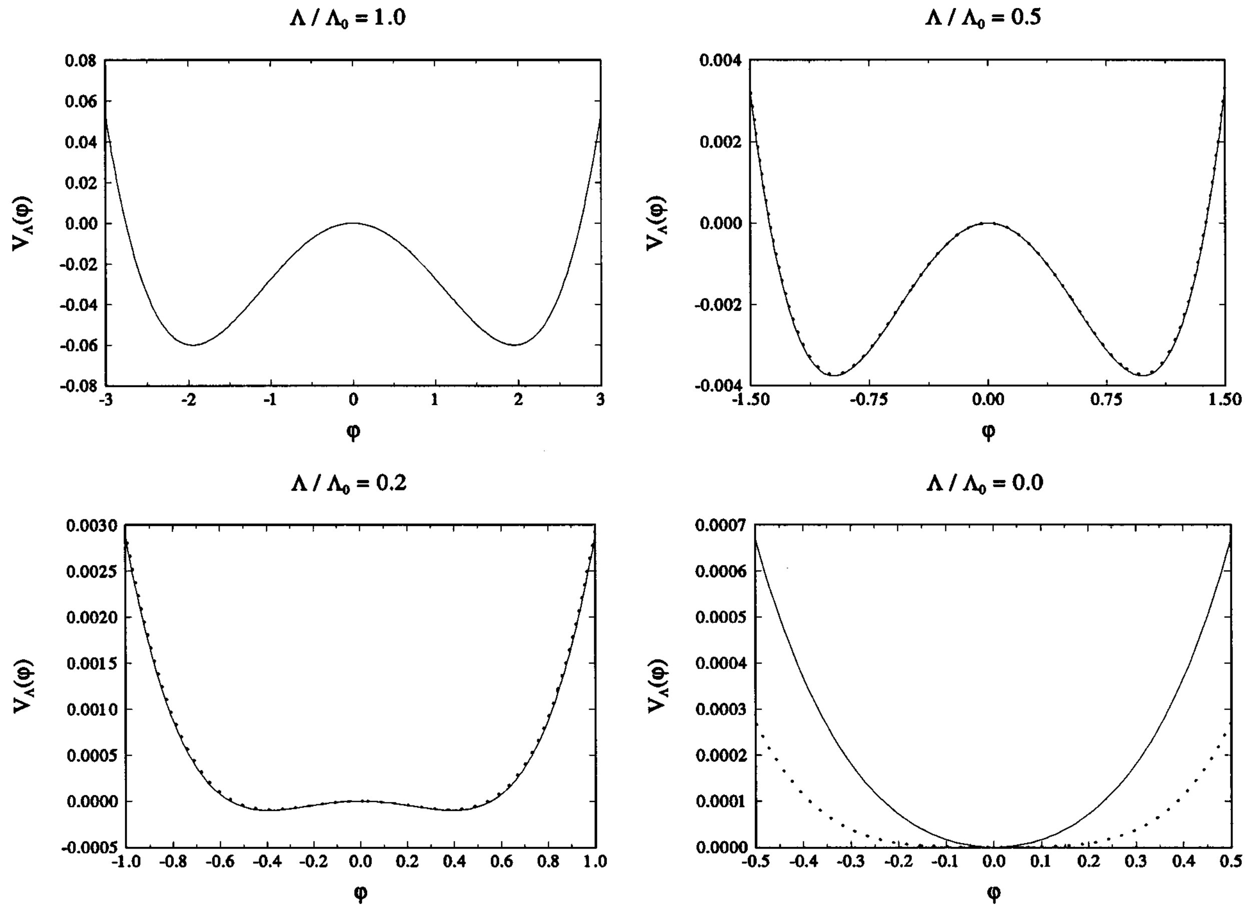

FIG. 1. The coarse-grained effective potential V_Λ for $\Lambda_0=10$, $m_R^2=10^{-4}$, and $\lambda_R=0.1$. Solid and dotted lines are the RG-improved and one-loop results, respectively.

The study of the potential V_Λ shows that it is possible to have a nontrivial domain structure even in the symmetrical phase of the theory ($m_R^2>0$) [27]. Indeed, for some range of the parameters of the theory, it may happen that, although $m_R^2>0$, the squared bare mass is negative. In this case, the potential has a double-well structure with two symmetrical nonzero minima for scales Λ greater than a critical one $\Lambda_{\rm cr}$, and has a unique minimum at $\phi=0$ for smaller values of Λ (see Fig. 1). The interpretation of this fact is that the average field fluctuates around zero for scales $\zeta>\Lambda_{\rm cr}^{-1}$ or around the nonzero minima for scales $\zeta<\Lambda_{\rm cr}^{-1}$. The symmetrical phase, therefore, contains domains of size $\zeta\approx\Lambda_{\rm cr}^{-1}$. We remark that this phenomenon takes place in the symmetrical phase of the theory, and should not be confused with spontaneous symmetry breaking (SSB).

On the other hand, when SSB takes place ($m_R^2<0$), both the renormalized one-loop potential and wave function renormalization develop an imaginary part for $\Lambda<\Lambda_{\rm SSB}\equiv\sqrt{-m_R^2-\frac{1}{2}\lambda_R\phi^2}$. These imaginary parts generate nonreal terms in the equations of motion, and are artifacts of the adiabatic approximation. They have nothing to do with the noise terms in the CTP effective action that we have mentioned before. It is worth noting that the wave function renormalization diverges as Λ approaches $\Lambda_{\rm SSB}$ from above. This clearly shows that the adiabatic approximation is not adequate to describe the temporal evolution of the order parameter neither in the vicinity of $\Lambda_{\rm SSB}$ nor for $\Lambda<\Lambda_{\rm SSB}$.

As pointed out in Ref. [28], the imaginary part in the effective potential is a signal for instabilities towards the formation of domains of size at least as great as $\sqrt{-m_R^2}$. This issue has been addressed using the Hartree approximation in Ref. [13] where it was shown that the size of the domains for very weakly coupled theories can be much larger than the zero temperature correlation length $\sqrt{-m_R^2}$.

III. THE EXACT CTP RENORMALIZATION GROUP EQUATION

In this section we shall derive an exact (nonperturbative) renormalization group equation for the flow of the CGEA. The approach follows that of Wegner and Houghton for Euclidean spacetime. We start by writing the CGEA for a scale $\Lambda-\delta\Lambda$, namely,

$$e^{iS_{\Lambda-\delta\Lambda}(\phi_+,\phi_-)}\equiv\int\prod_{\Lambda_0>|\vec{q}|>\Lambda-\delta\Lambda}\mathcal{D}[\phi_+(\vec{q},t)]\times\mathcal{D}[\phi_-(\vec{q},t)]e^{iS_{\rm cl}[\phi_+,\phi_-]}. \tag{16}$$

The modes to be integrated can be split into two parts: one within the shell $\Lambda>|\vec{q}|>\Lambda-\delta\Lambda$ and the other for modes with $\Lambda_0 > |\vec{q}| > \Lambda$. Expanding the action in powers of the modes within the shell, one has

$$e^{iS_{\Lambda-\delta\Lambda}(\phi_+,\phi_-)} = e^{iS_\Lambda(\phi_+,\phi_-)} \int \prod_{\Lambda>|\vec{q}|>\Lambda-\delta\Lambda} \mathcal{D}[\phi_+]\mathcal{D}[\phi_-] e^{i(S_1+S_2+S_3)} \times \exp\left[\frac{i}{2}\int{}' \frac{d^3q}{(2\pi)^3}\int dt \frac{d}{dt}(\phi_a(-\vec{q},t)\dot{\phi}_b(\vec{q},t)g_{ab})\right], \tag{17}$$

where

$$S_1 = \int dt \int{}' \frac{d^3q}{(2\pi)^3}\phi_a(\vec{q},t)\frac{\partial S_\Lambda}{\partial\phi_a(-\vec{q},t)},$$

$$S_2 = \frac{1}{2}\int dt dt' \int{}' \frac{d^3q}{(2\pi)^3}\phi_a(\vec{q},t) \times \frac{\partial^2 S_\Lambda}{\partial\phi_a(-\vec{q},t)\phi_b(\vec{q},t')}\phi_b(\vec{q},t'), \tag{18}$$

the prime in the momenta integrals meaning that integration is restricted to the shell. In the functional derivatives of S_Λ (which contains modes whose wave vectors satisfy $|\vec{q}|<\Lambda$) the modes within the shell are set to zero. We use the notation

$$\phi_a(\vec{q},t) = \begin{pmatrix}\phi_+(\vec{q},t)\\ \phi_-(\vec{q},t)\end{pmatrix}; \quad g_{ab} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{19}$$

The S_3 term is cubic in the modes within the shell, and it can be proved that it does not contribute in the limit $\delta\Lambda\to 0$ (basically, this is because one is doing a one-loop calculation for the shell modes). The functional integrals over the shell modes have the CTP boundary conditions. A comment about the last exponential factor in Eq. (17) is in order. Usually, one discards it because it is a surface term, but in the CTP formalism it must be kept since the boundary conditions are that $\phi_+(\vec{q},T) = \phi_-(\vec{q},T)$ with $T\to\infty$ for the modes $\vec{q}$ within the shell.

In order to evaluate the functional integrals we split the field as $\phi_a = \overline{\phi_a} + \varphi_a$ and impose the boundary conditions on the ''classical'' fields $\overline{\phi_\pm}$, i.e., they vanish in the past $-T$ (negative and positive frequencies, respectively) and match in the Cauchy surface at time T. The fluctuations φ_a vanish both in the past and in the future. The classical fields are solutions to

$$\left(-\frac{d^2}{dt^2} - q^2\right) g_{ab}\overline{\phi_a}(-\vec{q},t) + \int dt' \frac{\partial^2 S_{\rm int}}{\partial\varphi_a(-\vec{q},t)\partial\varphi_b(\vec{q}',t')}\overline{\phi_b}(-\vec{q}',t') = 0, \tag{20}$$

where we have split the CGEA as $S_\Lambda(\phi_\pm) = S_{\rm kin}(\phi_\pm) + S_{\rm int}(\phi_\pm)$ with

$$S_{\rm kin} = \int d^4x\left[\frac{1}{2}(\partial_\mu\phi_+)^2 + \frac{i\epsilon}{2}\phi_+^2\right] - \int d^4x\left[\frac{1}{2}(\partial_\mu\phi_-)^2 - \frac{i\epsilon}{2}\phi_-^2\right]. \tag{21}$$

As before, in the functional derivatives the modes within the shell are set to zero.

Let h_a be solutions to Eq. (20), vanishing in the past and satisfying an arbitrary normalization in the future, and let $\phi(\vec{q})$ be the common value the fields take in the future. We can then write

$$\overline{\phi_a}(\vec{q},t) = \phi(\vec{q})\frac{h_a(\vec{q},t)}{h_a(\vec{q},T)}. \tag{22}$$

We first integrate over the common value $\phi(\vec{q})$ and then we proceed with the functional integration over the fluctuations φ_a (both are Gaussian integrals with a ''source'' term). One finally gets

$$\Lambda\frac{\partial S_\Lambda}{\partial\Lambda} = \frac{i\Lambda}{2\delta\Lambda}\int{}' \frac{d^3q}{(2\pi)^3}\ln\left(\frac{\dot{h}_+(\vec{q},T)}{h_+(\vec{q},T)} - \frac{\dot{h}_-(\vec{q},T)}{h_-(\vec{q},T)}\right) + \frac{\Lambda}{2\delta\Lambda} \times \int{}' \frac{d^3q}{(2\pi)^3}\left(\frac{\dot{h}_+(\vec{q},T)}{h_+(\vec{q},T)} - \frac{\dot{h}_-(\vec{q},T)}{h_-(\vec{q},T)}\right)^{-1} \times\left(\int dt\frac{h_a(\vec{q},t)}{h_a(\vec{q},T)}\frac{\partial S_\Lambda}{\partial\varphi_a(-\vec{q},t)}\right)^2 - \frac{i\Lambda}{2\delta\Lambda} \times \ln\det{}'(A_{ab}) + \frac{\Lambda}{2\delta\Lambda}\int dt\, dt' \frac{d^3q}{(2\pi)^3}\frac{\partial S_\Lambda}{\partial\varphi_a(\vec{q},t')} \times A_{ab}^{-1}(-\vec{q},t;\vec{q}',t')\frac{\partial S_\Lambda}{\partial\varphi_b(\vec{q},t')}. \tag{23}$$

The 2×2 matrix A_{ab} has the elements

$$A_{++}(-\vec{q},t;\vec{q}',t') = \left(-\frac{d^2}{dt^2} - q^2 + i\epsilon\right)\delta(t-t')\delta^3(\vec{q}+\vec{q}') + \frac{\partial^2 S_{\rm int}}{\partial\varphi_+(-\vec{q},t)\partial\varphi_+(\vec{q}',t')},$$

$$A_{--}(-\vec{q},t;\vec{q}',t') = \left(\frac{d^2}{dt^2} + q^2 + i\epsilon\right)\delta(t-t')\delta^3(\vec{q}+\vec{q}') + \frac{\partial^2 S_{\rm int}}{\partial\varphi_-(-\vec{q},t)\partial\varphi_-(\vec{q}',t')},$$

$$A_{+-}(-\vec{q},t;\vec{q}',t') = A_{-+}(\vec{q}',t';-\vec{q},t) = \frac{\partial^2 S_{\rm int}}{\partial\varphi_+(-\vec{q},t)\partial\varphi_-(\vec{q}',t')}. \tag{24}$$

The primed determinant must be calculated as the product of the eigenvalues of A_{ab} in a space of functions with wave vectors within the shell $(\Lambda - \delta\Lambda < |\vec{q}| < \Lambda)$ and satisfying null conditions both in the past and in the future. Similar conditions are to be used to evaluate the inverse A_{ab}^{-1}.

The Eq. (23) is exact in the sense that no perturbative approximation has so far been used. It is the main result of the paper. It is similar to its Euclidean counterpart [16], but involves two fields and CTP boundary conditions. It contains *all* the information of the influence of the short wavelength modes on the long wavelength ones, and should be the starting point for a nonperturbative analysis of decoherence, dissipation, domain formation, and out of equilibrium evolution.

IV. DERIVATIVE EXPANSION

The overwhelming complexity of the exact renormalization group equation means that in practice one is compelled to use some sort of truncation. The usual ones are expansions in the number of powers of the fields (see Ref. [29] for a detailed analysis) or in derivatives of them [30–32]. In the following we shall make use of the derivative expansion approach.

We will prove that, within this approach, the exact RG equation (23) admits a solution of the form

$$S_\Lambda(\phi_+,\phi_-)=S_\Lambda(\phi_+)-S_\Lambda(\phi_-). \tag{25}$$

Clearly, this is not the most general form that can be imagined for the coarse-grained action because contributions involving mixing of both fields are not taken into account. The main drawback of this approach is, therefore, that as mentioned in Sec. II, we miss the stochastic aspects of the theory, such as dissipation and noise. However, the proposed form for the CGEA will be enough for studying the renormalization group flow of real time field theories.

The great technical advantage of the form equation (25) is that the second functional derivative of the action has no crossed terms, leading to a diagonal matrix A_{ab} whose determinant is easily computed as the product of two determinants, one for A_{++} and the other for A_{--}. Following Ref. [33], one can express both $\det' A_{++}$ and $\det' A_{--}$ as the product over momenta of a constant (momenta independent) times the mode $h(\vec{q},T)$ evaluated at the final time T. Therefore, the last term of the exact RG equation can be written as

$$\begin{aligned}\ln\det{}'(A_{ab})&=\ln[\det{}'(A_{++})\det{}'(A_{--})]\\&=\int{}'\frac{d^3q}{(2\pi)^3}\ln[h_+(\vec{q},T)h_-(\vec{q},T)].\end{aligned} \tag{26}$$

The first and the third terms can then be cast in the form of a single logarithm, and we arrive at

$$\begin{aligned}\Lambda\frac{\partial S_\Lambda}{\partial\Lambda}=&-\frac{i\Lambda}{2\delta\Lambda}\int{}'\frac{d^3q}{(2\pi)^3}\ln[h_-(\vec{q},T)\dot{h}_+(\vec{q},T)-h_+(\vec{q},T)\dot{h}_-(\vec{q},T)]+\frac{\Lambda}{2\delta\Lambda}\int{}'\frac{d^3q}{(2\pi)^3}\left(\frac{\dot{h}_+(\vec{q},T)}{h_+(\vec{q},T)}-\frac{\dot{h}_-(\vec{q},T)}{h_-(\vec{q},T)}\right)^{-1}\\&\times\left(\int dt\,\frac{h_a(\vec{q},t)}{h_a(\vec{q},T)}\frac{\partial S_\Lambda}{\partial\varphi_a(-\vec{q},t)}\right)^2+\frac{\Lambda}{2\delta\Lambda}\int dt\,dt'\int{}'\frac{d^3q}{(2\pi)^3}\frac{\partial S_\Lambda}{\partial\varphi_a(\vec{q},t)}A_{ab}^{-1}(-\vec{q},t;\vec{q},t')\frac{\partial S_\Lambda}{\partial\varphi_b(-\vec{q}t')}.\end{aligned} \tag{27}$$

Note that the equations for the two modes h_+ and h_- [Eq. (20)] simplify considerably, since the two equations are decoupled [a side point: it is easy to prove the one-loop result, Eq. (3), starting from the above equation (27)]. What we still have to prove is that the proposed form for the action makes the right-hand side (RHS) of the exact RG equation split in the same form.

Next, we make a derivative expansion of the interaction term. As our coarse graining explicitly breaks Lorentz invariance, we allow different coefficients for the temporal and spatial derivatives: namely,

$$\begin{aligned}S_{\rm int}(\phi_\pm)=\int d^4x[&-V_\Lambda(\phi_\pm)+\tfrac{1}{2}Z_\Lambda(\phi_\pm)\dot{\phi}_\pm^2-\tfrac{1}{2}Y_\Lambda(\phi_\pm)\\&\times(\vec{\nabla}\phi_\pm)^2+\cdots].\end{aligned} \tag{28}$$

We expand the fields around a time-dependent background: $\phi_\pm=\phi_\pm(t)+\varphi_\pm(\vec{x},t)$ and Fourier transform in space. We shall solve the Eq. (20) for the modes to zeroth order in the fluctuations, i.e., we equate terms in the equations for $h_\pm$ that are independent of $\varphi_\pm$'s. Since the first functional derivative of the CGEA (S') is linear in the fluctuations $\varphi_\pm$, we put $S'=0$ and keep the $\varphi_\pm$-independent contributions to $S''_{\rm int}$ After some little algebra and functional derivations, we get

$$\begin{aligned}&\frac{\partial^2 S_{\rm int}}{\partial\varphi(\vec{q},t)\partial\varphi(-\vec{q}',t')}\\&\quad=\left[-V''-\frac{1}{2}\dot{\phi}^2Z''-Yq^2-Z'\dot{\phi}\frac{d}{dt}-Z\frac{d^2}{dt^2}-\ddot{\phi}Z'+\cdots\right]\\&\qquad\times\delta(t-t')\delta^3(\vec{q}+\vec{q}'),\end{aligned} \tag{29}$$

where the primes denote derivation with respect to the field and the ellipsis denotes terms linear in the fluctuations. In this expression and hereafter, we omit (unless explicitly stated) the $\pm$ subscripts in the background fields $\phi_\pm(t)$, in the potential $V_\Lambda(\phi_\pm(t))$, and in the wave function factors $Z_\Lambda(\phi_\pm(t))$ and $Y_\Lambda(\phi_\pm(t))$. Note that the effective mass of the modes depends on the time-dependent background $\phi(t)$. The equations of motion for the modes h_a become localized and take the form of harmonic oscillators with variable frequency and a damping term. The boundary conditions to be imposed are the aforementioned CTP ones.

If one defines new modes as $f(\vec{q},t)=(1+Z_\Lambda)^{1/2}h(\vec{q},t)$, the damping terms cancel out and the new modes are harmonic oscillators with frequency

$$w_q^2(t)=q^2\frac{1+Y_\Lambda}{1+Z_\Lambda}+\frac{V''_\Lambda}{1+Z_\Lambda}+\frac{1}{4}\frac{Z'^2_\Lambda}{(1+Z_\Lambda)^2}\dot{\phi}^2+\frac{1}{2}\frac{Z'_\Lambda}{1+Z_\Lambda}\ddot{\phi}. \tag{30}$$

Using an adiabatic expansion for the modes,

$$h_\pm(\vec{q},t)=(1+Z_\Lambda)^{-1/2}\frac{1}{\sqrt{2W_\pm(\vec{q},t)}}e^{\pm i\int_{-T}^{t}W_\pm(\vec{q},t')dt'}, \tag{31}$$

we can evaluate the logarithmic term in the RHS of the exact RG equation (27). The other terms are quadratic in the fluctuations and do not contribute to the order we are working. We have

$$h_-(\vec{q},T)\dot{h}_+(\vec{q},T)-h_+(\vec{q},T)\dot{h}_-(\vec{q},T)=\exp\left\{+i\int_{-T}^{T}[W_+(\vec{q},t)-W_-(\vec{q},t)]dt\right\}$$
$$\times\left\{\frac{\left[-\frac{Z'_+}{2(1+Z_+)}\dot{\phi}_+-\frac{\dot{W}_+}{2W_+}+iW_+\right]-\left[-\frac{Z'_-}{2(1+Z_-)}\dot{\phi}_--\frac{\dot{W}_-}{2W_-}-iW_-\right]}{2\sqrt{W_+W_-(1+Z_+)(1+Z_-)}}\right\}_{t=T}. \tag{32}$$

Note that, as in the one-loop case, the plus and minus fields do decouple, up to a factor evaluated at $t=T$. This factor is just a surface contribution which, upon taking logarithm, is irrelevant for the equations of motion. On the contrary, the first factor does depend on the whole history of the fields and is consistent with the proposal for the CGEA, Eq. (25). Hence, the RGE can be casted in the form

$$\Lambda\int dt\left\{\left[-\frac{dV_\Lambda(\phi_+)}{d\Lambda}+\frac{1}{2}\frac{dZ_\Lambda(\phi_+)}{d\Lambda}\dot{\phi}_+^2\right]-\left[-\frac{dV_\Lambda(\phi_-)}{d\Lambda}+\frac{1}{2}\frac{dZ_\Lambda(\phi_-)}{d\Lambda}\dot{\phi}_-^2\right]\right\}$$
$$=\frac{\Lambda}{2\delta\Lambda}\int'\frac{d^3q}{(2\pi)^3}\int[W_+(\vec{q},t)-W_-(\vec{q},t)]dt. \tag{33}$$

In the adiabatic expansion, the W's read

$$W^2=A_\Lambda+B_\Lambda\dot{\phi}^2(t)+C_\Lambda\ddot{\phi}(t), \tag{34}$$

where the coefficients are

$$A_\Lambda=\Lambda^2\frac{1+Y_\Lambda}{1+Z_\Lambda}+\frac{V''_\Lambda}{1+Z_\Lambda},$$

$$B_\Lambda=\frac{Z'^2_\Lambda}{4(1+Z_\Lambda)^2}+\frac{5A'^2_\Lambda}{16A^2_\Lambda}-\frac{A''_\Lambda}{4A_\Lambda},$$

$$C_\Lambda=\frac{Z'_\Lambda}{2(1+Z_\Lambda)}-\frac{A'_\Lambda}{4A_\Lambda}. \tag{35}$$

Integrating by parts, we get

$$\int dt\left\{-\Lambda\frac{dV_\Lambda}{d\Lambda}+\frac{1}{2}\Lambda\frac{dZ_\Lambda}{d\Lambda}\dot{\phi}^2\right\}=\frac{\Lambda^3}{4\pi^2}\int dt\left\{\sqrt{A_\Lambda}+\frac{1}{2}\dot{\phi}^2\left[\frac{B_\Lambda}{\sqrt{A_\Lambda}}-\left(\frac{C_\Lambda}{\sqrt{A_\Lambda}}\right)'\right]\right\}. \tag{36}$$

Therefore, the dependence of the potential and the wave function renormalization on the infrared scale is given by

$$\Lambda\frac{dV_\Lambda}{d\Lambda}=-\frac{\Lambda^3}{4\pi^2}\sqrt{\Lambda^2\frac{1+Y_\Lambda}{1+Z_\Lambda}+\frac{V''_\Lambda}{1+Z_\Lambda}},$$

$$\Lambda\frac{dZ_\Lambda}{d\Lambda}=\frac{\Lambda^3}{4\pi^2}\left[\frac{B_\Lambda}{\sqrt{A_\Lambda}}-\left(\frac{C_\Lambda}{\sqrt{A_\Lambda}}\right)'\right]. \tag{37}$$

Remember that these equations are valid both for the ϕ_+ field and for the ϕ_- field.

The above equations are the main result of this section. They describe the flow of the coarse-grained action with the infrared scale in the derivative expansion of the exact CTP renormalization group equation. It is interesting to note that the higher derivative terms modify the differential equation for the effective potential.

We have obtained two equations for the three independent, unknown functions V_Λ, Z_Λ, and Y_Λ. In order to find an additional relation between the spatial and temporal wave funtion renormalization functions Z_Λ and Y_Λ, it is necessary to write the exact RGE up to quadratic order in the fluctuations. We will not present this long calculation here. For simplicity, we will assume that Z_Λ and Y_Λ are small numbers, and therefore, we will set them to zero on the RHS of Eq. (37). This assumption will be confirmed by the numerical calculations performed in the symmetric phase of the

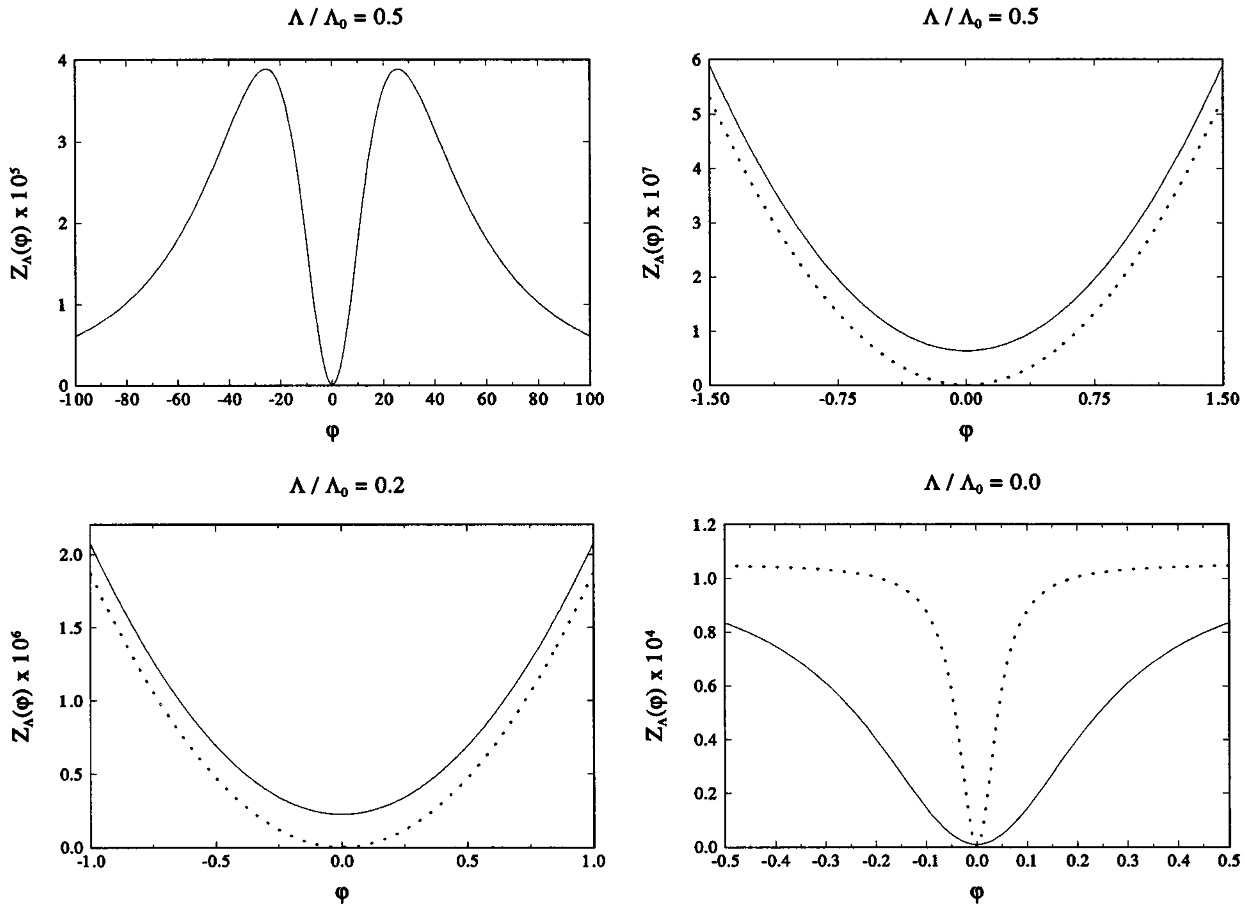

FIG. 2. The coarse-grained wave funtion renormalization Z_Λ for $\Lambda_0=10$, $m_R^2=10^{-4}$, and $\lambda_R=0.1$. Solid and dotted lines are the RG-improved and one-loop results, respectively.

theory. Note that in this approximation we recover the RG-improved equation proposed in Ref. [27] for the coarse-grained effective potential.

There are other points which are worth noting. The first one is that, when we substitute V_Λ, Z_Λ, and Y_Λ by their classical values $V_\Lambda=V$, $Z_\Lambda=Y_\Lambda=0$, on the RHS of both equations we obtain the one-loop evolution equations [Eq. (15)]. The second comment is about renormalization. In the one-loop calculation it was possible to take the limit $\Lambda_0\to\infty$. The infinities were absorbed into the bare mass and coupling constant. There was no need for wave function renormalization. However, in this nonperturbative calculation it is not possible to renormalize the theory (as is the case for Hartree, Gaussian, and $1/N$ approximations). For these reasons we keep Λ_0 as a large (compared with the mass) but finite number.

As an illustration we consider the $\lambda\phi^4$ field theory. The differential equations must be solved with the classical initial conditions $V_{\Lambda_0}=V$, $Z_{\Lambda_0}=0$, and $Y_{\Lambda_0}=0$. To this end we have written a simple code which evolves the equations from the UV scale down to the desired IR scale. We have plotted the results in Figs. 1 and 2, where we show the difference between the one-loop and the RG-improved solution for the effective potential and the wave function renormalization. Note that, at least in the symmetric phase of the theory, the results are consistent with the assumption $Z_\Lambda \ll 1$.

Once the functions V_Λ and Z_Λ are known, one can write the effective dynamical equations for the order parameter. These equations will be valid as long as ϕ is slowly varying and $Z_\Lambda \ll 1$. Therefore, as in the one-loop approximation, the derivative expansion may be inadequate in the case of SSB for scales in the vicinity of and lower than Λ_{SSB}.

V. CONCLUSIONS

The CTP coarse-grained effective action contains all the information about the influence of the short wavelength sector on the long wavelength sector of the theory. In principle, it is possible to derive from it a Langevin equation for the order parameter. This Langevin equation can be used to analyze domain formation and growth and, in general, the nonequilibrium aspects of phase transitions.

In this paper we have derived an exact evolution equation for the dependence of the CGEA with the coarse-graining scale. This renormalization group equation [Eq. (23)] is our main result. We expect this equation to be a useful tool to generate nonperturbative approximations for the CGEA.

In order to show a simple application of the exact CTP renormalization group equation, we have solved it using a derivative expansion. We have obtained a RG improvement to the effective potential $V_\Lambda(\phi)$, that coincides with the one proposed in Ref. [27], and an improvement to the one-loop wave function renormalization Z_Λ.

Within the derivative expansion approach, the CGEA is of the form $S_\Lambda(\phi_+,\phi_-)=S_\Lambda(\phi_+)-S_\Lambda(\phi_-)$, i.e., it contains neither dissipative nor noise terms. Only diffusive effects are included. As far as a calculation beyond the adiabatic approximation is concerned, we expect that, as soon as we decrease the scale from Λ_0, dissipative and noise terms will grow up: the CGEA will develop an imaginary part (related to noise) and a real part containing interactions between the $\phi_\pm$ fields (dissipation). This can be easily checked both in the one-loop approximation and from the exact RGE. Indeed, Eq. (3) shows that the one-loop CGEA is, in general, nonreal unless one uses the adiabatic approximation. On the other hand, the real and imaginary parts of the CGEA are not decoupled in Eq. (23), and a nonvanishing real part at $\Lambda=\Lambda_0$ will induce an imaginary part at lower scales.

We are currently investigating these issues. In particular, we are interested in finding a nonperturbative, Λ-dependent, fluctuation-dissipation relation from the exact RGE. Extensions to the cases of finite temperature and curved spaces are also under investigation.

ACKNOWLEDGMENTS

We would like to thank B. L. Hu, F. C. Lombardo, J. P. Paz, and F. M. Spedalieri for useful discussions. F.D.M. enjoyed the hospitality of the General Relativity Group at the University of Maryland, where part of this work was done. This work was supported by Universidad de Buenos Aires, Consejo Nacional de Investigaciones Cientificas y Técnicas (CONICET), and Fundación Antorchas.

[1] E. Kolb and M. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).
[2] G. Mazenko, W. Unruh, and R. Wald, Phys. Rev. D **31**, 273 (1985).
[3] D. Boyanovsky *et al.*, Phys. Rev. D **52**, 6805 (1995).
[4] B. Bergerhoff and C. Wetterich, Nucl. Phys. **B440**, 171 (1995).
[5] M. Gleiser and E. Kolb, Phys. Rev. Lett. **69**, 1304 (1992); Phys. Rev. D **48**, 1560 (1993).
[6] K. Rajagopal and F. Wilczek, Nucl. Phys. **B379**, 395 (1993).
[7] F. Cooper *et al.*, Phys. Rev. D **51**, 2377 (1995).
[8] D. Boyanovsky, H. J. de Vega, and R. Holman, Phys. Rev. D **51**, 734 (1995).
[9] J. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961); L. V. Keldysh, Zh. Éksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].
[10] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
[11] E. Calzetta and B. L. Hu, Phys. Rev. D **49**, 6636 (1994); **52**, 6770 (1995).
[12] D. Boyanovsky and H. J. de Vega, Phys. Rev. D **47**, 2343 (1993).
[13] D. Boyanovsky, D. S. Lee, and A. Singh, Phys. Rev. D **48**, 800 (1993).
[14] J. Borrill and M. Gleiser, Phys. Rev. D **51**, 4111 (1995).
[15] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1993); J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8.
[16] F. Wegner and A. Houghton, Phys. Rev. A **8**, 401 (1973).
[17] B. L. Hu, in *Relativity and Gravitation: Classical and Quantum*, Proceedings of SILARG VII, Cocoyoc, Mexico, 1990, edited by J. C. D'Olivo *et al.* (World Scientific, Singapore, 1991).
[18] F. Lombardo and F. D. Mazzitelli, Phys. Rev. D **53**, 2001 (1996).
[19] C. Wetterich, Nucl. Phys. **B352**, 529 (1991); Phys. Lett. B **301**, 90 (1993); Z. Phys. C **60**, 461 (1993).
[20] S. B. Liao and J. Polonyi, Ann. Phys. (N.Y.) **222**, 122 (1993).
[21] J. Polchinski, Nucl. Phys. **B231**, 269 (1984).
[22] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B270**, 687 (1986).
[23] T. R. Morris, Int. J. Mod. Phys. A **9**, 2411 (1994).
[24] M. Bonini, M. D'Attanasio, and G. Marchesini, Nucl. Phys. **B409**, 441 (1993).
[25] M. D'Attanasio and M. Pietroni, Nucl. Phys. **B472**, 711 (1996).
[26] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, London, 1982).
[27] S. B. Liao, J. Polonyi, and D. Xu, Phys. Rev. D **51**, 748 (1995); S. B. Liao and S. Strickland, *ibid.* **52**, 3653 (1995).
[28] E. J. Weinberg and A. Wu, Phys. Rev. D **36**, 2474 (1987).
[29] T. R. Morris, Phys. Lett. B **334**, 355 (1994).
[30] T. R. Morris, Phys. Lett. B **329**, 241 (1994).
[31] R. Ball *et al.*, Phys. Lett. B **347**, 80 (1995); P. Haagensen *et al.*, *ibid.* **323**, 330 (1994).
[32] J. Adams *et al.*, Mod. Phys. Lett. A **10**, 2367 (1995).
[33] I. Gel'fand and A. Yaglom, J. Math. Phys. (N.Y.) **1**, 48 (1960); S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985).