# Radiative decay of the massive neutrino in external electromagnetic fields

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The radiative decay of the massive neutrino $\nu_i \to \nu_j \gamma$ is investigated in the framework of the standard model in external electromagnetic fields of various configurations: constant crossed field, constant uniform magnetic field, plane monochromatic wave's field. The effect of significant enhancement of the neutrino decay probability by the external field (electromagnetic catalysis) is discussed. An especially strong enhancement occurs in the case of the ultrarelativistic neutrino decay, since in this case the decay probability does not contain suppression caused by the smallness of the decaying neutrino mass. The ultrarelativistic neutrino decay catalysis is significant even in a relatively weak external field ($F/F_e \ll 1$, where $F_e$ is the critical Schwinger value). The expression for the photon splitting probability into a neutrino pair $\gamma \to \nu_i \overline{\nu_j}$ in the wave field is given. The estimations of a number of $\gamma$ quanta produced in a volume filled with an electromagnetic field and the neutrino lifetime in a strong magnetic field are presented. [S0556-2821(96)02519-2]

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## I. INTRODUCTION

For quite a long time we have seen intensive theoretical studies of flavor-changing processes caused by the phenomenon of fermion mixing. The description of this phenomenon in the quark sector goes back to the pioneer work by Cabibbo [1] and presently is put into practice by introducing the unitary $3 \times 3$ matrix $V_{ij}$ (the so-called Cabibbo-Kobayashi-Maskawa matrix [2]). It should be noted that qualitative progress has been observed in experimental investigation of quark-mixing parameters. Except for a detailed examination of the ''old'' mixing angles related to the first two quark generations, important information has been obtained from the decays of $b$-quark-containing particles (ARGUS Collaboration [3], CLEO Collaboration [4]). On the other hand, thus far there is no experimental evidence in favor of the analogous mixing phenomenon in the lepton sector. This can be accounted for in a natural way by the fact that, because of the insufficiently high precision achieved in experimental studies of neutrino-involving processes, the neutrino mass spectrum appears degenerate (the neutrinos manifest themselves as massless particles [5]). The neutrino mass spectrum being degenerate, lepton mixing is known to be purely formal and unobservable. At the same time, with a massive neutrino the absence of lepton mixing seems unnatural and is virtually incompatible with attempts to somehow extend the standard model. Notice that lepton mixing may lead to some interesting physical phenomena such as (1) charged lepton radiative decays with lepton number violation of the type $\mu \to e\gamma$, $\mu \to 3e$ [6,7], and $\mu \to e\gamma\gamma$ [8], (2) neutrino radiative decays $\nu_i \to \nu_j \gamma$ [6] and $\nu_i \to \nu_j \gamma\gamma$ [9], (3) neutrino oscillations [10], and (4) the possible effect of massive neutrino mixing on the spectrum of $\beta$-decay-produced electrons [11].

Even such a short review of lepton-mixing effects shows that most of these are associated with the massive neutrino. Nowadays, the physics of the massive neutrino is becoming a vigorously growing and prospective line of investigation at the junction of elementary particles physics, astrophysics, and cosmology. It will suffice to mention the well-known problem of the solar neutrino [12] and the possibility of solving it (the mechanism of resonance enhancement of neutrino oscillations in substance [13]), the effect of the massive neutrino radiative decay on the spectrum of the relic radiation [14], and so on. The above-mentioned way of solving the solar neutrino problem using the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism[1] shows that the massive neutrino's properties are sensitive to the medium it propagates through. Substance is usually considered as the medium. We note, however, that the medium can also be represented by an external electromagnetic field, which can significantly influence both the properties of the massive neutrino itself [16] and the process of its decay [17] and even induce novel lepton transitions with flavor violation $\nu_i \leftrightarrow \nu_j$ $(i \neq j)$ [18], forbidden in vacuum. In our preliminary communication [17] we pointed out the probability of the massive neutrino radiative decay $\nu_i \to \nu_j \gamma$ $(i \neq j)$ being considerably enhanced in a constant uniform magnetic field. Such an enhancing influence of an external field can be illustrated with the straightforward example of a neutrino radiative decay in a weak (as compared with the Schwinger value $F_e = m_e^2/e \simeq 4.41 \times 10^{13}$ G) electromagnetic field. To this end we use the amplitude of the Compton-like process $\nu_i \gamma^* \to \nu_j \gamma^*$ with virtual photons [19], which, in particular, allows one to obtain the first term of the expansion of the radiative decay $\nu_i \to \nu_j \gamma$ amplitude in a weak external field. In the expression for the amplitude of the process

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[1] About the current situation around the solar neutrino problem see, for example, Ref. [15]

 

$\nu_i(p_1)+\gamma^*(q_1)\rightarrow\nu_j(p_2)+\gamma^*(q_2)$ it is sufficient to consider $\gamma(q_2)$ as a real photon, and to replace the field tensor of the virtual photon $\gamma^*(q_1)$ by the Fourier image of the external electromagnetic field tensor. Below we shall give the expression obtained in this way for the radiative decay amplitude in the simplest case of a uniform electromagnetic field, in which the decay kinematics $p_1+0=p_2+q_2$ is the same as in vacuum. The external-field-induced contribution $\Delta\mathcal{M}$ to the amplitude of the decay $\nu_i\rightarrow\nu_j\gamma$ can be represented in the form

$$\Delta\mathcal{M}\simeq\frac{e}{48\pi^2}\frac{G_F}{\sqrt{2}}(jq)[F\tilde{f}^*(q)]\left\langle\frac{1}{F_\ell}\right\rangle, \tag{1.1}$$

where $j_\mu=\overline{\nu_j}(p_2)\gamma_\mu(1+\gamma_5)\nu_i(p_1)$, $i,j=1,2,3$, enumerate the definite mass neutrino species, $p_1$, $p_2$, and $q$ are the four-momenta of the initial and final neutrinos and the photon, respectively, $F_{\mu\nu}$ is the external uniform electromagnetic field tensor, $\tilde{f}_{\alpha\beta}(q)=\epsilon_{\alpha\beta\mu\nu}q_\mu\epsilon_\nu(q)$, $\epsilon_\nu(q)$ is the polarization four-vector of the photon, and $F_\ell=m_\ell^2/e$ is the critical value of the strength of the electromagnetic field for the charged lepton with the mass $m_\ell$. We have introduced the designation

$$\langle A(m_\ell)\rangle=\sum_{\ell=e,\mu,\tau}K_{i\ell}K_{j\ell}^*A(m_\ell), \tag{1.2}$$

where $K_{i\ell}$ ($\ell=e,\mu,\tau$) is the lepton-mixing matrix of the Kobayashi-Maskawa type. For the sake of comparison we write the known expression for the amplitude of the neutrino radiative decay in vacuum [6] which can be represented as

$$\mathcal{M}_0\simeq-i\frac{3e}{32\pi^2}\frac{G_F}{\sqrt{2}}[j\tilde{f}^*(q)p]\left\langle\frac{m_\ell^2}{m_W^2}\right\rangle, \tag{1.3}$$

where $p=p_1+p_2$ and $m_\ell$ and $m_W$ are the masses of the virtual lepton and $W$ boson, respectively. In analyzing the amplitudes (1.1) and (1.3) in the case of the neutrino decay at rest, it is necessary to take account of $p_1$, $p_2$, $q$, and $j$ being of order of the mass of the decaying neutrino $m_\nu$. In this case, the expressions for the amplitudes (1.1) and (1.3) can be easily estimated (it is sufficient to allow for the order of the dimensional quantities):

$$\Delta\mathcal{M}\sim G_Fm_\nu^3(F/F_e), \tag{1.4}$$

$$\mathcal{M}_0\sim G_Fm_\nu^3(m_\tau/m_W)^2. \tag{1.5}$$

It follows here from that, given the condition

$$(F/F_e)^2\gg(m_\tau/m_W)^4 \tag{1.6}$$

[here $F$ stands for the strengths of the magnetic ($B$) and electric ($E$) fields], the probability of the decay $\nu_i\rightarrow\nu_j\gamma$ in an external field is much greater than that in vacuum, even for a relatively weak electromagnetic field $(10^{-3}\ll F/F_e\ll1)$. The catalyzing effect of an external field becomes even more substantial in the case of the ultrarelativistic neutrino decay ($E_\nu\gg m_\nu$). With the amplitudes (1.1) and (1.3), being Lorentz invariant, the analysis can be conveniently carried out in the rest frame of the decaying neutrino. In this case the the electromagnetic field in Eq. (1.4) is obtained by the Lorentz transformation from the laboratory frame, in which the external field $F$ is given, to the rest frame of the decaying neutrino:

$$F'\sim\frac{E_\nu}{m_\nu}F\gg F. \tag{1.7}$$

Comparing expressions (1.4) and (1.5), in view of Eq. (1.7), we notice that the catalyzing effect of the external field becomes appreciable under a much weaker condition, as compared to Eq. (1.6):

$$\frac{(p_1FFp_1)}{m_\nu^2F_e^2}\gg\left(\frac{m_\tau}{m_W}\right)^4. \tag{1.8}$$

In this case the ratio between the probabilities of the ultrarelativistic neutrino decay $w^{(F)}$ and the decay in vacuum $w^{(0)}$ is of the order

$$\frac{w^{(F)}}{w^{(0)}}\sim\left(\frac{F}{F_e}\right)^2\left(\frac{E_\nu}{m_\nu}\right)^2\left(\frac{m_W}{m_\tau}\right)^4\gg1. \tag{1.9}$$

Expression (1.9) shows that in the ultrarelativistic neutrino decay the enhancement is mainly due to a decrease in the decay probability suppression by the smallness of the neutrino mass $[w^{(F)}\sim m_\nu^4\ \ w^{(0)}\sim m_\nu^5(m_\nu/E_\nu)]$. It is natural to expect that in taking the account of further terms in the expansion of the amplitude of the radiative decay $\nu_i\rightarrow\nu_j\gamma$ with respect to the external field, the suppression mentioned above can be fully canceled. All this makes it interesting to calculate the amplitude with the external electromagnetic field taken into account exactly. An expression thus obtained will be valid in the case of the neutrino radiative decay $\nu_i\rightarrow\nu_j\gamma$ in an external electromagnetic field, which has not to be weak as against the Schwinger value $F_e$.

## II. CROSSED FIELD

At present the experimentally accessible strengths of electromagnetic fields are significantly below the critical strength $(F/F_e\ll1,\ F=B,\mathcal{E},\ F_e=m_e^2/e\simeq4.41\times10^{13}$ G). Because of this, field-induced effects are especially marked in the ultrarelativistic case with the dynamic parameter

$$\chi^2=\frac{e^2(pFFp)}{m^6}, \tag{2.1}$$

being not small even for a relatively weak field ($F_{\mu\nu}$ is the external field tensor, $p_\alpha$ is the four-momentum, and $m$ is the mass of the particle). This is due to the fact that in the relativistic particle rest frame the field may turn out to be of order of the critical one or even higher, appearing very close to the constant crossed field. Thus, the calculation in a constant crossed field ($\vec{\mathcal{E}}\perp\vec{B}$, $\mathcal{E}=B$) is the relativistic limit of the calculation in an arbitrary weak smooth field, possesses a great extent of generality, and acquires interest by itself. We note that, as $(FF=F\tilde{F}=0)$ in a crossed field, the dynamic parameter $\chi^2$, Eq. (2.1), is the single field invariant, by which the decay probability is expressed. Furthermore, the

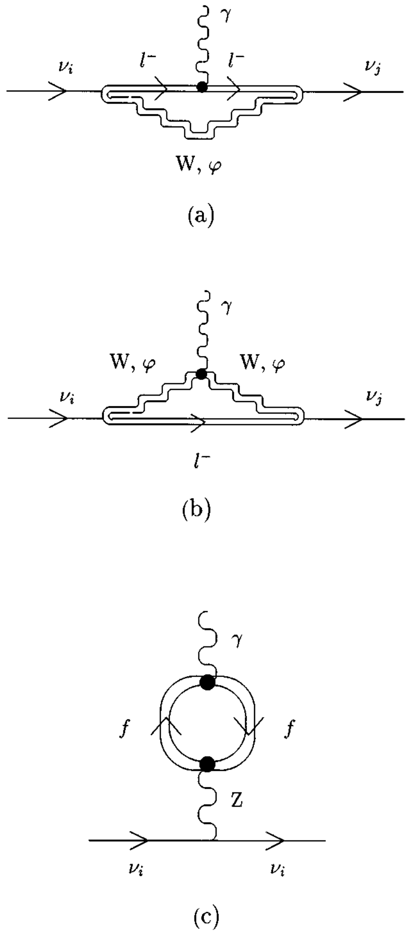


FIG. 1. Feynman diagrams of a matrix element of the radiative decay of the massive neutrino $\nu_i \to \nu_j \gamma$ $(i \neq j)$.

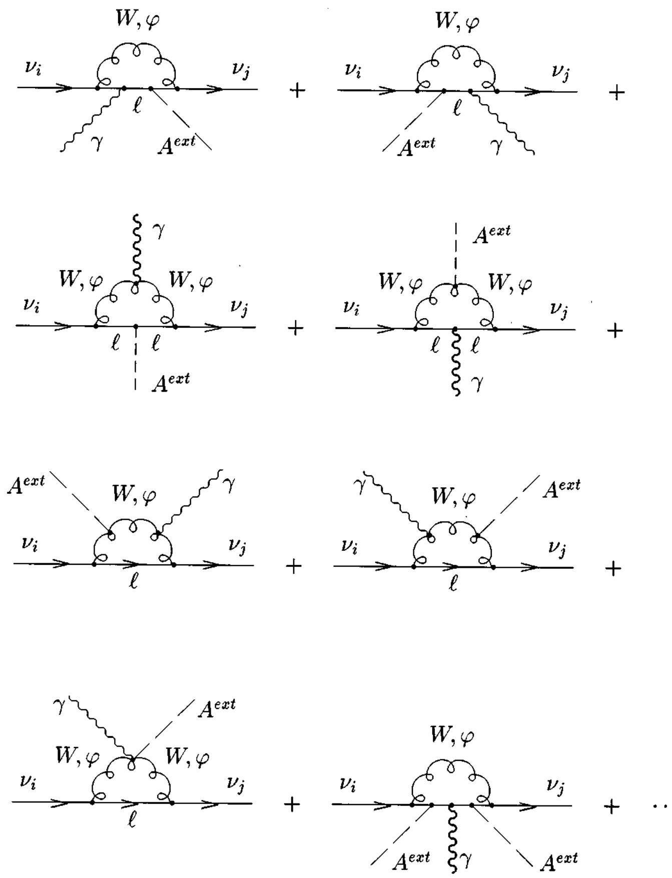

FIG. 2. Feynman diagrams of $\Delta\mathcal{M}^{(F)}$ expanded into a series in terms of the external field.

calculation in crossed field is the least cumbersome and, therefore, we consider this case first just to outline the calculation technique.

In the lowest order of the perturbation theory, a matrix element of the radiative decay of the massive neutrino $\nu_i \to \nu_j \gamma$ $(i \neq j)$ in the Feynman gauge is described by the diagrams, represented in Figs. 1(a) and 1(b) where double lines imply the influence of the external field in the propagators of intermediate particles. A summation is made over the virtual lepton $\ell$ in the loop $(\ell = e, \mu, \tau)$. Under the conditions $m_\ell^2/m_W^2 \ll 1$ and $eF/m_W^2 \ll 1$ the field-induced contribution $\Delta\mathcal{M}^{(F)} = \mathcal{M} - \mathcal{M}^{(0)}$ to the decay amplitude can be calculated in the local limit, in which the lines $W$ and $\varphi$ are contacted to a point. It is most easily seen, if $\Delta\mathcal{M}^{(F)}$ is expanded into a series in terms of the external field, as is shown in Fig. 2, where the dotted lines designate the external electromagnetic field $A^{\text{ex}}$. We note that the first seven diagrams in Fig. 2 coincide with the diagrams describing Compton-like process $\nu_i \gamma^* \to \nu_j \gamma^*$ and, as was pointed out in [19], this process amplitude is reduced to the contribution of the first two diagrams in the local limit. With the orthogonality of the mixing matrix $K_{ij}$ taken into account, this is due to the fact that the main contribution to the integral over momentum in the loop gives from the region of the virtual momenta $p \sim m_\ell \ll m_W$. We recall that we are investigating flavor-violating processes $(i \neq j)$, and, hence, $\langle A \rangle = 0$, if $A$ is independent of $m_\ell$. Thus, the dominant contribution of order $1/m_W^2 \sim G_F$ only comes from the diagrams with one $W$ propagator in Fig. 2.

Even such a simple analysis shows the following.

(1) The predominant contribution to the amplitude is made by the diagram (a) in Fig. 1. This diagram in the local limit of a $W$-boson propagator being contacted to a point, transforms to the diagram shown in Fig. 3.

(2) Since in calculating $\Delta\mathcal{M}$ the mass of the $W$ boson in the local limit appears in the weak interaction constant $G_F = g^2/8m_W^2$ only, the amplitude does not contain the known Glashow-Iliopoulos-Maiani (GIM) suppression factor of the decay $\nu_i \to \nu_j \gamma$ in vacuum $\sim m_\ell^2/m_W^2 \ll 1$ [see Eq. (1.3)].

The expression for the amplitude, corresponding to the diagram in Fig. 3, can be represented in the form

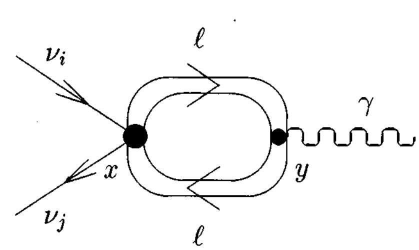

FIG. 3. Feynman diagram 1(a) transformed in the local limit of a $W$-boson propagator being contacted to a point.

$$\Delta\mathcal{M}=\frac{ieG_F}{\sqrt{2}}j_\beta\epsilon^*_\alpha(q)\langle J_{\alpha\beta}(q)\rangle-\mathcal{M}^{(0)}, \tag{2.2}$$

$$J_{\alpha\beta}(q)=\int d^4XSp[\gamma_\alpha\hat{S}(X)\gamma_\beta(1+\gamma_5)\hat{S}(-X)]e^{iqX}, \tag{2.3}$$

where $X=x-y$ and $\hat{S}(X)$ is the propagator of a charged lepton in the crossed field [see Appendix A, Eqs. (A1) and (A2)]. All the other quantities in Eq. (2.2) are defined above [see Eqs. (1.1)–(1.3)]. The details of the tensor $J_{\alpha\beta}(q)$ calculation may be found in Appendix A, while here we only give the result of the calculation:

$$\Delta\mathcal{M}=\frac{eG_F}{4\pi^2\sqrt{2}}\left\langle e(\widetilde{F}f^*)\frac{(qFFj)}{(qFFq)}I_1+\frac{e}{8m_\ell^2}(F\widetilde{f}^*)(qj)I_2+\frac{e^2}{24m_\ell^4}(F\widetilde{f}^*)(q\widetilde{F}j)I_3+\frac{e^2}{48m_\ell^4}(Ff^*)(qFj)I_4\right\rangle, \tag{2.4}$$

$$f_{\alpha\beta}=q_\alpha\epsilon_\beta-q_\beta\epsilon_\alpha,$$

$$\widetilde{f}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}f_{\alpha\beta},$$

where $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}=\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}/2$ are the tensor and the dual tensor of a constant field, $e>0$ is the elementary charge, and $G_F$ is the Fermi constant. In Eq. (2.4), $I_a$ $(a=1,\ldots,4)$ are integrals of the known Hardy-Stokes functions $f(u)$:

$$I_1=\int_0^1 dt\,uf(u),$$

$$I_2=\int_0^1 dt(1-t^2)uf(u), \tag{2.5}$$

$$I_3=\int_0^1 dt(1-t^2)(3-t^2)u^2\frac{df}{du},$$

$$I_4=\int_0^1 dt(1-t^2)(3+t^2)u^2\frac{df}{du},$$

$$f(u)=i\int_0^\infty dz\exp[-i(zu+\tfrac{1}{3}z^3)], \tag{2.6}$$

$$u=[e^2(qFFq)/16m_\ell^6]^{-1/3}(1-t^2)^{-2/3}.$$

As can be readily checked, the amplitude (2.4) is evidently gauge invariant, as it is expressed in terms of the tensors of the external field $F_{\mu\nu}$ and the photon field $f_{\mu\nu}$. In the weak field limit $(F/F_e\ll 1)$ the predominant contribution to the amplitude is made by the second term in expression (2.4). The integrals $I_a$ can be easily evaluated in this limit, taking into account the orthogonality of the lepton-mixing matrix $K_{i\ell}$:

$$\langle I_1\rangle=\langle 1+O[(F/F_e)^2]\rangle=O[(F/F_e)^2],$$

$$I_2\simeq 2/3,$$

$$I_3\simeq 28/15,$$

$$I_4=O[(F/F_e)^4]. \tag{2.7}$$

As expected, the amplitude (2.4) in view of Eq. (2.7) in the weak field limit coincides with expression (1.1). The amplitude of the process $\nu_i\to\nu_j\gamma$ in a crossed field (2.4) is simplified substantially in two cases: that of the decay of a neutrino at rest $(E_\nu=m_\nu)$ and that of the decay of an ultrarelativistic neutrino $(E_\nu\gg m_\nu)$.

### A. Neutrino at rest ($E_\nu=m_\nu$)

In this case, the dynamic parameter (2.1),

$$\chi_\ell^2=\frac{e^2(p_1FFp_1)}{m_\ell^6}=\left(\frac{m_\nu}{m_\ell}\frac{eF}{m_\ell^2}\right)^2,$$

is obviously small even when the field strength exceeds the critical values $(F\gg m_\ell^2/e,\ m_\nu\ll m_\ell,\ \chi_\ell\ll 1)$. The decay amplitude (2.4) and (2.5) can then be reduced to the form

$$\Delta\mathcal{M}\simeq\frac{eG_F}{60\pi^2\sqrt{2}}\{(\mathcal{F}\widetilde{f}^*)[(j\mathcal{F}\mathcal{F}q)+\tfrac{5}{4}(jq)-\tfrac{7}{6}(q\widetilde{\mathcal{F}}j)]-\tfrac{19}{24}(\mathcal{F}f^*)(q\mathcal{F}j)\}(K_{ie}K^*_{je}). \tag{2.8}$$

Here we have introduced the dimensionless field tensor $\mathcal{F}_{\mu\nu}=F_{\mu\nu}/F_e$, where $F_e=m_e^2/e$ is the critical strength value and $m_e$ is the electron mass. It is clear from Eq. (2.8) that the decay probability is represented by a polynomial of the sixth degree in field strength. In the limit $F\ll F_e$ (a weak field limit), the expression for the decay probability is determined by the lowest power of $F$ and has the form

$$w_{\text{weak}}\simeq\frac{\alpha}{18\pi}\frac{G_F^2}{192\pi^3}m_i^5\left(1-\frac{m_j^2}{m_i^2}\right)^5\left(\frac{F}{F_e}\right)^2|K_{ie}K^*_{je}|^2, \tag{2.9}$$

where $m_i$ and $m_j$ are the masses of the initial and final neutrinos. In the opposite case, $F\gg F_e$ (a strong field limit), we have

$$w_{\text{st}}\simeq\frac{\alpha}{4\pi}\frac{G_F^2}{(15\pi)^3}m_i^5\left(1-\frac{m_j^2}{m_i^2}\right)^5\left(1+5\frac{m_j^2}{m_i^2}\right)\left(\frac{F}{F_e}\right)^6|K_{ie}K^*_{je}|^2. \tag{2.10}$$

Expressions (2.9) and (2.10) should be compared with the well-known probability of the decay $\nu_i\to\nu_j\gamma$ in vacuum [6]:

$$w_0\simeq\frac{27\alpha}{32\pi}\frac{G_F^2}{192\pi^3}m_i^5\left(\frac{m_\tau}{m_W}\right)^4\left(1+\frac{m_j^2}{m_i^2}\right)\left(1-\frac{m_j^2}{m_i^2}\right)^3|K_{i\tau}K^*_{j\tau}|^2. \tag{2.11}$$

The comparison demonstrates the catalyzing effect of the external crossed field on the decay probabilities, as there is no suppression $\sim(m_\ell^2/m_W^2)$ in Eqs. (2.9) and (2.10). Actually, the enhancement influence of the external field takes place in the weak field limit $(F\ll F_e)$ under the condition

$F > 2\times 10^{-3}F_e$. Besides, the decay in the strong crossed field (2.10) is catalyzed by an additional factor of the form $\sim(F/F_e)^6 \gg 1$.

### B. Ultrarelativistic neutrino ($E_\nu \gg m_\nu$)

Notice that in the ultrarelativistic limit the kinematics of the decay $\nu_i(p_1)\to\nu_j(p_2)+\gamma(q)$ is such that the momentum four-vectors of the initial neutrino $p_1$ and the decay products $p_2$ and $q$ are almost parallel to each other. Therefore, the current four-vector $j_\alpha = \overline{\nu_j}(p_2)\gamma_\alpha(1+\gamma_5)\nu_i(p_1)$ is also proportional to these vectors ($j_\alpha \sim p_{1\alpha}\sim q_\alpha \sim p_{2\alpha}$). In this case, the expression for amplitude (2.4) can be simplified and reduced to the form

$$\Delta\mathcal{M} \simeq \frac{e^2 G_F}{\pi^2}(\epsilon^* \widetilde{F} p_1)\left[(1-x)+\frac{m_j^2}{m_i^2}(1+x)\right]^{1/2}\langle I_1\rangle, \qquad (2.12)$$

where $x=\cos\vartheta$; $\vartheta$ is the angle between the vectors $\vec{p}_1$ (the momentum of the decaying ultrarelativistic neutrino) and $\vec{q}\,'$ (the photon momentum in the decaying neutrino $\nu_i$ rest frame). The argument $u$ of the Hardy-Stokes function $f(u)$ in the integral $I_1$ [see Eq. (2.5)] in the ultrarelativistic case has the form

$$u = 4\left[(1+x)(1-t^2)\left(1-\frac{m_j^2}{m_i^2}\right)\chi_\ell\right]^{-2/3}. \qquad (2.13)$$

The Lorentz-invariant decay probability $wE_\nu$ can be expressed in terms of the integral of the squared amplitude over the variable $x$:

$$wE_\nu \simeq \frac{1}{16\pi}\left(1-\frac{m_j^2}{m_i^2}\right)\int_{-1}^{+1} dx |\Delta\mathcal{M}|^2$$
$$= \frac{\alpha}{4\pi}\frac{G_F^2}{\pi^3} m_e^6 \chi_e^2\left(1-\frac{m_j^2}{m_i^2}\right)\int_{-1}^{+1} dx$$
$$\times\left[(1-x)+\frac{m_j^2}{m_i^2}(1+x)\right]|\langle I_1\rangle|^2. \qquad (2.14)$$

For small values of the dynamic parameter ($\chi_\ell \ll 1$), the integral $I_1(\chi_\ell)$ is expanded into the series

$$I_1 \simeq 1+\tfrac{1}{15}\widetilde{\chi}_\ell^2+\tfrac{4}{63}\widetilde{\chi}_\ell^4+\cdots,$$
$$\widetilde{\chi}_\ell = \frac{1+x}{2}\left(1-\frac{m_j^2}{m_i^2}\right)\chi_\ell, \qquad (2.15)$$

and the probability (2.14) can be represented in the form

$$w \simeq \frac{\alpha}{4\pi}\frac{G_F^2 \; m_e^6}{(15\pi)^3 E_\nu}\chi_e^6\left(1-\frac{m_j^2}{m_i^2}\right)\left(1+5\frac{m_j^2}{m_i^2}\right)|K_{ie}K_{je}^*|^2. \qquad (2.16)$$

As the dynamic parameter $\chi_\ell \sim (E_\nu/m_\ell)(F/F_\ell)$ is proportional to the neutrino's energy, it is clear from Eq. (2.16) that, with increasing the energy of the decaying neutrino, the decay probability increases as $\sim E_\nu^5$

For great values of $\chi_\ell \gg 1$, using the asymptotic behavior of the Hardy-Stokes function both at large and at small values of the argument and also the unitarity of the mixing matrix $K_{i\ell}$, one can represent Eq. (2.14) in the form

$$w \simeq \frac{\alpha}{4\pi}\frac{G_F^2 m_e^6}{\pi^3 E_\nu}\chi_e^2\left(1-\frac{m_j^4}{m_i^4}\right)$$
$$\times\begin{cases}|K_{ie}K_{je}^*|^2, & \chi_e \gg 1,\ \chi_{\mu,\tau}\ll 1,\\ |K_{i\tau}K_{j\tau}^*|^2, & \chi_e\gg\chi_\mu\gg 1,\ \chi_\tau \ll 1.\end{cases} \qquad (2.17)$$

In this way, the decay probability increases linearly $\sim E_\nu$ ($\chi_e \gg 1$, $\chi_\tau \ll 1$) and finally ($\chi_\tau \gg 1$) becomes a constant:

$$w \simeq \frac{21.7\alpha}{\pi}\frac{G_F^2 m_\tau^6}{\pi^3 E_\nu}\chi_\tau |K_{i\tau}K_{j\tau}^*|^2,$$
$$\chi_\tau\left(1-\frac{m_j^2}{m_i^2}\right)\gg 1. \qquad (2.18)$$

Comparing the decay probabilities (2.16)–(2.18) in the crossed field with the vacuum decay probability (2.11), we notice that the catalyzing effect of the field on the ultrarelativistic neutrino decay becomes even more substantial compared to the situation with the neutrino at rest, because there is no suppression caused by the smallness of the mass of the decaying neutrino. Recall also that none of the expressions for the decay probability in the crossed field contain the well-known suppression GIM factor $(m_\ell/m_W)^4$ characteristic of the probability of the decay $\nu_i\to\nu_j\gamma$ in vacuum. The probability of neutrino decay at rest in a strong crossed field [see Eq. (2.10)] is enhanced by the additional factor $\sim(F/F_e)^6 \gg 1$.

Here we estimate a number of $\gamma$ quanta which can be result as the decay product from a neutrino beam of a high energy accelerator in a volume filled with an electromagnetic field. As the experimentally accessible strengths of electromagnetic fields are significantly below the critical strength ($F \ll F_e$) in the laboratory conditions we can use the expression (2.17) for the decay probability of a high energy neutrino ($E_\nu \gg m_e F/F_e$, $\chi_e \gg 1$).

The number of $\gamma$ quanta, $\Delta N^\gamma$, which are produced in a volume filled with a magnetic field of the strength $B$ transversal to the neutrino beam can be presented in the form

$$\Delta N^\gamma \sim 10^7\left(\frac{B}{B_e}\right)^2\left(\frac{L}{1m}\right)\left(\frac{W}{10^{19}\ \text{GeV}}\right)|K_{ie}K_{je}^*|^2, \quad (2.19)$$

where $L$ is a longitudinal dimension of the volume. The parameter $W$ has a simple meaning of a full energy of neutrinos passed through a ''target'' during the experiment. It can be presented as

$$W = \int E\frac{dN^\nu}{dE}dE,$$

where $dN^\nu/dE$ is the energy distribution of the neutrino beam. One can estimate this parameter from a number $N^{CC}$ of $\nu_\mu' s$ charge current (CC) events in detector with known design parameters during the time of neutrino experiment.

With the expected data on $N^{CC}$ from CERN Super Proton Synchrotron (SPS) neutrino beam, which is presented, for example, in Ref. [20], one can estimate

$$W \sim 10^{19} \text{ GeV}.$$

As one can see from Eq. (2.19), $\Delta N^{\gamma} \sim 1$ requires $B \sim 10^{10}$ G in a volume $\sim 1$ m$^3$.

## III. CONSTANT MAGNETIC FIELD

The probability of the massive neutrino decay $\nu_i \to \nu_j \gamma$ in a constant magnetic field having the strength $\vec{B}$ is described by two invariant parameters (1) the above-mentioned dynamical parameter

$$\chi_{\ell}^2 = \frac{e^2(p_1 FF p_1)}{m_{\ell}^6} = \frac{B^2}{B_{\ell}^2} \frac{p_{1\perp}^2}{m_{\ell}^2}, \tag{3.1}$$

where $B_{\ell} = m_{\ell}^2/e$ is the critical magnitude of the magnetic field and $p_{1\perp}$ is the initial neutrino's momentum component, normal to the magnetic field, and (2) the external magnetic field intensity parameter

$$\eta_{\ell}^2 = -\frac{e^2(FF)}{2m_{\ell}^4} = \frac{B^2}{B_{\ell}^2}, \tag{3.2}$$

where $F_{\mu\nu}$ is the constant uniform magnetic field tensor, $(FF) = -2B^2$. Since magnetic fields in up-to-date superconductive magnets range up to strengths $B \leqslant 10^5$ G, the hope of obtaining marked quantum effects induced by a magnetic field, as it seems, should be related only to the ultrarelativistic neutrino decay, when the dynamic parameter $\chi_{\ell}$ may be not small, while the intensity parameter $\eta_e \ll 1$. In this case we have the crossed field limit which was considered in detail in the preceding section. The decay in a strong magnetic field ($B \geqslant B_e = 4.41 \times 10^{13}$ G) is probably of interest in astrophysics or in early Universe cosmology. By mentioning astrophysics, we, first of all, mean intensive magnetic fields, ''frozen-in'' neutron star substance. The primordial magnetic fields, ''frozen in'' the cosmological plasma, could also have had the strenghts $B \geqslant B_e$ at some stages of the early Universe evolution [21].

The amplitude of the decay $\nu_i \to \nu_j \gamma$ in a magnetic field is described by the same effective Feinman diagram as that shown in Fig. 3. The expression for the lepton propagator and the calculation details are given in Appendix A; here, we only present the external-field-induced contribution $\Delta\mathcal{M}^{(B)}$ to the decay amplitude:

$$\Delta\mathcal{M}^{(B)} \simeq \frac{e}{(4\pi)^2} \frac{G_F}{\sqrt{2}} [i(f\tilde{\varphi})(qj)\langle Y_1\rangle + (f\tilde{\varphi})(q\tilde{\varphi}j)\langle Y_2\rangle + (f\varphi)(q\varphi j)\langle Y_3\rangle + i(f\tilde{\varphi})(q\Lambda j)\langle Y_4\rangle]. \tag{3.3}$$

We recall that here $j_\mu = \overline{\nu_j}(p_2)\gamma_\mu(1+\gamma_5)\nu_i(p_1)$ is the neutrino current,

$$\langle Y(m_{\ell})\rangle = \sum_{\ell = e,\mu,\tau} K_{i\ell} K_{j\ell}^* Y(m_{\ell}), \tag{3.4}$$

$p_1$, $p_2$, and $q$ are the four-momenta of the initial and final neutrino and the photon, respectively, and $f_{\alpha\beta} = q_\alpha \epsilon_\beta - q_\beta \epsilon_\alpha$ is the photon field tensor. We have introduced the dimensionless tensors of the external magnetic field $\varphi_{\alpha\beta} = F_{\alpha\beta}/B$, $\tilde{\varphi}_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}\varphi_{\mu\nu}/2$, and $\Lambda_{\alpha\beta} = (\varphi\varphi)_{\alpha\beta}$. The double integrals $Y_i$ ($i$ = 1, 2, 3, 4), entering Eq. (3.3), have the form

$$Y_i = \int_0^1 dt \int_0^\infty \frac{dz}{\sin z} e^{\ i\Omega(z,t)} y_i(z,t),$$

$$y_1 = (1-t^2)\sin z + t \sin zt - \frac{1 - \cos z \cos zt}{\sin z},$$

$$y_2 = \cos zt - (1-t^2)\cos z - \frac{t \sin zt}{\tan z},$$

$$y_3 = -\cos zt + \frac{t \sin zt}{\tan z} + 2\frac{\cos zt - \cos z}{\sin^2 z},$$

$$y_4 = 2y_1 - (1-t^2)\sin z,$$

$$\Omega(z,t) = \frac{1}{2}\left(\frac{z(1-t^2)}{2} + \frac{1}{\tan z} - \frac{\cos zt}{\sin z}\right)\frac{(q\Lambda q)}{eB} - \frac{z}{\eta_{\ell}}. \tag{3.5}$$

It should be noted that in Eqs. (3.5) the integration with respect to $z$ is performed over the complex $z$ plane. The integrand's singularities are bypassed in a usual way [30] in the lower semiplane ( $\mathrm{Im}\, z < 0$) of the complex $z$ plane. In the weak field approximation $B \ll B_{\ell}$ ($\eta_{\ell} \ll 1$) the integrands in Eqs. (3.5) become rapidly oscillating functions of the variable $z$ [since at $z \simeq 1$ we have $\Omega(z,t) \simeq 1/\eta_{\ell} \gg 1$]. Therefore, the predominant contribution to the integrals comes from a narrow region of small $z \simeq \eta_{\ell}$, in which the integrals can be considerably simplified. By straightforward calculation, it can be shown that the amplitude (3.3) in this limit [$\eta_{\ell} = B/B_{\ell} \ll 1$; however, $\chi_{\ell} = (B/B_{\ell})(p_\perp/m_{\ell})$ has not to be small], as should be expected, coincides with the amplitude of the neutrino radiative decay in the crossed field (2.4). Note that this is the necessary (but, certainly, not sufficient) condition for our combersome calculations of the magnetic-field-induced amplitude $\Delta\mathcal{M}^{(B)}$ to be correct.

In the strong field approximation $B \gg B_{\ell}$ ($\eta_{\ell} \gg 1$), to evaluate the integrals (3.5), it is convenient to rotate the integration loop clockwise by $\pi/2$ in the complex $z$ plane. In this case the amplitude $\Delta\mathcal{M}^{(B)}$, Eqs. (3.3) and (3.5), is significantly simplified and can be represented as

$$\Delta\mathcal{M}_{\text{st}}^{(B)} \simeq \frac{e}{24\pi^2} \frac{G_F}{\sqrt{2}} (f\tilde{\varphi})[(jq) + (j\tilde{\varphi}q) + (j\Lambda q)] \times \langle \eta_{\ell} H(4m_{\ell}^2/q_\perp^2)\rangle, \tag{3.6}$$

where $q_\perp^2 = (q\Lambda q)$ is the square of the photon momentum, which is normal to the magnetic field,

$$H(x) = \frac{3}{2}x\left(\frac{x}{\sqrt{x-1}}\arctan\frac{1}{\sqrt{x-1}} - 1\right), \quad x > 1,$$

$$H(x)=-\frac{3}{4}x\left[\frac{x}{\sqrt{1-x}}\ln\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)+2+i\pi\frac{x}{\sqrt{1-x}}\right],$$

$$x<1. \tag{3.7}$$

Note that, if $q_\perp^2=4m_\ell^2$ $(x=1)$, the amplitude (3.6) has a root singularity associated with the known root singularity of the probability of the $e^+e^-$ pair generation by a photon in an external magnetic field [22]. Such a behavior of the decay amplitude in the vicinity of $q_\perp^2\to 4m_\ell^2$ exactly corresponds to the singularity of the imaginary part of the photon polarization operator in a magnetic field [23].

On the basis of the expression (3.6) for the amplitude we obtained the following expression for the probability of the ultrarelativistic neutrino of a moderate-energy ($p_\perp^2<4m_\ell^2$) in a strong magnetic field ($\eta_\ell \gg 1$):

$$w_{st}\simeq\frac{2\alpha}{\pi}\frac{G_F^2}{\pi^3}\frac{m_e^6}{E_\nu}\left(\frac{B}{B_e}\right)^2|K_{ie}K_{je}^*|^2 J(z),$$

$$J(z)=\int_0^z dy(z-y)\left[\frac{1}{y\sqrt{1-y^2}}\arctan\frac{y}{\sqrt{1-y^2}}-1\right]^2,$$

$$z=\frac{E_\nu\sin\theta}{2m_e}. \tag{3.8}$$

To estimate the enhancement of the decay by the external magnetic field numerically, it is sufficient to compare expression (3.8) for the decay probability in a magnetic field with expression (2.11) for the probability of the decay of the massive neutrino in vacuum.

As we can see, in the relativistic neutrino decay probability (3.8) there is no suppression caused by the smallness of the neutrino mass, because the probability is virtually independent of the neutrino mass (if $m_j^2/m_i^2 \ll 1$). Finally, the decay in a strong field is also catalyzed by a factor $\sim(B/B_e)^2$. Especially impressive is the comparison of the moderate energy ($E_\nu^2<4m_e^2$) relativistic neutrino's lifetime $\tau^{(B)}$ in the radiative decay $\nu_i\to\nu_j\gamma$ in a strong magnetic field ($B\gg B_e$),

$$\tau^{(B)}\simeq\frac{2\times10^7}{|K_{je}^*K_{ie}|^2}\left(\frac{B_e}{B}\right)^2\left(\frac{1\ \text{MeV}}{E_\nu}\right)\text{sec}, \tag{3.9}$$

and the lifetime $\tau^{(0)}$ in the radiative decay in vacuum:

$$\tau^{(0)}\simeq\frac{10^{50}}{|K_{j\tau}^*K_{i\tau}|^2}\left(\frac{1\ \text{eV}}{m_\nu}\right)^6\left(\frac{E_\nu}{1\ \text{MeV}}\right)\text{sec}. \tag{3.10}$$

So the magnetic catalysis of the massive neutrino radiative decay might solve the problem of whether a heavy enough ($m_\nu \gg 20$ eV) neutrino exists in the Universe. In fact, if the magnetic field fluctuations in an early enough ($t \gg 1$ sec) Universe would, for some reasons, reach such values as $B\sim10^3B_e$, the massive neutrino's lifetime under such fields could, according to Eq. (3.9), reduce to values of order of 1 sec.

## IV. PLANE MONOCHROMATIC WAVE

The development of intensive electromagnetic field generation techniques and the current possibility to obtain waves of high strength of electromagnetic field, namely, $\mathcal{E}\sim10^9$ V/cm, have stimulated the investigation of quantum processes in strong external wave fields. Indeed, the so-called wave intensity parameter

$$\ae_\ell^2=-\frac{e^2a^2}{m_\ell^2} \tag{4.1}$$

(where $a$ is the amplitude of wave, $m_l$ is the lepton mass, and $e$ is an elementary charge), characterizing the effect of the electromagnetic wave, should not be neglected. It is also worth noting that the energy-momentum conservation law for the radiative decay in the wave also contains, along with the four-momenta $p_1$, $p_2$, and $q$, the four-wave-vector $k$. The process amplitude is calculated by standard Feynman rules, in which for the propagators of intermediate fermions (see Appendix B) exact solutions are used of the corresponding wave equations. In this section we consider the external field of a monochromatic circularly polarized wave with the four-potential

$$A_\mu=a_{1\mu}\cos\varphi+\xi a_{2\mu}\sin\varphi,\quad \varphi=kx, \tag{4.2}$$

where $k^\mu=(\omega,\vec{k})$ is the four-wave-vector, $k^2=(a_1k)=(a_2k)=(a_1a_2)=0$, and $a_1^2=a_2^2=a^2$; the parameter $\xi=\pm1$ indicates the direction of the circular polarization (leftward or rightward). Note that vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{k}$ form a right-handed coordinate system. The $S$-matrix element of the given process, just as previously, can be represented in the form

$$S=S_0+\Delta S, \tag{4.3}$$

where $S_0$ is the matrix element of the radiative decay of the massive neutrino in a vacuum and $\Delta S$ is the contribution induced by the wave field:

$$\Delta S=\frac{i(2\pi)^4}{\sqrt{2E_1V2E_2V2q_0V}}\sum_{n=-2}^{+2}\mathcal{M}^{(n)}\delta^{(4)}(nk+p_1-p_2-q). \tag{4.4}$$

Here $p_1$, $p_2$, $q$ and $E_1$, $E_2$, $q_0$ are the four-momenta and the energies of the initial and final neutrinos and the photon, respectively, and $n=0,\pm1,\pm2$ is the difference between the numbers of absorbed and emitted photons of the wave field.

Note that the matrix element of some process in the field of an electromagnetic wave has usually the form of summation of $n$ type (4.4), where $-\infty<n<\infty$ [24]. That only five values of $n$ in our case are possible is extraordinary and is due to the following reasons. The process $\nu_i\to\nu_j\gamma$ is local with the typical scale $\Delta x \ll 1/m_f$ ($m_f$ is the mass of the virtual fermion). In this case the angular momentum conservation degenerates to spin conservation. Since the total spin of the particles participating in this process is no greater than 2, $|n|_{\max}=2$ is the maximum difference between the numbers of absorbed and emitted photons of the external field (the photons of a monochromatic circularly polarized wave have a definite spin $\xi=\pm1$). The direct calculation supports this conclusion. A similar phenomenon has been discovered before [18] in studies of the effect of a circularly polarized wave on flavor, changing transitions of the massive neutrinos $\nu_i \leftrightarrow \nu_j$ $(i \neq j)$ with $|n|_{\max}=1$.

Notice that in the uniform constant fields the decay $\nu_i \to \nu_j \gamma$ with $m_i > m_j$ is valid only. This is due to the fact that the energy-momentum conservation law in this fields coincides with the one in vacuum. On the other hand, as follows from Eq. (4.4), the external electromagnetic wave field can also induce radiative transition with $m_i \leqslant m_j$ forbidden without the field. Indeed, from the energy-momentum conservation law in the wave field,

$$nk + p_1 = p_2 + q,$$

the relation

$$m_i^2 - m_j^2 \geqslant -2n(kp_1)$$

follows. In such a manner, the radiation decay $\nu_i \to \nu_j \gamma$ with $m_i \leqslant m_j$ is possible on condition that $n>0$.

In the lowest order of perturbation theory, a matrix element of the radiative transition $\nu_i \to \nu_j \gamma$ is described by the effective diagram represented in Fig. 3, in which not only the $W$ boson, but also (for $i=j$) the $Z$ boson is exchanged at the point $x$. The expressions for the fermion propagator in the wave's field, as well as some details of the awkward calculation of the invariant amplitudes $\mathcal{M}^{(n)}$, are given in Appendix B. Here we only present the final result:

$$\mathcal{M}^{(0)} = -\frac{e}{16\pi^2}\frac{G_F}{\sqrt{2}}\frac{4}{(kq)}\Big\{\sum_{\ell} (K_{i\ell}K_{j\ell}^* + \tfrac{1}{2}\delta_{ij}g_\ell)\text{æ}_\ell^2 m_\ell^2[(jfk)J_1(m_\ell) + (j\tilde{f}k)J_2(m_\ell)]$$
$$-\tfrac{3}{2}\delta_{ij}\sum_q Q_q g_q\, \text{æ}_q^2 m_q^2[(jfk)J_1(m_q) + (j\tilde{f}k)J_2(m_q)]\Big\},$$

$$\mathcal{M}^{(\sigma)} = -\frac{e}{16\pi^2}\frac{G_F e\sigma(\tilde{f}F^\sigma)}{\sqrt{2}\;\;(kq)}\Big\{\sum_{\ell} (K_{i\ell}K_{j\ell}^* - \tfrac{1}{2}\delta_{ij})\Big[[(p_1-p_2)j]J_3^{(\sigma)}(m_\ell) + 8\frac{m_\ell^2\text{æ}_\ell^2}{(kq)}(jk)J_4^{(\sigma)}(m_\ell)\Big]$$
$$+\tfrac{3}{2}\delta_{ij}\sum_q (2T_{3q}Q_q^2)\Big[[(p_1-p_2)j]J_3^{(\sigma)}(m_q) + 8\frac{m_q^2\,\text{æ}_q^2}{(kq)}(jk)J_4^{(\sigma)}(m_q)\Big]\Big\},$$

$$\mathcal{M}^{(2\sigma)} = -\frac{e}{16\pi^2}\frac{G_F e^2(fF^\sigma)(jF^\sigma q)}{\sqrt{2}\;\;(kq)^2}\Big\{\sum_{\ell} (K_{i\ell}K_{j\ell}^* + \tfrac{1}{2}\delta_{ij}g_\ell)J_5^{(\sigma)}(m_\ell) - \tfrac{3}{2}\delta_{ij}\sum_q (Q_q^3 g_q)J_5^{(\sigma)}(m_q)\Big\},$$

$$F_{\mu\nu}^{(\sigma)} = k_\mu a_\nu^{(\sigma)} - k_\nu a_\mu^{(\sigma)}, \quad a_\mu^{(\sigma)} = (a_1 + i\xi\sigma a_2)_\mu,$$

$$g_f = 2T_{3f} - 4Q_f \sin^2\theta_w, \quad f=\ell, q. \tag{4.5}$$

Here $\sigma$ is the sign of the summation index $n$ in Eq. (4.4) $(\sigma=\pm 1)$, the index $\ell$ indicates charged leptons $(\ell = e,\mu,\tau)$, the index $q$ indicates quark flavors $(q=u,c,t,d,s,b)$, $T_{3f}$ is the third component of the weak isospin, $Q_f$ is the electric charge in units of the elementary charge, $m_\ell$ and $m_q$ are the masses of the virtual leptons and quarks, and $K_{ij}$ is the unitary lepton mixing matrix,

$$J_1(m_f) = \int_0^1 dy \int_0^\infty d\tau\Big[\tau\Big(j_1^2 + \frac{3+y^2}{1-y^2}j_0^2\Big) - \frac{4y^2}{1-y^2}j_0 j_1\Big]e^{i\Phi(m_f)},$$

$$J_2(m_f) = \int_0^1 dy\,\frac{1+y^2}{1-y^2}\int_0^\infty d\tau\,\tau j_0 j_1 e^{i\Phi(m_f)},$$

$$J_3^{(\sigma)}(m_f) = \int_0^1 dy\int_0^\infty d\tau (j_0 - i\sigma j_1)e^{i[\Phi(m_f)-\sigma\tau]},$$

$$J_4^{(\sigma)}(m_f) = \int_0^1 \frac{dy}{1-y^2}\int_0^\infty d\tau\,\tau^2 j_0(j_0^2 + j_1^2)e^{i[\Phi(m_f)-\sigma\tau]},$$

$$J_5^{(\sigma)}(m_f) = \int_0^1 dy\int_0^\infty d\tau\,\tau(j_0^2+j_1^2)e^{i[\Phi(m_f)-2\sigma\tau]},$$

$$\Phi(m_f) = -\frac{4\tau}{1-y^2}\frac{m_f^2}{(kq)}[1 + \text{æ}_f^2(1-j_0^2)],$$

$$\text{æ}_f^2 = -Q_f^2\frac{e^2a^2}{m_f^2}. \tag{4.6}$$

Here $j_0 = \sin\tau/\tau$, $j_1 = -dj_0/d\tau$ are so-called Bessel spherical harmonics; the other denotations and quantities in Eqs. (4.5) and (4.6) have been introduced above. It is easy to see that the amplitudes $\mathcal{M}^{(n)}$ are explicitly gauge invariant and do not contain divergences. Note that the expressions we have obtained for the amplitudes $\mathcal{M}^{(n)}$ of the radiative transition $\nu_i \to \nu_j\gamma$ in the field of a monochromatic wave allow a simple check. Indeed, if in Eqs. (4.5) and (4.6) the wave frequency $\omega$ tends to zero (i.e., $k_\mu \to 0$), the strengths $\vec{\mathcal{E}}$ and $\vec{B}$ of the electric and magnetic fields being fixed, then in this limit the amplitude of the decay $\nu_i \to \nu_j \gamma$ in crossed field must be obtained [see Eqs. (2.4)–(2.6)]:

$$\Delta \mathcal{M}[\text{Eq. (2.4)}] = \sum_{n=-2}^{+2} \mathcal{M}^{(n)} \Big|_{\substack{k_\mu \to (0,\vec{0}) \\ \mathcal{E}, B, \text{fix}}} . \qquad (4.7)$$

To prove the conclusion (4.7), it is necessary to take into account that (1) in the above limit, the tensor $F_{\mu\nu}^{(\sigma)}$ is expressed in terms of the strength tensor $F_{\mu\nu}$ and dual tensor $\widetilde{F}_{\mu\nu}$ of the crossed field,

$$F_{\mu\nu}^{(\sigma)} \big|_{\mathcal{E},B,\text{fix}}^{k\to 0} = i\sigma (F_{\mu\nu} + i\sigma \xi \widetilde{F}_{\mu\nu}). \qquad (4.8)$$

(2) If we take advantage of the identity

$$(A_1 \widetilde{A}_2)_{\alpha\beta} + (A_2 \widetilde{A}_1)_{\alpha\beta} = \tfrac{1}{2}(A_1 \widetilde{A}_2) g_{\alpha\beta},$$

$$(A_1 \widetilde{A}_2)_{\alpha\beta} = A_{1\alpha\rho} \widetilde{A}_{2\rho\beta}, \quad (A_1 \widetilde{A}_2) = A_{1\alpha\rho} \widetilde{A}_{2\rho\alpha},$$

where $A_{1\mu\nu}$ and $A_{2\mu\nu}$ are arbitrary antisymmetric four-tensors, then the wave intensity parameter $æ_{\mathscr{l}}^2$ Eq. (4.1), can be related to the dynamic parameter $\chi_{\mathscr{l}}^2$ Eq. (2.1), by the expression

$$\left[ æ_{\mathscr{l}}^2 \left( \frac{p_1 k}{m_{\mathscr{l}}^2} \right)^2 \right] \Big|_{k\to 0} = \chi_{\mathscr{l}}^2 . \qquad (4.9)$$

(3) To remove the undeterminacy arising at $k \to 0$ in expressions (4.5) for $\mathcal{M}^{(n)}$ it is useful to take advantage of the limit relationship

$$\frac{k_\alpha}{(ka)} \Big|_{k\to 0} = \frac{(bFF)_\alpha}{(bFFa)}, \qquad (4.10)$$

where $a$ and $b$ are arbitrary four-vectors (however, $b$ is a timelike vector), and (4) in the crossed field limit the terms in $M^{(n)}$, Eq. (4.5), proportional to $\delta_{ij}$ do not make any contribution (because of kinematics reasons). In passing to the limit using Eqs. (4.8)–(4.10) the result (4.7) is reproduced immediately.

The probability of the transition $\nu_i \to \nu_j \gamma$ in the wave field

$$w = \sum_{n=-2}^{+2} w^{(n)} \qquad (4.11)$$

is, in general, rather awkward. We shall give it only in the most interesting, from the physical point of view, case of the transition $\nu_i \to \nu_j \gamma$ with the initial neutrino $\nu_i$ being ultrarelativistic ($E_\nu \gg m_\nu$). Despite the fact that the wave intensity parameter $æ_{\mathscr{l}}^2$ Eq. (4.1), under laboratory conditions cannot be great (e.g., for laser fields $\omega \sim 1$ eV, $\mathcal{E} \leqslant 10^9$ V/sm, $æ_e^2 \leqslant 10^{-3}$), substantial enhancement of the transition probability is possible. The main effect of the enhancement connects with the decrease and even complete disappearance (for $n>0$ as we shall see below) of the suppression factor caused by the smallness of the neutrino mass. Recall that an analogous result was obtained for uniform and constant fields and was discussed in the above sections [see Eqs. (2.16)–(2.18) and (3.8)]. In the ultrarelativistic limit, the probabilities $w^{(n)}$ at $n \leqslant 0$ remain suppressed:

$$E_\nu w^{(-2)} = O\left( \alpha \frac{G_F^2 m_\nu^{10}}{m_e^4} æ_e^4 \right),$$

$$E_\nu w^{(-1)} = O\left( \alpha \frac{G_F^2 m_\nu^{8}}{m_e^2} æ_e^2 \right)$$

$$E_\nu w^{(0)} = O(\alpha G_F^2 m_\nu^2 m_e^4 æ_e^4). \qquad (4.12)$$

The other probabilities $w^{(n)}$ ($n = +1, +2$) in the limit $E_\nu \gg m_\nu$ are substantially simplified to be represented in the form

$$E_\nu w^{(+1)} \simeq \frac{4\alpha}{\pi} \frac{G_F^2}{\pi^3} m_e^6 æ_e^6 |K_{ie} K_{je}^* - \tfrac{1}{2}\delta_{ij}|^2 \times \int_{-1}^{+1} dx \frac{1-x}{(1+x)^2} |J_4^{(+1)}(m_e)|^2,$$

$$E_\nu w^{(+2)} \simeq \frac{\alpha\, G_F^2}{4\pi \pi^3} (p_1 k) m_e^4 æ_e^4 |K_{ie} K_{je}^* + \tfrac{1}{2}\delta_{ij} g_e|^2 \times \int_{-1}^{+1} \frac{dx}{1+x} \left[ \frac{(1-\xi)}{2} + \frac{(1-x)^2}{4} \frac{(1+\xi)}{2} \right] \times |J_5^{(+1)}(m_e)|^2. \qquad (4.13)$$

Here, the integration is performed with respect to $x = \cos\vartheta$, $\vartheta$ being the angle between the photon momentum $\vec{q}$ and the wave vector $\vec{k}$ in the center of mass of the final neutrino $\nu_j$ and photon $\gamma$. Consequently, in the ultrarelativistic case in the integrals $J_4^{(+1)}$ and $J_5^{(+1)}$ [see Eqs. (4.6)] the substitution $(qk) \simeq (1+x)(p_1 k)/2$ is needed. The comparison of the probabilities $w^{(n)}$, Eqs. (4.12) and (4.13) of the transition $\nu_i \to \nu_j \gamma$ in the wave field and the probability $w_0$ Eq. (2.11), of the decay $\nu_i \to \nu_j \gamma$ in vacuum shows that the probabilities $w^{(n)}$ at $n = +1, +2$ do not contain suppression associated with the smallness of the neutrino mass (recall that the probability of the decay of an ultrarelativistic neutrino in vacuum is $w_0 \sim m_\nu^6 / E_\nu$). Below we estimate the ratio of the probability $w$, Eq. (4.11), of the transition $\nu_i \to \nu_j \gamma$ for a neutrino from a high energy accelerator in the wave field of the laser type and the probability $w_0$, Eq. (2.11), of the decay in vacuum:

$$R = \frac{w}{w_0} \sim 10^{33} \left( \frac{1\,\text{eV}}{m_\nu} \right)^6 \left( \frac{E_\nu \omega}{m_e^2} \right)^5 (10^3\, æ_e^2)^2, \qquad (4.14)$$

where the wave intensity parameter $æ_e^2$ for the laser type fields is

$$æ_e^2 \simeq 10^{-3} \left( \frac{\mathcal{E}}{10^9\ \text{V}/sm} \right)^2 \left( 1 \frac{\text{eV}}{\omega} \right). \qquad (4.15)$$

Such a strong enhancement of the $\nu_i \to \nu_j \gamma$ transition probability, even at relatively small wave intensity ($æ_e^2 \leqslant 10^{-3}$), appears rather impressive.

The results obtained in this section may be of interest for astrophysics and cosmology. In particular, in the wave field the process of the photon splitting into the neutrino pair $\gamma \to \nu_i \widetilde{\nu}_j$ becomes possible. This process probability has the form

$$w_{\gamma \to \nu_i \overline{\nu_j}} \simeq \frac{\alpha}{3\pi} \frac{G_F^2}{8\pi^3} \frac{m_e^4}{q_0} æ_e^4 \{ 8 m_e^2 æ_e^2 |K_{ie} K_{je}^* - \tfrac{1}{2}\delta_{ij}|^2 \times |J_4^{(+1)}(m_e)|^2 + (qk) |K_{ie} K_{je}^* + \tfrac{1}{2}\delta_{ij}|^2 |J_5^{(+1)}(m_e)|^2 \}. \tag{4.16}$$

As is easily seen from Eq. (4.16), this process probability, in the same way, is not suppressed by the smallness of the neutrino mass. It can be treated as an additional mechanism of the energy loss by stars.

## V. CONCLUSION

In this work, in the framework of the standard model with fermion mixing, we have investigated the effect on the process $\nu_i \to \nu_j \gamma$ of the massive neutrino radiative decay of external electromagnetic fields of various configurations: constant crossed field (Sec. II), constant uniform magnetic field (Sec. III), and plane monochromatic wave's field (Sec. IV). The analysis of the decay amplitudes and probabilities obtained leads to the following conclusion, which is the same for all the field configurations covered: An external electromagnetic field catalyses the massive neutrino radiative decay. An especially strong enhancement occurs in the case of the ultrarelativistic neutrino radiative decay, since in this case the decay probability does not contain suppression caused by the smallness of the neutrino's mass.

In Sec. II [see Eq. (2.19)] we estimated a number of $\gamma$ quanta which could be resulted as the neutrino decay in a volume filled with a magnetic field. Let us give here the estimation in the case of limiting in the laboratory conditions values of $B$ and $W$ ($B \sim 10^9$ G, $W \sim 10^{19}$ GeV):

$$\Delta N^{\gamma} \sim 10^{-2} \left( \frac{B}{10^9 \text{ G}} \right)^2 \left( \frac{L}{1 \text{m}} \right) |K_{ie} K_{je}^*|^2.$$

It is worth noting that the estimation we have presented is numerically small and seems likely that there is no possibility to carry out such neutrino experiment in the near future.

Nevertheless, the mentioned above mechanism of the electromagnetic catalysis of the massive neutrino radiative decay is of interest in astrophysics where gigantic neutrino fluxes and strong magnetic fields can take place simultaneously (a process of a coalescence of neutron stars [25], an explosion of a supernova of the type SN 1987A [26]). Let us estimate a relative flux of $\gamma$ quanta which traverses a domain filled with a strong magnetic field ($B \gg B_e$):

$$\frac{\Phi^{\gamma}}{\Phi^{\nu}} \sim 10^{-12} \left( \frac{B}{B_e} \right)^2 \left( \frac{L}{10 \text{ km}} \right) |K_{ie} K_{je}^*|^2,$$

where $\Phi^{\nu}$ is the neutrino flux with the average energy $E_\nu \sim 1$ MeV transversal to the magnetic field strength and $L$ is the characteristic dimension of the domain. $\gamma$ quanta produced from the neutrino decay can be observed in astrophysical experiments provided that the domains with such strong magnetic fields exist.

On the other hand, the results presented in the Sec. III are, in our opinion, of interest in the cosmology of the early Universe. Indeed, in recent papers the possibility of the generation of primordial strong magnetic fields through thermal fluctuation in the primordial plasma with magnetic field strengths of order of $10^{12}$–$10^{15}$ G [27] or $10^{13}$–$10^{18}$ G [28] and coherence lengths of order of 10–100 cm was pointed out. Let us estimate the neutrino lifetime in the case of the existence of primordial small scale magnetic field strengths of order of $\sim 10^{17}$–$10^{15}$ G. For this purpose we use expression (3.9) we obtained for the moderate energy neutrino ($E_\nu \sim kT \sim 1$ MeV) lifetime $\tau^{(B)}$ in a strong magnetic field ($B \gg B_e$) and get the following estimation: $\tau^{(B)} \sim 0.1$–100 sec. This may be of interest in connection with the cosmological problem, concerning the contradiction between the Cosmic Background Explorer (COBE) data on the cosmic microwave background anisotropies and the observed power spectrum of the large-scale structure [29].

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## APPENDIX A: $J_{\alpha\beta}$ CALCULATION IN CONSTANT ELECTROMAGNETIC FIELD

The amplitude corresponding to the diagram in Fig. 3 is calculated according to the conventional Feinman rules. In doing so, for propagators of intermediate charged leptons exact solutions are used of the corresponding wave equations in the constant electromagnetic field. With the crossed field, the propagator of the charged lepton $\hat{S}^{(F)}(x,y)$ in the proper time formalism [30] has the form

$$\hat{S}^{(F)}(x,y) = e^{i\Phi(x,y)} \hat{S}(X), \tag{A1}$$

$$\hat{S}(X) = -\frac{i}{16\pi^2} \int_0^\infty \frac{ds}{s^2} \left[ \frac{1}{2s}(X\gamma) + \frac{ie}{2}(X\widetilde{F}\gamma)\gamma_5 - \frac{se^2}{3}(XFF\gamma) + m - \frac{sme}{2}(\gamma F \gamma) \right] \times \exp\left( -i \left[ m^2 s + \frac{1}{4s} X^2 + \frac{se^2}{12}(XFFX) \right] \right), \tag{A2}$$

where $X_\mu = (x-y)_\mu$, $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ are the field tensor and field dual tensor, $e > 0$ is the elementary charge, $\gamma_\mu$ and $\gamma_5$ are Dirac $\gamma$ matrices (the metric, the convensional representation of Dirac $\gamma$ matrices, etc., correspond to the book [24]), $m$ is the mass of the charge lepton, and the phase $\Phi(x,y)$ is determined in the following way:

$$\Phi(x,y)=e\int_y^x d\xi_\mu K_\mu(\xi),$$

$$K_\mu(\xi)=A_\mu(\xi)+\tfrac{1}{2}F_{\mu\nu}(\xi-y)_\nu. \tag{A3}$$

Owing to $\partial_\mu K_\nu-\partial_\nu K_\mu=0$, the path of integration from $y$ to $x$ in (A3) is arbitrary and, therefore,

$$\Phi(x,y)+\Phi(y,x)=0. \tag{A4}$$

Using (A4), the integration of $J_{\alpha\beta}$ with respect to $x$ and $y$ (see Fig. 3) can be easily redused to an integration with respect to $X=x-y$ [see Eq. (2.3)]. From Eqs. (2.3) and (A2) it is clear that the integrals with respect to $X$ are Gaussian, so that they can be readily calculated:

$$G=\int d^4X e^{-i(XRX/4+qX)}=-(4\pi)^2(\det R)^{-1/2}e^{(iqR^{-1}q)},$$

$$G_\mu=\int d^4X X_\mu e^{-i(XRX/4+qX)}=i\frac{\partial G}{\partial q^\mu},$$

$$G_{\mu\nu}=\int d^4X X_\mu X_\nu e^{-i(XRX/4+qX)}=-\frac{\partial^2 G}{\partial q^\mu \partial q^\nu}. \tag{A5}$$

In the remaining double integral with respect to the proper times $s_1$ and $s_2$, it is convenient to pass to the dimensionless variables $z$ and $t$:

$$z=m^2(s_1+s_2),\quad t=\frac{s_1-s_2}{s_1+s_2},\quad ds_1 ds_2=\frac{1}{2m^4}z\,dz\,dt,$$

$$0\leqslant z\leqslant\infty,\quad -1\leqslant t\leqslant 1. \tag{A6}$$

The substitution into the amplitude (2.2) of the expression for $J_{\alpha\beta}$ in the form of a double integral with respect to $z$ and $t$ results in the final expression (2.4) and (2.5).

In the case of a constant uniform magnetic field $\vec{B}$ the propagator of the charged lepton $\hat{S}^{(B)}(x,y)$ in the proper time formalism has the form

$$\hat{S}^{(B)}(x,y)=-\frac{i\beta}{2(4\pi)^2}e^{i\Phi(x,y)}\int_0^\infty \frac{ds}{s\sin(\beta s)}\Bigg\{\frac{1}{s}[\cos(\beta s)(X\tilde{\Lambda}\gamma)+i\sin(\beta s)(X\tilde{\varphi}\gamma)\gamma_5] - \frac{\beta}{\sin(\beta s)}(X\Lambda\gamma)+m[2\cos(\beta s)-\sin(\beta s)(\gamma\varphi\gamma)]\Bigg\}\exp\left(-i\left[m^2 s+\frac{X\tilde{\Lambda}X}{4s}-\frac{\beta}{4\tan(\beta s)}(X\Lambda X)\right]\right), \tag{A7}$$

where $\varphi_{\mu\nu}=F_{\mu\nu}/B$ and $\tilde{\varphi}_{\mu\nu}=\tilde{F}_{\mu\nu}/B$ are the dimensionless field tensor and dual field tensor of the constant magnetic field, $\Lambda_{\alpha\beta}=(\varphi\varphi)_{\alpha\beta}$, $\tilde{\Lambda}_{\alpha\beta}=(\tilde{\varphi}\tilde{\varphi})_{\alpha\beta}$, $\beta=eB$, $X_\mu=(x-y)_\mu$, and the phase $\Phi(x,y)$ is described in Eq. (A3). Note that the propagator (A7) can be represented in a fully covariant form, because the parameter $\beta$ in a purely magnetic field can be rewritten as $\beta=eB=\sqrt{-F^2/2}$. The calculation procedure for the tensor $J_{\alpha\beta}$ Eq. (2.3), in the case of a constant magnetic field, though more awkward, does not, in principle, differ from the case of a crossed field.

## APPENDIX B: ON THE CALCULATION OF THE $S$-MATRIX ELEMENT IN THE FIELD OF A MONOCHROMATIC CIRCULARLY POLARIZED WAVE

The propagator of the charged fermion in the field of a plane wave with a four-potential $A_\mu=A_\mu(\varphi)$ of the general form can be obtained by method, given in [31], and has the form

$$\hat{S}(x,y)=\int\frac{d^4p}{(2\pi)^4}\left(1-\frac{e_f\hat{k}\hat{A}'}{2(kp)}\right)\frac{\hat{p}+m_f}{p^2-m_f^2}\left(1-\frac{e_f\hat{A}\hat{k}}{2(kp)}\right)\times\exp\left\{i\left[-p(y-x)+\frac{1}{(kp)}\times\int_\varphi^{\varphi'}d\varphi(e_f(pA)+\tfrac{1}{2}e_f^2A^2)\right]\right\}, \tag{B1}$$

where $A_\mu=A_\mu(\varphi)$, $\varphi=kx$, $A'_\mu=A_\mu(\varphi')$, $\varphi'=ky$, $k$ is the four-wave-vector ($k^2=0$) and $e_f$ and $m_f$ are the charge and the mass of fermion, respectively.

In the case of the circularly polarized wave with four-potential

$$A_\mu(\varphi)=a_{1\mu}\cos(\varphi)+a_{2\mu}\sin(\varphi), \tag{B2}$$

where four-vectors $a_{1\mu}$ and $a_{2\mu}$ are orthogonal to the four-wave-vector $k_\mu$,

$$(a_1a_2)=(a_1k)=(a_2k)=0, \tag{B3}$$

the expression for propagator may be represented in the form

$$\hat{S}(x,y)=\int \frac{d^4p}{(2\pi)^4}\left(1-\frac{e_f\hat{k}\hat{A}'}{2(kp)}\right)\frac{\hat{p}+m_f}{p^2-m_f^2}\left(1-\frac{e_f\hat{A}\hat{k}}{2(kp)}\right) \times \exp\left\{ i\left[ -p(y-x)-\frac{e_f(a_1p)}{(kp)}[\sin(\varphi')-\sin(\varphi)] -\frac{e_f(a_2p)}{(kp)}[\cos(\varphi')-\cos(\varphi)]-\frac{e_f^2a^2}{2(kp)}(\varphi'-\varphi)\right]\right\}. \quad \text{(B4)}$$

Since the power of the exponent of the propagator (B4) contains nonlinear functions of coordinates $x$ and $y$ [$\sin(\varphi)$, $\sin(\varphi')$, $\cos(\varphi)$, and $\cos(\varphi')$], it is convenient to expand the corresponding part of an exponent in the Fourier expansion with the coefficients of expansion been proportional to the Bessel functions [24].

Given the integration over one of the momenta in the loop $d^4q$ it is convenient to use the basis $p_\mu$, $h_{\mu\nu}p_\mu$, $\tilde{h}_{\mu\nu}p_\nu$, $h_{\mu\nu}h_{\nu\beta}p_\beta$, where

$$h_{\mu\nu}=k_\mu a_{1\nu}-k_\nu a_{1\mu}, \quad \tilde{h}_{\mu\nu}=k_\mu a_{2\nu}-k_\nu a_{2\mu}. \quad \text{(B5)}$$

By using the known relation

$$J_0(\sqrt{b^2-2bc\cos\alpha+c^2}) =J_0(b)J_0(c)+2\sum_{s=1}^{\infty} J_s(b)J_s(c)\cos s\alpha, \quad \text{(B6)}$$

one can reduce the remaining infinite series to five terms, which may be brought to the form (4.4).


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