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Dimensional renormalization: Ladders and rainbows

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Renormalization factors are most easily extracted by going to the massless limit of the quantum field theory and retaining only a single momentum scale. We derive the factors and renormalized Green's functions to *all* orders in perturbation theory for rainbow graphs and vertex (or scattering) diagrams at zero momentum transfer, in the context of dimensional regularization, and we prove that the correct anomalous dimensions for those processes emerge in the limit $D\to 4$. [S0556-2821(96)00320-7]

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I. INTRODUCTION

The connection between knot theory and renormalization theory [1] is one of the more exciting developments of field theory in recent years because it relates apparently different Feynman diagrams through the common topology of the associated knots. Thus it serves to explain why transcendental numbers for the renormalization constants Z occur in some diagrams [2] and not in others, thereby allowing the Feynman graphs to be grouped into equivalence classes. Sometimes, in gauge theories the Z factors within a particular class may cancel because of the existence of Ward identities, leaving a non-transcendental result for Z; this happens in electrodynamics of scalar and spinor particles to fourth order in the quenched limit and in chromodynamics to third order [3].

The class of graphs which correspond to ladders and rainbows are especially simple in this connection because they possess trivial knot topologies. Thus one may anticipate that Z factors for them are particularly easy to evaluate. Kreimer [1] has provided rules for extracting them within the framework of dimensional regularization, through the standard expedient of finding the simple $1/\epsilon=2/(4-D)$ pole term arising in products of functions, after removing lower-order pole terms connected with subdivergences. Thus vertex diagrams bring in function factors of the type

$$ {}_j\Delta(\epsilon)\equiv(p^2)^{\epsilon(j+1)}\int\frac{d^{4-2\epsilon}k}{(k^2)^{1+j\epsilon}(k+p)^2}, $$

while rainbow graphs lead to products of

$$ {}_j\Omega(\epsilon)\equiv(p^2)^{3+j\epsilon-D/2}\int[d^Dk/(k^2)^{2+j\epsilon}(k+p)^2]. $$

It has to be said that, although the procedure is straightforward, extracting the $1/\epsilon$ term in nth order requires considerable graft. Kreimer has proven that the simple pole in ϵ is free of Riemann zeta functions.

In this paper we shall show that the problem can be solved to *all* orders in perturbation theory for ladders and rainbows [4], in the context of renormalization in dimensional regularization because of two fortuitous circumstances: (i) the Green's function satisfies a differential equation and (ii) this equation is actually soluble in terms of Bessel functions. The limit as $D\to 4$ may then be taken at the end and, as a useful check, the anomalous dimension properly emerges. (It is a rather delicate limit, requiring a saddle-point analysis of the integral representation of the Bessel function, since it looks quite singular.) We have successfully carried out this program for meson-fermion theories, both for vertex functions and rainbow diagrams; however we have not succeeded in solving the problem near $D=6$ for ϕ^3 theory because the differential equation is of fourth order and cannot be expressed in terms of standard functions; nevertheless we can obtain the answer in the limit $x\to 0$ or $p\to\infty$ for $D=6$.

In the next section we treat the vertex diagrams for scalar mesons, while the following section contains the analysis of the rainbow diagrams. The Appendix contains details of the vector meson case, which are rather more complicated.

II. VERTEX DIAGRAMS

We shall consider a theory of massless fermions ψ and mesons ϕ in D dimensions since the purpose of our work is to investigate the behavior of the Green's function as D

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tends to 4. Let $\gamma_{[r]}$ signify the product of $r\gamma$ matrices (of size $2^{D/2}\times 2^{D/2}$) normalized to unity, namely $\gamma_{[\mu_1\mu_2\cdots\mu_r]}$ so that we can write the meson-fermion interaction in the form

$$\mathcal{L}_{\rm int}=g\,\overline{\psi}\gamma_{[r]}\psi\phi^{[r]},$$

where $\phi^{[r]}$ is the corresponding tensor meson field. The equation for the renormalized tensor vertex function $\Gamma_{[s]}$ at zero meson momentum, taking out the factor g, is

$$\Gamma_{[s]}(p)=Z\gamma_{[r]}\delta_s^r-ig^2\int \overline{d^D}q\,\gamma_{[r]}\frac{1}{\gamma\cdot q}\times\Gamma_{[s]}(q)\frac{1}{\gamma\cdot q}\gamma_{[r']}\Delta^{rr'}(p-q), \tag{1}$$

where $\overline{d^D}q\equiv d^Dq/(2\pi)^D$. We shall assume that the massless meson propagator above, $\Delta^{rr'}(p-q)$, can be chosen in a Fermi-Feynman gauge so

$$\Delta^{rr'}=(-1)^r\eta^{rr'}/(p-q)^2,$$

where η stands for the diagonal Minkowskian metric pertaining to the tensor structure, specifically $\eta^{[\mu_1\nu_1}\cdots\eta^{\mu_r]\nu_{r'}}$.

To make further progress we utilize the nonamputated Green's function,

$$G_{[r]}(p)=(1/\gamma\cdot p)\,\Gamma_{[r]}(p)\,(1/\gamma\cdot p),$$

to remain with the ''simpler'' linear integral equation

$$\gamma\cdot pG_{[s]}(p)\gamma\cdot p=Z\delta_s^r\gamma_{[r]}+i(-)^rc_r^sg^2\int\overline{d^D}q\frac{G_{[s]}(q)}{(p-q)^2}, \tag{2}$$

The nature of the couplings in massless theories means that the Green's function always stays proportional to $\gamma_{[s]}$ and can be decomposed into just two pieces [5,6]:

$$G_{[s]}(p)=\gamma_{[s]}A(p^2)+\gamma\cdot p\,\gamma_{[s]}\gamma\cdot pB(p^2). \tag{3}$$

On the right-hand side of Eq. (2), $c_r^s\gamma_{[s]}=\gamma_{[r]}\gamma_{[s]}\gamma^{[r]}$, is essentially an element of the Fierz transformation matrix for any D, given by [5]

$$c_r^s=(-1)^{rs}\sum_q(-1)^q\binom{D-r}{s-q}\binom{r}{q}.$$

We shall convert the integral equation (2) into a differential equation by taking the Fourier transform. (In fact we could almost have done this from the start by writing the equation for the full Green's function in coordinate space.) This maneuver produces

$$\gamma\cdot\partial G_{[s]}(x)\gamma\cdot\tilde{\partial}=Z\delta_s^r\gamma_{[r]}\delta^D(x)+ic_r^s(-)^rg^2\Delta_c(x)G_{[s]}(x), \tag{4}$$

where the massless meson propagator is $i\Delta_c(x)=\Gamma(D/2-1)(-x^2+i\epsilon)^{1-D/2}/4\pi^{D/2}$. Because the coupling constant is a dimensionful quantity, we can define a dimensionless strength $a=$ fine structure constant $/4\pi$ via

$$(-)^rc_r^sg^2\Gamma(D/2-1)\equiv 16\pi^{D/2}\mu^{4-D}a$$

upon introducing a mass scale μ. This simplifies the resulting expressions, as we can see in the purely scalar case, where there is but a single term and equation:

$$[\partial^2-(4a/x^2)(-\mu^2x^2)^{2-D/2}]G(x)=-Z\delta^D(x). \tag{5}$$

The equation is readily solved in dimension $D=4$ yielding $G\propto(-x^2+i\epsilon)^{(-1-\sqrt{1+4a})/2}$. For $D\neq 4$ we can make progress by passing to a Euclidean metric ($r^2\equiv -x^2$):

$$\left[\frac{d^2}{dr^2}+\frac{D-1}{r}\frac{d}{dr}-\frac{4a}{r^2}(\mu r)^{4-D}\right]G(r)=Z\delta^D(r).$$

The whole point of the manipulation is that one is fortunately able to solve this equation (for $r\neq 0$ at first) in terms of known functions, namely Bessel functions. The correct choice of solution, up to an overall factor, is

$$G(r)\propto r^{\epsilon-1}J_{1-1/\epsilon}[\sqrt{-4a}(\mu r)^{\epsilon}/\epsilon];\quad D\equiv 4-2\epsilon$$

because in the limit as $a\to 0$ we recover the free-field solution r^{2D}. For dimensional reasons, let us carry out our renormalization so that $G(1/\mu)=\mu^{2-2\epsilon}$. With that convention, the vertex function reduces to

$$G(r)=(r/\mu)^{\epsilon-1}J_{1-1/\epsilon}[\sqrt{-4a}(\mu r)^{\epsilon}/\epsilon]/J_{1-1/\epsilon}(\sqrt{-4a}/\epsilon). \tag{6}$$

Furthermore the correct singularity for the time-ordered function $\delta(x)$ emerges if we reinterpret $r^2=-x^2+i\epsilon$ above. We shall not worry at this stage whether a is positive or negative, the sign can vary with the model anyway, since we can easily continue the function from J to I as needed.

The problem presents itself: how does the four-dimensional result, with its anomalous scale $\gamma=\sqrt{1+4a}-1$, emerge from Eq. (6) as $\epsilon\to 0^-$, say? This is clearly a delicate limit because both the index and the argument of the Bessel function become infinitely large. Before answering this question, let us note that, in a perturbative expansion of Eq. (4), viz., a small argument expansion of J,

$$G(r)=r^{2\epsilon-2}\frac{[1+a(\mu r)^{2\epsilon}/\epsilon(2\epsilon-1)+a^2(\mu r)^{4\epsilon}/2\epsilon^2(2\epsilon-1)(3\epsilon-1)+\cdots]}{[1+a/\epsilon(2\epsilon-1)+a^2/2\epsilon^2(2\epsilon-1)(3\epsilon-1)+\cdots]},$$

the poles in ϵ cancel out to any particular order in a. For instance up to order a^2 we obtain, as $\epsilon\to 0$,

$$G(r)=r^{-2}[1+2(-a+a^2)\ln\mu r+2a^2(\ln\mu r)^2+\cdots]$$

which agrees precisely with the expansion of the anomalous dimension in the logarithmic terms. Returning to the limit of small ϵ, we will make use of the saddle-point method of obtaining asymptotic expansions of integrals. Suppose that $f(t)$ has a minimum at $t=\tau$ in the integral representation:

$$F=(-i/2\pi)\int_C \exp[f(t)]dt.$$

Then the saddle point method gives

$$F=\exp f(\tau)\frac{1}{\sqrt{2\pi f''(\tau)}}\left[1+\frac{f''''(\tau)}{8[f''(\tau)]^2}-\frac{5[f'''(\tau)]^2}{24[f''(\tau)]^3}+\frac{35[f''''(\tau)]^2}{384[f''(\tau)]^4}+\cdots\right].$$

As confirmation of the correctness of the terms above we can verify that the Debye expansion [9] of the Bessel function is properly reproduced:

$$J_\nu(\nu/\cosh\tau)=\frac{-i}{2\pi}\int_{-i\pi+\infty}^{i\pi+\infty}dt e^{\nu(\sinh t/\cosh\tau-t)}=\frac{e^{\nu(\tanh\tau-\tau)}}{\sqrt{2\pi\nu\tanh\tau}}\times\left[1+\frac{\coth\tau}{8\nu}\left(1-\frac{5}{3}(\coth\tau)^2\right)+\cdots\right],$$

because the integrand minimum occurs at $t=\tau$. Our case (6) is a variant of this. Working only to first order in ϵ, and taking a negative initially, the integrand exponent is

$$(\sqrt{-4a}\sinh t+t)/\epsilon+\sqrt{-4a}\ln\mu r\sinh t-t$$

and is stationary at the complex value $t=\tau$, where

$$\cosh\tau=[(\epsilon-1)/\sqrt{-4a}(1+\epsilon\ln\mu r)].$$

Following through the mathematical steps, and omitting straightforward details, to order ϵ we end up with

$$G(r)=r^{2\epsilon-2}(\mu r)^{1-\epsilon-\sqrt{1+4a}}\left[1-\frac{2a\epsilon}{1+4a}\ln(\mu r)+\cdots\right]. \quad (7)$$

It is satisfying that this produces the all-orders (in coupling, a) result at four dimensions when $\epsilon\to 0$, with the correct anomalous scale.

The problem can be treated in much the same way for a pseudoscalar meson field. The only possible difference is a change in sign of a, because of ''γ_5'' matrix anticommutation. As for the vector case ($r=s=1$), the procedure produces a pair of coupled equations for the two scalar components A and B of the Green's function, $G_\mu(p)=\gamma_\mu A(p^2)+\gamma\cdot p\gamma_\mu\gamma\cdot pB(p^2)$. A discussion of this case is given in the Appendix, where it is shown that the only easy limit is $D=4$; one finds after Euclidean rotation that

$$A=ar^\beta,\quad B=br^{\beta+2},\quad a/b=c(\beta+2)/(\beta-2),$$

where

$$\beta=-1+\sqrt{5+\sqrt{16+4c+c^2}}\quad\text{and}\quad c=g^2/2\pi^2,$$

in four dimensions; we have chosen the root which reduces to the free field solution when $g=0$ although one can contemplate strictly nonperturbative solutions [10].

The difficulty is symptomatic of what happens in ϕ^3 theory near six dimensions; in that case the Green's function, $G(p)=\Gamma(p)/p^4$ obeys the Fourier transformed equation [7],

$$(\partial^4-4a[(\mu r)^{6-D}/r^4])G(x)=Z\delta^D(r). \quad (8)$$

This is a differential equation of fourth order in r and its solution cannot readily be expressed in terms of familiar transcendental functions. However in the limit as $D\to 6$, it is quite simple to find the (power law) solution:

$$G(r)\propto r^\beta,\quad \beta=-1-\sqrt{5-2\sqrt{4+a}},$$

which correctly reduces to the free field solution $\beta=-2$ when the coupling vanishes.

III. RAINBOW DIAGRAMS

We wish to treat the corrections to the fermion propagator in a similar manner, by considering the rainbow corrections. In such an approximation the rainbow graphs give rise to a self-energy, which is self-consistently determined according to

$$\Sigma_R(p)=-ig^2\int\frac{\overline{d^Dq}}{(p-q)^2}\left[\frac{1}{\gamma\cdot q}-\frac{1}{\gamma\cdot q}\Sigma_R(q)\frac{1}{\gamma\cdot q}\right],$$

with the unrenormalized propagator determined by

$$S_R(p)=1/\gamma\cdot p-(1/\gamma\cdot p)\Sigma_R(p)(1/\gamma\cdot p)$$

at this level [8]. This leads to the renormalized rainbow corrected propagator equation:

$$\gamma\cdot pS_R(p)\gamma\cdot p=Z_\psi\gamma\cdot p+ig^2\int\frac{\overline{d^Dq}}{(p-q)^2}S(q). \quad (9)$$

The nature of the massless problem is that one can always write $S_R(p)=\gamma\cdot p\sigma(p)$, and by Fourier transformation, convert Eq. (9) from an integral equation to a differential equation:

$$-i\gamma\cdot\partial\partial^2\sigma(x)=i\gamma\cdot\partial Z_\psi\delta^D(x)+ig^2\Delta_c(x)i\gamma\cdot\partial\sigma(x)$$

or

$$[\partial^2+ig^2\Delta_c(x)]\partial_\mu\sigma(x)=-Z_\psi\partial_\mu\delta^D(x). \quad (10)$$

Now for any function $f(\sqrt{x^2})$, using the two lemmas,

$$\partial_\mu f=x_\mu f'/\sqrt{x^2}, \quad (11a)$$

$$\partial_\mu\partial_\nu f=\left(\eta_{\mu\nu}-\frac{x_\mu x_\nu}{x^2}\right)\frac{f'}{\sqrt{x^2}}+\frac{x_\mu x_\nu}{x^2}f'', \quad (11b)$$

we can carry out an Euclidean rotation in order to arrive at the differential equation for the scalar function $\mathcal{S}\equiv d\sigma/dr$:

$$\left[\frac{d}{dr}\left(\frac{D-1}{r}+\frac{d}{dr}\right)+\frac{4a(\mu r)^{4-D}}{r^2}\right]\mathcal{S}(r)=-Z_\psi\delta^D(r). \quad (12)$$

In 4D this has the simple solution $\mathcal{S}(r)\propto r^{-1-2\sqrt{1+a}}$, in turn implying $S(p)\propto\gamma\cdot pp^{-4+2\sqrt{1+a}}$. However it is in fact possible to solve Eq. (12) for any D. The proper solution, normalized to $\mathcal{S}(1/\mu)=\mu^{3-2\epsilon}$ is

$$\mathcal{S}(r)=r(r/\mu)^{\epsilon-2}\frac{J_{1-2/\epsilon}[\sqrt{-4a}(\mu r)^{\epsilon}/\epsilon]}{J_{1-2/\epsilon}(\sqrt{-4a}/\epsilon)}. \qquad (13)$$

To get the rainbow propagator, we must first integrate, $\sigma(r)=\int^r \mathcal{S}(r)dr \propto \Sigma_{m=0}^{\infty} J_{2-2\epsilon+2m}(\sqrt{-4a}(\mu r)^{\epsilon}/\epsilon)$, and then Fourier transform to obtain $S_R(p)=\gamma\cdot p\,\sigma(p)$.

If the mesons are neither scalar nor pseudoscalar, but tensor, the coupling constant is multiplied by the factor c_1^r; that is all.

IV. CONCLUSIONS

We have demonstrated that it is possible to work out the all-orders solution of Green's functions for ladder and rainbow diagrams for any dimension D and that, in the limit as D approaches the physical dimension, the correct scaling dimension is obtained. We have exhibited fully how this happens for scalar theories, but have succeeded only to a limited extent in vector theories, because the equations are coupled and end up as fourth-order ones, with no transparent expression in terms of standard functions of mathematical physics. In any event, it is clear from the form of the Green's function that there are no transcendental constants in sight, even when we expand the answers perturbatively in terms of $\ln(\mu r)$, so that the renormalization constants are free of them. This confirms the finding of Kreimer for arbitrary ladder or rainbow order [2] and does not come as a surprise.

One can extend the ideas here to scattering processes which contain a single momentum scale, such as fermion-fermion scattering (again ladder graphs) for any D. It is a simple matter of taking the Fourier transform in particular channels and converting the momentum integral equations to differential ones in coordinate space. We shall not labor the issue in this paper since the steps are fairly obvious and can easily be filled in by the reader. What we have not solved for any D is the case of crossed ladders, when the kernel will presumably lead to transcendental Z constants; that is a task for the future.

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APPENDIX: THE VECTOR CASE

The vector vertex function

$$G_\mu(p)=A(p^2)\gamma_\mu+\gamma\cdot p\,\gamma_\mu\gamma\cdot pB(p^2),$$

upon Fourier transformation and tracing with γ_ν, produces the coordinate space equation,

$$(\partial^2\eta_{\mu\nu}-2\partial_\mu\partial_\nu)A+\partial^4\eta_{\mu\nu}B = Z\eta_{\mu\nu}\delta(x)+ig^2(D-2)\Delta_c(x) \times[\eta_{\mu\nu}A+(\partial^2\eta_{\mu\nu}-2\partial_\mu\partial_\nu)B].$$

Using lemmas (11), and identifying the terms multiplying $\eta_{\mu\nu}$ and $x_\mu x_\nu$, we arrive at the pair of coupled equations

$$\mathcal{O}[-A+\mathcal{O}B]+\frac{2}{2}\frac{dA}{dr}=ig^2(D-2)\Delta_c\left[A+\frac{2}{r}\frac{dB}{dr}-\mathcal{O}B\right],$$

$$\mathcal{Q}A=ig^2(D-2)\Delta_c\mathcal{Q}B,$$

where $\mathcal{O}\equiv[d^2/dr^2+(D-1)/r(d/dr)]$ and $\mathcal{Q}\equiv[d^2/dr^2-(1/r)(d/dr)]$. We have not suceeded in solving these equations in terms of familiar functions for $D\neq 4$. However in 4D, one can make considerable progress by looking for a power-law solution of the type $A(r)=ar^\beta$, $B(r)=br^{\beta+2}$. Simple calculation reveals that a solution exists provided that

$$a/b=c(\beta+2)/(\beta-2),$$

$$c^2+4c-\beta(\beta-2)(\beta+2)(\beta+4)=0,$$

$$c\equiv g^2/2\pi^2.$$

The quartic in the power exponent β is fortunately simple to solve in terms of the coupling (or c), the answer being

$$\beta=-1+\sqrt{5+\sqrt{16+4c+c^2}}=2+c/3-c^2/54\cdots,$$

so that $a/b\simeq 12+5c/3+\cdots$.

[1] D. Kreimer, Phys. Lett. B **354**, 111 (1995); ''Renormalization and Knot Theory,'' Report No. UTAS-PHYS-94-25 and hep-th/9412045 (unpublished).

[2] D. Broadhurst and D. Kreimer, Int. J. Mod. Phys. C **6**, 51 (1995).

[3] D. Broadhurst, R. Delbourgo, and D. Kreimer, Phys. Lett. B **366**, 421 (1996).

[4] S. F. Edwards, Phys. Rev. **90**, 284 (1953). In particular see the appendix, which discusses the case of massive mesons (with massless particle interchange); the vertex turns out to be a hypergeometric function in that case.

[5] K. M. Case, Phys. Rev. **97**, 810 (1955); R. Delbourgo and V. Prasad, Nuovo Cimento A **21**, 32 (1974).

[6] Equation (3) is merely the most convenient way of writing out the decomposition; it can be transformed into the form $A\gamma_{[\mu_1\cdots\mu_t]}+C\Sigma_{j=1}^{t}p_{\mu_j}p^\lambda\gamma_{[\mu_1\ldots\mu_{j-1}\lambda\mu_{j+1}\cdots\mu_t]}$, by using the γ-matrix commutation relations. Here the tensor structure is more explicit. Note that C or B terms do not have an independent existence when $r=0$ (scalar) or when $r=D$ (pseudoscalar).

[7] Remember that for an interaction $g\phi^3$ in D dimensions, g has mass dimension $3-D/2$, while the propagator behaves as $1/r^4$. That is why the strength a is dimensionless in Eq. (8).

[8] Chains of self-energies are *not* being summed here. Were one to attempt that, it would be necessary to tackle the full Dyson-Schwinger equation $S^{-1}(p)=\gamma\cdot p+\Sigma(p)$, which would lead to a nonlinear equation for the propagator.

[9] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965). See especially Sec. 9.3.7.

[10] B. A. Arbuzov and A. T. Fillipov, Nuovo Cimento **38**, 284 (1965).