Constraints on the spectral index from primordial black holes

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We construct the mass spectrum of primordial black holes (PBH's) using the Press-Schechter formalism and study the contribution of PBH's to the density of the universe, $\Omega_{\rm BH}$, when the power spectrum of the density fluctuation is a simple power-law spectrum. From the condition $\Omega_{\rm BH}<1$, the constraints on the spectral index n are found for both cases of possible end states of black hole evaporation, one in which black holes evaporate completely without relics and the other in which black hole evaporation ends with relics with mass of the order of $M_{\rm Pl}$. With the normalization of the fluctuation amplitude to COBE's quadrupole anisotropy measurement, we find that the upper limit of n is somewhat largely deviated from the Harrison-Zel'dovich spectrum. [S0556-2821(96)05122-3]

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I. INTRODUCTION

To explain how the large scale structures today such as galaxies, clusters, superclusters, etc., can be formed, some kinds of density fluctuations in the early homogeneous isotropic universe are needed to offer seeds for those structure formations. Needless to say, overdense regions in the fluctuated universe are responsible for the structure formation. They can grow via the gravitational instability into the large scale structures we observe today. In addition, it is possible that some overdense regions can collapse to black holes [primordial black holes (PBH's) [1]].

If PBH formation is due to the density fluctuations at the radiation-dominated era, PBH's form when the evolved size of the overdense regions is about the horizon size at that time so PBH's have masses about the horizon mass [1]. Since the initial horizon mass when the density fluctuations develop is very small, PBH formation can offer the possibility of small black holes with masses far less than $M_\odot \sim 10^{33}$ g that is about the lower limit of the masses of the black holes which are formed by ordinary stellar collapses. Specially, PBH's with $M_{\rm BH} \lesssim 10^{15}$ g are interesting. Because of the black hole evaporation suggested by Hawking [2], such black holes have evaporated away by the present. But the particle emission by such black holes can cause many interesting phenomena in the early universe. Of course, such phenomena should not violate the well-founded theoretical and observational results, and many have investigated the upper limit of the density fraction of PBH's to the critical density of the universe, $\Omega_{\rm BH}$, by considering the roles of the particle emission in relations with the nucleosynthesis, baryogenesis, cosmic microwave background radiation (CMBR) distortion, entropy production, diffusive γ-ray background, and so on [3]. Their results show $\Omega_{\rm BH}$ should be far less than 1 in order that PBH formation do not hamper the history of the universe. For $M_{\rm BH} \gtrsim 10^{15}$ g, the black hole evaporation is not efficient and such PBH's can remain today without significant mass losses. So they can contribute to the density of the present universe and can be a dark matter candidate provided $\Omega_{\rm BH}\sim 1$. If the PBH formation lasts until the matter-radiation equal time $t_{\rm eq}$, very large PBH's can be formed during this period ($M_{\rm BH}\sim 10^{15} M_\odot$ at $t_{\rm eq}$). So they can bound a large region of surrounding matter through their gravitational forces. For example, a PBH with $M_{\rm BH}\sim 10^7 M_\odot$ can bound the mass of a typical galaxy since decoupling [4].

As mentioned above, PBH's have many interesting roles in very broad mass ranges, so it is important to know the mass spectrum of PBH's, that is, how many PBH's with what masses can be formed. If the PBH formation is due to the density fluctuation in the early universe, the mass spectrum largely depends on the power spectrum of the density fluctuation. The first work on the mass spectrum of PBH's in relation with the power spectrum was done by Carr [5]. He considered the simple power-law spectrum having a horizon crossing amplitude $\delta_{\rm H} \propto k^{(n-1)/2}$ where n is the spectral index and k is the comoving wave number. He then showed that, for $n=1$ (the Harrison-Zel'dovich spectrum), the mass spectrum falls off as a power of mass and it is possible that some large PBH's with $M_{\rm BH} \gtrsim 10^{15}$ g exist today in some significant amount. However, with the amplitude $\delta_{\rm H}\sim 5\times 10^{-6}$ which is obtained from the anisotropy measurement of CMBR by the Cosmic Background Explorer (COBE), PBH formation is practically impossible when $n=1$. With such a value, PBH formation needs some $\delta_{\rm H}$ decreasing with increasing mass scale corresponding to the $n>1$ spectrum which is called ''blue perturbation spectrum.'' Recently, some have derived inflationary potentials for which this kind of power spectrum is produced [6,7]. Specially in Ref. [7], constraints on the inflationary model from PBH formation have been studied. Also, Carr, Gilbert, and Lidsey [8] have studied the constraint on the spectral index for the case where PBH's that are produced under the blue spectrum do not completely evaporate but leave stable relics with masses about $M_{\rm Pl}$, and found that the upper limit of n is about 1.4 if PBH's form with a hard equation of state at $T=10^{16}$ GeV. On the other hand, inspired by the observed massive compact halo objects (MACHO's [9]) with masses $\sim 0.1 M_\odot$, PBH formation also has been studied in the case of inflationary potential having a plateau in some ranges [10], and in the inflationary model with multiple scalar fields [11].

In this paper, we will review the mass spectrum of PBH's with a somewhat different view than Carr's. We adopt the Press-Schechter formalism [12] which is widely used in calculating the number density of bound objects such as galaxies. Then, we calculate Ω_{BH} from the mass spectrum and find the constraint on the spectral index from the condition $\Omega_{BH}<1$ when the power spectrum is simple power law.

II. PBH FORMATION AND THEIR MASS SPECTRUM

First, we briefly introduce the mechanism and criteria of PBH formation by reviewing Ref. [5] in which most of the equations can be found. If the universe experiences the adiabatic density fluctuations, there exist overdense regions with various sizes which are larger than the horizon size at that time. In the Friedmann universe, regions separated by distance larger than the horizon size behave just like parts of different Friedmann universe [13]. So, the evolutions of overdense regions can be deduced from the locally closed Friedmann model. If we consider a spherical overdense region with the initial radius R when the background universe has a hard equation of state $p=\gamma\rho$ with the sound velocity $v_s \equiv \sqrt{\gamma} \sim 1$, for the region to collapse by overcoming the pressure forces, the radius at their maximum expansion R_c should be larger than the Jean's length $R_J \simeq v_s R_h$, where R_h is the horizon size at that time. Also, R_c has an upper limit of the order of the horizon length in order that the overdense region should not be disconnected from the universe. So, the size of the region at the moment of recollapse must be of order of that of the horizon. The condition that the region should be within its Schwarzchild radius requires that the size of the region should be larger than the order of the horizon length, so a black hole forms at about the time when the region begins to recollapse and the black hole mass is of the order of the horizon mass.

Now, we can set the conditions for PBH formation as

$$\beta R_h \lesssim R_c \lesssim \alpha R_h, \tag{1}$$

where α and β are constants of the order of $\sqrt{\gamma}$. From the calculations in the evolution of a spherical overdense region with the radius R and the initial density contrast δ_i, the size and time at the maximum expansion, R_c and t_c, are related with δ_i as

$$R_c \sim R\,\delta_i^{-1/2}, \quad t_c \sim t_i \delta_i^{-1}, \tag{2}$$

where t_i is the time when the density fluctuations develop (from here, we take the radiation equation of state, i.e., $\gamma=1/3$). So, we can convert Eq. (1) into the form of the initial density contrast as

$$\beta^2 \left(\frac{M}{M_i}\right)^{-(2/3)} \lesssim \delta_i \lesssim \alpha^2 \left(\frac{M}{M_i}\right)^{-(2/3)}, \tag{3}$$

where M is the mass contained in the region with radius R at t_i, and M_i is the horizon mass at t_i. Since $M \propto R^3$ and $R \propto k^{-1}$, we will use M, R, and k interchangeably in representing the initial mass, size or comoving wavelength. Since a PBH that forms at the time t has a mass about $\rho_i (t/t_i)^{-2} R_c^3$, where ρ_i is the background density at t_i, its mass M_{BH} can be written as

$$M_{BH} \simeq \gamma^{3/2} M \frac{t}{t_i}. \tag{4}$$

Sometimes, we shall use M_{BH} as denoting the time t. Also, M_{BH} is related with M as

$$M_{BH} \simeq \gamma^{1/2} M^{2/3} M_i^{1/3}. \tag{5}$$

As the universe expands, larger PBH can be formed, so PBH's with mass less than M_{BH} coexist in the universe at t.

To go to the next stage of constructing the mass spectrum, we assume the density fluctuation has a Gaussian distribution. If one surveys the universe with the window having a size R, the smoothed density field $\delta_R(\mathbf{x})$ is defined by

$$\delta_R(\mathbf{x}) = \int d^3\mathbf{y}\, \delta(\mathbf{x}+\mathbf{y}) W_R(\mathbf{y}). \tag{6}$$

Here, $\delta(\mathbf{x}) = (\rho(\mathbf{x}) - \rho_b)/\rho_b$ where ρ_b is the background energy density and $W_R(\mathbf{x})$ is the window function of size R. And the dispersion σ_R which is the standard deviation of the density contrast of the regions with R is defined by

$$\sigma_R^2 = \frac{1}{V_W^2} \langle \delta_R^2(\mathbf{x}) \rangle = \frac{1}{V_W^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\delta_\mathbf{k}|^2 W_\mathbf{k}^2(R), \tag{7}$$

where $V_W \sim R^3$ is the effective volume filtered by W_R, and $\delta_\mathbf{k}$ and $W_\mathbf{k}$ are the Fourier transforms of $\delta(\mathbf{x})$ and $W_R(\mathbf{x})$, respectively. Then, the probability that the region with R has the density contrast in the range $(\delta + d\delta, \delta)$ is

$$P(M,\delta)\, d\delta = \frac{1}{\sqrt{2\pi}\sigma_R} \exp\left(-\frac{\delta^2}{2\sigma_R^2}\right) d\delta. \tag{8}$$

Carr's mass spectrum was initiated by the integration of the above equation in the range given in Eq. (3), and he interpreted it as the probability that the region with M collapses to a black hole, P_{BH}:

$$P_{BH}(M) = \int_{\mathcal{B}}^{\mathcal{A}} P(M,\delta)\, d\delta, \tag{9}$$

where

$$\mathcal{A} = \alpha^2 \left(\frac{M}{M_i}\right)^{-(2/3)}, \quad \mathcal{B} = \beta^2 \left(\frac{M}{M_i}\right)^{-(2/3)}. \tag{10}$$

After further consideration on the case where the regions containing the small black holes collapse to larger black holes at later times, he found the final mass spectrum of ultimate PBH's (PBH's not included in other PBH's) for $n=1$ power spectrum. For other cases of power spectrum, however, the mass spectrum was not shown explicitly and $P_{BH}(M)$ is regarded as the density fraction of PBH's with M_{BH} to the background density of the universe in some papers by neglecting the probability that small PBH's are included in larger PBH's at later times.

With a more rigorous view, however, $P_{\rm BH}$ does not represent the probability that the region with M collapses to a black hole. There are contributions from PBH's larger than $M_{\rm BH}$ in $P_{\rm BH}(M)$. Because the filtered density contrast under the window W_R does see fluctuations larger than R, some of the points satisfying $\delta_R > \mathcal{B}$ lie in the fluctuated regions larger than R. So, they should be substracted. Substraction of such parts is the main step of the Press-Schechter formalism and we will briefly introduce this formalism [14]. Since $\delta_R(\mathbf{x})$ see structures larger than R, the integration of Eq. (8) from some critical value δ_c to infinity

$$F(M,\delta_c)=\int_{\delta_c}^{\infty}P(M,\delta)d\delta=\frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R}\right), \tag{11}$$

where erfc(x) is the complementary error function, represents the fraction of regions having scales larger than R and $\delta > \delta_c$. Substraction can be done by differentiating $F(M,\delta_c)$ with M. So, the fraction of regions with $\delta > \delta_c$ in the range $M+dM,M$ is

$$f(M,\delta_c)dM=-\frac{\partial F}{\partial M}dM=\frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_R^2}\frac{\partial\sigma_R}{\partial M}\exp\left(-\frac{\delta_c^2}{2\sigma_R^2}\right)dM. \tag{12}$$

Then, the number density of regions with masses between M and $M+dM$, that is, the mass spectrum $n(M,\delta_c)dM$ can be found by multiplying $f(M,\delta_c)$ by ρ_i/M:

$$n(M,\delta_c)dM=\frac{\rho_i}{M}f(M)dM. \tag{13}$$

So, for given power spectrum of $|\delta_k|^2$ and the window function, one can calculate the number density of the regions with M and $\delta > \delta_c$. When δ_c is the value of which the regions evolve into the nonlinear regime, $n(M,\delta_c)$ can be used to find the number density of gravitationally bound objects. However, this expression is not complete. The integral of $f(M,\delta_c)$ over all M does not give unity but a half. This arises because the fraction of the underdense region is not considered correctly. If a point in space has $\delta > \delta_c$ when filtered at scale R, then that point should correspond to a region with mass greater than M. However, for those points having $\delta < \delta_c$ under this filtering, there is a nonzero probability that such a point will have $\delta > \delta_c$ when the density field is filtered with a radius $R_1 > R$. Many intensive works have been done to count correctly those underdense regions embedded within overdense regions and have given some modifications to the original Press-Schechter formalism [15]. The failure of normalization can be cured by using the sharp k-space filtering [16] but the result may change when different window functions are considered. In this paper, we simply use the number density which is obtained by multiplying Eq. (13) by 2 for normalization. Then $n(M,\delta_c)$ is given by

$$n(M,\delta_c)dM=-\frac{\rho_i}{M}\sqrt{2/\pi}\frac{\delta_c}{\sigma_R^2}\frac{\partial\sigma_R}{\partial M}\exp\left(-\frac{\delta_c^2}{2\sigma_R^2}\right)dM. \tag{14}$$

From the above procedure, we conceive that the mass spectrum of PBH's can be given by

$$n_{\rm BH}(M)dM=n(M,\mathcal{B})dM-n(M,\mathcal{A})dM. \tag{15}$$

Also, $n(M,\mathcal{A})$ can be ignored because it experiences more exponential damping than $n(M,\mathcal{B})$. So, the mass spectrum can be approximated as $n_{\rm BH}(M)dM\simeq n(M,\mathcal{B})dM$.

For the actual calculation, we should choose the window function. But whatever window functions we choose, $\sigma_R \simeq \delta(k)$ where

$$\delta(k)=\frac{k^{3/2}}{\sqrt{2\pi}}|\delta_k|. \tag{16}$$

We therefore set $\sigma_R=\delta(k)$. From the linear perturbation theory, we know that $\delta(k)\propto a^2(t)$ in the radiation-dominated era, where $a(t)$ is the cosmic scale factor. So $\delta(k)$ is related to $\delta_{\rm H}$, which is the amplitude when the fluctuation with k or M enter into the horizon, as

$$\delta(k)=\delta(M)=\left(\frac{M}{M_i}\right)^{-(2/3)}\delta_{\rm H}. \tag{17}$$

Thus, $n_{\rm BH}(M)$ can be rewritten as

$$n_{\rm BH}(M)dM=-\sqrt{2/\pi}\frac{\rho_i}{M}\gamma\left[\frac{1}{\delta_{\rm H}^2}\frac{\partial\delta_{\rm H}}{\partial M}-\frac{2}{3}\frac{\delta_{\rm H}^{-1}}{M}\right]\times\exp\left(-\frac{\gamma^2}{2\delta_{\rm H}^2}\right)dM. \tag{18}$$

Finally, from Eq. (5), the mass spectrum can be expressed as a function of the PBH mass $M_{\rm BH}$ as

$$n_{\rm BH}(M_{\rm BH})dM_{\rm BH}=-\sqrt{2/\pi}\gamma^{7/4}\rho_i M_i^{1/2}M_{\rm BH}^{-(3/2)}\times\left[\frac{1}{\delta_{\rm H}^2}\frac{\partial\delta_{\rm H}}{\partial M_{\rm BH}}-\frac{\delta_{\rm H}^{-1}}{M_{\rm BH}}\right]\times\exp\left(-\frac{\gamma^2}{2\delta_{\rm H}^2}\right)dM_{\rm BH}. \tag{19}$$

In Carr's approach, there are some difficulties in formulating the mass spectrum for which the power spectrum is scale-dependent ($n\neq 1$) but we can obtain rather simply the mass spectrum under the given power spectrum. Incidentally, if $n=1$, the mass spectrum is the same as Carr's mass spectrum of ultimate PBH's except for the numeric factor. However, there is one more complication to be considered, that is that the condition on the initial density contrast δ_i for PBH formation, Eq. (3) depends on the mass (fluctuation scale) of the region. Therefore there is the possibility that some small sized black holes are eventually trapped into a larger region whose average density contrast lies outside the condition for the original small sized black hole formation but which still develops into a larger size black hole. The above substraction procedure does not treat this case properly. But, this process is highly nonlocal and very difficult to treat analytically. For $n>1$ power spectrum that we have a concern, however, the large PBH formation is suppressed due to the exponential term in Eq. (19), so, we will not consider it further, assuming the modification caused by it is not very significant, but only take mass spectrum given in Eq. (19) in calculating $\Omega_{\rm BH}$.

Since PBH's with the masses from roughly M_i to $M_{\rm BH}$ given in Eq. (4) coexist in the universe at the time t, and the number density falls off as a^{-3}, the cumulative number density $n_c(t)$ and energy density $\rho_{\rm BH}(t)$ can be obtained from

$$n_c(t)=\left(\frac{a(t)}{a(t_i)}\right)^{-3}\int_{M_i}^{M_{\rm BH}} n_{\rm BH}(M'_{\rm BH})dM'_{\rm BH} \tag{20}$$

and

$$\rho_{\rm BH}(t)=\left(\frac{a(t)}{a(t_i)}\right)^{-3}\int_{M_i}^{M_{\rm BH}} \mathcal{M}(M'_{\rm BH},t)n_{\rm BH}(M'_{\rm BH})dM'_{\rm BH}. \tag{21}$$

Here $\mathcal{M}(m,t)$ is the mass of PBH at time t whose initial mass m and it is dependent on the details of evaporation effects. Then, the density fraction of PBH's at t, $\Omega_{\rm BH}$ can be found from

$$\Omega_{\rm BH}=\frac{\rho_{\rm BH}(t)}{\rho_c(t)}, \tag{22}$$

where ρ_c is the critical density of the universe.

III. CONSTRAINTS ON THE SPECTRAL INDEX

Qualitative behavior of mass spectrum is largely dependent on the exponential factor in Eq. (19). Thus, with $\delta_{\rm H}\sim 5\times 10^{-6}$ at the quadrupole scale, PBH formation in the early universe can be realized only in the cases of the power spectrum with $n>1$ or $\delta_{\rm H}$ decreasing as the fluctuation scale increases. This spectrum has not been widely studied because it has rare theoretical foundations. In the inflationary model it is generally predicted that $\delta_{\rm H}$ is nearly scale independent [17] or occasionally decreases as a power of k [18]. In the hybrid inflation model [19], however, the spectral index is generically larger than 1. Some have investigated the inflationary potential which results in the exact power-law spectrum with constant n and showed that it can be realized approximately in the hybrid inflation [6,8]. On the observational grounds, even though the Harrison-Zel'dovich spectrum is well fitted to the COBE data, the spectral index is not definitely determined. And there are some new observations which are in better agreement with $n>1$ spectrum (for more details, see Refs. [6,8]). On this ground, we will find the constraint on the spectral index n from the PBH overproduction constraints by taking $n>1$ power-law spectrum.

Under the power-law spectrum, the horizon crossing amplitude $\delta_{\rm H}$ is given by

$$\delta_{\rm H}=\delta_0\left(\frac{M}{M_0}\right)^{(1-n)/6}, \tag{23}$$

where $M_0\sim 10^{57}$ g is today's horizon mass which corresponds to the quadrupole scale and $\delta_0\sim 5\times 10^{-6}$ is the amplitude at the quadrupole measured by the COBE. In general, tensor perturbations (gravitational waves) are also produced in the inflationary scenarios and contribute to the CMBR anisotropy but considering the tensor perturbation does not alter significantly the mass spectrum of PBH if the fraction of tensor perturbation is not dominant, so we assume the anisotropy is only due to the scalar fluctuation. If we represent $\delta_{\rm H}$ as a function of $M_{\rm BH}$, then

$$\delta_{\rm H}=D\left(\frac{M_{\rm BH}}{M_i}\right)^{(1-n)/4}, \tag{24}$$

where

$$D=\delta_0\gamma^{(n-1)/8}\left(\frac{M_i}{M_0}\right)^{(1-n)/6}. \tag{25}$$

And the mass spectrum is given by

$$n_{\rm BH}(M_{\rm BH})dM_{\rm BH}=\frac{n+3}{4}\sqrt{2/\pi}\,\gamma^{7/4}\rho_i M_i^{1/2}M_{\rm BH}^{-(5/2)}\delta_{\rm H}^{-1}\times\exp\left(-\frac{\gamma^2}{2\delta_{\rm H}^2}\right)dM_{\rm BH}. \tag{26}$$

What we determine next is the epoch that density fluctuations develop. In noninflationary cosmology, t_i is the time when the adiabatic fluctuations take place in the early radiation era. On the other hand, if the universe fall into the inflating phase, the expansion effect is so large that PBH formation is impossible. PBH's begin to form only after the universe return to the ordinary radiation-dominated era. This can be achieved by some kinds of thermalizations called reheating [20]. After reheating, the universe reaches the equilibrium temperature $T_{\rm RH}$ and then evolves as the standard model. So, we can set t_i as

$$t_i=0.301g_*^{-1/2}\frac{M_{\rm Pl}}{T_{\rm RH}^2}\sim\left(\frac{T_{\rm RH}}{\rm MeV}\right)^{-2}\ \text{sec}, \tag{27}$$

where $g_*\sim 100$ is the degrees of freedom of the constituents in the early universe. Actual value of $T_{\rm RH}$ is very model-dependent, so we do not take the specific value but treat t_i as a free parameter. There may exist a dust phase by coherent oscillations of inflation field after the inflating phase [21] and PBH formation under such an equation of state can constrain the power spectrum [8]. In the recent reheating theory [22], it is shown that the equation of state of the universe changes instantaneously into the radiation type due to the parametric resonance after the inflating phase. In this view, we do not consider the dust stage after inflation.

Now, with two parameters t_i and n, we will calculate $\Omega_{\rm BH}$ and find the constraint on the spectral index from the condition $\Omega_{\rm BH}<1$ during the evolution of the universe. First, we consider the standard black hole evaporation scenario in which black holes evaporate completely with no relics (case I). In this case, the mass loss rate is given by

$$\frac{dM_{\rm BH}}{dt}\simeq -\epsilon M_{\rm Pl}^4 M_{\rm BH}^{-2}, \tag{28}$$

where $\epsilon\sim 3.6\times 10^{-4}$ is the numerically determined constant [23]. From Eq. (28), the evaporated PBH mass $M_{\rm evap}$ at the time $t\gg t_{\rm BH}$ where $t_{\rm BH}$ is the time when the black hole forms, is easily shown to be

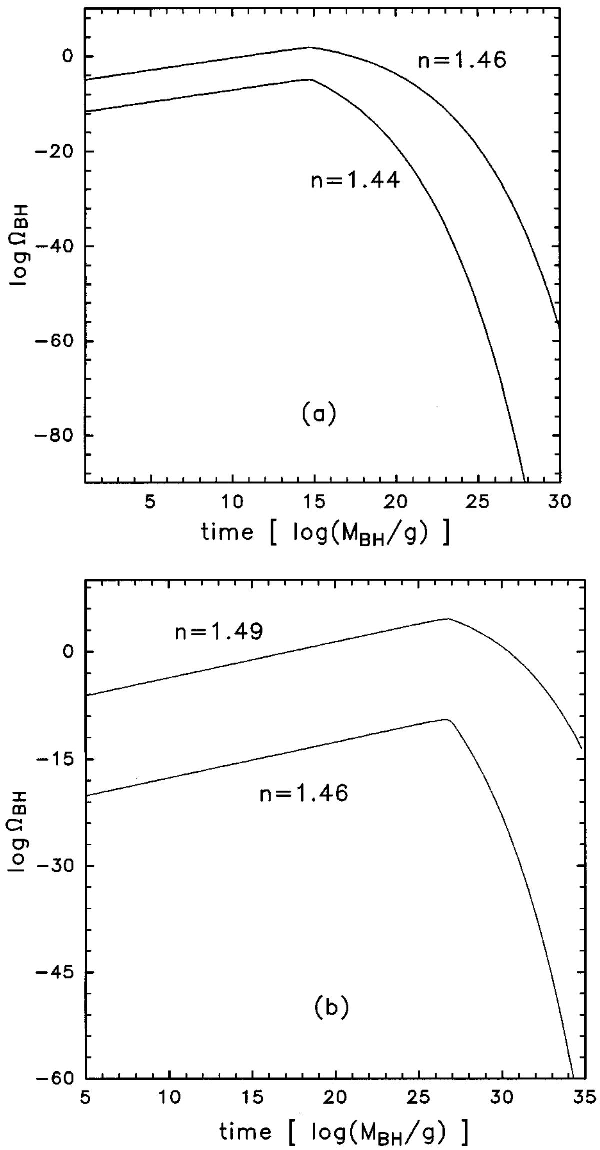

FIG. 1. Evolution of $\Omega_{\rm BH}$ when PBH's leave no relics (case I): (a) $\Omega_{\rm BH}$ for $T_{\rm RH}=10^{16}$ GeV; (b) $\Omega_{\rm BH}$ for $T_{\rm RH}=10^{14}$ GeV with $M_0=10^{57}$ g and $\delta_0=3.6\times10^{-6}$.

$$M_{\rm evap}\simeq(M_{\rm ini}^3-3\epsilon\gamma^{-3/2}M_{\rm Pl}^2M_{\rm BH})^{1/3}, \tag{29}$$

where $M_{\rm ini}$ is the initial PBH mass and $M_{\rm BH}$ is PBH mass formed at t. So, PBH's with the mass below

$$M_*\simeq(\epsilon\gamma^{-3/2}M_{\rm Pl}^2M_{\rm BH})^{1/3} \tag{30}$$

have evaporated away and do not exist at t. Then, $\mathcal{M}(M'_{\rm BH},t)$ in Eq. (21) can be set as

$$\mathcal{M}(M'_{\rm BH},t)\simeq(M'^3_{\rm BH}-3\epsilon\gamma^{-3/2}M_{\rm Pl}^2M_{\rm BH})^{1/3}\theta(M'_{\rm BH}-M_*) \tag{31}$$

where $\theta(x)$ is the step function.

We performed the numerical calculations of $\Omega_{\rm BH}$ using Eqs. (26) and (31). Figure 1 shows the evolution of $\Omega_{\rm BH}$ for some parameters. Since δ_0 is very small, the exponential factor in the mass spectrum decreases very rapidly as $M_{\rm BH}$ increases and most of PBH's form during a very short period of time after t_i. So, the mass spectrum is nearly δ-function type. $\Omega_{\rm BH}$ roughly grows as $a(t)\propto t^{1/2}$ because $\rho_{\rm BH}$ effectively falls off as $a^{-3}(t)$ whereas $\rho_c\propto a^{-4}(t)$ in the radiation-dominated era. Then, $\Omega_{\rm BH}$ goes to zero very rapidly after PBH's formed at the first stage have evaporated away. We follow the evolution of $\Omega_{\rm BH}$ for given parameters t_i and n, and find the upper limit of n satisfying $\Omega_{\rm BH}<1$ for a given t_i (the solid line in Fig. 2). So, the upper limit of the spectral index corresponds to the value of n for which the peaks in Fig. 1 coincide with zero. One can see that the constraint is generally very small. Since the reheating temperature is constrained to be less than about 10^{16} GeV from the CMBR anisotropy, we take 10^{16} GeV as the initial reheating temperature. Smallness of the exponential factor also causes fine-tuning in constraint value of n. With n somewhat below the constraint value, PBH formation is insignificant and with n above the line, PBH density easily overcloses the universe.

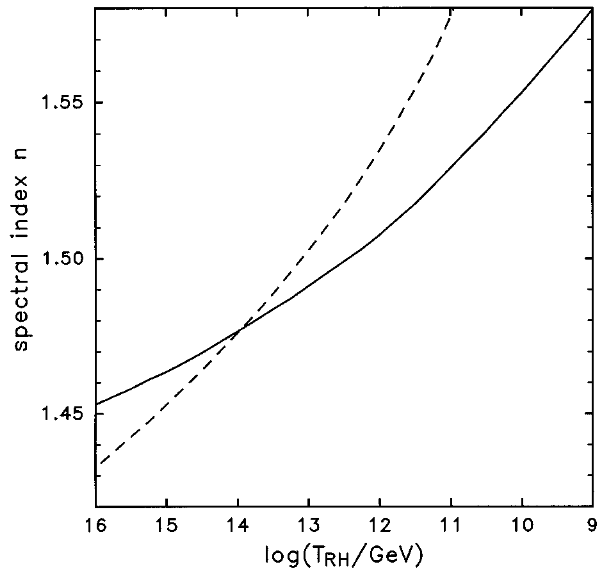

FIG. 2. The upper limits of n against the initial fluctuation time t_i: The solid line shows the upper limit of n in the case where PBH's leave no relics, obtained by $\Omega_{\rm BH}<1$ at all times in the history of the universe. The dotted line shows the upper limit of n obtained by the condition that the today's relic density fraction $\Omega_{\rm rel}<1$ in the case where PBH's leave stable relics with $M_{\rm rel}=M_{\rm Pl}$. Figures are obtained for $t_{\rm eq}=10^{10}$ sec, $M_0=10^{57}$ g, and $\delta_0=3.6\times10^{-6}$.

Next, we consider the case (case II) where black holes leave relics with masses about $M_{\rm rel}\simeq M_{\rm Pl}\sim10^3M_{\rm Pl}$ as is predicted in many non-Einstein gravity models (references are in Ref. [8]). Of course, black hole relics, if any, can contribute to the density of the universe and play a role in dark matter [24,25]. Specially, in Ref. [8], the constraints on n have been studied on the assumption that $\Omega_{\rm BH}\simeq P_{\rm BH}$.

It is convenient to consider today's relic density fraction $\Omega_{\rm rel}$ to find the upper limit of n. Since the number density and evaporation of PBH's heavier than 10^{15} g are negligible, $\Omega_{\rm BH}$ is dominated by $\Omega_{\rm rel}$ which is given by

$$\Omega_{\rm rel}=\frac{1}{\rho_c}\left(\frac{a(t_0)}{a(t_i)}\right)^{-3}\int_{M_i}^{M_*}M_{\rm rel}n_{\rm BH}(M'_{\rm BH})dM'_{\rm BH}, \tag{32}$$

where

$$\left(\frac{a(t_0)}{a(t_i)}\right)^{-3}=\left(\frac{t_i}{t_0}\right)^2\left(\frac{t_{\rm eq}}{t_i}\right)^{1/2} \tag{33}$$

and t_0 is the present time and $M_*=10^{15}$ g.

The constraints on n from the condition that today's relic density should be less than the critical density, $\Omega_{\rm rel}<1$, are shown as the dotted line in Fig. 2. The results are not much different from those of Ref. [8] because our mass spectrum also becomes effectively δ-function-like under the normalization of fluctuation amplitude to the COBE. With $M_0=10^{57}$ g and $\delta_0=3.6\times10^{-6}$, the limit on the spectral index is $n\lesssim1.43$ for $T_{\rm RH}=10^{16}$ GeV, and $n\lesssim1.48$ for $T_{\rm RH}=10^{14}$ GeV. The upper limit for the spectral index n increases as t_i and more easily blows up than in case I. And this upper limit line coincides well with the plots of Eq. (4.10) in Ref. [8]. After $T_{\rm RH}\lesssim10^{14}$ GeV, the relic constraint becomes weaker than the constraint of case I. However, the effect of PBH relics will appear only after $\Omega_{\rm BH}$ has a peak as shown in Fig. 1. The evolution of $\Omega_{\rm BH}$ in case II will not be much different with that of case I until the relic mass makes $\Omega_{\rm BH}$ increase again after $\Omega_{\rm BH}$ falls off for some period. So the spectral index above the upper limit in case I means that the PBH's overclose the universe not depending on whether PBH's leave relics or not. Therefore, in the region bounded by both upper limit lines in the ranges of $T_{\rm RH}\lesssim10^{14}$ GeV in Fig. 2, the parameters n and $T_{\rm RH}$ are not allowed because PBH's with them overclose the universe in the past even though they do not lead to the case where the today's relic density exceeds the critical density.

In summary, we derived the mass spectrum of PBH's and investigated the constraints on the spectral index from PBH's when the density fluctuations have simple power-law spectrum. Due to the smallness of the amplitude of density fluctuations in CMBR anisotropy, the spectral index should be largely blue-shifted from the Harrison-Zel'dovich spectrum for significant amount of PBH formation and the mass spectrum is nearly δ-function type. So, with such normalization and power spectrum, it is generally difficult to achieve PBH formation which needs to explain the astrophysical effects. In the range, $M_{\rm BH}\lesssim10^5$ g corresponding to $T_{\rm RH}\gtrsim10^{14}$ GeV, our results show that PBH relics give the most severe constraint on the spectral index. For 10^5 g$\lesssim M_{\rm BH}\lesssim10^{15}$ g, however, some cosmological and astrophysical effects of particle emission from PBH's may constrain the spectral index more. Since the upper limit of n in this range is obtained by the condition that the density fraction of PBH's at their end stage of evaporations should be less than one, the constraint given in this range means that the density fraction of PBH with $M_{\rm BH}$ at their formation should be less than about $10^{-2}(M_{\rm BH}/M_{\rm Pl})^{-1}$. This limit generally gives stronger or same amounts of constraints than those given from the particle emission effects from PBH's except for the ranges, $M_{\rm BH}\sim10^{10}$ g (photodissociation of deuterium by photons emitted from PBH's [26]) and $M_{\rm BH}\sim10^{15}$ g (diffusive γ-ray background [27]). In these regions, the effects of particle emission is 10^2–10^4 times stronger than the constraint given by $\Omega_{\rm BH}<1$. Therefore, the effects of black hole evaporations give stronger constraints on the spectral index. However, since the spectral index is fine-tuned with very small fluctuation amplitude, the upper limit on n does not change significantly under such considerations.

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