Non-Abelian Weizsäcker-Williams field and a two-dimensional effective color charge density for a very large nucleus

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We consider a very large ultrarelativistic nucleus. Assuming a simple model of the nucleus and weak coupling we find a classical solution for the gluon field of the nucleus and construct the two-dimensional color charge density for McLerran-Venugopalan model out of it. We prove that the density of states distribution, as a function of color charge density, is Gaussian, confirming the assumption made by McLerran and Venugopalan. [S0556-2821(96)01721-3]

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I. INTRODUCTION

Consider a very large nucleus, probably larger than can be physically realized. The nucleons are distributed homogeneously inside the nucleus. Recently McLerran and Venugopalan proposed a program of computing the gluon distribution fuctions for such a nucleus at small x [1,2].

One of the interesting problems in the McLerran-Venugopalan model for the small-x part of the gluon distribution of such a large nucleus [1,2] is finding the classical solution for the gluon field, treating the valence quarks of the nucleons in the nucleus as recoiless sources, which are δ functions along the light cone when the nucleus is moving near the velocity of light. A convenient way to deal with the problem is by working in the light-cone gauge. The source is characterized by a two-dimensional color charge density $\rho(\underline{x})$, where $\underline{x}$ is a vector in the transverse direction. The proposed model assumes that in order to find the average value of any observable having longitudinal coherence length long compared to the nucleus, one calculates this observable for a given $\rho(\underline{x})$, and then averages it over all ρ with the measure

$$\int [d\rho]\exp\left(-\frac{1}{2\mu^2}\int d^2x\rho^2(\underline{x})\right), \qquad (1)$$

where μ^2 is the average charge density squared.

We consider a large nucleus consisting of ''nucleons,'' which for simplicity of description are chosen to be just quark-antiquark pairs (see Fig. 1). Valence quark and antiquark are treated as point particles free to move inside of the nucleon, but unable to get out.

We are interested in the gluon field of the ultrarelativistic nucleus viewed in the laboratory frame. We assume that the field in each individual nucleon is not large. This allows us to approximate the covariant gauge potential of each quark by a single-gluon exchange. In a covariant gauge the classical field of a single ultrarelativistic particle is proportional to a δ function in the x_- direction [3]. Since in our model of a ultrarelativistic nucleus different quarks have different x_- coordinates in the laboratory frame, the fields of individual quarks do not overlap, which allows us to superimpose them and justifies our single-gluon exchange approximation. Then the total field of the nucleus in the covariant gauge is the sum of the quark fields. We make a gauge transformation which changes the total potential to the light-cone gauge. So we get a solution to the classical equations of motion in the light-cone gauge, where the field A_μ is directly related to the gluon distribution in the small-x region [1].

Following McLerran and Venugopalan the nucleus is considered to be very large; thus, although the field of each individual nucleon is weak, the total field is strong at low momentum in the light-cone gauge due to the overlap of the fields of a huge number of nucleons. Still we can neglect the contributions of several nucleons, without changing the answer; i.e., we work in the leading power of the number of nucleons.

We may treat the source as classical only when we are at sufficiently small momenta that the individual quarks cannot be resolved. It was shown in [1] that this requires that $\underline{k}^2 \ll \mu^2$, where $\underline{k}$ is the typical momentum scale. The weak coupling approximation is valid when $\underline{k}^2 \gg \Lambda^2_{\mathrm{QCD}}$ Then the momentum range we consider is $\Lambda^2_{\mathrm{QCD}} \ll \underline{k}^2 \ll \mu^2$.

Now the task is to construct the two-dimensional charge density, giving the correct classical solution. This is done by just substituting the classical solution in the equations of motion. The density we find this way happens to satisfy the Gaussian distribution. We show this by calculating the correlation functions of the densities at different transverse points and proving that they are exactly what one would expect for the Gaussian distribution (1). That is, we justify the method for averaging the observables proposed by McLerran and Venugopalan.

In Sec. II we calculate the solution of the classical equations of motion, the non-Abelian Weizsäcker-Williams field.

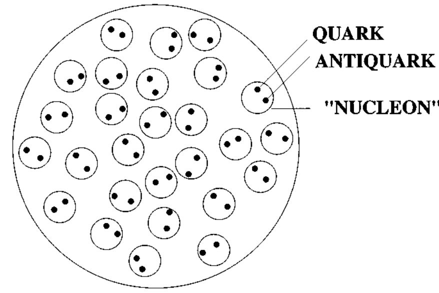

FIG. 1. Nucleus with ''nucleons'' being quark-antiquark pairs.

In Sec. III we construct the two-dimensional charge density. In Sec. IV we show that the charge density has a Gaussian distribution by calculating the correlation functions. In Sec. V we confirm the techniques proposed in [1].

II. APPROXIMATE SOLUTION

We start with some random distribution of nucleons in the nucleus and quarks and antiquarks in the nucleons. The nucleons, in the rest frame of the nucleus, are assumed to be spheres with equal radius, and the quarks and antiquarks are distributed randomly inside each sphere, with equal probability to be at any place inside the sphere, but with zero probability to get outside. The density in the rest frame is given by

$$\rho(\vec{x}) = \sum_{a=1}^{8} T^a \rho^a(\vec{x}), \tag{2}$$

with

$$\rho^a(\vec{x}) = g\sum_{i=1}^{N} (T_i^a)[\delta(\vec{x}-\vec{x}_i) - \delta(\vec{x}-\vec{x}_i')], \tag{3}$$

where $\vec{x}_i$ is the coordinate of a quark in the ith nucleon (there are N nucleons in the nucleus), $\vec{x}_i'$ is the coordinate of the antiquark, T^a are generators of SU(3) in color space, and (T_i^a) are similar generators in the color space of each nucleon. The reason we separate them is because ρ^a comes from the current $j_\mu^a = g\overline{q}_\alpha \gamma_\mu (T^a)_{\alpha\beta} q_\beta$, and so the expression for ρ^a should include a T^a acting in each individual nucleon's color space.

In the laboratory frame for the ultrarelativistic nucleus the density is

$$\rho(\underline{x},x_-) = \frac{g}{\sqrt{2}} \sum_{a=1}^{8}\sum_{i=1}^{N} T^a(T_i^a)[\delta(x_- - x_{-i})\,\delta(\underline{x}-\underline{x}_i) - \delta(x_- - x'_{-i})\,\delta(\underline{x}-\underline{x}_i')]. \tag{4}$$

Assuming that the coupling is weak and using the expression for the potential of a single particle in a covariant gauge [3, Appendix A] we approximate the field of the nucleus by superposition as

$$A'_+ = -\frac{g}{2\pi} \sum_{a=1}^{8}\sum_{i=1}^{N} T^a(T_i^a)[\delta(x_- - x_{-i})\ln(|\underline{x}-\underline{x}_i|\lambda) - \delta(x_- - x'_{-i})\ln(|\underline{x}-\underline{x}_i'|\lambda)], \quad \underline{A}'=0, \quad A'_-=0, \tag{5}$$

where λ is some infrared cutoff. The prime at the field or the field strength denotes the covariant gauge. The field strength in the covariant gauge is then

$$F'_{+\perp} = \frac{g}{2\pi} \sum_{a=1}^{8}\sum_{i=1}^{N} T^a(T_i^a)\left(\delta(x_- - x_{-i})\frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2} - \delta(x_- - x'_{-i})\frac{\underline{x}-\underline{x}'_i}{|\underline{x}-\underline{x}_i'|^2}\right). \tag{6}$$

From here on the subscript $\perp$ will mean that the object is a vector in the transverse space over this index.

We now perform a gauge transformation to transorm this field into the light cone gauge. The potential in a new gauge is

$$A_\mu = SA'_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}. \tag{7}$$

Requiring the new gauge to be the light cone gauge, $A_+ = 0$, we obtain

$$S(\underline{x},x_-) = P\exp\left(-ig\int_{-\infty}^{x_-} dx'_- A'_+(\underline{x},x'_-)\right). \tag{8}$$

Then the field in the light-cone gauge is

$$\underline{A}(\underline{x},x_-) = \int_{-\infty}^{x_-} dx'_- F_{+\perp}(\underline{x},x'_-) = \int_{-\infty}^{x_-} dx'_- S(\underline{x},x'_-)F'_{+\perp}(\underline{x},x'_-)S^{-1}(\underline{x},x'_-). \tag{9}$$

Only transverse components are non-zero. Substituting $F'_{+\perp}(\underline{x},x'_-)$ from Eq. (6) we get

$$\underline{A}(\underline{x},x_-) = \frac{g}{2\pi}\sum_{a=1}^{8}\sum_{i=1}^{N}(T_i^a) \times\left(S(\underline{x},x_{-i})T^a S^{-1}(\underline{x},x_{-i})\frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2}\theta(x_- - x_{-i}) - S(\underline{x},x'_{-i})T^a S^{-1}(\underline{x},x'_{-i})\frac{\underline{x}-\underline{x}'_i}{|\underline{x}-\underline{x}_i'|^2}\theta(x_- - x'_{-i})\right). \tag{10}$$

This is our estimate of the solution of the classical equations of motion for a given configuration of quarks inside the nucleons and nucleons inside the nucleus. Formula (10) gives us the non-Abelian Weizsäcker-Williams field generated by the valence quarks.

III. TWO-DIMENSIONAL COLOR CHARGE DENSITY

The equation of motion is

$$D_\mu F^{\mu\nu} = J^\nu. \tag{11}$$

In the McLerran-Venugopalan model [1] the classical current J^μ only has components in the + direction and is proportional to a δ function of x_-:

$$J^\mu(x) = \delta^{\mu +}\delta(x_-)\rho(\underline{x}). \tag{12}$$

This can be treated as a definition of the two-dimensional color density. Our goal now is to construct $\rho(\underline{x})$. Integrating both sides of Eq. (11) over x_- and using Eq. (12) gives

$$\rho(\underline{x})=\int_{-\infty}^{+\infty}dx_- D_i F_{+i}(\underline{x},x_-)$$
$$=\int_{-\infty}^{+\infty}dx_-\{\partial_i F_{+i}(\underline{x},x_-)-ig[A_i(\underline{x},x_-),F_{+i}(\underline{x},x_-)]\}, \tag{13}$$

where $i=1,2$ (transverse direction). Using $\underline{A}(\underline{x},x_-)$ from Eq. (10) , we can calculate $F_{+\perp}(\underline{x},x_-)$. Substituting both in Eq. (13) we end up with the following expression for the density:

$$\rho(\underline{x})=g\sum_{a=1}^{8}\sum_{i=1}^{N}(T_i^a)[S(\underline{x}_i,x_{-i})T^aS^{-1}(\underline{x}_i,x_{-i})\,\delta(\underline{x}-\underline{x}_i)$$
$$-S(\underline{x}_i',x_{-i}')T^aS^{-1}(\underline{x}_i',x_{-i}')\,\delta(\underline{x}-\underline{x}_i')]. \tag{14}$$

The details of calculations are presented in Appendix B. We can see now that our expression for two-dimensional density is just a rotation of the three-dimensional density (in the laboratory frame) we started with:

$$\rho(\underline{x})=\int_{-\infty}^{+\infty}dx_- S(\underline{x},x_-)\sqrt{2}\rho(\underline{x},x_-)S^{-1}(\underline{x},x_-). \tag{15}$$

In the expression for the light cone potential we had an infrared cutoff λ, and so it may seem that this cutoff will appear in the $S(\underline{x},x_-)$ and, consequently, in $\rho(\underline{x})$. However, this is not the case, because, although the light cone potential is cutoff dependent, $S(\underline{x},x_-)$ is not. To see this let us perform an explicit calculation of $S(\underline{x},x_-)$: Substituting Eq. (5) into Eq. (8) and using the definition of the path-ordered exponential we obtain

$$S(\underline{x},x_-)=\prod_{i=1}^{N}[\theta(x_{-i}'-x_{-i})e^{\Sigma_i}e^{\Sigma_i'}+\theta(x_{-i}-x_{-i}')e^{\Sigma_i'}e^{\Sigma_i}], \tag{16}$$

with

$$\Sigma_i=\frac{ig^2}{2\pi}\sum_{a=1}^{8}T^a(T_i^a)\ln(|\underline{x}-\underline{x}_i|\lambda)\,\theta(x_- - x_{-i}), \tag{17}$$

$$\Sigma_i'=-\frac{ig^2}{2\pi}\sum_{a=1}^{8}T^a(T_i^a)\ln(|\underline{x}-\underline{x}'_i|\lambda)\,\theta(x_- - x'_{-i}).$$

But the matrices we exponentiate (Σ_i and Σ_i') commute (for the same i), and so

$$S(\underline{x},x_-)=\prod_{i=1}^{N}\exp\left[\frac{ig^2}{2\pi}\sum_{a=1}^{8}T^a(T_i^a)\ln\left(\frac{|\underline{x}-\underline{x}_i|}{|\underline{x}-\underline{x}_i'|}\right)\times\theta(x_- - x_{-i})\right]. \tag{18}$$

Here we neglected the contribution of the ''last'' nucleon, i.e., the nucleon (or several nucleons) whose quarks or antiquarks may overlap the point x_- at which we calculate $S(\underline{x},x_-)$. These nucleons may potentially cause us some trouble, but the philosophy of the large nucleus approximation implies that the fields of individual nucleons are small, and we construct a strong field out of a large number of nucleons. The contribution of each individual nucleon is negligible; it is their sum which matters. That means we can neglect these ''last'' nucleons.

Another way to say this is that we want to perform a calculation keeping the leading powers of N only. Then dropping a few nucleons will not change our result.

From Eq. (18) we see that $S(\underline{x},x_-)$ is cutoff independent, and so is the density $\rho(\underline{x})$.

IV. CALCULATION OF CORRELATION FUNCTIONS

Now that we found the charge density, let us show that its distribution is Gaussian by calculating the density correlation functions. First we note that the average density is zero, as expected: $\langle\rho^a(\underline{x})\rangle=0$, where $\langle\cdots\rangle$ denotes the averaging over all possible positions of quarks and antiquarks in the nucleons, and nucleons in the nucleus, as well as averaging over all possible colors (keeping each nucleon color neutral).

For two densities correlation function we have

$$\langle\rho^a(\underline{x})\rho^b(\underline{y})\rangle=\prod_{k=1}^{N}\int\frac{d^3r_k}{(4/3)\pi R^3}\frac{d^3x_kd^3x_k'}{[(4/3)\pi a^3]^2}\times(\alpha\overline{\alpha}|\rho^a(\underline{x})\rho^b(\underline{y})|\beta\overline{\beta}), \tag{19}$$

where R is the radius of the nucleus, a is the radius of the nucleons, r_k is the position of the center of the kth nucleon in the nucleus (in the rest frame), and x_k and x_k' are positions of the quarks in the nucleons; $(\alpha\overline{\alpha}|\cdots|\beta\overline{\beta})$ implies an average over all color-neutral states of the nucleons.

Using Eq. (14) we obtain (since $\rho^a(\underline{x})=2\,\mathrm{Tr}[T^a\rho(\underline{x})]$)

$$\langle\rho^a(\underline{x})\rho^b(\underline{y})\rangle=g^2\prod_{k=1}^{N}\int\frac{d^3r_k}{(4/3)\pi R^3}\frac{d^3x_kd^3x_k'}{[(4/3)\pi a^3]^2}(\alpha\overline{\alpha}|\sum_{c,d=1}^{8}\sum_{i,j=1}^{N}(T_i^c)(T_j^d)$$
$$\times\{2\,\mathrm{Tr}[T^aS(\underline{x}_i,x_{-i})T^cS^{-1}(\underline{x}_i,x_{-i})]\delta(\underline{x}-\underline{x}_i)-2\,\mathrm{Tr}[T^aS(\underline{x}_i',x_{-i}')T^cS^{-1}(\underline{x}_i',x_{-i}')]\delta(\underline{x}-\underline{x}_i')\}$$
$$\times\{2\,\mathrm{Tr}[T^bS(\underline{x}_j,x_{-j})T^dS^{-1}(\underline{x}_j,x_{-j})]\delta(\underline{y}-\underline{x}_j)-2\,\mathrm{Tr}[T^bS(\underline{x}_j',x_{-j}')T^dS^{-1}(\underline{x}_j',x_{-j}')]\delta(\underline{y}-\underline{x}_j')\}|\beta\overline{\beta}). \tag{20}$$

In $S(\underline{x}_i, x_{-i})$ the ''last'' nucleon is the ith nucleon. Applying the same arguments we had before we can drop this ''last'' nucleon. Then there will be no (T_i^c) matrices in $S(\underline{x}_i, x_{-i})$ and $S(\underline{x}_i', x_{-i}')$. It is convenient to label nucleons according to their coordinates along the x_- axis. The greater the coordinate, the greater the number of the nucleon. Without any loss of generality we can assume that $i \geqslant j$. Then, if $i > j$ the (T_i^c) matrix in front will be the only matrix in the ith nucleon color space for the term corresponding to fixed i and j in our expression. So when we do the color averaging it will give zero [$\mathrm{Tr}(T_i^c)=0$]. That means that $i=j$. Then color averaging in the ith nucleon space gives

$$\frac{1}{N_c}\,\mathrm{Tr}[(T_i^c)(T_i^d)] = \frac{1}{2N_c}\,\delta^{cd}.$$

Each density is a difference of the quark and antiquark parts. In Eq. (20) one can easily see that the product of quark (and antiquark) components gives a δ function of $\underline{x}$ and $\underline{y}$, while the product of quark and antiquark components gives some smooth function. In most of the physical applications we will be looking at scales much smaller than the nucleon's radius a [4]. At these scales we can neglect this cross terms with respect to the δ function terms:

$$\langle \rho^a(\underline{x})\rho^b(\underline{y})\rangle = \frac{g^2}{N_c}\,\frac{1}{\pi a^2}\,\delta(\underline{x}-\underline{y})\prod_{l=1}^{N}\int \frac{d^3r_l}{(4/3)\pi R^3}\sum_{i=1}^{N}\frac{3}{2}\sqrt{1-\frac{(\underline{x}-\underline{r}_i)^2}{a^2}}\prod_{k=1}^{i-1}\int\frac{d^3x_k d^3x_k'}{[(4/3)\pi a^3]^2}$$
$$\times (\alpha\overline{\alpha}|\sum_{c=1}^{8} 4\,\mathrm{Tr}[T^a S(\underline{x}_i, x_{-i}) T^c S^{-1}(\underline{x}_i, x_{-i})]\,\mathrm{Tr}[T^b S(\underline{x}_i, x_{-i}) T^c S^{-1}(\underline{x}_i, x_{-i})]|\beta\overline{\beta}). \tag{21}$$

After dropping the ''last'' nucleon $S(\underline{x}, x_{-i}) = S(\underline{x}, x_{-i}')$.

For two traceless 3×3 matrices M and N the following formula is true:

$$\sum_{a=1}^{8}\,\mathrm{Tr}[MT^a]\,\mathrm{Tr}[NT^a] = \frac{1}{2}\,\mathrm{Tr}[MN]. \tag{22}$$

Using this, and making some approximation when integrating over r_i ($a \ll R$), we get

$$\langle \rho^a(\underline{x})\rho^b(\underline{y})\rangle = \delta^{ab}\,\frac{3g^2}{2N_c}\,\frac{N}{\pi R^2}\,\delta(\underline{x}-\underline{y})\,\sqrt{1-\frac{\underline{x}^2}{R^2}}. \tag{23}$$

Here, for the first time we made an assumption about the geometry of the nucleus and nucleons in it — when doing the average over the positions of quarks and nucleons we assumed that the nucleus and nucleons are spherical in the rest frame. For a cylindrical nucleus (in the z direction) one would get

$$\langle \rho^a(\underline{x})\rho^b(\underline{y})\rangle = \delta^{ab}\,\frac{g^2}{N_c}\,\frac{N}{\pi R^2}\,\delta(\underline{x}-\underline{y}).$$

By employing a similar technique of dropping the ''last'' nucleon in the $S(\underline{x}_i, x_{-i})$ and keeping the leading powers of N only, we can show that the four-density correlation function is

$$\langle \rho^a(\underline{x})\rho^b(\underline{y})\rho^c(\underline{z})\rho^d(\underline{w})\rangle$$
$$= \langle \rho^a(\underline{x})\rho^b(\underline{y})\rangle\langle \rho^c(\underline{z})\rho^d(\underline{w})\rangle + \langle \rho^a(\underline{x})\rho^c(\underline{z})\rangle$$
$$\times\langle \rho^b(\underline{y})\rho^d(\underline{w})\rangle + \langle \rho^a(\underline{x})\rho^d(\underline{w})\rangle\langle \rho^b(\underline{y})\rho^c(\underline{z})\rangle \tag{24}$$

and prove a similar formula for a correlation function with any even number of densities, i.e., that it can be represented as

$$\langle \rho^{a_1}(\underline{x}^1)\cdots\rho^{a_{2n}}(\underline{x}^{2n})\rangle$$
$$= \langle \rho^{a_1}(\underline{x}^1)\rho^{a_2}(\underline{x}^2)\rangle\cdots\langle \rho^{a_{2n-1}}(\underline{x}^{2n-1})\rho^{a_{2n}}(\underline{x}^{2n})\rangle$$
$$+\ \text{permutations}, \tag{25}$$

where $\underline{x}^1, \ldots, \underline{x}^{2n}$ are just $2n$ arbitrary points in the nucleus. Also we can show that a correlation function with an odd number of densities is zero to all orders in N (for details see Appendix C):

$$\langle \rho^{a_1}(\underline{x}^1)\cdots\rho^{a_{2n+1}}(\underline{x}^{2n+1})\rangle = 0. \tag{26}$$

V. CONCLUSIONS

By calculating the two-density correlation function (23) and proving Eq. (25) and (26) we showed that, for any n points $\underline{x}^1\cdots\underline{x}^n$,

$$\langle \rho^{a_1}(\underline{x}^1)\cdots\rho^{a_n}(\underline{x}^n)\rangle$$
$$= \frac{\int[d\rho]\rho^{a_1}(\underline{x}^1)\cdots\rho^{a_n}(\underline{x}^n)\exp[-\int d^2x\rho^2(\underline{x})/2\mu^2(\underline{x})]}{\int[d\rho]\exp[-\int d^2x\rho^2(\underline{x})/2\mu^2(\underline{x})]}, \tag{27}$$

with

$$\mu^2(\underline{x}) = \frac{3g^2}{2}\,\frac{C_F}{N_c}\,\frac{N}{\pi R^2}\sqrt{1-\frac{\underline{x}^2}{R^2}}$$
$$= \frac{3g^2}{2}\,\frac{C_F}{N_c}\,\frac{1}{\pi a^2}N^{1/3}\sqrt{1-\frac{\underline{x}^2}{R^2}} \tag{28}$$

in our model. Note that our μ^2 goes as $N^{1/3}$, as expected. The average on the left hand side is understood as

$$\langle\cdots\rangle=\prod_{k=1}^{N}\int\frac{d^3r_k}{(4/3)\pi R^3}\frac{d^3x_kd^3x'_k}{[(4/3)\pi a^3]^2}(\alpha\overline{\alpha|}\cdots|\beta\overline{\beta}).$$

For a spherical nucleus μ^2 is a function of $\underline{x}$.

The most general observable in our system can be represented as some functional of $\int K(x',\underline{x})\rho(\underline{x})d^2x$; namely, the value of this observable for a given $\rho(\underline{x})$ is

$$\mathcal{O}_\rho=F\left(\int K(x',\underline{x})\rho(\underline{x})d^2x\right), \tag{29}$$

where $F(f)$ is some functional and $K(x',\underline{x})$ is some kernel independent of $\rho(\underline{x})$. For instance, one can consider the Wilson loop $W=\langle\text{ Tr Pexp}(-ig\int dx\cdot\mathbf{A})\rangle$. It is a functional of $\mathbf{A}(x)$. The field $\mathbf{A}(x)$ can be represented as $\int K(x',\underline{x})\rho(\underline{x})d^2x$. Then, using the definition of a path-ordered integral, we can expand the Wilson loop in powers of $\mathbf{A}(x)$:

$$W=\left\langle\text{ Tr}\prod_k[1-igdx_k\cdot\mathbf{A}(x_k)]\right\rangle$$
$$=\left\langle\text{ Tr}\sum_n c_n\mathbf{A}(x_1)\cdots\mathbf{A}(x_n)\right\rangle, \tag{30}$$

with some coefficients c_n. Writing $\mathbf{A}(x)$ as $\mathbf{A}(x)=\Sigma_{a=1}^{8}T^a\int K^a(x,\underline{x}')\rho(\underline{x}')d^2x'$, we achieve

$$W=\sum_n c_n\sum_{a_1\cdots a_n=1}^{8}\text{Tr}[T^{a_1}\cdots T^{a_n}]$$
$$\times\int d^2x'_1\cdots d^2x'_nK^{a_1}(x_1,\underline{x}'_1)\cdots K^{a_n}(x_n,\underline{x}'_n)$$
$$\times\langle\rho(\underline{x}'_1)\cdots\rho(\underline{x}'_n)\rangle. \tag{31}$$

Now we can use Eq. (27), obtaining

$$W=\frac{\int[d\rho]\exp\left[-\int d^2x\frac{\rho^2(\underline{x})}{2\mu^2(\underline{x})}\right]\text{Tr Pexp}\left(-ig\int dx\cdot\mathbf{A}\right)}{\int[d\rho]\exp\left[-\int d^2x\frac{\rho^2(\underline{x})}{2\mu^2(\underline{x})}\right]}. \tag{32}$$

A similar treatment can be applied to any observable [which can be represented as Eq. (29)] to prove that

$$\langle\mathcal{O}_\rho\rangle=\frac{\int[d\rho]\exp[-\int d^2x\rho^2(\underline{x})/2\mu^2(\underline{x})]\mathcal{O}_\rho}{\int[d\rho]\exp[-\int d^2x\rho^2(\underline{x})/2\mu^2(\underline{x})]}. \tag{33}$$

This confirms the assumption made by McLerran and Venugopalan in their model[1] [1].

[1] I have recently learned [L. McLerran to A. Mueller (private communication)] that similar conclusions have been reached by Professor L. McLerran and collaborators.

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APPENDIX A

From [3] the light cone potential of a charge e moving with a velocity v is

$$A_+=\frac{\sqrt{2}e}{4\pi}\frac{1}{\sqrt{2x_-^2+(1-v^2)\underline{x}^2}},\quad \underline{A}=A_-=0. \tag{A1}$$

If we perform a Fourier transform

$$A_\mu(k)=\frac{1}{(2\pi)^2}\int dx_+dx_-d^2\underline{x}e^{ik_+x_-+ik_-x_+-i\underline{k}\cdot\underline{x}}A_\mu(x) \tag{A2}$$

and take the limit $v\to1$, we get, for the ultrarelativistic particle,

$$A_+(k)=\frac{e\delta(k_-)}{2\pi\underline{k}^2}. \tag{A3}$$

If we go back to the coordinate space, by performing an inverse Fourier transform of Eq. (A3), we end up with

$$A_+(x)=-\frac{e}{2\pi}\delta(x_-)\ln(|\underline{x}|\lambda), \tag{A4}$$

which is different form the $v\to1$ limit of the original expression by a gauge transformation.

APPENDIX B

Defining

$$f_i(\underline{x},x_-)=\frac{g}{2\pi}\sum_{a=1}^{8}(T_i^a)S(\underline{x},x_-)T^aS^{-1}(\underline{x},x_-), \tag{B1}$$

we can rewrite Eq. (10) as

$$\underline{A}(\underline{x},x_-)=\sum_{i=1}^{N}\left(f_i(\underline{x},x_{-i})\frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2}\theta(x_--x_{-i})\right.$$
$$\left.-f_i(\underline{x},x'_{-i})\frac{\underline{x}-\underline{x}'_i}{|\underline{x}-\underline{x}'_i|^2}\theta(x_--x'_{-i})\right). \tag{B2}$$

Then the field strength is

$$F_{+\perp}(\underline{x},x_-)=\frac{\partial}{\partial x_-}\underline{A}(\underline{x},x_-)$$
$$=\sum_{i=1}^{N}\left(f_i(\underline{x},x_{-i})\frac{\underline{x}-\underline{x}_i}{|\underline{x}-\underline{x}_i|^2}\delta(x_--x_{-i})\right.$$
$$\left.-f_i(\underline{x},x'_{-i})\frac{\underline{x}-\underline{x}'_i}{|\underline{x}-\underline{x}'_i|^2}\delta(x_--x'_{-i})\right). \tag{B3}$$

Substituting Eqs. (B2) and (B3) into Eq. (13) we get

$$\rho(\underline{x}) = \sum_{i=1}^{N} [f_i(\underline{x}, x_{-i}) 2\pi\delta(\underline{x} - \underline{x}_i) - f_i(\underline{x}, x'_{-i}) 2\pi\delta(\underline{x} - \underline{x}'_i)]$$
$$+ \sum_{i=1}^{N} \left(\nabla f_i(\underline{x}, x_{-i}) \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^2} - \nabla f_i(\underline{x}, x'_{-i}) \frac{\underline{x} - \underline{x}'_i}{|\underline{x} - \underline{x}'_i|^2} \right)$$
$$- ig \sum_{i,j=1}^{N} \left([f_i(\underline{x}, x_{-i}), f_j(\underline{x}, x_{-j})] \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^2} \frac{\underline{x} - \underline{x}_j}{|\underline{x} - \underline{x}_j|^2} \theta(x_{-j} - x_{-i}) \right.$$
$$- [f_i(\underline{x}, x_{-i}), f_j(\underline{x}, x'_{-j})] \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^2} \frac{\underline{x} - \underline{x}'_j}{|\underline{x} - \underline{x}'_j|^2} \theta(x'_{-j} - x_{-i})$$
$$- [f_i(\underline{x}, x'_{-i}), f_j(\underline{x}, x_{-j})] \frac{\underline{x} - \underline{x}'_i}{|\underline{x} - \underline{x}'_i|^2} \frac{\underline{x} - \underline{x}_j}{|\underline{x} - \underline{x}_j|^2} \theta(x_{-j} - x'_{-i})$$
$$\left. + [f_i(\underline{x}, x'_{-i}), f_j(\underline{x}, x'_{-j})] \frac{\underline{x} - \underline{x}'_i}{|\underline{x} - \underline{x}'_i|^2} \frac{\underline{x} - \underline{x}'_j}{|\underline{x} - \underline{x}'_j|^2} \theta(x'_{-j} - x'_{-i}) \right). \tag{B4}$$

A straightforward calculation yields

$$\nabla f_i(\underline{x}, x_{-i}) = \frac{ig^3}{(2\pi)^2} \sum_{a=1}^{8} \sum_{b=1}^{8} \sum_{j=1}^{N} (T_i^a)(T_j^b) \left\{ \frac{\underline{x} - \underline{x}_j}{|\underline{x} - \underline{x}_j|^2} \theta(x_{-i} - x_{-j}) \right.$$
$$\times [S(\underline{x}, x_{-j}) T^b W(\underline{x}, x_{-i}, x_{-j}) T^a S^{-1}(\underline{x}, x_{-i}) - S(\underline{x}, x_{-i}) T^a W(\underline{x}, x_{-j}, x_{-i})$$
$$\times T^b S^{-1}(\underline{x}, x_{-j})] - \frac{\underline{x} - \underline{x}'_j}{|\underline{x} - \underline{x}'_j|^2} \theta(x_{-i} - x'_{-j}) [S(\underline{x}, x'_{-j}) T^b W(\underline{x}, x_{-i}, x'_{-j}) T^a$$
$$\left. \times S^{-1}(\underline{x}, x_{-i}) - S(\underline{x}, x_{-i}) T^a W(\underline{x}, x'_{-j}, x_{-i}) T^b S^{-1}(\underline{x}, x'_{-j})] \right\} \tag{B5}$$

and

$$[f_i(\underline{x}, x_{-i}), f_j(\underline{x}, x_{-j})] = \frac{g^2}{(2\pi)^2} \sum_{a=1}^{8} \sum_{b=1}^{8} (T_i^a)(T_j^b) [S(\underline{x}, x_{-i}) T^a W(\underline{x}, x_{-j}, x_{-i})$$
$$\times T^b S^{-1}(\underline{x}, x_{-j}) - S(\underline{x}, x_{-j}) T^b W(\underline{x}, x_{-i}, x_{-j}) T^a S^{-1}(\underline{x}, x_{-i})], \tag{B6}$$

where

$$W(\underline{x}, x_{-i}, x_{-j}) = P \exp\left(-ig \int_{x_{-j}}^{x_{-i}} dx_- A'_+(\underline{x}, x_-) \right). \tag{B7}$$

Plugging this back into Eq. (B4) we end up with

$$\rho(\underline{x}) = \sum_{i=1}^{N} [f_i(\underline{x}, x_{-i}) 2\pi\delta(\underline{x} - \underline{x}_i) - f_i(\underline{x}, x'_{-i}) 2\pi\delta(\underline{x} - \underline{x}'_i)], \tag{B8}$$

which is equivalent to Eq. (14).

APPENDIX C

Let us prove that the three-density correlation function is zero. Then the proof for an arbitrary correlation function of an odd number of densities will become obvious. Similar to Eq. (20) we write

$$\langle\rho^a(\underline{x})\rho^b(\underline{y})\rho^c(\underline{z})\rangle = g^3 \prod_{l=1}^{N} \int \frac{d^3 r_l}{(4/3)\pi R^3} \frac{d^3 x_l d^3 x_l'}{[(4/3)\pi a^3]^2} (\alpha\overline{\alpha}| \sum_{a',b',c'=1}^{8} \sum_{i,j,k=1}^{N} (T_i^{a'})(T_j^{b'})(T_k^{c'})$$
$$\times\{2\,\mathrm{Tr}[T^a S(\underline{x}_i, x_{-i}) T^{a'} S^{-1}(\underline{x}_i, x_{-i})]\delta(\underline{x}-\underline{x}_i) - 2\,\mathrm{Tr}[T^a S(\underline{x}_i', x_{-i}') T^{a'} S^{-1}(\underline{x}_i', x_{-i}')]\delta(\underline{x}-\underline{x}_i')\}$$
$$\times\{2\,\mathrm{Tr}[T^b S(\underline{x}_j, x_{-j}) T^{b'} S^{-1}(\underline{x}_j, x_{-j})]\delta(\underline{y}-\underline{x}_j) - 2\,\mathrm{Tr}[T^b S(\underline{x}_j', x_{-j}') T^{b'} S^{-1}(\underline{x}_j', x_{-j}')]\delta(\underline{y}-\underline{x}_j')\}$$
$$\times\{2\,\mathrm{Tr}[T^c S(\underline{x}_k, x_{-k}) T^{c'} S^{-1}(\underline{x}_k, x_{-k})]\delta(\underline{z}-\underline{x}_k) - 2\,\mathrm{Tr}[T^c S(\underline{x}_k', x_{-k}') T^{c'} S^{-1}(\underline{x}_k', x_{-k}')]\delta(\underline{z}-\underline{x}_k')\}|\beta\overline{\beta}). \tag{C1}$$

After dropping the ''last'' nucleon and averaging over the colors of this nucleon, we get

$$\langle\rho^a(\underline{x})\rho^b(\underline{y})\rho^c(\underline{z})\rangle = g^3 8 \prod_{l=1}^{N} \int \frac{d^3 r_l}{(4/3)\pi R^3} \frac{d^3 x_l d^3 x_l'}{[(4/3)\pi a^3]^2} (\alpha\overline{\alpha}| \sum_{a',b',c'=1}^{8} \sum_{i=1}^{N} \mathrm{Tr}[(T_i^{a'})(T_i^{b'})(T_i^{c'})]$$
$$\times\,\mathrm{Tr}[T^a S_x T^{a'} S_x^{-1}]\,\mathrm{Tr}[T^b S_y T^{b'} S_y^{-1}]\,\mathrm{Tr}[T^c S_z T^{c'} S_z^{-1}]|\beta\overline{\beta})$$
$$\times[\delta(\underline{x}-\underline{x}_i) - \delta(\underline{x}-\underline{x}_i')][\delta(\underline{y}-\underline{x}_i) - \delta(\underline{y}-\underline{x}_i')][\delta(\underline{z}-\underline{x}_i) - \delta(\underline{z}-\underline{x}_i')], \tag{C2}$$

where $S_x = S(\underline{x}, x_{-i}) = S(\underline{x}, x'_{-i})$ (after dropping the ''last'' nucleon) and independent of $\underline{x}_i$. The product of three brackets with δ functions integrated over $\underline{x}_i$ and $\underline{x}_i'$ obviously gives zero. So $\langle\rho^a(\underline{x})\rho^b(\underline{y})\rho^c(\underline{z})\rangle = 0$, as advertised.

Similar techniques can be applied to an arbitrary odd number of densities to show that their correlation function is zero.

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