Fractal dimensions and scaling laws in the interstellar medium: A new field theory approach

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We develop a field theoretical approach to the cold interstellar medium (ISM). We show that a nonrelativistic self-gravitating gas in thermal equilibrium with a variable number of atoms or fragments is exactly equivalent to a field theory of a single scalar field $\phi(\vec{x})$ with an exponential self-interaction. We analyze this field theory perturbatively and nonperturbatively through the renormalization group approach. We show a *scaling* behavior (critical) for a continuous range of the temperature and of the other physical parameters. We derive in this framework the scaling relation $\Delta M(R) \sim R^{d_H}$ for the mass on a region of size R, and $\Delta v \sim R^q$ for the velocity dispersion where $q = \frac{1}{2}(d_H - 1)$. For the density-density correlations we find a power-law behavior for large distances $\sim |\vec{r}_1 - \vec{r}_2|^{2d_H - 6}$. The fractal dimension d_H turns out to be related with the critical exponent ν of the correlation length by $d_H = 1/\nu$. The renormalization group approach for a single component scalar field in three dimensions states that the long-distance critical behavior is governed by the (nonperturbative) Ising fixed point. The corresponding values of the scaling exponents are $\nu = 0.631\ldots$, $d_H = 1.585\ldots$, and $q = 0.293\ldots$. Mean field theory yields for the scaling exponents $\nu = 1/2$, $d_H = 2$, and $q = 1/2$. Both the Ising and the mean field values are compatible with the present ISM observational data: $1.4 \leqslant d_H \leqslant 2$, $0.3 \leqslant q \leqslant 0.6$. As typical in critical phenomena, the scaling behavior and critical exponents of the ISM can be obtained without dealing with the dynamical (time-dependent) behavior. [S0556-2821(96)02422-8]

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I. INTRODUCTION AND RESULTS

The interstellar medium (ISM) is a gas essentially formed by atomic (H I) and molecular (H_2) hydrogen, distributed in cold ($T \sim$ 5–50 K) clouds, in a very inhomogeneous and fragmented structure. These clouds are confined in the galactic plane and, in particular, along the spiral arms. They are distributed in a hierarchy of structures, of observed masses from 1 $M_\odot$ to 10^6 $M_\odot$. The morphology and kinematics of these structures are traced by radio astronomical observations of the H I hyperfine line at the wavelength of 21 cm, and of the rotational lines of the CO molecule (the fundamental line being at 2.6 mm in wavelength), and many other less abundant molecules. Structures have been measured directly in emission from 0.01–100 pc, and there is some evidence in VLBI (very long-based interferometry) H I absorption of structures as low as 10^{-4} pc = 20 AU (3×10^{14} cm). The mean density of structures is roughly inversely proportional to their sizes, and vary between 10 and 10^5 atoms/cm^3 (significantly above the mean density of the ISM which is about 0.1 atoms/cm^3 or 1.6×10^{-25} g/cm^3). Observations of the ISM revealed remarkable relations between the mass, the radius, and velocity dispersion of the various regions, as first noticed by Larson [1], and since then confirmed by many other independent observations (see, for example, Ref. [2]). From a compilation of well-established samples of data for many different types of molecular clouds of maximum linear dimension (size) R, mass fluctuation ΔM, and internal velocity dispersion Δv in each region:

$$\Delta M(R) \sim R^{d_H}, \quad \Delta v \sim R^q, \tag{1.1}$$

over a large range of cloud sizes, with 10^{-4}–10^{-2} pc$\leqslant R \leqslant 100$ pc,

$$1.4 \leqslant d_H \leqslant 2, \quad 0.3 \leqslant q \leqslant 0.6. \tag{1.2}$$

These *scaling* relations indicate a hierarchical structure for the molecular clouds which is independent of the scale over the above-cited range; above 100 pc in size, corresponding to giant molecular clouds, larger structures will be destroyed by galactic shear.

These relations appear to be *universal*, the exponents d_H, q are almost constant over all scales of the Galaxy, and whatever be the observed molecule or element. These properties of interstellar cold gas are supported first of all from observations (and for many different tracers of cloud structures: dark globules using ^{13}CO, since the more abundant isotopic species ^{12}CO is highly optically thick, dark cloud cores using HCN or CS as density tracers, giant molecular clouds using ^{12}CO, H I to trace more diffuse gas, and even cold dust emission in the far infrared). Nearby molecular clouds are observed to be fragmented and self-similar in projection over a range of scales and densities of at least 10^4, and perhaps up to 10^6.

The physical origin as well as the interpretation of the scaling relations (1.1) are not theoretically understood. The theoretical derivation of these relations has been the subject of many proposals and controversial discussions. It is not our aim here to account for all the proposed models of the ISM and we refer the reader to Refs. [2] for a review.

The physics of the ISM is complex, especially when we consider the violent perturbations brought by star formation. Energy is then poured into the ISM either mechanically through supernovae explosions, stellar winds, bipolar gas flows, etc. or radiatively through star light, heating or ionizing the medium, directly or through heated dust. Relative velocities between the various fragments of the ISM exceed their internal thermal speeds, shock fronts develop, and are highly dissipative; radiative cooling is very efficient, so that globally the ISM might be considered isothermal on large scales. Whatever the diversity of the processes, the universality of the scaling relations suggests a common mechanism underlying the physics. We propose that self-gravity is the main force at the origin of the structures, that can be perturbed locally by heating sources. Observations are compatible with virialized structures at all scales. Moreover, it has been suggested that the molecular cloud ensemble is in isothermal equilibrium with the cosmic background radiation at $T \sim 3$ K in the outer parts of galaxies, devoid of any star and heating sources [3]. This colder isothermal medium might represent the ideal frame to understand the role of self-gravity in shaping the hierarchical structures. Our aim is to show that the scaling laws obtained are then quite stable to perturbations.

Till now, no theoretical derivation of the scaling laws [Eq. (1.1)] has been provided in which the values of the exponents are *obtained* from the theory (and not just taken from outside or as a starting input or hypothesis).

The aim of these authors is to develop a theory of the cold ISM. A first step in this goal is to provide a theoretical derivation of the scaling laws [Eq. (1.1)], in which the values of the exponents d_H, q are *obtained* from the theory. For this purpose, we will implement for the ISM the powerful tool of field theory and the Wilson's approach to critical phenomena [4,13].

We consider a gas of nonrelativistic atoms interacting with one another through Newtonian gravity and which are in thermal equilibrium at temperature T. We work in the grand canonical ensemble, allowing for a variable number of particles N.

Then, we show that this system is exactly equivalent to a field theory of a single scalar field $\phi(\vec{x})$ with exponential interaction. We express the grand canonical partition function $\mathcal{Z}$ as

$$\mathcal{Z} = \int \int \mathcal{D}\phi e^{-S[\phi(.)]}, \qquad (1.3)$$

where

$$S[\phi(.)] \equiv \frac{1}{T_{\text{eff}}} \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 - \mu^2 e^{\phi(\vec{x})} \right],$$

$$T_{\text{eff}} = 4\pi \frac{Gm^2}{T}, \quad \mu^2 = \sqrt{\frac{2}{\pi}} z G m^{7/2} \sqrt{T}, \qquad (1.4)$$

m stands for the mass of the atoms and z for the fugacity. We show that in the ϕ-field language, the particle density expresses as

$$\langle \rho(\vec{r}) \rangle = -\frac{1}{T_{\text{eff}}} \langle \nabla^2 \phi(\vec{r}) \rangle = \frac{\mu^2}{T_{\text{eff}}} \langle e^{\phi(\vec{r})} \rangle, \qquad (1.5)$$

where angular brackets mean a functional average over $\phi(.)$ with statistical weight $e^{S[\phi(.)]}$. Density correlators are written as

$$C(\vec{r}_1, \vec{r}_2) \equiv \langle \rho(\vec{r}_1) \rho(\vec{r}_2) \rangle - \langle \rho(\vec{r}_1) \rangle \langle \rho(\vec{r}_2) \rangle$$
$$= \frac{\mu^4}{T_{\text{eff}}^2} [\langle e^{\phi(\vec{r}_1)} e^{\phi(\vec{r}_2)} \rangle - \langle e^{\phi(\vec{r}_1)} \rangle \langle e^{\phi(\vec{r}_2)} \rangle]. \qquad (1.6)$$

The ϕ field defined by Eqs. (1.3) and (1.4) has remarkable properties under scale transformations

$$\vec{x} \to \vec{x}_\lambda \equiv \lambda \vec{x},$$

where λ is an arbitrary real number. For any solution $\phi(\vec{x})$ of the stationary point equations,

$$\nabla^2 \phi(\vec{x}) + \mu^2 e^{\phi(\vec{x})} = 0, \qquad (1.7)$$

there is a family of dilated solutions of the same Eq. (1.7), given by

$$\phi_\lambda(\vec{x}) \equiv \phi(\lambda \vec{x}) + \ln \lambda^2.$$

In addition, $S[\phi_\lambda(.)] = \lambda^{2-D} S[\phi(.)]$.

We study the field theory (1.3) and (1.4) both perturbatively and nonperturbatively.

The computation of the thermal fluctuations through the evaluation of the functional integral equation (1.3) is quite nontrivial. We use the scaling property as a guiding principle. In order to build a perturbation theory in the dimensionless coupling $g \equiv \sqrt{\mu T_{\text{eff}}}$ we look for stationary points of Eq. (1.4). We compute the density correlator equation (1.6) to leading order in g. For large distances it behaves as

$$C(\vec{r}_1, \vec{r}_2) \underset{|\vec{r}_1 - \vec{r}_2| \to \infty}{\simeq} \frac{\mu^4}{32\pi^2 |\vec{r}_1 - \vec{r}_2|^2}$$
$$+ O(|\vec{r}_1 - \vec{r}_2|^{-3}). \qquad (1.8)$$

We analyze further this theory with the renormalization group approach. Such nonperturbative approach is the more powerful framework to derive scaling behaviors in field theory [4,6,7].

We show that the mass contained in a region of volume $V = R^3$ scales as

$$\langle M(R) \rangle = m \int^R \langle e^{\phi(\vec{x})} \rangle d^3x$$
$$\simeq m V a + m \frac{K}{1-\alpha} R^{1/\nu} + \cdots,$$

and the mass fluctuation, $[\Delta M(R)]^2 = \langle M^2 \rangle - \langle M \rangle^2$, scales as

$$\Delta M(R) \sim R^{d_H}.$$

Here, ν is the correlation length critical exponent for the ϕ theory (1.3) and a and K are constants. Moreover,

$$\langle\rho(\vec{r})\rangle = ma + m\frac{K}{4\pi\nu(1-\alpha)}\ r^{(1/\nu)-3}$$
$$\text{for } r \text{ of order } \sim R. \tag{1.9}$$

The scaling exponent ν can be identified with the inverse Haussdorf (fractal) dimension d_H of the system

$$d_H = \frac{1}{\nu}.$$

In this way, $\Delta M \sim R^{d_H}$ according to the usual definition of fractal dimensions [8].

From the renormalization group analysis, the density-density correlators (1.6) result to be

$$C(\vec{r}_1,\vec{r}_2) \sim |\vec{r}_1 - \vec{r}_2|^{(2/\nu)-6}. \tag{1.10}$$

Computing the average gravitational potential energy and using the virial theorem yields, for the velocity dispersion,

$$\Delta v \sim R^{1/2[(1/\nu)-1]}.$$

This gives a new scaling relation between the exponents d_H and q:

$$q = \frac{1}{2}\left(\frac{1}{\nu}-1\right) = \frac{1}{2}(d_H - 1).$$

The perturbative calculation (1.8) yields the mean field value for ν [9]. That is,

$$\nu = \frac{1}{2}, \quad d_H = 2, \quad \text{and } q = \frac{1}{2}. \tag{1.11}$$

We find scaling behavior in the ϕ theory for a *continuum set* of values of μ^2 and T_{eff}. The renormalization group transformation amounts to replace the parameters μ^2 and T_{eff} in βH and $S[\phi(.)]$ by the effective ones at the scale L in question.

The renormalization group approach applied to a *single* component scalar field in three space dimensions indicates that the long-distance critical behavior is governed by the (nonperturbative) Ising fixed point [4,6,7]. Very probably, there are no further fixed points [10]. The scaling exponents associated to the Ising fixed point are

$$\nu = 0.631\ldots, \quad d_H = 1.585\ldots, \quad \text{and } q = 0.293\ldots. \tag{1.12}$$

Both the mean field (1.11) and the Ising (1.12) numerical values are compatible with the present observational values (1.1) and (1.2).

The theory presented here also predicts a power-law behavior for the two-point ISM density correlation function [see Eq. (1.10), $2d_H - 6 = -2.830\ldots$, for the Ising fixed point and $2d_H - 6 = -2$ for the mean field exponents], that should be compared with observations. Previous attempts to derive correlation functions from observations were not entirely conclusive, because of lack of dynamical range [11], but much more extended maps of the ISM could be available soon to test our theory. In addition, we predict an independent exponent for the gravitational potential correlations ($\sim r^{-1-\eta}$, where $\eta_{\mathrm{Ising}} = 0.037\ldots$ and $\eta_{\text{mean field}} = 0$ [6]), which could be checked through gravitational lens observations in front of quasars.

The mass parameter μ [see Eq. (1.4)] in the ϕ theory turns to coincide at the tree level with the inverse of the Jeans length

$$\mu = \sqrt{\frac{12}{\pi}}\frac{1}{d_J}.$$

We find that in the scaling domain the Jeans distance d_J grows as $\langle d_J \rangle \sim R$. This shows that the Jeans distance *scales* with the *size* of the system and, therefore, the instability is present for all sizes R. Had d_J been of order larger than R, the Jeans instability would be absent.

The gravitational gas in thermal equilibrium explains quantitatively the observed scaling laws in the ISM. This fact does not exclude turbulent phenomena in the ISM. Fluid flows (including turbulent regimes) are probably relevant in the dynamics (time-dependent processes) of the ISM. As usual in critical phenomena [4,6], the equilibrium scaling laws can be understood for the ISM without delving into the dynamics. A further step in the study of the ISM will be to include the dynamical (time-dependent) description within the field theory approach presented in this paper.

If the ISM is considered as a flow, the Reynolds number Re_{ISM} on scales $L \sim 100$ pc has a very high value of the order of 10^6. This led to the suggestion that the ISM (and the Universe in general) could be *modeled* as a turbulent flow [12]. (Larson [1] first observed that the exponent in the power-law relation for the velocity dispersion is not greatly different from the Kolmogorov value 1/3 for subsonic turbulence.)

It must be noticed that the turbulence hypothesis for the ISM is based on the comparison of the ISM with the results known for incompressible flows. However, the physical conditions in the ISM are very different from those of incompressible flows in the laboratory. (And the study of ISM turbulence needs more complete and enlarged investigation than those performed until now based in the concepts of flow turbulence in the laboratory.) In addition to the fact that the ISM exhibits large density fluctuations on all scales, and the observed fluctuations are highly supersonic (thus the ISM cannot be viewed as an ‘‘incompressible’’ and ‘‘subsonic’’ flow), and in addition to other differences, an essential feature to point out is that the long-range, self-gravity interaction present in the ISM is completely absent in the studies of flow turbulence. In any case, in a satisfactory theory of the ISM, it should be possible to extract the behaviors of the ISM (be turbulent or whatever) from the theory as a result, instead to be introduced as a starting input or hypothesis.

This paper is organized as follows. In Sec. II we develop the field theory approach to the gravitational gas. A short-distance cutoff is naturally present here and prevents zero-distance gravitational collapse singularities (which would be unphysical in the present case). Here, the cutoff theory is physically meaningful. The gravitational gas is also treated in a D-dimensional space.

In Sec. III we study the scaling behavior and thermal fluctuations both in perturbation theory and nonperturbatively (renormalization group approach). $g^2 \equiv \mu T_{\mathrm{eff}}$ acts as the dimensionless coupling constant for the nonlinear fluctuations of the field ϕ. We show that these fluctuations are massless and that the theory scales (behaves critically) for a continuous range of values $\mu^2 T_{\rm eff}$. Thus, changing μ^2 and $T_{\rm eff}$ keeps the theory at *criticality*. The renormalization group analysis made in Sec. III confirms such results. We also treat (Sec. III E) the two-dimensional case making contact with random surfaces and their fractal dimensions.

Discussion and remarks are presented in Sec. IV. External gravity forces to the gas, like stars, are shown *not* to affect the scaling behavior of the gas. That is, the scaling exponents q, d_H are solely governed by fixed points and hence, they are stable under gravitational perturbations. In addition, we generalize the ϕ theory to a gas formed by several types of atoms with different masses and fugacities. Again, the scaling exponents are shown to be identical to the gravitational gas formed of identical atoms.

The differences between the critical behavior of the gravitational gas and those in spin models (and other statistical models in the same universality class) are also pointed out in Sec. IV.

II. FIELD THEORY APPROACH TO THE GRAVITATIONAL GAS

Let us consider a gas of nonrelativistic atoms with mass m interacting only through Newtonian gravity and which are in thermal equilibrium at temperature $T \equiv \beta^{-1}$. We shall work in the grand canonical ensemble, allowing for a variable number of particles N.

The grand partition function of the system can be written as

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int \cdots \int \prod_{l=1}^{N} \frac{d^3 p_l d^3 q_l}{(2\pi)^3} e^{-\beta H_N}, \tag{2.1}$$

where

$$H_N = \sum_{l=1}^{N} \frac{p_l^2}{2m} - G m^2 \sum_{1 \le l < j \le N} \frac{1}{|\vec{q}_l - \vec{q}_j|}, \tag{2.2}$$

G is Newton's constant, and z is the fugacity.

The integrals over the momenta $p_l, (1 \le l \le N)$ can be performed explicitly in Eq. (2.1) using

$$\int \frac{d^3 p}{(2\pi)^3} \exp\left(-\frac{\beta p^2}{2m}\right) = \left(\frac{m}{2\pi\beta}\right)^{3/2}.$$

We thus find,

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \left[z \left(\frac{m}{2\pi\beta}\right)^{3/2} \right]^N \int \cdots \int \prod_{l=1}^{N} d^3 q_l \times \exp\left(\beta G m^2 \sum_{1 \le l < j \le N} \frac{1}{|\vec{q}_l - \vec{q}_j|} \right). \tag{2.3}$$

We proceed now to recast this many-body problem into a field-theoretical form [13–16].

Let us define the density

$$\rho(\vec{r}) = \sum_{j=1}^{N} \delta(\vec{r} - \vec{q}_j), \tag{2.4}$$

such that we can rewrite the potential energy in Eq. (2.3) as

$$\frac{1}{2} \beta G m^2 \sum_{1 \le l \ne j \le N} \frac{1}{|\vec{q}_l - \vec{q}_j|} = \frac{1}{2} \beta G m^2 \int_{|\vec{x} - \vec{y}| > a} \frac{d^3 x d^3 y}{|\vec{x} - \vec{y}|} \rho(\vec{x}) \rho(\vec{y}). \tag{2.5}$$

The cutoff a in the right-hand side (RHS) is introduced in order to avoid self-interacting divergent terms. However, such divergent terms would contribute to $\mathcal{Z}$ by an infinite multiplicative factor that can be factored out.

By using

$$\nabla^2 \frac{1}{|\vec{x} - \vec{y}|} = -4\pi \delta(\vec{x} - \vec{y})$$

and partial integration, we can now represent the exponent of the potential energy equation (2.5) as a functional integral [14]

$$\exp\left(\tfrac{1}{2} \beta G m^2 \int \frac{d^3 x d^3 y}{|\vec{x} - \vec{y}|} \rho(\vec{x}) \rho(\vec{y}) \right) = \int \int \mathcal{D}\xi \exp\left(-\tfrac{1}{2} \int d^3 x (\nabla \xi)^2 + 2m\sqrt{\pi G \beta} \int d^3 x\, \xi(\vec{x}) \rho(\vec{x}) \right). \tag{2.6}$$

Inserting this expression into Eq. (2.3) and using Eq. (2.4) yields

$$\begin{aligned}
\mathcal{Z} &= \sum_{N=0}^{\infty} \frac{1}{N!} \left[z \left(\frac{m}{2\pi\beta}\right)^{3/2} \right]^N \int \int \mathcal{D}\xi \exp\left[-\frac{1}{2} \int d^3 x (\nabla \xi)^2 \right] \int \cdots \int \prod_{l=1}^{N} d^3 q_l \exp\left[2m\sqrt{\pi G \beta} \sum_{l=1}^{N} \xi(\vec{q}_l) \right] \\
&= \int \int \mathcal{D}\xi \exp\left[-\frac{1}{2} \int d^3 x (\nabla \xi)^2 \right] \sum_{N=0}^{\infty} \frac{1}{N!} \left[z \left(\frac{m}{2\pi\beta}\right)^{3/2} \right]^N \left[\int d^3 q \exp[2m\sqrt{\pi G \beta}\, \xi(\vec{q})] \right]^N \\
&= \int \int \mathcal{D}\xi \exp\left\{ -\int d^3 x \left[\frac{1}{2} (\nabla \xi)^2 - z \left(\frac{m}{2\pi\beta}\right)^{3/2} e^{2m\sqrt{\pi G \beta}\, \xi(\vec{x})} \right] \right\}.
\end{aligned} \tag{2.7}$$

It is convenient to introduce the dimensionless field

$$\phi(\vec{x}) \equiv 2m\sqrt{\pi G\beta}\,\xi(\vec{x}). \tag{2.8}$$

Then,

$$\mathcal{Z} = \int\int \mathcal{D}\phi \exp\left\{-\frac{1}{T_{\rm eff}}\int d^3x\left[\frac{1}{2}(\nabla\phi)^2 - \mu^2 e^{\phi(\vec{x})}\right]\right\}, \tag{2.9}$$

where

$$\mu^2 = \sqrt{\frac{2}{\pi}}zGm^{7/2}\sqrt{T}, \quad T_{\rm eff} = 4\pi\frac{Gm^2}{T}. \tag{2.10}$$

The partition function for the gas of particles in gravitational interaction has been transformed into the partition function for a single scalar field $\phi(\vec{x})$ with *local* action

$$S[\phi(.)] \equiv \frac{1}{T_{\rm eff}}\int d^3x\left[\frac{1}{2}(\nabla\phi)^2 - \mu^2 e^{\phi(\vec{x})}\right]. \tag{2.11}$$

The ϕ field exhibits an exponential self-interaction $-\mu^2 e^{\phi(\vec{x})}$.

Notice that the effective temperature $T_{\rm eff}$ for the ϕ-field partition function turns out to be *inversely* proportional to T whereas the characteristic length μ^{-1} behaves as $\sim T^{-1/4}$. This is a duality-type mapping between the two models.

It must be noticed that the term $-\mu^2 e^{\phi(\vec{x})}$ makes the ϕ-field energy density unbounded from below. Actually, the initial Hamiltonian (2.1) is also unbounded from below. This unboundedness physically originates in the attractive character of the gravitational force. Including a short-distance cutoff (see Sec. II A, below) eliminates the zero-distance singularity and hence the possibility of zero-distance collapse which is unphysical in the present context. We, therefore, expect meaningful physical results in the cutoff theory. Moreover, assuming zero-boundary conditions for $\phi(\vec{r})$ at $r\to\infty$ shows that the derivatives of ϕ must also be large if e^{ϕ} is large. Hence, the term $\frac{1}{2}(\nabla\phi)^2$ may stabilize the energy.

The action (2.11) defines a nonrenormalizable field theory for any number of dimensions $D>2$ [see Eq. (2.33) below]. This is a further reason to keep the short-distance cutoff nonzero.

Let us compute now the statistical average value of the density $\rho(\vec{r})$ which in the grand canonical ensemble is given by

$$\langle\rho(\vec{r})\rangle = \mathcal{Z}^{-1}\sum_{N=0}^{\infty}\frac{1}{N!}\left[z\left(\frac{m}{2\pi\beta}\right)^{3/2}\right]^N\int\ldots\int\prod_{l=1}^{N}d^3q_l\,\rho(\vec{r})$$
$$\times\exp\left(\frac{1}{2}\beta Gm^2\sum_{1\le l\ne j\le N}\frac{1}{|\vec{q}_l-\vec{q}_j|}\right). \tag{2.12}$$

As usual in the functional integral calculations, it is convenient to introduce sources in the partition function (2.9) in order to compute average values of fields

$$\mathcal{Z}[J(.)] \equiv \int\int\mathcal{D}\phi\exp\left\{-\frac{1}{T_{\rm eff}}\int d^3x\left[\frac{1}{2}(\nabla\phi)^2-\mu^2e^{\phi(\vec{x})}\right]\right.$$
$$\left.+\int d^3x\,J(\vec{x})\phi(\vec{x})\right\}. \tag{2.13}$$

The average value of $\phi(\vec{r})$ then writes as

$$\langle\phi(\vec{r})\rangle = \frac{\delta\ln\mathcal{Z}}{\delta J(\vec{r})}. \tag{2.14}$$

In order to compute $\langle\rho(\vec{r})\rangle$ it is useful to introduce

$$\mathcal{V}[J(.)] \equiv \frac{1}{2}\beta Gm^2\int_{|\vec{x}-\vec{y}|>a}\frac{d^3x\,d^3y}{|\vec{x}-\vec{y}|}[\rho(\vec{x})+J(\vec{x})]$$
$$\times[\rho(\vec{y})+J(\vec{y})]. \tag{2.15}$$

Then, we have

$$\rho(\vec{r})e^{\mathcal{V}[0]} = -\frac{1}{T_{\rm eff}}\nabla^2_{\vec{x}}\left(\frac{\delta}{\delta J(\vec{r})}e^{\mathcal{V}[J(.)]}\right)\bigg|_{J=0}.$$

By following the same steps as in Eqs. (2.6) and (2.7), we find

$$\langle\rho(\vec{r})\rangle = -\frac{1}{T_{\rm eff}}\nabla^2_{\vec{r}}\left(\frac{\delta}{\delta J(\vec{r})}\sum_{N=0}^{\infty}\frac{1}{N!}\left[z\left(\frac{m}{2\pi\beta}\right)^{3/2}\right]^N\mathcal{Z}[0]^{-1}\int\int\mathcal{D}\xi\exp\left\{-\int d^3x\left[\frac{1}{2}(\nabla\xi)^2-2m\sqrt{\pi G\beta}\,\xi(\vec{x})J(\vec{x})\right]\right\}\right.$$
$$\left.\times\int\ldots\int\prod_{l=1}^{N}d^3q_l\exp\left[2m\sqrt{\pi G\beta}\sum_{l=1}^{N}\xi(\vec{q}_l)\right]\right)\bigg|_{J=0} = -\frac{1}{T_{\rm eff}}\nabla^2_{\vec{r}}\left(\frac{\delta}{\delta J(\vec{r})}\ln\mathcal{Z}[J(.)]\right)\bigg|_{J=0}. \tag{2.16}$$

Performing the derivatives in the last formula yields

$$\langle\rho(\vec{r})\rangle = -\frac{1}{T_{\rm eff}}\int\int\mathcal{D}\phi\,\nabla^2\phi(\vec{r})\exp\left\{-\frac{1}{T_{\rm eff}}\int d^3x\left[\frac{1}{2}(\nabla\phi)^2-\mu^2e^{\phi(\vec{x})}\right]\right\}\mathcal{Z}[0]^{-1}. \tag{2.17}$$

One can analogously prove that $\rho(\vec{r})$ inserted in any correlator becomes $-(1/T_{\rm eff})\nabla^2\phi(\vec{r})$ in the ϕ-field language. Therefore, we can express the particle density operator as

$$\rho(\vec{r}) = -\frac{1}{T_{\rm eff}}\nabla^2\phi(\vec{r}). \tag{2.18}$$

Let us now derive the field-theoretical equations of motion. Since the functional integral of a total functional derivative identically vanishes, we can write

$$\int\int \mathcal{D}\phi\left[-\frac{\delta S}{\delta\phi(\vec{r})} + J(\vec{r})\exp\left\{-S[\phi(.)] + \int d^3x J(\vec{x})\phi(\vec{x})\right\}\right] = 0.$$

We get, from Eq. (2.11),

$$\frac{\delta S}{\delta\phi(\vec{r})} = -\frac{1}{T_{\rm eff}}[\nabla^2\phi(\vec{r}) + \mu^2 e^{\phi(\vec{r})}].$$

Thus, setting $J(\vec{r})\equiv 0$,

$$\langle\nabla^2\phi(\vec{r})\rangle + \mu^2\langle e^{\phi(\vec{r})}\rangle = 0. \tag{2.19}$$

Now, combining Eqs. (2.18) and (2.19), yields

$$\langle\rho(\vec{r})\rangle = \frac{\mu^2}{T_{\rm eff}}\langle e^{\phi(\vec{r})}\rangle. \tag{2.20}$$

By using Eq. (2.18), the gravitational potential at the point $\vec{r}$,

$$U(\vec{r}) = -Gm\int\frac{d^3x}{|\vec{x}-\vec{r}|}\rho(\vec{x}),$$

can be expressed as

$$U(\vec{r}) = -\frac{T}{m}\phi(\vec{r}). \tag{2.21}$$

We can analogously express the correlation functions as

$$C(\vec{r}_1,\vec{r}_2) \equiv \langle\rho(\vec{r}_1)\rho(\vec{r}_2)\rangle - \langle\rho(\vec{r}_1)\rangle\langle\rho(\vec{r}_2)\rangle = \left(\frac{1}{T_{\rm eff}}\right)^2\nabla^2_{\vec{r}_1}\nabla^2_{\vec{r}_2}\left(\frac{\delta}{\delta J(\vec{r}_1)}\frac{\delta}{\delta J(\vec{r}_2)}\ln\mathcal{Z}[J(.)]\right)\Bigg|_{J=0}. \tag{2.22}$$

This can be also written as

$$C(\vec{r}_1,\vec{r}_2) = \frac{\mu^4}{T_{\rm eff}}2[\langle e^{\phi(\vec{r}_1)}e^{\phi(\vec{r}_2)}\rangle - \langle e^{\phi(\vec{r}_1)}\rangle\langle e^{\phi(\vec{r}_2)}\rangle]. \tag{2.23}$$

A. Short distances cutoff

A simple short-distance regularization of the Newtonian force for the two-body potential is

$$v_a(\vec{r}) = -\frac{Gm^2}{r}[1-\theta(a-r)],$$

$\theta(x)$ being the step function. The cutoff a can be chosen of the order of atomic distances but its actual value is unessential.

The N-particle-regularized Hamiltonian then takes the form

$$H_N = \sum_{l=1}^{N}\frac{p_l^2}{2m} + \frac{1}{2}\sum_{1\leqslant l,j\leqslant N} v_a(\vec{q}_l - \vec{q}_j). \tag{2.24}$$

Notice that now we can include in the sum terms with $l=j$ since $v_a(0)=0$.

The steps from Eqs. (2.2)–(2.9) can be just repeated by using now the regularized $v_a(\vec{r})$. Notice that we must use now the inverse operator of $v_a(\vec{r})$ instead of that of $1/r,[-(1/4\pi)\nabla^2]$, previously used.

We now find

$$\mathcal{Z}_a = \int\int \mathcal{D}\phi\exp\left\{-\frac{1}{T_{\rm eff}}\int d^3x\left[\frac{1}{2}\phi K_a\phi - \mu^2 e^{\phi(\vec{x})}\right]\right\}, \tag{2.25}$$

i.e.,

$$S_a[\phi(.)] = \frac{1}{T_{\rm eff}}\int d^3x\left[\frac{1}{2}\phi K_a\phi - \mu^2 e^{\phi(\vec{x})}\right], \tag{2.26}$$

where K_a is the inverse operator of v_a,

$$K_a\phi(\vec{r}) = \int K_a(\vec{r}-\vec{r}\,')\phi(\vec{r}\,')d^3r',$$

$$\int K_a(\vec{r}-\vec{r}\,'')\frac{1}{4\pi}\frac{1-\theta(a-|\vec{r}\,''-\vec{r}\,'|)}{|\vec{r}\,''-\vec{r}\,'|}d^3r'' = \delta(\vec{r}-\vec{r}\,').$$

$K_a(\vec{r})$ admits the Fourier representation

$$K_a(\vec{r}) = \mathrm{PV}\int\frac{d^3p}{(2\pi)^3}\frac{p^2 e^{i\vec{p}.\vec{r}}}{\cos pa}.$$

where PV denotes principal value. Actually, $K_a(\vec{r})=0$ for $r\neq 0$. $K_a(\vec{r})$ has the following asymptotic expansion in powers of the cutoff a^2:

$$K_a(\vec{r}) = -\nabla^2\delta(\vec{r}) + \frac{a^2}{2}\nabla^4\delta(\vec{r}) + O(a^4), \tag{2.27}$$

and then

$$S_a[\phi(.)] = S[\phi(.)] + \frac{a^2}{2}\int d^3x(\nabla^2\phi)^2 + O(a^4). \tag{2.28}$$

As we see, the high orders in a^2 are irrelevant operators which do not affect the scaling behavior, as is well known from renormalization group arguments. For $a\to 0$, the action (2.11) is recovered.

B. D-dimensional generalization

This approach generalizes immediately to D-dimensional space where the Hamiltonian (2.2) then takes the form

$$H_N=\sum_{l=1}^{N}\frac{p_l^2}{2m}-Gm^2\sum_{1\leq l<j\leq N}\frac{1}{|\vec{q}_l-\vec{q}_j|^{D-2}}\quad\text{for } D\neq 2, \tag{2.29}$$

and

$$H_N=\sum_{l=1}^{N}\frac{p_l^2}{2m}-Gm^2\sum_{1\leq l<j\leq N}\ln\frac{1}{|\vec{q}_l-\vec{q}_j|}\quad\text{at } D=2\,. \tag{2.30}$$

The steps from Eqs. (2.1)–(2.9) can be trivially generalized with the help of the relation

$$\nabla^2\frac{1}{|\vec{x}-\vec{y}|^{D-2}}=-C_D\delta(\vec{x}-\vec{y}) \tag{2.31}$$

in D dimensions and

$$\nabla^2\ln\frac{1}{|\vec{x}-\vec{y}|}=-C_2\delta(\vec{x}-\vec{y})$$

at $D=2$.

Here,

$$C_D\equiv(D-2)\frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\quad\text{for } D\neq 2\quad\text{and } C_2\equiv 2\pi. \tag{2.32}$$

We finally obtain, as a generalization of Eq. (2.9),

$$\mathcal{Z}=\int\int\mathcal{D}\phi\exp\left\{-\frac{1}{T_{\rm eff}}\int d^Dx\left[\frac{1}{2}(\nabla\phi)^2-\mu^2e^{\phi(\vec{x})}\right]\right\}, \tag{2.33}$$

where

$$\mu^2=\frac{C_D}{(2\pi)^{D/2}}\,zGm^{2+D/2}T^{D/2-1},\quad T_{\rm eff}=C_D\frac{Gm^2}{T}. \tag{2.34}$$

We have then transformed the partition function for the D-dimensional gas of particles in gravitational interaction into the partition function for a scalar field ϕ with exponential interaction. The effective temperature $T_{\rm eff}$ for the ϕ-field partition function is *inversely* proportional to T for *any* space dimension. The characteristic length μ^{-1} behaves as $\sim T^{-(D-2)/4}$.

III. SCALING BEHAVIOR

We derive here the scaling behavior of the ϕ field following the general renormalization group arguments in the theory of critical phenomena [4,6].

A. Classical scale invariance

Let us investigate how the action (2.11) transforms under scale transformations

$$\vec{x}\to\vec{x}_\lambda\equiv\lambda\vec{x}, \tag{3.1}$$

where λ is an arbitrary real number.

In D dimensions the action takes the form

$$S[\phi(.)]\equiv\frac{1}{T_{\rm eff}}\int d^Dx\left[\frac{1}{2}(\nabla\phi)^2-\mu^2e^{\phi(\vec{x})}\right]. \tag{3.2}$$

We define the scale-transformed field $\phi_\lambda(\vec{x})$ as

$$\phi_\lambda(\vec{x})\equiv\phi(\lambda\vec{x})+\ln\lambda^2. \tag{3.3}$$

Hence,

$$[\nabla\phi_\lambda(\vec{x})]^2=\lambda^2[\nabla_{x_\lambda}\phi(\vec{x}_\lambda)]^2,\quad e^{\phi_\lambda(\vec{x})}=\lambda^2e^{\phi(\vec{x}_\lambda)}.$$

We find, upon changing the integration variable in Eq. (3.2) from $\vec{x}$ to $\vec{x}_\lambda$,

$$S[\phi_\lambda(.)]=\lambda^{2-D}S[\phi(.)]. \tag{3.4}$$

We thus see that the action (3.2) *scales* under dilatations in spite of the fact that it contains the dimensionful parameter μ^2. This remarkable scaling property is, of course, a consequence of the scale behavior of the gravitational interaction (2.29).

In particular, in $D=2$ the action (3.2) is scale invariant. In such a special case, it is, moreover, conformal invariant.

The (Noether) current associated to the scale transformations (3.1) is

$$J_i(\vec{x})=x_jT_{ij}(\vec{x})+2\nabla_i\phi(\vec{x}), \tag{3.5}$$

where $T_{ij}(\vec{x})$ is the stress tensor

$$T_{ij}(\vec{x})=\nabla_i\phi(\vec{x})\nabla_j\phi(\vec{x})-\delta_{ij}L$$

and $L\equiv\frac{1}{2}(\nabla\phi)^2-\mu^2e^{\phi(\vec{x})}$ stands for the action density. That is,

$$J_i(\vec{x})=(\vec{x}.\nabla\phi+2)\nabla_i\phi(\vec{x})-x_i[\tfrac{1}{2}(\nabla\phi)^2-\mu^2e^{\phi(\vec{x})}].$$

By using the classical equation of motion (3.6), we then find

$$\nabla_iJ_i(\vec{x})=(2-D)L.$$

This nonzero divergence is due to the variation of the action under dilatations [Eq. (3.4)].

If $\phi(\vec{x})$ is a stationary point of the action (3.2):

$$\nabla^2\phi(\vec{x})+\mu^2e^{\phi(\vec{x})}=0, \tag{3.6}$$

then $\phi_\lambda(\vec{x})$ [defined by Eq. (3.3)] is also a stationary point:

$$\nabla^2\phi_\lambda(\vec{x})+\mu^2e^{\phi_\lambda(\vec{x})}=0.$$

A rotationally invariant stationary point is given by

$$\phi^c(r) = \ln\frac{2(D-2)}{\mu^2 r^2}. \tag{3.7}$$

This singular solution is *invariant* under the scale transformations (3.3). That is,

$$\phi^c_\lambda(r) = \phi^c(r).$$

Equation (3.7) is dilatation and rotation invariant. It provides the *most symmetric* stationary point of the action. Notice that there are no constant stationary solutions besides the singular solution $\phi_0 = -\infty$.

The introduction of the short-distance cutoff a, Eq. (2.24), spoils the scale behavior (3.4). For the cutoff theory from Eqs. (2.26) and (3.1)–(3.3), we have instead

$$S_a[\phi_\lambda(.)] = \lambda^{2-D} S_{\lambda a}[\phi(.)].$$

For $r \sim a$, Eq. (3.7) does not hold anymore for the spherically symmetric solution $\phi^c(r)$. For small r and a, using Eqs. (2.26)–(2.28) we have

$$\phi^c(r) \underset{r\to 0}{\sim} -\frac{\mu^2 r^2}{2D} + O(r^2, r^2 a^2). \tag{3.8}$$

That is, $\phi^c(r)$ is regular at $r=0$ in the presence of the cutoff a.

B. Thermal fluctuations

In this section we compute the partition function equations (2.9) and (2.13) by saddle-point methods.

Equation (3.6) admits only one constant stationary solution

$$\phi_0 = -\infty. \tag{3.9}$$

In order to make such solution finite we now introduce a regularization term $\epsilon\mu^2\phi(\vec{x})$ with $\epsilon \ll 1$ in the action S [Eq. (2.11)]. This corresponds to an action density

$$L = \tfrac{1}{2}(\nabla\phi)^2 + u(\phi), \tag{3.10}$$

where

$$u(\phi) = -\mu^2 e^{\phi(\vec{x})} + \epsilon\mu^2\phi(\vec{x}).$$

This extra term can be obtained by adding a small constant term $-\epsilon\mu^2/T_{\rm eff}$ to $\rho(\vec{x})$ in Eqs. (2.4) – (2.6). This is a simple way to make ϕ_0 finite.

We get in this way a constant stationary point at $\phi_0 = \ln\epsilon$ where $u'(\phi_0) = 0$. However, scale invariance is broken since $u''(\phi_0) = -\epsilon\mu^2 \neq 0$. We can add a second regularization term to $\frac{1}{2}\delta\mu^2\phi(\vec{x})^2$ to L, (with $\delta \ll 1$) in order to enforce $u''(\phi_0) = 0$. This quadratic term amounts to a long-range shielding of the gravitational force. We finally set

$$u(\phi) = -\mu^2[e^{\phi(\vec{x})} - \epsilon\phi(\vec{x}) - \tfrac{1}{2}\,\delta\phi(\vec{x})^2],$$

where the two regularization parameters ϵ and δ are related by

$$\epsilon(\delta) = \delta[1 - \ln\delta],$$

and the stationary point has the value

$$\phi_0 = \ln\delta.$$

Expanding around ϕ_0,

$$\phi(\vec{x}) = \phi_0 + g\chi(\vec{x}),$$

where $g \equiv \sqrt{\mu^{D-2} T_{\rm eff}}$ and $\chi(\vec{x})$ is the fluctuation field, yields

$$\frac{1}{g^2} L = \frac{1}{2}(\nabla\chi)^2 - \frac{\mu^2\delta}{g^2}\left[e^{g\chi} - 1 - g\chi - \frac{1}{2}g^2\chi^2\right]. \tag{3.11}$$

We see perturbatively in g that $\chi(\vec{x})$ is a *massless* field.

Concerning the boundary conditions, we must consider the system inside a large sphere of radius $R(10^{-4}$–10^{-2} pc$\leq R \leq 100$ pc). That is, all integrals are computed over such large sphere.

Using Eq. (2.18) the particle density now takes the form

$$\rho(\vec{r}) = -\frac{1}{T_{\rm eff}}\nabla^2\phi(\vec{r}) = -\frac{g}{T_{\rm eff}}\nabla^2\chi(\vec{r})$$
$$= \frac{\mu^2\delta}{T_{\rm eff}}[e^{g\chi(\vec{r})} - 1 - g\chi(\vec{r})].$$

It is convenient to renormalize the particle density by its stationary value $\delta = e^{\phi_0}$,

$$\rho(\vec{r})_{\rm ren} \equiv \frac{1}{\delta}\rho(\vec{r}) = \frac{\mu^D}{g^2}[e^{g\chi(\vec{r})} - 1 - g\chi(\vec{r})]. \tag{3.12}$$

We see that in the $\delta \to 0$ limit the interaction in Eq. (3.11) vanishes. No infrared divergences appear in the Feynman graph calculations, since we work on a very large but finite volume of size R. Hence, in the $\delta \to 0$ limit, the whole perturbation series around ϕ_0 reduces to the zeroth order term.

The constant saddle point ϕ_0 fails to catch up to the whole field theory content. In fact, more information arises perturbing around the stationary point $\phi^c(r)$ given by Eq. (3.7) [17].

Using Eqs. (2.23), (2.31), (3.11), and (3.12) we obtain, for the density correlator in the $\delta \to 0$ limit,

$$C(\vec{r}_1, \vec{r}_2) = \frac{\mu^{2D}}{g^4}\left\{\exp\left[\frac{g^2}{C_D(\mu|\vec{r}_1 - \vec{r}_2|)^{D-2}}\right] - 1 - \frac{g^2}{C_D(\mu|\vec{r}_1 - \vec{r}_2|)^{D-2}}\right\}.$$

For large distances, we find

$$C(\vec{r}_1, \vec{r}_2) \underset{|\vec{r}_1 - \vec{r}_2|\to\infty}{\sim} \frac{\mu^4}{2\,C_D^2|\vec{r}_1 - \vec{r}_2|^{2(D-2)}} + O(|\vec{r}_1 - \vec{r}_2|^{-3(D-2)}). \tag{3.13}$$

That is, the ϕ-field theory *scales*. Namely, the theory behaves critically for a *continuum set* of values of μ and $T_{\rm eff}$.

Notice that the density correlator $C(\vec{r}_1,\vec{r}_2)$ behaves for large distances as the correlator of $\chi(\vec{r})^2$. This stems from the fact that $\chi(\vec{r})^2$ is the most relevant operator in the series expansion of the density (3.12):

$$\rho(\vec{r})_{\rm ren}=\tfrac{1}{2}\,\mu^D\chi(\vec{r})^2+O(\chi^3). \tag{3.14}$$

As remarked above, the constant stationary point $\phi_0=\ln\delta\to-\infty$ only produces the zeroth order of perturbation theory. More information arises perturbing around the stationary point $\phi^c(r)$ given by Eq. (3.7) [17].

C. Renormalization group finite size scaling analysis

As is well known [4,6,7], physical quantities for *infinite* volume systems diverge at the critical point as Λ to a negative power. Λ measures the distance to the critical point. (In condensed matter and spin systems, Λ is proportional to the temperature minus the critical temperature [6,7].) One has, for the correlation length ξ,

$$\xi(\Lambda)\sim\Lambda^{-\nu},$$

and, for the specific heat (per unit volume) $\mathcal{C}$,

$$\mathcal{C}\sim\Lambda^{-\alpha}. \tag{3.15}$$

Correlation functions scale at criticality. For example, the scalar field ϕ (which in spin systems describes the magnetization) scales as

$$\langle\phi(\vec{r})\phi(0)\rangle\sim r^{-1-\eta}.$$

The critical exponents ν,α, and η are pure numbers that depend only on the universality class [4,6,7].

For a *finite*-volume system, all physical quantities are *finite* at the critical point. Indeed, for a system whose size R is large, the physical magnitudes take large values at the critical point. Thus, for large R, one can use the infinite-volume theory to treat finite-size systems at criticality. In particular, the correlation length provides the relevant physical length $\xi\sim R$. This implies that

$$\Lambda\sim R^{-1/\nu}. \tag{3.16}$$

We can apply these concepts to the ϕ theory since, as we have seen in the previous section, it exhibits scaling in a finite volume $\sim R^3$. Namely, the two point correlation function exhibits a powerlike behavior in perturbation theory as shown by Eq. (3.13). This happens for a *continuum set* of values of $T_{\rm eff}$ and μ^2. Therefore, changing $\mu^2/T_{\rm eff}$ keeps the theory in the scaling region. At the point $\mu^2/T_{\rm eff}=0$, the partition function $\mathcal{Z}$ is singular. From Eq. (2.10), we shall thus identify

$$\Lambda\equiv\frac{\mu^2}{T_{\rm eff}}=z\left(\frac{mT}{2\pi}\right)^{3/2}. \tag{3.17}$$

Notice that the critical point $\Lambda=0$ corresponds to zero fugacity.

Thus, the partition function in the scaling regime can be written as

$$\mathcal{Z}(\Lambda)=\int\int\mathcal{D}\phi\exp\left[-S^*+\Lambda\int d^Dx\,e^{\phi(\vec{x})}\right], \tag{3.18}$$

where S^* stands for the action (2.11) at the critical point $\Lambda=0$.

We define the renormalized mass density as

$$m\rho(\vec{x})_{\rm ren}\equiv m e^{\phi(\vec{x})} \tag{3.19}$$

and we identify it with the energy density in the renormalization group. [Also called the ‘‘thermal perturbation operator.’’]. This identification follows from the fact that they are the most relevant positive definite operators. Moreover, such identification is supported by the perturbative result (3.14).

In the scaling regime we have [6] for the logarithm of the partition function

$$\frac{1}{V}\ln\mathcal{Z}(\Lambda)=\frac{K}{(2-\alpha)(1-\alpha)}\Lambda^{2-\alpha}+F(\Lambda), \tag{3.20}$$

where $F(\Lambda)$ is an analytic function of Λ around the origin

$$F(\Lambda)=F_0+a\Lambda+\tfrac{1}{2}\,b\Lambda^2+\cdots.$$

$V=R^D$ stands for the volume and F_0, K, a, and b are constants.

Calculating the logarithmic derivative of $\mathcal{Z}(\Lambda)$ with respect to Λ from Eqs. (3.18) and (3.20) and equating the results, yields

$$\frac{1}{V}\frac{\partial}{\partial\Lambda}\ln\mathcal{Z}(\Lambda)=a+\frac{K}{1-\alpha}\Lambda^{1-\alpha}+\cdots=\frac{1}{V}\int d^Dx\langle e^{\phi(\vec{x})}\rangle, \tag{3.21}$$

where we used the scaling relation $\alpha=2-\nu D$ [6,7].

We can apply here finite-size scaling arguments and replace Λ by $\sim R^{-(1/\nu)}$ [Eq. (3.16)],

$$\frac{\partial}{\partial\Lambda}\ln\mathcal{Z}(\Lambda)=Va+\frac{K}{1-\alpha}R^{1/\nu}+\cdots.$$

Recalling Eq. (3.19), we can express the mass contained in a region of size R as

$$M(R)=m\int^R e^{\phi(\vec{x})}d^Dx. \tag{3.22}$$

Using Eq. (3.21) we find

$$\langle M(R)\rangle=mVa+m\frac{K}{1-\alpha}R^{1/\nu}+\cdots$$

and

$$\langle\rho(\vec{r})\rangle=ma+m\frac{K}{\nu(1-\alpha)\Omega_D}r^{(1/\nu)-D}$$

$$\text{for } r \text{ of order } \sim R, \tag{3.23}$$

where Ω_D is the surface of the unit sphere in D dimensions.

The energy density correlation function is known in general in the scaling region (see Refs. [6], [7]). We can, therefore, write for the density-density correlators (2.22) in D space dimensions

$$C(\vec{r}_1,\vec{r}_2)\sim|\vec{r}_1-\vec{r}_2|^{(2/\nu)-2D}, \tag{3.24}$$

where both $\vec{r}_1$ and $\vec{r}_2$ are inside the finite volume $\sim R^D$.

The perturbative calculation (3.13) matches with this result for $\nu=\frac{1}{2}$. That is, the mean field value for the exponent ν.

Let us now compute the second derivative of $\ln\mathcal{Z}(\Lambda)$ with respect to Λ in two ways. We find, from Eq. (3.20),

$$\frac{\partial^2}{\partial\Lambda^2}\ln\mathcal{Z}(\Lambda)=V[\Lambda^{-\alpha}K+b+\cdots].$$

We get, from Eq. (3.18),

$$\frac{\partial^2}{\partial\Lambda^2}\ln\mathcal{Z}(\Lambda)=\int d^Dx\,d^Dy\,C(\vec{x},\vec{y})\sim R^D\int^R\frac{d^3x}{x^{2D-2d_H}}\sim\Lambda^{-2}$$
$$\sim R^D\Lambda^{-\alpha}, \tag{3.25}$$

where we used Eqs. (3.16) and (3.24) and the scaling relation $\alpha=2-\nu D$ [6,7]. We conclude that the scaling behaviors, Eq. (3.20) for the partition function, Eq. (3.15) for the specific heat, and Eq. (3.24) for the two point correlator are consistent. In addition, Eqs. (3.22) and (3.25) yield for the mass fluctuations squared

$$[\Delta M(R)]^2\equiv\langle M^2\rangle-\langle M\rangle^2\sim\int d^Dx\,d^Dy\,C(\vec{x},\vec{y})\sim R^{2d_H}.$$

Hence,

$$\Delta M(R)\sim R^{d_H}. \tag{3.26}$$

The scaling exponent ν can be identified with the inverse Haussdorf (fractal) dimension d_H of the system

$$d_H=\frac{1}{\nu}.$$

In this way, $\Delta M\sim R^{d_H}$ according to the usual definition of fractal dimensions [8].

Using Eq. (3.24) we can compute the average potential energy in three space dimensions as

$$\langle\mathcal{V}\rangle=\tfrac{1}{2}\,\beta G m^2\int_{|\vec{x}-\vec{y}|>a}^{R}\frac{d^3x\,d^3y}{|\vec{x}-\vec{y}|}\;C(\vec{x},\vec{y})\sim R^{(2/\nu)-1}.$$

From here and Eq. (3.26) we get as virial estimate for the atomic kinetic energy

$$\langle v^2\rangle=\frac{\langle\mathcal{V}\rangle}{\langle\Delta M(R)\rangle}\sim R^{(1/\nu)-1}.$$

This corresponds to a velocity dispersion

$$\Delta v\sim R^{1/2\,[(1/\nu)-1]}. \tag{3.27}$$

That is, we predict [see Eq. (1.1)] a new scaling relation

$$q=\frac{1}{2}\left(\frac{1}{\nu}-1\right)=\frac{1}{2}(d_H-1).$$

The calculation of the critical amplitudes [that is, the coefficients in front of the powers of R in Eqs. (3.24), (3.26), and (3.27)] is beyond the scope of the present paper [17].

D. Values of the scaling exponents and the fractal dimensions

The scaling exponents ν,α considered in Sec. III C can be computed through the renormalization group approach. The case of a *single*-component scalar field has been extensively studied in the literature [6,7,10]. Very probably, there is a unique, infrared-stable, fixed point in three-space dimensions: the Ising model-fixed point. Such nonperturbative fixed point is reached in the long-scale regime independently of the initial shape of the interaction $u(\phi)$ [Eq. (3.10)] [10].

The numerical values of the scaling exponents associated to the Ising model-fixed point are

$$\nu=0.631\ldots,\quad d_H=1.585\ldots,\quad \eta=0.037\ldots,\quad\text{and}$$
$$\alpha=0.107\ldots. \tag{3.28}$$

In the ϕ-field model there are two dimensionful parameters: μ and $T_{\rm eff}$. The dimensionless combination

$$g^2=\mu T_{\rm eff}=(8\pi)^{3/4}\sqrt{z}\,\frac{G^{3/2}m^{15/4}}{T^{3/4}} \tag{3.29}$$

acts as the coupling constant for the nonlinear fluctuations of the field ϕ.

Let us consider a gas formed by neutral hydrogen at thermal equilibrium with the cosmic microwave background. We set $T=2.73$ K and estimate the fugacity z using the ideal gas value

$$z=\left(\frac{2\pi}{mT}\right)^{3/2}\rho.$$

Here, we use $\rho=\delta_0$ atoms/cm^{-3} for the ISM density and $\delta_0\simeq10^{10}$. Equation (2.10) yields

$$\frac{1}{\mu}=2.7\frac{1}{\sqrt{\delta_0}}\text{AU}\sim30\ \text{AU}\quad\text{and}$$
$$g^2=\mu T_{\rm eff}=4.9\times10^{-58}\sqrt{\delta_0}\sim5\times10^{-53}. \tag{3.30}$$

This extremely low value for g^2 suggests that the perturbative calculation [Sec. III B] may apply here yielding the mean field values for the exponents: i.e.,

$$\nu=1/2,\quad d_H=2,\quad \eta=0,\quad\text{and}\ \ \alpha=0. \tag{3.31}$$

That is, the effective coupling constant grows with the scale according to the renormalization group flow (towards the Ising-fixed point). Now, if the extremely low value of the initial coupling [Eq. (3.30)] applies, the perturbative result (mean field) will hold for many scales (the effective g grows roughly as the length).

μ^{-1} indicates the order of the smallest distance where the scaling regime applies. A safe lower bound supported by observations is around 20 AU $\sim 3\times 10^{14}$ cm, in agreement with our estimate.

Our theoretical predictions for $\Delta M(R)$ and Δv [Eqs. (3.26) and (3.27)] both for the Ising equation (3.28) and for the mean field value equation (3.31), are in agreement with the astronomical observations [Eq. (1.1)]. The present observational bounds on the data are larger than the difference between the mean field and Ising values of the exponents d_H and q.

Further theoretical work in the ϕ theory will determine whether the scaling behavior is given by the mean field or by the Ising-fixed point [17].

E. The two-dimensional gas and random surface fractal dimensions

In the two-dimensional case ($D=2$) the partition function (2.33) describes the Liouville model that arises in string theory [18] and in the theory of random surfaces (also called two-dimensional quantum gravity). For strings in c-dimensional Euclidean space the partition function takes the form [18]

$$\mathcal{Z}_c=\int\int \mathcal{D}\phi \exp\left\{-\frac{26-c}{24\pi}\int d^2x\left[\frac{1}{2}(\nabla\phi)^2+\mu^2 e^{\phi(\vec{x})}\right]\right\}. \tag{3.32}$$

This coincides with Eq. (2.33) at $D=2$ provided we flip the sign of μ^2 and identify the parameters (2.34) as

$$T=Gm^2\frac{26-c}{12},\quad \mu^2=zGm^3. \tag{3.33}$$

Reference [19] states that $d_H=4$ for $c\leqslant 1$, $d_H=3$ for $c=2$, and $d_H=2$ for $c\geqslant 4$. In our context this means

$$d_H=2 \quad \text{for } T\leqslant\frac{25}{12}Gm^2,\quad d_H=3 \quad \text{for } T=2\,Gm^2,$$

$$\text{and } d_H=4 \quad \text{for } T\geqslant\frac{11}{6}Gm^2.$$

For $c\to\infty$, $g^2\to 0$ and we can use the perturbative result (3.13) yielding $\nu=\frac{1}{2}$, $d_H=2$ in agreement with the above discussion for $c\geqslant 4$.

F. Stationary points and the Jeans length

The stationary points of the ϕ-field partition function (2.9) are given by the nonlinear partial differential equation

$$\nabla^2\phi=-\mu^2 e^{\phi(\vec{x})}.$$

In terms of the gravitational potential $U(\vec{x})$ [see Eq. (2.21)], this takes the form

$$\nabla^2 U(\vec{r})=4\pi Gzm\left(\frac{mT}{2\pi}\right)^{3/2}e^{-(m/T)U(\vec{r})}. \tag{3.34}$$

This corresponds to the Poisson equation for a thermal matter distribution satisfying an ideal gas in hydrostatic equilibrium, as can be seen as follows [20]. The hydrostatic equilibrium condition

$$\nabla P(\vec{r})=-m\rho(\vec{r})\nabla U(\vec{r}),$$

where $P(\vec{r})$ stands for the pressure, combined with the equation of state for the ideal gas

$$P=T\rho,$$

yields for the particle density

$$\rho(\vec{r})=\rho_0 e^{-(m/T)U(\vec{r})},$$

where ρ_0 is a constant. Inserting this relation into the Poisson equation

$$\nabla^2 U(\vec{r})=4\pi Gm\rho(\vec{r})$$

yields Eq. (3.34) with

$$\rho_0=z\left(\frac{mT}{2\pi}\right)^{3/2}. \tag{3.35}$$

For large r, Eq. (3.34) gives a density decaying as r^{-2} :

$$\rho(\vec{r})\underset{r\to\infty}{\sim}\frac{T}{2\pi Gm}\frac{1}{r^2}\left[1+O\left(\frac{1}{\sqrt{r}}\right)\right],$$

$$U(\vec{r})\underset{r\to\infty}{\sim}\frac{T}{m}\ln\left[\frac{2\pi G\rho_0}{T}r^2\right]+O\left(\frac{1}{\sqrt{r}}\right). \tag{3.36}$$

Notice that this density, which describes a single stationary solution, decays for large r *faster* than the density (3.23) governed by thermal fluctuations.

Spherically symmetric solutions of Eq. (3.34) have been studied in detail [21]. The small fluctuations around such isothermal spherical solutions as well as the stability problem were studied in [16].

The Jeans distance is, in this context,

$$d_J\equiv\sqrt{\frac{3T}{m}}\frac{1}{\sqrt{Gm\rho_0}}=\frac{\sqrt{3}(2\pi)^{3/4}}{\sqrt{zGm^{7/4}T^{1/4}}}. \tag{3.37}$$

This distance precisely coincides with μ^{-1} [see Eq. (2.10)] up to an inessential numerical coefficient ($\sqrt{12/\pi}$). Hence, μ, the only dimensionful parameter in the ϕ theory, can be interpreted as the inverse of the Jeans distance.

We want to notice that in the critical regime, d_J grows as

$$d_J\sim R^{d_H/2}, \tag{3.38}$$

since $\rho_0=\Lambda\sim R^{-d_H}$ vanishes as can be seen from Eqs. (3.16), (3.17), and (3.37). In this tree-level estimate we should use for consistency the mean field value $d_H=2$, which yields $d_J\sim R$.

This shows that the Jeans distance is of the order of the *size* of the system. The Jeans distance *scales* and the instability is, therefore, present for all sizes R.

Had d_J been of order larger than R, the Jeans instability would be absent.

The fact that the Jeans instability is present *precisely* at $d_J \sim R$ is probably essential to the scaling regime and to the self-similar (fractal) structure of the gravitational gas.

The dimensionless coupling constant g^2 can be written from Eqs. (3.17) and (3.29) as

$$g^2 = \left(2m \sqrt{\frac{\pi G}{T}} \right)^3 \sqrt{\Lambda}.$$

Hence, the tree-level coupling scales as

$$g^2 \sim R^{-1}.$$

Direct perturbative calculations explicitly exhibit such scaling behavior [17].

We can express g^2 in terms of d_J and ρ_0 as

$$g^2 = \frac{(12\pi)^{3/2}}{\rho_0 d_J^3} = \frac{\pi^2 \mu^3}{\rho_0}.$$

This shows that g^2 is, at the tree level, the inverse of the number of particles inside a Jeans volume.

Equation (3.38) applies to the tree-level Jeans length or tree level μ^{-1}. We can, furthermore, estimate the Jeans length using the renormalization group behavior of the physical quantities derived in Sec. III C. Setting

$$\langle d_J \rangle = \frac{\langle \Delta v \rangle}{\sqrt{G m \langle \Delta \rho \rangle}},$$

we find, from Eqs. (3.23) and (3.27),

$$\langle d_J \rangle \sim R.$$

Namely, we find again that the Jeans length grows as the size R.

IV. DISCUSSION

In previous sections we ignored gravitational forces external to the gas, like stars, etc. Adding a fixed external mass density $\rho_{\text{ext}}(\vec{r})$ amounts to introducing an external source

$$J(\vec{r}) = -T_{\text{eff}} \rho_{\text{ext}}(\vec{r}),$$

in Eq. (2.13). Such term will obviously affect correlation functions, the mass density, etc. except when we look at the scaling behavior which is governed by the critical point. That is, the values we find for the scaling exponents d_H and q are *stable* under external perturbations.

We considered all atoms with the same mass in the gravitational gas. It is easy to generalize the transformation into the ϕ field presented in Sec. II for a mixture of several kinds of atoms. Let us consider n species of atoms with masses $m_a, 1 \leq a \leq n$. Repeating the steps from Eqs. (2.1)–(2.11), yields again a field theory with a single scalar field but the action now takes the form

$$S[\phi(.)] \equiv \frac{1}{T_{\text{eff}}} \int d^3x \left[\frac{1}{2} (\nabla \phi)^2 - \sum_{a=1}^{n} \mu_a^2 \exp\left(\frac{m_a}{m} \phi(\vec{x}) \right) \right], \tag{4.1}$$

where

$$\mu_a^2 = \sqrt{\frac{2}{\pi}} z_a G m_a^{3/2} m^2 \sqrt{T},$$

and m is just a reference mass.

Correlation functions, mass densities, and other observables will obviously depend on the number of species, their masses, and fugacities but it is easy to see that the fixed points and scaling exponents are exactly the *same* as for the ϕ-field theory [Eqs. (2.9) and (2.10)].

We want to notice that there is an important difference between the behavior of the gravitational gas and the spin models (and all other statistical models in the same universality class). For the gravitational gas we find scaling behavior for a *full range* of temperatures and couplings. For spin models scaling only appears at the critical value of the temperature. At the critical temperature the correlation length ξ is infinite and the theory is massless. For temperatures near the critical one, i.e., in the critical domain, ξ is finite (although very large compared with the lattice spacing) and the correlation functions decrease as $\sim e^{-r/\xi}$ for large distances r. Fluctuations of the relevant operators support perturbations which can be interpreted as massive excitations. Such (massive) behavior does not appear for the gravitational gas. The ISM correlators scale exhibiting power-law behavior. This feature is connected with the scale-invariant character of the Newtonian force and its infinite range.

The hypothesis of strict thermal equilibrium does not apply to the ISM as a whole where temperatures range from 5–50 K and even 1000 K. However, since the scaling behavior is independent of the temperature, it applies to *each* region of the ISM in thermal equilibrium. Therefore, our theory applies provided thermal equilibrium holds in regions or clouds.

We have developed here the theory of a gravitationally interacting ensemble of bodies at a fixed temperature. In a real situation such as the ISM, gravitational perturbations from external masses, as well as other perturbations, are present. We have shown that the scaling solution is stable with respect to the gravitational perturbations. It is well known that solutions based on a fixed point are generally quite robust.

Our theory especially applies to the interstellar medium far from star-forming regions, which can be locally far from thermal equilibrium, and where ionized gas at 10^4 K together with coronal gas at 10^6 K can coexist with the cold interstellar medium. In the outer parts of galaxies, devoid of any star formation, the ideal isothermal conditions are met [3]. Inside the Galaxy, large regions satisfy also the near isothermal criterion, and these are precisely the regions where scaling laws are the best verified. Globally over the Galaxy, the fraction of the gas in the hot ionized phase represents a negligible mass, a few percents, although occupying a significant volume. Hence, this hot ionized gas is a perturbation which may not change the fixed-point behavior of the thermal self-gravitating gas.

In Ref. [22] a connection between a gravitational gas of galaxies in an expanding Universe and the Ising model is conjectured. However, the unproven identification made in Ref. [22] of the mass density contrast with the Ising spin leads to scaling exponents different from ours.

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