Gauge-invariant resummation formalism for two-point correlation functions

Joannis Papavassiliou*
Centre de Physique Théorique, Centre National de la Recherche Scientifique, Luminy, F-13288 Marseille CEDEX 09, France

Apostolos Pilaftsis†
Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, OX11 0QX, United Kingdom
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The consistent description of unstable particles, renormalons, or other Schwinger-Dyson-type solutions within the framework of perturbative gauge field theories necessitates the definition and resummation of off-shell Green's functions, which must respect several crucial physical requirements. A formalism is presented for resummation of off-shell two-point correlation functions, which is mainly based on arguments of analyticity, unitarity, gauge invariance, and renormalizability. The analytic results obtained with various methods, including the background field gauges and the pinch technique, are confronted with the physical requirements imposed; to one-loop order the pinch technique approach satisfies all of them. Using renormalization group arguments, we discuss the issues of the uniqueness of the resummation procedure related to the latter method. [S0556-2821(96)02120-0]

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I. INTRODUCTION

It is well known that in non-Abelian gauge theories individual off-shell Green's functions are, in general, plagued with various pathologies, such as gauge dependences, bad high-energy behavior, or lack of renormalizability, which, strictly speaking, render them void of any physical meaning. To the extend that the physical issues at hand can be dealt with within the confines of conventional perturbation theory, the aforementioned pathologies pose no real problem. Indeed, when combined together to form observables, the individually pathological Green's functions conspire in such a way as to give a physically meaningful answer, order by order, in perturbation theory. A classic example of the subtle cancellation mechanisms in effect is the computation of electroweak S-matrix elements in the unitary gauge; there, even though the conventional two-, three-, and four-point functions are not even renormalizable, the final S-matrix element turns out to be well defined.

There is, however, a plethora of physically important questions, which cannot be treated in the framework of conventional perturbation theory. In quantum chromodynamics (QCD) for example, the only known way to study in the continuum phenomena, such as chiral symmetry breaking or gluon mass generation, is by means of the Schwinger-Dyson equations [1]. Here, the pathologies of the Green's functions start playing a role. Indeed, the Schwinger-Dyson equations are built up by off-shell Green's functions; if one could solve these equations exactly, the Green's functions obtained would again conspire to yield physically meaningful answers. However, since the Schwinger-Dyson series constitutes an infinite set of coupled nonlinear integral equations, a truncation is necessary, which, if carried out casually, may give rise to physically meaningless answers, such as gauge-dependent expressions for ostensibly gauge-independent, physical quantities.

Even though the need for a self-consistent scheme for constructing off-shell Green's functions is more or less expected when dealing with a strongly coupled theory such as QCD, perhaps the most compelling physical circumstances advocating its necessity have been encountered in the context of a ''weakly'' coupled theory, namely, the electroweak $SU(2)_L \otimes U(1)_Y$ model [2–4]. Indeed, the presence of unstable particles makes it impossible to compute physical amplitudes for arbitrary values of the kinematic parameters, unless a resummation has first taken place. Simply stated, perturbation theory breaks down in the vicinity of resonances, and information about the dynamics to ''all orders'' needs be encoded already at the level of Born amplitudes. As was already pointed out in [2], if one attempts to naively promote Veltman's formalism for scalar theories [5] to the case of gauge theories, one is invariably led to gross violations of gauge invariance and unitarity. As explained in [4], resumming the conventional two-point function of a gauge boson in order to construct a Breit-Wigner-type propagator, takes into account higher order corrections for only certain parts of the Born amplitude, whereas crucial contributions originating from box and vertex graphs are not included properly. As a result, the subtle cancellation mechanism alluded to before, even though in reality is still in effect, gets distorted by the casual resummation, resulting in artifacts, which thwart the predictive power of S-matrix perturbation theory.

Given the subtle nature of the problem, the question naturally arises, what set of physical criteria must be satisfied by a resummation algorithm, in order for it to qualify as ''physical.'' In other words, what are the guiding principles, which will allow one to determine whether or not the resummed quantity carries any physically meaningful information, and

*Electronic address: papavass@cpt.univ-mrs.fr
†Address after 1st of October: Max-Planck-Institute, Föhringer Ring 6, D-80805 München, FRG. Electronic address: pilaftsis@v2.rl.ac.uk

to what extent it captures the essential underlying dynamics? To address these questions in this paper, we postulate a set of field-theoretical requirements that we consider crucial when attempting to define a proper resummed propagator. Our considerations propose an answer to the question of how to analytically continue the Lehmann–Symanzik-Zimmermann (LSZ) formalism [6] in the off-shell region of Green's functions in a way which is manifestly gauge invariant and consistent with unitarity. In addition, we demonstrate that the off-shell Green's functions obtained by the pinch technique (PT) [7–10] satisfy all these requirements. In fact, these requirements are, in a way, inherent within the PT approach, as we will see in detail in what follows.

In particular, the following is required from an off-shell, one-particle irreducible (1PI), effective two-point function.

(i) *Resummability*. The effective two-point functions must be resummable. For the conventionally defined two-point functions, the resummability can be formally derived from the path integral. In the S matrix PT approach, the resummability of the effective two-point functions is more involved and must be based on a careful analysis of the structure of the S-matrix to higher orders in perturbation theory [4].

(ii) *Analyticity of the off-shell Green's function*. An analytic two-point function has the property that its real and imaginary parts are related by a dispersion relation (DR), up to a maximum number of two subtractions. The latter is a necessary condition when considering renormalizable Green's functions, as we will discuss in Sec. II.

(iii) *Unitarity and the optical relation*. In the conventional framework, unitarity is defined only for on-shell S-matrix elements, leading to the familiar optical theorem (OT) for the forward scattering. Here, we postulate the validity of the optical relation for the off-shell Green's function, when embedded in an S-matrix element, in a way which will become clear in what follows. An important consequence of this requirement is that the imaginary part of the off-shell Green's function should not contain any unphysical thresholds. As a counterexample, in Sec. VII, it will be shown that this pathology is, in fact, induced by the quantum fields in the background-field-gauge (BFG) method [11] for $\xi_Q \neq 1$.

(iv) *Gauge invariance*. As has been mentioned above, one has to require that the effective Green's functions are gauge-fixing parameter (GFP) independent and satisfy Ward identities (WI's) in compliance with the classical action. For instance, the latter is guaranteed in the BFG method but not the former. This condition also guarantees that gauge invariance does not get spoiled after Dyson summation of the GFP-independent self-energies. In some of the recent literature, the terms of gauge invariance and gauge independence have been used for two different aspects. For example, in the BFG the classical background fields respect gauge invariance in the classical action. However, this fact does not ensure that the quantum fields respect some form of quantum gauge invariance, neither does imply that some kind of a Becchi-Rouet-Stora (BRS) symmetry [12] is present for the fields inside the quantum loops after fixing the gauge of the theory [13,14]. In our discussion, when referring to gauge invariance, we will encompass both meanings, i.e., gauge invariance of the tree-level classical particles as well as BRS invariance of the quantum fields. A direct but nontrivial consequence of the gauge invariance and of the Abelian-type WI's that the effective off-shell Green's functions satisfy is that for large asymptotic momenta transfers ($s \to \infty$) the self-energy under construction must capture the running of the gauge coupling, as it happens in quantum electrodynamics (QED). Because of the Abelian-type WI's and on account of resummation, the above argument can be generalized to n-point functions. In addition, the off-shell n-point transition amplitudes should display the correct high-energy limit as is dictated by the equivalence theorem [15].

(v) *Multiplicative renormalization*. Since we are interested in renormalizable theories, i.e., theories containing operators of dimension no higher than four, the off-shell Green's functions calculated within an approach should admit renormalization. However, this requirement alone is not sufficient when resummation is considered. The appearance of a two-point function in the denominator of a resummed propagator makes it unavoidable to demand that renormalization be *multiplicative*; otherwise, the analytic expressions will suffer from spurious ultraviolet (UV) divergences. Particular examples of the kind are some ghost-free gauges, such as the light cone or planar gauge [16].

(vi) *Position of the pole*. Since the position of the pole is the only gauge-invariant quantity that one can extract from conventional self-energies, any acceptable resummation procedure should give rise to an effective self-energies which do not shift the position of the pole. This requirement drastically reduces the arbitrariness in constructing an effective two-point correlation function.

A closer look at these requirements reveals that they are, in fact, very tightly interwoven; relaxing even one of them could give rise to unphysical results, sometimes in rather subtle ways. As an example of the subtleties involved, we investigate the BFG [11,17] in Sec. VIII. Despite the fact that the background fields of the BFG obey the WI's of the classical Lagrangian, even after quantizing the theory, the BFG expressions for the self-energies depend explicitly on the quantum gauge parameter ξ_Q; in turn, in theories with spontaneous symmetry breaking (SSB), this dependence on ξ_Q gives rise to *unphysical* threshold channels for $\xi_Q \neq 1$. Obviously, such unphysical absorptive contributions should not be resummed to all orders. In fact, we find that the subamplitudes containing physical Landau singularities, and those which do not, satisfy the same BFG WI's. Only the case of BFG with $\xi_Q = 1$ is free from unphysical poles, and the results of the Green's functions collapse to these of the PT. Evidently, relaxing the requirement of GFP independence, by allowing ξ_Q to survive, interferes with unitarity in a nontrivial way.

We now present a roadmap of our paper: In Sec. II, we review the crucial properties of analyticity of two-point correlation functions. We then derive some important consequences arising from dispersion relations (DR's), which should be satisfied by a consistent analytic approach. The results of this analysis may also be applied to eliminate a large degree of arbitrariness in defining off-shell transition amplitudes. Issues of renormalization are also discussed.

In Sec. III, we discuss the rôle of unitarity and OT and elucidate their connection with gauge invariance. In Sec. IV, we show how to employ unitarity, analyticity, and elementary tree-level WI's (EWI's), in order to obtain a self-consistent picture in the context of QCD. In particular, we

work with the right-hand side (RHS) of the OT, where only physical particles (no ghosts) appear as intermediate states. In Sec. V, we focus again on the same process as in the previous section and present a different (equivalent but nontrivial) point of view. In particular, we start again from the RHS of the OT and show how the unitarity of an on-shell transition amplitude and the BRS symmetry [12] of the quantum action can be exploited to reinforce gauge invariance and GFP independence for off-shell Green's functions. In the context of one-loop QCD, these properties rigorously prove the independence of the PT on the gauge-fixing procedure.

In Sec. VI, the analysis of Sec. V is extended to the case of the minimal standard model (SM). We concentrate on a charged process with *nonconserved* external currents and resort again to the (slightly more involved) EWI's. The propagatorlike expression obtained by working with the RHS of the OT is then fed into a twice subtracted DR. The result obtained is identical to the real part of the PT W-boson self-energy, already known from previous considerations. This example convincingly demonstrates the combined power of unitarity and analyticity. In Sec. VII, we take a different point of view and work directly with the left-hand-side (LHS) of the OT, where ''unphysical'' degrees of freedom, such as ghosts and would-be Goldstone bosons, appear now as intermediate states. Using the usual Cutkosky rules, and exploiting again the EWI's of the theory to the fullest, we arrive at the imaginary part of the PT W-boson self-energy. This constitutes a highly nontrivial self-consistency check, demonstrating that as long as one fully exploits the elementary symmetries of the theory, one can work freely with either side of the optical relation, arriving at the same physically consistent results.

In Sec. VIII, we turn our attention to the BFG and show that the dependence of the resummed BFG two-point functions on the ''quantum'' GFP ξ_Q is far from innocuous, leading to the violation of unitarity, because of the appearance of unphysical thresholds. Furthermore, the physical and unphysical expressions are found to satisfy exactly the same tree-level WI's. This fact demonstrates beyond any doubt that a combination of requirements need be imposed in order to arrive at a physically reliable result. Indeed, satisfying external tree-level WI's is a necessary but not sufficient requirement in this context.

In Sec. IX, we show under mild assumptions that the PT resummation gives rise to ''unique'' results. By ''unique'' we mean that at the end of the PT rearrangement, and after renormalization has been completed, no further pieces may be moved around without leading to a violation of some of the physical properties characterizing the PT Green's functions. Here, we should mention that in the framework of the S-matrix perturbation theory, the evaluation of an S-matrix element or a gauge-invariant operator at a given order of loop expansion is not unique in general, in the sense that the analytic result depends on the renormalization prescription used to remove the UV divergences. Of course, the summation of all infinite perturbative contributions should formally yield a unique result independent of the choice of renormalization. Furthermore, to a given order of perturbation theory, one can invoke the renormalization group equation (RGE) in order to show that this uniqueness of the final expression gets spoiled by terms which are of higher order in the coupling constants. The latter notion of RGE invariance is to be adopted throughout this paper.

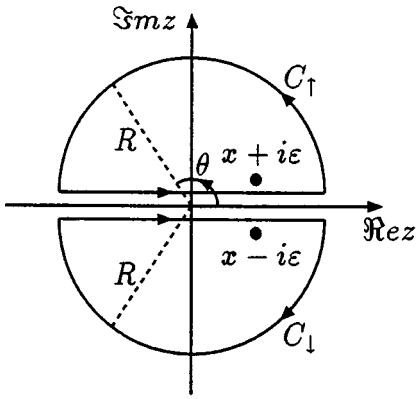

FIG. 1. Contours of complex integration.

Finally, we present our conclusions in Sec. X.

II. ANALYTICITY AND RENORMALIZATION

Analyticity is one of the most important properties that govern physical transition amplitudes. Correlation functions are considered to be analytic in their kinematic variables, which is expressed by means of the so-called DR's [18–20]. In this section, we briefly review some important facts about DR's and renormalization and discuss the subtleties encountered in non-Abelian gauge theories.

If a complex function $f(z)$ is analytic in the interior of and upon a closed curve $C_\uparrow$, say in Fig. 1, and $x+i\varepsilon$ (with $x,\varepsilon \in \mathbf{R}$ and $\varepsilon>0$) is a point within the closed curve $C_\uparrow$, we then have the Cauchy's integral form

$$f(x+i\varepsilon)=\frac{1}{2\pi i}\oint_{C_\uparrow} dz \frac{f(z)}{z-x-i\varepsilon}, \tag{2.1}$$

where $\oint$ denotes that the path $C_\uparrow$ is singly wound. Using Schwartz's reflection principle, one also obtains

$$f(x-i\varepsilon)=-\frac{1}{2\pi i}\oint_{C_\downarrow} dz \frac{f(z)}{z-x+i\varepsilon}. \tag{2.2}$$

Note that $C_\uparrow^*=C_\downarrow$. Sometimes, an analytic function is called holomorphic; both terms are equivalent for complex functions.

Of significant importance in the discussion of physical processes is a DR, which relates the imaginary part of an analytic function $f(x)$ to its real part, and vice versa. We assume for the moment that the analytic function $f(z)$ has the asymptotic behavior, $|f(z)|\leqslant C/R^k$, for large radii R as shown in Fig. 1, where C is a real non-negative constant and $k>0$; this assumption will be relaxed later on, giving rise to more involved DR. Taking now the limit $\varepsilon\to 0$, it is easy to evaluate $\mathrm{Re}f(x)$ through

$$2\mathrm{Re}f(x)=\text{``}\lim_{\varepsilon\to 0}\text{''}[f(x+i\varepsilon)+f^*(x-i\varepsilon)]$$
$$=\text{``}\lim_{\varepsilon\to 0}\text{''}\frac{1}{\pi}\int_{-\infty}^{+\infty}dx'\,\mathrm{Im}\left(\frac{f(x')}{x'-x-i\varepsilon}\right)+\Gamma_\infty . \tag{2.3}$$

Here, ``$\lim_{\varepsilon\to 0}$'' means that the limit should be taken *after* the integration has been performed, and

$$\Gamma_\infty=\frac{1}{\pi}\lim_{R\to\infty}\mathrm{Re}\int_0^\pi d\theta f(\mathrm{Re}^{i\theta}). \tag{2.4}$$

Because of the assumed asymptotic behavior of $f(z)$ at infinity, the integral over the upper infinite semicircle in Fig. 1, Γ_∞, can be easily shown to vanish. Employing the well-known identity for distributions,

$$\text{``}\lim_{\varepsilon\to 0}\text{''}\frac{1}{x'-x-i\varepsilon}=\mathrm{P}\frac{1}{x'-x}+i\pi\delta(x'-x),$$

we arrive at the unsubtracted DR,

$$\mathrm{Re}f(x)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty}dx'\,\frac{\mathrm{Im}f(x')}{x'-x}. \tag{2.5}$$

In Eq. (2.5) the symbol P in front of the integral stands for principle value integration. Following a similar line of arguments, one can express the imaginary part of $f(x)$ as an integral over $\mathrm{Re}f(x)$.

In the previous derivation, the assumption that $|f(z)|$ approaches zero sufficiently fast at infinity has been crucial, since it guarantees that $\Gamma_\infty\to 0$. However, if we were to relax this assumption, additional subtractions need be included in order to arrive at a finite expression. For instance, for $|f(z)|\leqslant CR^k$ with $k<1$, it is sufficient to carry out a single subtraction at a point $x=a$. In this way, one has

$$\mathrm{Re}f(x)=\mathrm{Re}f(a)+\frac{(x-a)}{\pi}\mathrm{P}\int_{-\infty}^{+\infty}dx'\,\frac{\mathrm{Im}f(x')}{(x'-a)(x'-x)}. \tag{2.6}$$

From Eq. (2.6) it is obvious that $\mathrm{Re}f(x)$ can entirely be obtained from $\mathrm{Im}f(x)$, up to a unknown, real constant $\mathrm{Re}f(a)$. Usually, the point a is chosen in a way such that $\mathrm{Re}f(a)$ takes a specific value on account of some physical requirement. For example, if $\mathrm{Im}f(q^2)$ is the imaginary part of the magnetic form factor of an electron with photon virtuality q^2, one can prescribe that the physical condition $\mathrm{Re}f(0)=0$ should hold true in the Thomson limit.

We next focus on the study of some crucial analytic properties of off-shell transition amplitudes within the context of renormalizable field theories. In such theories, one is allowed to have at most two subtractions for a two-point correlation function. If $\Pi(s)$ is the self-energy function of a scalar particle with mass m and off-shell momentum q $(s=q^2)$, the fermionic or vector case is analogous, then the real (or dispersive) part of this amplitude can be fully determined by its imaginary (or absorptive) part via the expression

$$\mathrm{Re}\,\Pi(s)=\mathrm{Re}\Pi(m^2)+(s-m^2)\mathrm{Re}\,\Pi'(m^2)$$
$$+\frac{(s-m^2)^2}{\pi}\mathrm{P}\int_0^{+\infty}ds'\frac{\mathrm{Im}\Pi(s')}{(s'-m^2)^2(s'-s)}. \tag{2.7}$$

From Eq. (2.7) one can readily see that the two subtractions, $\mathrm{Re}\Pi(m^2)$ and the derivative $\mathrm{Re}\Pi'(m^2)$, correspond to the mass and wave-function renormalization constants, respectively, in the on-mass shell (OS) scheme. At higher orders, internal renormalizations of $\mathrm{Im}\Pi(s)$, due to counterterms (CT's) coming from lower orders, should also be taken into account. Then, Eq. (2.7) is still valid, i.e., it holds to order n provided $\mathrm{Im}\Pi(s)$ is renormalized to order $n-1$. In general, the function $\mathrm{Im}\Pi(s)$ has its support in the non-negative real axis, i.e., for $s\geqslant 0$. This can be attributed to the semiboundness of the spectrum of the Hamiltonian, $\mathrm{Spec}H\geqslant 0$ [21]. Note that for spectrally represented two-point correlation functions, we have the additional condition $\mathrm{Im}\Pi(m^2)\geqslant 0$ [22,23].

As has been mentioned above, in renormalizable field theories it is required that $\Pi(s)$ should be finite after two subtractions have been performed. This implies that

$$|\Pi(s)|\leqslant Cs^k,\quad \text{with}\ \ k<2, \tag{2.8}$$

as $s\to\infty$. Obviously, the same inequality holds true for the real as well as the imaginary part of $\Pi(s)$. In pure non-Abelian Yang-Mills theories, such as quarkless QCD, the transverse part $\Pi_T(s)$ of the gluon vacuum polarization behaves asymptotically as

$$\Pi_T(s)\to Cs\left(\ln\frac{s}{\mu^2}\right)^n.$$

This result is consistent with Eq. (2.8) for any $n<\infty$. Furthermore, we mention that the Froissart–Martin bound [24],

$$|\Pi(s)|\leqslant Cs^3\left(\ln\frac{s}{s_0}\right)^2, \tag{2.9}$$

at $s\to\infty$, which may be derived from axiomatic methods of field theory, is weaker than Eq. (2.8). In fact, the Froissart-Martin bound [24] refers to the asymptotic behavior of a total cross section $\sigma(s)$ in the limit $s\to\infty$. This is expressed as $\sigma(s)\leqslant C[\ln(s/s_0)]^2$. Furthermore, the OT gives the relation $s\sigma(s)=\mathrm{Im}T(s)$, where $T(s)$ is the forward-scattering amplitude. If one assumes the absence of accidental cancellations between the two-point function $\Pi(s)$, and higher n-point functions within the expression $\mathrm{Im}T(s)$, one can derive that

$$|\mathrm{Im}\Pi(s)|\leqslant Cs^2\mathrm{Im}T(s)\leqslant Cs^3[\ln(s/s_0)]^2.$$

Because of analyticity, the s dependence of $\mathrm{Im}\Pi(s)$ will affect the high-s behavior of $|\Pi(s)|$. Even if we assume that the s dependence thus induced on $|\Pi(s)|$ is the most modest possible, i.e., $|\Pi(s)|\sim\mathrm{Im}\Pi(s)$ as $s\to\infty$, still the tightest upper bound one could obtain from these considerations is that of Eq. (2.9). The analytic expression of gluon vacuum polarization satisfies Eq. (2.9). As a counterexample to this situation, we may consider the Higgs self-energy in the unitary

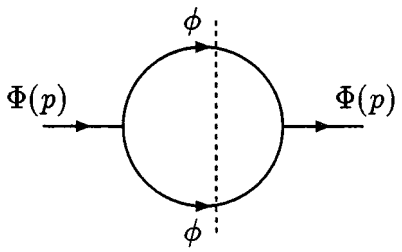

FIG. 2. Two-point correlation function $\Pi_\Phi(s)$ at one loop.

gauge; the absorptive part of the Higgs self-energy has an s^2 dependence at high energies, and its resummation [25] is, therefore, not justified.

We will now illustrate how DR's work in practice in the context of a scalar field theory. As an example, we consider a toy model with interaction Lagrangian,

$$\mathcal{L}_{\rm int}=\frac{\lambda}{2}\phi^2\Phi, \tag{2.10}$$

where λ is a nonvanishing coupling constant of dimensions of mass. We denote the mass of the scalar ϕ by m and the one of the Φ by M and assume that $M\geqslant m$.

One can calculate the imaginary part of the one-loop self-energy $\Pi_\Phi(s)$ by using Cutkosky rules. The self-energy $\Pi_\Phi(s)$ develops a branch cut for $s=p^2>4m^2$, which arises from the on-shell ϕ-pair contribution shown in Fig. 2. Thus, it is not difficult to obtain

$$\mathrm{Im}\Pi_\Phi(s)=\frac{\lambda^2}{32\pi}\left(1-\frac{4m^2}{s}\right)^{1/2}\theta(s-4m^2). \tag{2.11}$$

On the other hand, adopting dimensional regularization in dimensions $D=4-2\epsilon$, we have

$$\Pi_\Phi(s)=\frac{\lambda^2}{32\pi^2}\left\{\frac{1}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+2 -\left(1-\frac{4m^2}{s}\right)^{1/2}\ln\left[\frac{\left(1-\frac{4m^2}{s}\right)^{1/2}+1}{\left(1-\frac{4m^2}{s}\right)^{1/2}-1}\right]\right\}, \tag{2.12}$$

where s should be analytically continued to $s+i\varepsilon$. In fact, for $s>4m^2$, the logarithmic function in Eq. (2.12) assumes the form

$$\ln\left[\frac{1+\left(1-\frac{4m^2}{s}\right)^{1/2}}{1-\left(1-\frac{4m^2}{s}\right)^{1/2}}\right]-i\pi\theta(s-4m^2).$$

Evidently, the absorptive part of $\Pi_\Phi(s)$ obtained from Eq. (2.12) is equal to $\mathrm{Im}\Pi_\Phi(s)$ in Eq. (2.11). Furthermore, one can verify the validity of a DR of Eq. (2.6), singly subtracted at $s=0$. Since

$$\mathrm{Re}\Pi_\Phi(0)=\frac{\lambda^2}{32\pi^2}\left[\frac{1}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}\right], \tag{2.13}$$

one can check that indeed,

$$\frac{s}{\pi}\mathrm{P}\int_{4m^2}^{\infty}ds'\,\frac{\mathrm{Im}\Pi_\Phi(s')}{s'(s'-s)}=\mathrm{Re}\Pi_\Phi(s)-\mathrm{Re}\Pi_\Phi(0).$$

This simple example explicitly demonstrates the analytic nature of a two-point correlation function.

In the context of gauge field theories, one should anticipate a similar analytic structure for two-point correlation functions. However, an extra complication appears in such theories when off-shell transition amplitudes are considered. In a theory with SSB, such as the SM for example, this complication originates from the fact that, in addition to the physical particles of the spectrum of the Hamiltonian, unphysical, gauge-dependent degrees of freedom, such as would-be Goldstone bosons and ghost fields, make their appearance. Although, on-shell transition amplitudes contain only the physical degrees of freedom of the particles involved on account of unitarity, their continuation to the off-shell region is ambiguous, because of the presence unphysical Landau poles, introduced by the aforementioned unphysical particles. A reasonable prescription for accomplishing such an off-shell continuation, which is very close in spirit to the previous example of the scalar theory, would be to continue analytically an off-shell amplitude by taking only *physical* Landau singularities into account.

Consider for example the off-shell propagator of a gauge particle in the conventional R_ξ gauges or BFG's, which runs inside a quantum loop, viz.,

$$\Delta^{(\xi_Q)}_{0\mu\nu}(q)=t_{\mu\nu}(q)\frac{1}{q^2-M^2}-\ell_{\mu\nu}(q)\frac{\xi_Q}{q^2-\xi_Q M^2}, \tag{2.14}$$

with

$$t_{\mu\nu}(q)=-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2},\quad \ell_{\mu\nu}(q)=\frac{q_\mu q_\nu}{q^2}.$$

One can write two separate DR's for the transverse self-energy Π_T of a massive gauge boson, which crucially depend on the pole structure of Eq. (2.14): namely,

$$\mathrm{Re}\overline{\Pi}_T(s)=\mathrm{Re}\overline{\Pi}_T(M^2)+(s-M^2)\mathrm{Re}\overline{\Pi}'_T(M^2) +\frac{(s-M^2)^2}{\pi}\mathrm{P}\int_{\{M^2_{\rm phys}\}}^{+\infty}ds'\,\frac{\mathrm{Im}\overline{\Pi}_T(s')}{(s'-M^2)^2(s'-s)}, \tag{2.15}$$

$$\mathrm{Re}\overline{\Pi}_T^{(\xi_Q)}(s) = (s-M^2)\mathrm{Re}\overline{\Pi}'^{(\xi_Q)}_T(M^2) + \frac{(s-M^2)^2}{\pi}\mathrm{P}\int_{\{M^2_{\rm unphys}\}}^{+\infty} ds' \frac{\mathrm{Im}\overline{\Pi}_T^{(\xi_Q)}(s')}{(s'-M^2)^2(s'-s)}. \tag{2.16}$$

In the first DR given in Eq. (2.15), the real part of Π_T, $\mathrm{Re}\overline{\Pi}_T$, is determined from branch cuts induced by physical poles, where the masses of the real on-shell particles in the loop are collectively denoted by $\{M^2_{\rm phys}\}$. In what follows we refer to such a DR as *physical* DR. Note that $\mathrm{Re}\overline{\Pi}_T$ depends only implicitly on the gauge choice. In fact, $\mathrm{Re}\Pi_T$ can be viewed as the truncated part of the self-energy that will survive if $\mathrm{Re}\Pi_T$ is embedded in a S-matrix element. In Eq. (2.16), the dispersive part of the two-point function depends explicitly on ξ_Q-dependent unphysical thresholds, collectively denoted by $\{M^2_{\rm unphys}\}$, which are induced by the longitudinal parts of the gauge propagators contained in $\mathrm{Im}\overline{\Pi}_T^{(\xi_Q)}$ Evidently, one has the decomposition

$$\mathrm{Im}\Pi_T(s) = \mathrm{Im}\overline{\Pi}_T(s) + \mathrm{Im}\overline{\Pi}_T^{(\xi_Q)}(s),$$

$$\mathrm{Re}\Pi_T(s) = \mathrm{Re}\,\overline{\Pi}_T(s) + \mathrm{Re}\overline{\Pi}_T^{(\xi_Q)}(s). \tag{2.17}$$

From Eq. (2.14) one can now isolate that part of the propagator that should be used in a physical DR. For $\xi_Q \neq 1$, one has

$$\Delta^{(\xi_Q)}_{0\mu\nu} \to U_{\mu\nu}(q) \equiv \Delta^{(\infty)}_{0\mu\nu}(q). \tag{2.18}$$

It is, therefore, obvious that the ''physical'' sector of an off-shell transition amplitude in BFG (for $\xi_Q \neq 1$), or equivalently, the part of the off-shell matrix element that satisfies a *physical* DR, is effectively obtained by considering all the internal propagators in the unitary gauge ($\xi_Q \to \infty$), but leaving the Feynman rules for the vertices in the general ξ_Q gauge.

In view of a physical DR, the gauge $\xi_Q = 1$ is very specific, since the physical and unphysical poles coincide in such a case, making them indistinguishable. At one-loop order, the results of this gauge are found to collapse to those obtained via the PT [17]. Finally, we remark in passing that, if $\overline{\Pi}_T$ in $\xi_Q \neq 1$ is used for a definition of a ''physical'' self-energy, one encounters problems with the high-energy unitarity behavior, even though the full $\Pi(\xi_Q)$ is asymptotically well behaved. In the case of the one-loop Z self-energy for example, for $\xi_Q \neq 1$ [17], $\overline{\Pi}_T$ contains terms proportional to q^4; all such terms eventually cancel in the entire $\Pi(\xi_Q)$ against the part that contains the unphysical poles. Incidentally, it is interesting to note that the recovery of the correct asymptotic behavior is the more delayed (i.e., it happens for larger values of q^2) the larger the value of ξ_Q. However, if one was to resum only the Π_T part, the terms proportional to q^4 would survive, leading to bad high-energy behavior. If, on the other hand, one had resummed the full $\Pi(\xi_Q)$, then one would have introduced unphysical poles, as explained above.

III. UNITARITY AND GAUGE INVARIANCE

In this section, we will briefly discuss the basic field-theoretical consequences resulting from the unitarity of the S-matrix theory, and establish its connection with gauge invariance. In addition to the requirement of explicit gauge invariance, the necessary conditions derived from unitarity will constitute our guiding principle to analytically continue n-point correlation functions in the off-shell region. Furthermore, we arrive at the important conclusion that the resummed self-energies, in addition to being GFP independent, must also be ''unitary,'' in the sense that they do not spoil unitarity when embedded in an S-matrix element.

The T-matrix element of a reaction $i \to f$ is defined via the relation

$$\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4\delta^{(4)}(P_f - P_i)\langle f|T|i\rangle, \tag{3.1}$$

where P_i (P_f) is the sum of all initial (final) momenta of the $|i\rangle$ ($|f\rangle$) state. Furthermore, imposing the unitarity relation $S^\dagger S = 1$ leads to the OT:

$$\langle f|T|i\rangle - \langle i|T|f\rangle^* = i\sum_{i'} (2\pi)^4\delta^{(4)}(P_{i'} - P_i) \times \langle i'|T|f\rangle^*\langle i'|T|i\rangle. \tag{3.2}$$

In Eq. (3.2), the sum $\Sigma_{i'}$ should be understood to be over the entire phase space and spins of all possible on-shell intermediate particles i'. A corollary of this theorem is obtained if $i = f$. In this particular case, we have

$$\mathrm{Im}\langle i|T|i\rangle = \frac{1}{2}\sum_f (2\pi)^4\delta^{(4)}(P_f - P_i)|\langle f|T|i\rangle|^2. \tag{3.3}$$

In the conventional S-matrix theory with stable particles, Eqs. (3.2) and (3.3) hold also perturbatively. To be precise, if one expands the transition $T = T^{(1)} + T^{(2)} + \cdots + T^{(n)} + \cdots$, to a given order n, one has

$$T^{(n)}_{fi} - T^{(n)*}_{if} = i\sum_{i'} (2\pi)^4\delta^{(4)}(P_{i'} - P_i)\sum_{k=1}^{n-1} T^{(k)*}_{i'f} T^{(n-k)}_{i'i}. \tag{3.4}$$

There are two important conclusions that can be drawn from Eq. (3.4). First, the anti-Hermitian part of the LHS of Eq. (3.4) contains, in general, would-be Goldstone bosons or ghost fields [26]. Such contributions manifest themselves as Landau singularities at unphysical points, e.g., $q^2 = \xi_Q M_W^2$ for a W propagator in a general BFG. However, unitarity requires that these unphysical contributions should vanish, as can be read off from the RHS of Eq. (3.4). Second, the RHS explicitly shows the connection between gauge invariance and unitarity at the quantum loop level. To lowest order for example, the RHS consists of the product of GFP-independent on-shell tree amplitudes, thus enforcing the gauge invariance of the imaginary part of the one-loop amplitude on the LHS.

The above powerful constraints imposed by unitarity will be in effect as long as one computes *full* amplitudes to a finite order in perturbation theory. However, for resummation purposes, a certain subamplitude, i.e., a part of the full amplitude, must be singled out and subsequently undergo a

Dyson summation, while the rest of the S matrix is computed to a finite order n. Therefore, if the resummed amplitude contains gauge artifacts and/or unphysical thresholds, the cancellations imposed by Eq. (3.4) will only operate up to order n, introducing unphysical contributions of order $n+1$ or higher. To avoid the contamination of the physical amplitudes by such unphysical artifacts, we impose the following two requirements on the effective Green's functions, when we attempt to continue them analytically in the off-shell region for the purpose of resummation.

(i) The off-shell n-point correlation functions ought to be derivable from or embeddable into S-matrix elements.

(ii) The off-shell Green's functions should not display unphysical thresholds induced by unphysical Landau singularities, as has been described above.

Even though property (i) is automatic for Green's functions generated by the functional differentiation of the conventional path-integral functional, in general the off-shell amplitudes so obtained fail to satisfy property (ii). In the PT framework instead, both conditions are satisfied: effective Green's functions are directly derived from the S-matrix amplitudes [so condition (i) is satisfied by construction] and contain only physical thresholds, so that unitarity is not explicitly violated [4].

In our discussion of unitarity at one loop, we will make extensive use of the following two-body Lorentz-invariant phase-space (LIPS) integrals: The scalar integral

$$\begin{aligned}\int dX_{\rm LIPS} &= \frac{1}{(2\pi)^2}\int d^4k_1 \int d^4k_2 \delta_+(k_1^2-m_1^2) \\ &\quad \times \delta_+(k_2^2-m_2^2)\,\delta^{(4)}(q-k_1-k_2) \\ &= \theta(q^0)\,\theta[q^2-(m_1+m_2)^2] \\ &\quad \times \frac{1}{8\pi q^2}\lambda^{1/2}(q^2,m_1^2,m_2^2), \end{aligned} \tag{3.5}$$

where $\lambda(x,y,z)=(x-y-z)^2-4yz$ and $\delta_+(k^2-m^2) \equiv \theta(k^0)\,\delta(k^2-m^2)$, and the tensor integral:

$$\begin{aligned}\int dX_{\rm LIPS}(k_1-k_2)_\mu(k_1-k_2)_\nu \\ = \left\{\frac{\lambda(q^2,m_1^2,m_2^2)}{3q^2}t_{\mu\nu}(q)+\left[\frac{\lambda(q^2,m_1^2,m_2^2)}{q^2}-q^2\right.\right. \\ \left.\left.+2(m_1^2+m_2^2)\right]\ell_{\mu\nu}(q)\right\}\int dX_{\rm LIPS}. \end{aligned} \tag{3.6}$$

The Lorentz projection tensors, $t_{\mu\nu}(q)$ and $\ell_{\mu\nu}(q)$, have been defined after Eq. (2.14).

IV. THE CASE OF QCD

In this section, we show that a self-consistent picture may be obtained by resorting to such fundamental properties of the S matrix as unitarity and analyticity, using as additional input only EWI's for tree-level, on-shell processes, and tree-level vertices and propagators. It is important to emphasize that the GFP independence of the results emerges *automatically* from the previous considerations.

We begin from the RHS of the optical relation given in Eq. (3.3). The RHS involves on-shell physical processes, which satisfy the EWI's. It turns out that the full exploitation of those EWI's leads unambiguously to a decomposition of the tree-level amplitude into propagator-, vertex-, and box-like structures. The propagatorlike structure corresponds to the imaginary part of the effective propagator under construction. By imposing the additional requirement that the effective propagator be an analytic function of q^2, one arrives at a DR, which, up to renormalization-scheme choices, leads to a unique result for the real part.

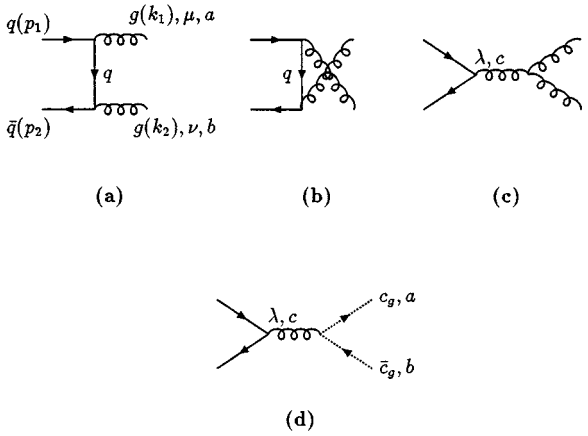

FIG. 3. Diagrams (a)–(c) contribute to $\mathcal{T}^{ab}_{\mu\nu}$ and diagram (d) to $\mathcal{S}^{ab}$.

Consider the forward-scattering process $q\overline{q}\to q\overline{q}$. From the OT, we then have

$$\mathrm{Im}\langle q\overline{q}|T|q\overline{q}\rangle = \frac{1}{2}\left(\frac{1}{2}\right)\int dX_{\rm LIPS}\langle q\overline{q}|T|gg\rangle\langle gg|T|q\overline{q}\rangle^*. \tag{4.1}$$

In Eq. (4.1), the statistical factor 1/2 in parentheses arises from the fact that the final on-shell gluons should be considered as identical particles in the total rate. We now set $\mathcal{M}=\langle q\overline{q}|T|q\overline{q}\rangle$ and $\mathcal{T}=\langle q\overline{q}|T|gg\rangle$, and focus on the RHS of Eq. (4.1). Diagrammatically, the amplitude $\mathcal{T}$ consists of two distinct parts: t- and u-channel graphs that contain an internal quark propagator $\mathcal{T}^{ab}_{t\mu\nu}$ as shown in Figs. 3(a) and 3(b), and an s-channel amplitude $\mathcal{T}^{ab}_{s\mu\nu}$ which is given in Fig. 3(c). The subscripts ''s'' and ''t'' refer to the corresponding Mandelstam variables, i.e., $s=q^2=(p_1+p_2)^2=(k_1+k_2)^2$, and $t=(p_1-k_1)^2=(p_2-k_2)^2$. Defining

$$V^c_\rho = g\overline{v}(p_2)\frac{\lambda^c}{2}\gamma_\rho u(p_1), \tag{4.2}$$

we have that

$$\mathcal{T}^{ab}_{\mu\nu}=\mathcal{T}^{ab}_{s\mu\nu}(\xi)+\mathcal{T}^{ab}_{t\mu\nu}, \tag{4.3}$$

with

$$\mathcal{T}^{ab}_{s\mu\nu}(\xi) = -gf^{abc}\Delta_0^{(\xi),\rho\lambda}(q)\Gamma_{\lambda\mu\nu}(q,-k_1,-k_2)V^c_\rho, \tag{4.4}$$

$$\mathcal{T}^{ab}_{t\mu\nu} = -ig^2\overline{v}(p_2)\left(\frac{\lambda^b}{2}\gamma^\nu\frac{1}{\not p_1-\not k_1-m}\frac{\lambda^a}{2}\gamma^\mu + \frac{\lambda^a}{2}\gamma^\mu\frac{1}{\not p_1-\not k_2-m}\gamma^\nu\frac{\lambda^b}{2}\right)u(p_1), \tag{4.5}$$

where

$$\Gamma_{\lambda\mu\nu}(q,-k_1,-k_2) = (k_1-k_2)_\lambda g_{\mu\nu} + (q+k_2)_\mu g_{\lambda\nu} - (q+k_1)_\nu g_{\lambda\nu}. \tag{4.6}$$

Note that $\mathcal{T}_s$ depends explicitly on the GFP ξ, through the tree-level gluon propagator $\Delta^{(\xi)}_{0\mu\nu}(q)$, whereas $\mathcal{T}_t$ does not. The explicit expression of $\Delta^{(\xi)}_{0\mu\nu}(q)$ depends on the specific gauge-fixing procedure chosen. In addition, we define the quantities $\mathcal{S}^{ab}$ and $\mathcal{R}^{ab}_\mu$ as

$$\mathcal{S}^{ab} = gf^{abc}\frac{k_1^\sigma}{q^2}V^c_\sigma = -gf^{abc}\frac{k_2^\sigma}{q^2}V^c_\sigma \tag{4.7}$$

and

$$\mathcal{R}^{ab}_\mu = gf^{abc}V^c_\mu. \tag{4.8}$$

Clearly,

$$k_1^\sigma\mathcal{R}^{ab}_\sigma = -k_2^\sigma\mathcal{R}^{ab}_\sigma = q^2\mathcal{S}^{ab}. \tag{4.9}$$

We then have

$$\begin{aligned}\mathrm{Im}\mathcal{M} &= \frac{1}{4}\mathcal{T}^{ab}_{\mu\nu}P^{\mu\sigma}(k_1,\eta_1)P^{\nu\lambda}(k_2,\eta_2)\mathcal{T}^{ab*}_{\sigma\lambda}\\ &= \frac{1}{4}[\mathcal{T}^{ab}_{s\mu\nu}(\xi)+\mathcal{T}^{ab}_{t\mu\nu}]P^{\mu\sigma}(k_1,\eta_1)\\ &\quad\times P^{\nu\lambda}(k_2,\eta_2)[\mathcal{T}^{ab*}_{s\sigma\lambda}(\xi)+\mathcal{T}^{ab*}_{t\sigma\lambda}],\end{aligned} \tag{4.10}$$

where the polarization tensor $P^{\mu\nu}(k,\eta)$ is given by

$$P_{\mu\nu}(k,\eta) = -g_{\mu\nu} + \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta k} - \eta^2\frac{k_\mu k_\nu}{(\eta k)^2}. \tag{4.11}$$

Moreover, we have that on shell, i.e., for $k^2=0$, $k^\mu P_{\mu\nu}=0$. By virtue of this last property, we see immediately that if we write the three-gluon vertex of Eq. (4.6) in the form

$$\begin{aligned}\Gamma_{\lambda\mu\nu}(q,-k_1,-k_2) &= [(k_1-k_2)_\lambda g_{\mu\nu} + 2q_\mu g_{\lambda\nu} - 2q_\nu g_{\lambda\mu}]\\ &\quad + (-k_{1\mu}g_{\lambda\nu} + k_{2\nu}g_{\lambda\mu})\\ &= \Gamma^F_{\lambda\mu\nu}(q,-k_1,-k_2)\\ &\quad + \Gamma^P_{\lambda\mu\nu}(q,-k_1,-k_2),\end{aligned} \tag{4.12}$$

the term $\Gamma^P_{\rho\mu\nu}$ dies after hitting the polarization vectors $P_{\mu\sigma}(k_1,\eta_1)$ and $P_{\nu\lambda}(k_2,\eta_2)$. Therefore, if we denote by $\mathcal{T}^F_s(\xi)$ the part of $\mathcal{T}_s$ which survives, Eq. (4.10) becomes

$$\mathrm{Im}\mathcal{M} = \frac{1}{4}[\mathcal{T}^F_s(\xi)+\mathcal{T}_t]^{ab}_{\mu\nu}P^{\mu\sigma}(k_1,\eta_1) \times P^{\nu\lambda}(k_2,\eta_2)[\mathcal{T}^F_s(\xi)+\mathcal{T}_t]^{ab*}_{\sigma\lambda}. \tag{4.13}$$

The next step is to verify that any dependence on the GFP inside the propagator $\Delta^{(\xi)}_{0\mu\nu}(q)$ of the off-shell gluon will disappear. This is indeed so, because the longitudinal parts of $\Delta_{0\mu\nu}$ either vanish because the external quark current is conserved, or because they trigger the following EWI:

$$q^\mu\Gamma^F_{\mu\alpha\beta}(q,-k_1,-k_2) = (k_1^2-k_2^2)g_{\alpha\beta}, \tag{4.14}$$

which vanishes on shell. This last EWI is crucial, because in general, current conservation alone is not sufficient to guarantee the GFP independence of the final answer. In the covariant gauges for example, the gauge-fixing term is proportional to $q^\mu q^\nu$; current conservation kills such a term. But if we had chosen an axial gauge instead, i.e.,

$$\Delta^{(\widetilde{\eta})}_{0\mu\nu}(q) = \frac{P_{\mu\nu}(q,\widetilde{\eta})}{q^2}, \tag{4.15}$$

where $\widetilde{\eta}\neq\eta$ in general, then only the term $\widetilde{\eta}_\nu q_\mu$ vanishes because of current conservation, whereas the term $\widetilde{\eta}_\nu q_\mu$ can only disappear if Eq. (4.14) holds. So, Eq. (4.13) becomes

$$\mathrm{Im}\mathcal{M} = \frac{1}{4}(\mathcal{T}^F_s+\mathcal{T}_t)^{ab}_{\mu\nu}P^{\mu\sigma}(k_1,\eta_1)P^{\nu\lambda}(k_2,\eta_2)(\mathcal{T}^F_s+\mathcal{T}_t)^{ab*}_{\sigma\lambda}, \tag{4.16}$$

where the GFP-*independent* quantity $\mathcal{T}^F_s$ is given by

$$\mathcal{T}^{F,ab}_{s\mu\nu} = -gf^{abc}\frac{g^{\rho\lambda}}{q^2}\Gamma^F_{\lambda\mu\nu}(q,-k_1,-k_2)V^c_\rho. \tag{4.17}$$

Next, we want to show that the dependence on η_μ and η^2 stemming from the polarization vectors disappears. Using the on-shell conditions $k_1^2=k_2^2=0$, we can easily verify the EWI's

$$k_1^\mu\mathcal{T}^{F,ab}_{s\mu\nu} = 2k_{2\nu}\mathcal{S}^{ab} - \mathcal{R}^{ab}_\nu, \tag{4.18}$$

$$k_2^\nu\mathcal{T}^{F,ab}_{s\mu\nu} = 2k_{1\mu}\mathcal{S}^{ab} + \mathcal{R}^{ab}_\mu, \tag{4.19}$$

$$k_1^\mu\mathcal{T}^{ab}_{t\mu\nu} = \mathcal{R}^{ab}_\nu, \tag{4.20}$$

$$k_2^\nu\mathcal{T}^{ab}_{t\mu\nu} = -\mathcal{R}^{ab}_\mu, \tag{4.21}$$

from which we have that

$$k_1^\mu k_2^\nu\mathcal{T}^{F,ab}_{s\mu\nu} = q^2\mathcal{S}^{ab}, \tag{4.22}$$

$$k_1^\mu k_2^\nu\mathcal{T}^{ab}_{t\mu\nu} = -q^2\mathcal{S}^{ab}. \tag{4.23}$$

Using the above EWI's, it is now easy to check that indeed, all dependence on both η_μ and η^2 cancels in Eq. (4.16), as it should, and we are finally left with (omitting the fully contracted color and Lorentz indices)

$$\mathrm{Im}\mathcal{M}=\frac{1}{4}[(\mathcal{T}_s^F\mathcal{T}_s^{F*}-8\mathcal{S}\mathcal{S}^*)+(\mathcal{T}_s^F\mathcal{T}_t^*+\mathcal{T}_s^{F*}\mathcal{T}_t)+\mathcal{T}_t\mathcal{T}_t^*]$$
$$=\mathrm{Im}\widehat{\mathcal{M}}_1+\mathrm{Im}\widehat{\mathcal{M}}_2+\mathrm{Im}\widehat{\mathcal{M}}_3. \quad (4.24)$$

The first part is the genuine propagatorlike piece, the second is the vertex, and the third the box. Employing the fact that

$$\Gamma^F_{\rho\mu\nu}\Gamma^{F,\mu\nu}_{\lambda}=-8q^2t_{\rho\lambda}(q)+4(k_1-k_2)_\rho(k_1-k_2)_\lambda \quad (4.25)$$

and

$$\mathcal{S}\mathcal{S}^*=g^2c_AV^c_\rho\frac{k_1^\rho k_1^\lambda}{(q^2)^2}V^c_\lambda=\frac{g^2}{4}c_AV^c_\rho\frac{(k_1-k_2)^\rho(k_1-k_2)^\lambda}{(q^2)^2}V^c_\lambda, \quad (4.26)$$

where c_A is the eigenvalue of the Casimir operator in the adjoint representation [$c_A=N$ for SU(N)], we obtain, for $\mathrm{Im}\widehat{\mathcal{M}}_1$,

$$\mathrm{Im}\widehat{\mathcal{M}}_1=\frac{g^2}{2}c_AV^c_\mu\frac{1}{q^2}[-4q^2t^{\mu\nu}(q)$$
$$+(k_1-k_2)^\mu(k_1-k_2)^\nu]\frac{1}{q^2}V^c_\nu. \quad (4.27)$$

This last expression must be integrated over the available phase space. With the help of Eqs. (3.5) and (3.6), we arrive at the final expression

$$\mathrm{Im}\widehat{\mathcal{M}}_1=V^c_\mu\frac{1}{q^2}\mathrm{Im}\hat{\Pi}^{\mu\nu}(q)\frac{1}{q^2}V^c_\nu, \quad (4.28)$$

with

$$\mathrm{Im}\hat{\Pi}_{\mu\nu}(q)=-\frac{\alpha_s}{4}\ \frac{11c_A}{3}q^2t_{\mu\nu}(q), \quad (4.29)$$

and $\alpha_s=g^2/(4\pi)$.

Before we proceed, we make the following remark. It is well known that the vanishing of the longitudinal part of the gluon self-energy is an important consequence of gauge invariance. One might naively expect that even if a nonvanishing longitudinal part had been induced by some contributions which do not respect gauge invariance, it would not have contributed to physical processes, since the gluon self-energy couples to conserved fermionic currents, thus projecting out only the transverse degrees of the gluon vacuum polarization. However, this expectation is not true in general. Indeed, if one uses, for example, the tree-level gluon propagator in the axial gauge, as given in Eq. (4.15), then there will be residual η-dependent terms induced by the longitudinal component of the gluon vacuum polarization, which would not vanish, despite the fact that the external quark currents are conserved. Such terms are obviously gauge dependent. Evidently, projecting out only the transverse parts of Green's functions will not necessarily render them gauge invariant.

The vacuum polarization of the gluon within the PT is given by [7]

$$\hat{\Pi}_{\mu\nu}(q)=\frac{\alpha_s}{4\pi}\frac{11c_A}{3}t_{\mu\nu}(q)q^2\left[\ln\left(-\frac{q^2}{\mu^2}\right)+C_{UV}\right]. \quad (4.30)$$

Here, $C_{UV}=1/\epsilon-\gamma_E+\ln4\pi+C$, with C being some constant and μ is a subtraction point. In Eq. (4.30), it is interesting to note that a change of $\mu^2\to\mu'^2$ gives rise to a variation of the constant C by an amount $C'-C=\ln\mu'^2/\mu^2$. Thus, a general μ-scheme renormalization yields

$$\hat{\Pi}^R_T(s)=\hat{\Pi}_T(s)-(s-\mu^2)\mathrm{Re}\hat{\Pi}'_T(\mu^2)-\mathrm{Re}\hat{\Pi}_T(\mu^2)$$
$$=\frac{\alpha_s}{4\pi}\frac{11c_A}{3}s\left[\ln\left(-\frac{s}{\mu^2}\right)-1+\frac{\mu^2}{s}\right]. \quad (4.31)$$

From Eq. (2.7), one can readily see that $\mathrm{Re}\hat{\Pi}^R_T(s)$ can be calculated by the following double-subtracted DR:

$$\mathrm{Re}\hat{\Pi}^R_T(s)=\frac{(s-\mu^2)^2}{\pi}\mathrm{P}\int_0^\infty ds'\frac{\mathrm{Im}\hat{\Pi}_T(s')}{(s'-\mu^2)^2(s'-s)}. \quad (4.32)$$

Inserting Eq. (4.29) into Eq. (4.32), it is not difficult to show that it leads to the result given in Eq. (4.31), a fact that demonstrates the analytic power of the DR.

It is important to emphasize that the above derivation rigorously proves the GFP independence of the one-loop PT effective Green's functions, for *every* gauge-fixing procedure. Indeed, in our derivation, we have solely relied on the RHS of the OT, which we have rearranged in a well-defined way, *after* having explicitly demonstrated its GFP independence. The proof of the GFP independence of the RHS presented here is, of course, expected on physical grounds, since it only relies on the use of EWI's, triggered by the longitudinal parts of the gluon tree-level propagators. Note that the tree-level tri-gluon coupling $\Gamma_{\lambda\mu\nu}$ is uniquely given by Eq. (4.6). Since the GFP dependence is carried entirely by the longitudinal parts of the gluon tree-level propagator in *any* gauge-fixing scheme whereas the $g^{\mu\nu}$ part is GFP-independent and universal, the proof presented here is generally true. Obviously, the final step of reconstructing the real part from the imaginary one by means of a DR does not introduce any gauge dependences.

V. THE QCD ANALYSIS FROM BRS CONSIDERATIONS

In this section, we will show how we can obtain the same answer by resorting only to the EWI's that one obtains as a direct consequence of the BRS symmetry of the quantum Lagrangian.

If we consider $\mathcal{T}^{ab}_{\mu\nu}$ as before, it is easy to show that it satisfies the BRS identities [27]

$$k_1^\mu\mathcal{T}^{ab}_{\mu\nu}=k_{2\nu}\mathcal{S}^{ab},\quad k_2^\nu\mathcal{T}^{ab}_{\mu\nu}=k_{1\mu}\mathcal{S}^{ab},\quad k_1^\mu k_2^\nu\mathcal{T}^{ab}_{\mu\nu}=0\,, \quad (5.1)$$

where S^{ab} is the ghost amplitude shown in Fig. 3(d); its closed form is given in Eq. (4.7).

Note that the BRS identities of Eq. (5.1) are different from those listed in Eqs. (4.18)–(4.23), because the term $\Gamma^P_{\mu\nu\rho}$ had been removed in the latter group. Here, we follow

a different sequence and do not cancel the term $\Gamma^P_{\mu\nu\rho}$; instead, we will exploit the *exact* BRS identities from the very beginning.

We start again with the expression for $\mathrm{Im}\mathcal{M}$ given in Eq. (4.10). First of all, it is easy to verify again that the dependence on the GFP of the off-shell gluon vanishes. This is so because of the tree-level EWI, involving the *full* vertex $\Gamma_{\mu\nu\rho}$:

$$q^\lambda \Gamma_{\lambda\mu\nu}(q,-k_1,-k_2)=k_2^2 t_{\mu\nu}(k_2)-k_1^2 t_{\mu\nu}(k_1). \quad (5.2)$$

The RHS of Eq. (5.2) vanishes after contracting with the polarization vectors, and employing the on-shell condition $k_1^2=k_2^2=0$. Again, by virtue of the BRS identities and the on-shell condition $k_1^2=k_2^2=0$, the dependence of $\mathrm{Im}\mathcal{M}$ on the parameters η_μ and η^2 cancels, and we eventually obtain

$$\begin{aligned}\mathrm{Im}\mathcal{M}&=\frac{1}{4}\mathcal{T}_{\mu\nu}P^{\mu\rho}(k_1,\eta_1)P^{\nu\sigma}(k_2,\eta_2)\mathcal{T}^*_{\rho\sigma}\\&=\frac{1}{4}(\mathcal{T}^{\mu\nu}\mathcal{T}^*_{\mu\nu}-2\mathcal{S}\mathcal{S}^*)\\&=\frac{1}{4}[(\mathcal{T}_s^F+\mathcal{T}_s^P+\mathcal{T}_t)^{\mu\nu}(\mathcal{T}_s^F+\mathcal{T}_s^P+\mathcal{T}_t)^*_{\mu\nu}-2\mathcal{S}\mathcal{S}^*],\end{aligned} \quad (5.3)$$

where

$$\mathcal{T}^{P,ab}_{s\mu\nu}=-gf^{abc}\frac{g^{\rho\lambda}}{q^2}\Gamma^P_{\lambda\mu\nu}(q,-k_1,-k_2)V^c_\rho. \quad (5.4)$$

At this point, one must recognize that due to the four-momenta of the trilinear vertex Γ^P inside $\mathcal{T}_s^P$ one can further trigger the EWI's, exactly as one did in order to derive from Eq. (4.9) the last step of Eq. (5.3). In fact, only the process-independent terms contained in $\mathrm{Im}\mathcal{M}$ will be projected out on account of the BRS identities of Eq. (5.1). It is important to emphasize that $\mathcal{T}_s^F$ and $\mathcal{T}_t$ do not contain any pinching momenta. This is particular to this example, where we have only two gluons as final states, but is not true for more gluons. To further exploit the EWI's derived from BRS symmetries, we rewrite the RHS of Eq. (5.3) in the following way (we omit the fully contracted Lorentz indices):

$$\begin{aligned}\mathrm{Im}\mathcal{M}&=\frac{1}{4}[(\mathcal{T}_t+\mathcal{T}_s^P+\mathcal{T}_s^F)(\mathcal{T}_t+\mathcal{T}_s^P+\mathcal{T}_s^F)^*-2\mathcal{S}\mathcal{S}^*]\\&=\frac{1}{4}[(\mathcal{T}_s^F\mathcal{T}_s^{F*}-\mathcal{T}_s^P\mathcal{T}_s^{P*}+\mathcal{T}_s^P\mathcal{T}^*+\mathcal{T}\mathcal{T}_s^{P*}-2\mathcal{S}\mathcal{S}^*)\\&\quad+(\mathcal{T}_t\mathcal{T}_s^{F*}+\mathcal{T}_s^F\mathcal{T}_t^*)+\mathcal{T}_t\mathcal{T}_t^*]\\&=\mathrm{Im}\widehat{\mathcal{M}}_1+\mathrm{Im}\widehat{\mathcal{M}}_2+\mathrm{Im}\widehat{\mathcal{M}}_3.\end{aligned} \quad (5.5)$$

In Eq. (5.5), the reader may recognize the rearrangement characteristic of the ''intrinsic'' PT, presented in [28].

Inserting the explicit form of $\mathcal{T}_s^P$ given in Eq. (5.4) into Eq. (5.5) and using the BRS identities,

$$\mathcal{T}_s^P\mathcal{T}^*=-2\mathcal{S}\mathcal{S}^*, \quad \mathcal{T}_s^P\mathcal{T}_s^{P*}=2\mathcal{S}\mathcal{S}^*, \quad (5.6)$$

we obtain

$$\begin{aligned}\mathrm{Im}\widehat{\mathcal{M}}_1&=\frac{1}{4}(\mathcal{T}_s^F\mathcal{T}_s^{F*}-\mathcal{T}_s^P\mathcal{T}_s^{P*}+\mathcal{T}_s^P\mathcal{T}^*+\mathcal{T}_s^{P*}\mathcal{T}-2\mathcal{S}\mathcal{S}^*)\\&=\frac{1}{4}(\mathcal{T}_s^F\mathcal{T}_s^{F*}-8\mathcal{S}\mathcal{S}^*),\end{aligned} \quad (5.7)$$

which is the same result found in the previous section, i.e., Eq. (4.24).

An interesting by-product of the above analysis is that one is able to show the independence of the PT results of the number of the external fermionic currents. An explicit proof of the process independence of the PT self-energy at one loop has already been presented in [10], through the detailed study of all possible combinations of on-shell states one can consider, including fermions, gluons, and scalars. In this analysis, the number and type of incoming (outgoing) particles is such that they can all merge to produce an off-shell gluon through only *one* elementary interaction vertex. Here, instead, we pay more attention to the independence of the results on the number of fermionic currents used to produce the intermediate two-gluon state. In such a case, several elementary vertices are needed in general. The main ingredient of the proof is the observation that the BRS identities in Eqs. (5.1), as well as those given in Eq. (5.6), will still hold for any transition amplitude of n-fermionic currents to two gluons. By analogy, one can decompose the transition amplitude into $\mathcal{T}_t$ and $\mathcal{T}_s$ structures. Similarly, the forms of the substructures $\mathcal{T}_s^F$ and $\mathcal{T}_s^P$ will then change accordingly. In fact, the only modification will be that the vector current V^c_ρ, contained in Eqs. (4.17) and (5.4), will now represent the transition of one gluon to n-fermionic currents. Making use of the ''intrinsic'' PT, one then obtains the result given in Eq. (5.7). Hence, we can conclude that the PT does not depend on the number of the external fermionic currents attached to gluons.

VI. THE ELECTROWEAK CASE

In this section, we will show how the same considerations apply directly to the case of the electroweak sector of the SM. We consider the charged current process $e^-\nu\to e^-\nu$ and assume that the electron mass m_e is nonzero, so that the external current is not conserved. We focus on the part of the amplitude which has a threshold at $q^2=M_W^2$ This corresponds the virtual process $W^-\to W^-\gamma$, where γ is the photon. From the OT, we have

$$\begin{aligned}\mathrm{Im}\langle e^-\nu|T|e^-\nu\rangle=\frac{1}{2}\int dX_{\mathrm{LIPS}}&\langle e^-\nu|T|W^-\gamma\rangle\\&\times\langle W^-\gamma|T|e^-\nu\rangle^*.\end{aligned} \quad (6.1)$$

We set again $\mathcal{M}=\langle e^-\nu|T|e^-\nu\rangle$ and $\mathcal{T}=\langle e^-\nu|T|W^-\gamma\rangle$. As in the case of QCD, the amplitude consists of two distinct parts, a part that contains an electron propagator [Fig. 4(a)] and a part that does not, which is shown in Figs. 4(b) and 4(c). As before, we denote them by $\mathcal{T}_t$ and $\mathcal{T}_s(\xi_w)$, respectively. We first define

$$V_L^\mu=\frac{g_w}{2\sqrt{2}}\overline{v}(p_2)\gamma^\mu(1-\gamma_5)u(p_1) \quad (6.2)$$

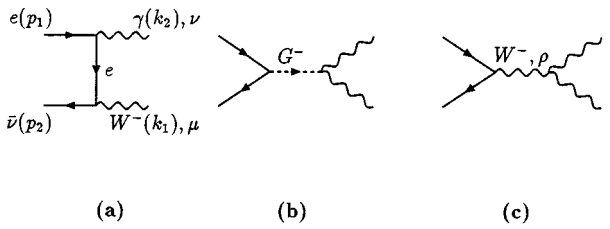

FIG. 4. Amplitudes contributing to the reaction $e^- \overline{\nu} \to W^- \gamma$.

and

$$S_R = \frac{g_w}{2\sqrt{2}} \frac{m_e}{M_W} \overline{v}(p_2)(1+\gamma_5)u(p_1). \tag{6.3}$$

Clearly, one has the EWI

$$q_\mu V_L^\mu = M_W S_R . \tag{6.4}$$

The amplitude $\mathcal{T}_s$ can then be written down in the closed form

$$\mathcal{T}_{s\mu\nu}(\xi_w) = iV_L^\lambda \Delta_{0\lambda}^{(\xi_w),\rho}(q) \Gamma_{\nu\rho\mu}^{\gamma W^- W^+} + iS_R D_0^{(\xi_w)}(q) \Gamma_{\nu\mu}^{\gamma G^- W^+}, \tag{6.5}$$

where $D_0^{(\xi_w)}(q) = 1/(q^2 - \xi_w M_W^2)$, $\Gamma_{\nu\rho\mu}^{\gamma W^- W^+} = e\Gamma_{\nu\rho\mu}(-k_2, q, -k_1)$ is the tree-level $\gamma W^- W^+$ vertex, and $\Gamma_{\nu\mu}^{\gamma G^- W^+} = eM_W g_{\mu\nu}$ is the tree-level $\gamma G^- W^+$ vertex. In the expression (6.5), we explicitly display the dependence on the GFP ξ_w. In addition, the amplitude $\mathcal{T}_t$ is given by

$$\mathcal{T}_t^{\mu\nu} = \frac{ieg_w}{2\sqrt{2}} \overline{v}(p_2) \gamma^\mu (1-\gamma_5) \frac{1}{\not p_1 - \not k_2 - m_e} \gamma^\nu u(p_1). \tag{6.6}$$

Note that $\mathcal{T}_t^{\mu\nu}$ does not depend on ξ_w. Denoting by k_1 the four-momentum of the W and by k_2 that of the photon, Eq. (6.1) becomes

$$\mathrm{Im}\mathcal{M} = \mathcal{T}_{\mu\nu} Q^{\mu\rho}(k_1) P^{\nu\sigma}(k_2, \eta) \mathcal{T}_{\rho\sigma}^* , \tag{6.7}$$

where $P^{\mu\nu}$ is the photon polarization tensor given in Eq. (4.11), and

$$Q^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2} \tag{6.8}$$

is the W polarization tensor. The polarization tensor $Q^{\mu\nu}(k)$ shares the property that, on shell, i.e., for $k^2 = M_W^2$ $k^\mu Q_{\mu\nu}(k) = 0$. Furthermore, in Eq. (6.7), we omit the integration measure $1/2 \int dX_{\mathrm{LIPS}}$.

First, we will show how the dependence on the GFP ξ_w cancels. To that end, we employ the usual decomposition

$$\Delta_{0\mu\nu}^{(\xi_w)}(q) = U_{\mu\nu}(q) - \frac{q_\mu q_\nu}{M_W^2} D_0^{(\xi_w)}(q^2), \tag{6.9}$$

the EWI

$$q^\rho \Gamma_{\nu\rho\mu}^{\gamma W^- W^+}(-k_2, q, -k_1) Q^{\mu\lambda}(k_1) P^{\nu\sigma}(k_2, \eta) = M_W \Gamma_{\mu\nu}^{\gamma G^- W^+} Q^{\mu\lambda}(k_1) P^{\nu\sigma}(k_2, \eta), \tag{6.10}$$

and the EWI of Eq. (6.4), and we obtain the following ξ_w-independent expression for $\mathcal{T}_s^{\mu\nu}$:

$$\mathcal{T}_s^{\mu\nu} = ieV_L^\lambda U_{\lambda\rho}(q) \Gamma^{\nu\rho\mu}(-k_2, q, -k_1) = ieV_L^\lambda U_{\lambda\rho}(q) \Gamma^{F,\nu\rho\mu}(-k_2, q, -k_1) = \mathcal{T}_s^{F,\mu\nu}, \tag{6.11}$$

where contraction over the polarization tensors $Q_{\mu\nu}$ and $P_{\mu\nu}$ is implied. In the last step of Eq. (6.11), we have used the fact that the Γ^P part of the vertex, defined in Eq. (4.12), vanishes when contracted with the polarization tensors.

Next, following arguments analogous to the QCD case, it is straightforward to show that any dependence on the four-vector η_μ and the parameter η^2 vanishes, and that finally $\mathrm{Im}\mathcal{M}$ takes on the form

$$\begin{aligned} \mathrm{Im}\mathcal{M} &= -(\mathcal{T}_s^F + \mathcal{T}_t)_{\mu\nu} Q^{\mu\rho}(k_1) (\mathcal{T}_s^F + \mathcal{T}_t)_\rho^{*\nu} \\ &= (\mathcal{T}_s^F + \mathcal{T}_t)^{\mu\nu} (\mathcal{T}_s^F + \mathcal{T}_t)_{\mu\nu}^* \\ &\quad - (\mathcal{T}_s^F + \mathcal{T}_t)^{\mu\nu} \frac{k_{1\mu} k_1^\rho}{M_W^2} (\mathcal{T}_s^F + \mathcal{T}_t)_{\rho\nu}^* \\ &= \mathrm{Im}\mathcal{M}^a + \mathrm{Im}\mathcal{M}^b . \end{aligned} \tag{6.12}$$

The absorptive subamplitude $\mathrm{Im}\mathcal{M}^a$ consists of three terms:

$$\begin{aligned} \mathrm{Im}\mathcal{M}^a &= \mathcal{T}_s^F \mathcal{T}_s^{F*} + (\mathcal{T}_s^F \mathcal{T}_t^* + \mathcal{T}_t \mathcal{T}_s^{F*}) + \mathcal{T}_t \mathcal{T}_t^* \\ &= \mathrm{Im}\widehat{\mathcal{M}}_1^a + \mathrm{Im}\widehat{\mathcal{M}}_2^a + \mathrm{Im}\widehat{\mathcal{M}}_3^a . \end{aligned} \tag{6.13}$$

The first term $\mathrm{Im}\widehat{\mathcal{M}}_1^a$ can easily be identified with a propagatorlike contribution. In particular, using Eq. (4.25), we find

$$\mathrm{Im}\widehat{\mathcal{M}}_1^a = e^2 V_L^\rho U_{\rho\mu}(q) [-8q^2 t^{\mu\nu}(q) + 4(k_1 - k_2)^\mu (k_1 - k_2)^\nu] U_{\nu\lambda}(q) V_L^\lambda . \tag{6.14}$$

The amplitudes $\mathrm{Im}\widehat{\mathcal{M}}_2^a$ and $\mathrm{Im}\widehat{\mathcal{M}}_3^a$ are vertex- and boxlike contributions, respectively, and they will not be considered any further here.

We must now isolate the corresponding propagatorlike piece from $\mathrm{Im}\mathcal{M}^b$. It is easy to find the relation

$$k_1^\mu \mathcal{T}_{s\,\mu\nu}^F = -ieV_{L\nu} - ieV_{L\lambda} U^{\lambda\rho}(q) [(k_1 - k_2)_\rho k_{2\nu} - 2M_W^2 g_{\rho\nu}]. \tag{6.15}$$

In addition, we evaluate the EWI

$$\begin{aligned} k_1^\mu \mathcal{T}_{t\mu\nu} &= ieV_{L\nu} + M_W \frac{ieg_w m_e}{2\sqrt{2} M_W} \overline{v}(p_2)(1+\gamma_5) \\ &\quad \times \frac{1}{\not p_1 - \not k_2 - m_e} \gamma_\nu u(p_1) \\ &= ieV_{L\nu} + M_W \mathcal{L}_\nu , \end{aligned} \tag{6.16}$$

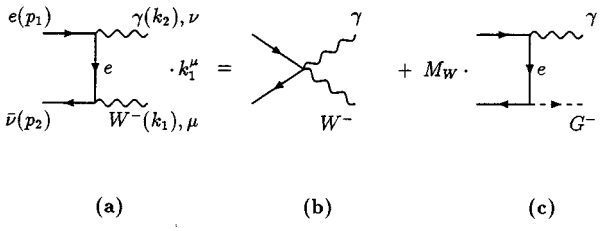

FIG. 5. Elementary BRS identity for the e-dependent amplitude $\mathcal{T}_t^{\mu\nu}$

which is shown diagrammatically in Fig. 5. Adding Eqs. (6.15) and (6.16) by parts, we obtain

$$k_1^\mu(\mathcal{T}_s^F+\mathcal{T}_t)_{\mu\nu}=-ieV_{L\lambda}U^{\lambda\rho}(q)[(k_1-k_2)_\rho k_{2\nu}-2M_W^2 g_{\rho\nu}] + M_W\mathcal{L}_\nu \tag{6.17}$$

and after a straightforward calculation, we eventually arrive at

$$\begin{aligned}\mathrm{Im}\mathcal{M}^b=&-e^2V_{L\rho}U^{\rho\mu}(q)\\ &\times[4M_W^2 g_{\mu\nu}+2(k_1-k_2)_\mu(k_1-k_2)_\nu]U^{\nu\lambda}(q)V_{L\lambda}\\ &-2ieM_W[V_{L\rho}U^{\rho\nu}(q)\mathcal{L}_\nu^*-\mathcal{L}_\nu U^{\nu\lambda}(q)V_{L\lambda}]-\mathcal{L}^\nu\mathcal{L}_\nu^*\\ =&\,\mathrm{Im}\widehat{\mathcal{M}}_1^b+\mathrm{Im}\widehat{\mathcal{M}}_2^b+\mathrm{Im}\widehat{\mathcal{M}}_3^b.\end{aligned} \tag{6.18}$$

Adding the two propagatorlike parts $\mathrm{Im}\widehat{\mathcal{M}}_1^a$ and $\mathrm{Im}\widehat{\mathcal{M}}_1^b$ from Eqs. (6.14) and (6.18), respectively, we find

$$\begin{aligned}\mathrm{Im}\widehat{\mathcal{M}}_1=&\,\mathrm{Im}\widehat{\mathcal{M}}_1^a+\mathrm{Im}\widehat{\mathcal{M}}_1^b\\ =&\,e^2V_L^\rho U_{\rho\mu}(q)[-8q^2t^{\mu\nu}(q)-4M_W^2 g^{\mu\nu}\\ &+2(k_1-k_2)^\mu(k_1-k_2)^\nu]U_{\nu\lambda}(q)V_L^\lambda.\end{aligned} \tag{6.19}$$

Next, we carry out the phase-space integration over $1/2\int dX_{\mathrm{LIPS}}$, using the formulas given in Eqs. (3.5) and (3.6), and the fact that $\lambda^{1/2}(q^2,M_W^2,0)=q^2-M_W^2>0$. In this way, we have

$$\mathrm{Im}\widehat{\mathcal{M}}_1=V_{L\rho}U^{\rho\mu}(q)\mathrm{Im}\hat{\Pi}_{\mu\nu}^W U^{\nu\lambda}(q)V_{L\lambda}, \tag{6.20}$$

with

$$\begin{aligned}\mathrm{Im}\hat{\Pi}_{\mu\nu}^W(q)=&\,\mathrm{Im}\hat{\Pi}_T^W(q^2)t_{\mu\nu}(q)+\mathrm{Im}\hat{\Pi}_L^W(q^2)\ell_{\mu\nu}(q),\\ \mathrm{Im}\hat{\Pi}_T^W(q^2)=&\,\frac{\alpha_{em}}{2}(q^2-M_W^2)\\ &\times\left(-\frac{11}{3}+\frac{4M_W^2}{3q^2}+\frac{M_W^4}{3q^4}\right),\\ \mathrm{Im}\hat{\Pi}_L^W(q^2)=&\,\frac{\alpha_{em}}{2}(q^2-M_W^2)\left(-\frac{2M_W^2}{q^2}+\frac{M_W^4}{q^4}\right).\end{aligned} \tag{6.21}$$

Here, $\alpha_{em}=e^2/(4\pi)$ is the electromagnetic fine structure constant. The real part of the transverse, on-shell renormalized, W-boson self-energy, $\mathrm{Re}\hat{\Pi}_T^{W,R}(s)$, can be determined by means of a doubly subtracted DR given in Eq. (2.7). Furthermore, we have to assume a fictitious photon mass μ_γ in order to regulate the infrared (IR) divergences. More explicitly, the DR of our interest reads

$$\begin{aligned}\mathrm{Re}\hat{\Pi}_T^{W,R}(s)=&\,\mathrm{Re}\hat{\Pi}_T^W(s)\\ &-(s-M_W^2)\mathrm{Re}\hat{\Pi}_T^{W\prime}(M_W^2)-\mathrm{Re}\hat{\Pi}_T^W(M_W^2)\\ =&\lim_{\Lambda\to\infty}\lim_{\mu_\gamma\to0}\frac{(s-M_W^2)^2}{\pi}\mathrm{P}\\ &\times\int_{(M_W+\mu_\gamma)^2}^{\Lambda}\frac{ds'\,\mathrm{Im}\hat{\Pi}_T^W(s')}{(s'-M_W^2)^2(s'-s)}.\end{aligned} \tag{6.22}$$

To obtain the analytic form of $\mathrm{Re}\hat{\Pi}_T^{W,R}(s)$, we first evaluate the integrals

$$\begin{aligned}F_0(s)=&(s-M_W^2)\mathrm{P}\int_{(M_W+\mu_\gamma)^2}^{\infty}ds'\,\frac{1}{(s'-M_W^2)(s'-s)}\\ =&-\ln\left(\frac{|s-M_W^2|}{2M_W\mu_\gamma}\right),\end{aligned} \tag{6.23}$$

$$\begin{aligned}F_1(s)=&(s-M_W^2)\mathrm{P}\int_{(M_W+\mu_\gamma)^2}^{\infty}ds'\,\frac{1}{(s'-M_W^2)(s'-s)}\,\frac{M_W^2}{s'}\\ =&-\frac{M_W^2}{s}\ln\left(\frac{|s-M_W^2|}{2M_W\mu_\gamma}\right)-\left(1-\frac{M_W^2}{s}\right)\ln\left(\frac{M_W}{2\mu_\gamma}\right),\end{aligned} \tag{6.24}$$

$$\begin{aligned}F_2(s)=&(s-M_W^2)\mathrm{P}\int_{(M_W+\mu_\gamma)^2}^{\infty}ds'\,\frac{1}{(s'-M_W^2)(s'-s)}\,\frac{M_W^4}{s'^2}\\ =&-\frac{M_W^4}{s^2}\ln\left(\frac{|s-M_W^2|}{2M_W\mu_\gamma}\right)-\ln\left(\frac{M_W}{2\mu_\gamma}\right)+1-\frac{M_W^2}{s},\end{aligned} \tag{6.25}$$

Armed with the integrals defined in Eqs. (6.23)–(6.25), one then obtains

$$\mathrm{Re}\hat{\Pi}_T^W(s)=\frac{\alpha_{em}}{2}(s-M_W^2)\left(-\frac{11}{3}F_0+\frac{4}{3}F_1+\frac{1}{3}F_2\right). \tag{6.26}$$

Equation (6.26) coincides with the PT W-boson self-energy [8] or equivalently, with the W-boson self-energy computed in the BFG for $\xi_Q=1$ [17].

VII. CUTKOSKY CONSIDERATIONS

In this section, we focus on the LHS of the OT and present a different point of view and a self-consistency check. In particular, we consider the one-loop S-matrix element for a given process and compute its imaginary part by direct application of the Cutkosky rules. The expressions so obtained consist of the product of the tree-level amplitudes, with the important difference that now ''unphysical'' degrees of freedom appear as intermediate states, giving, in turn, rise to ''unphysical'' thresholds. These tree-level amplitudes are related by EWI's. We show that, when fully

exploited, these EWI's give rise to propagator-, vertex-, and boxlike expressions, which contain physical thresholds only, whereas all the unphysical thresholds disappear completely. The expressions so derived are identical to the imaginary parts of the corresponding PT Green's functions, which one can obtain directly from the S matrix. Also, both real and imaginary parts are related via a DR, as has been discussed in Sec. II.

For the rest of this section, we focus on the particular process $l\nu_l \to W^- H$, corresponding to the virtual process $W \to WH$. We choose this process since it captures all the crucial field-theoretical features involved, without being excessively cumbersome from the calculational point of view. In addition, by treating a process other than the technically more involved $e^-\nu \to e^-\nu$ of the previous section, we want to emphasize the general validity of our method. A lengthier but straightforward calculation can convince the reader that one arrives at precisely the same physical conclusions also in the case of the virtual $W \to W\gamma$ process, or any other process for that matter.

For the process $l\nu_l \to W^-(p)H(p_H)$, we have in an arbitrary ξ gauge

$$\frac{p^\mu}{M_W} T^{(\xi)}_{(a)\mu} = T^{(\xi)}_{(b)} + \frac{ig_w}{2M_W} S_R, \tag{7.1}$$

$$\frac{p^\mu}{M_W} T^{(\xi)}_{(c)\mu} = T^{(\xi)}_{(d)} - \frac{ig_w}{2M_W} S_R. \tag{7.2}$$

We will carry out an explicit calculation of the $\mathrm{Im}\hat{\mathcal{M}}_1$ of the process $e\overline{\nu_e} \to e\overline{\nu_e}$ at the one-loop electroweak order, working on the LHS of the OT. To simplify the algebra, we will assume that only the W and H particles can come kinematically on the mass shell, as shown in Fig. 6. In what follows, we omit the common integration measure of the loop, $1/[2(2\pi)^4]\int d^4p\, d^4p_H \delta^{(4)}(p_H + p - p_e - p_\nu)$. Then, the absorptive amplitude $\mathrm{Im}\mathcal{M}$ for the aforementioned process may be written as (suppressing contraction over Lorentz indices, and using the on-shell conditions $p_H^2 = M_H^2$ $p^2 = M_W^2$)

$$\begin{aligned}\mathrm{Im}\mathcal{M} = \widetilde{\Delta}_{0H}(p_H)[& T^{(\xi)}_{(a)} \widetilde{\Delta}^{(\xi)}_0(p) T^{(\xi)*}_{(a)} + T^{(\xi)}_{(b)} \widetilde{D}^{(\xi)}_0(p) T^{(\xi)*}_{(b)} \\ & + T^{(\xi)}_{(c)} \widetilde{\Delta}^{(\xi)}_0(p) T^{(\xi)*}_{(a)} + T^{(\xi)}_{(a)} \widetilde{\Delta}^{(\xi)}_0(p) T^{(\xi)*}_{(c)} \\ & + T^{(\xi)}_{(d)} \widetilde{D}^{(\xi)}_0(p) T^{(\xi)*}_{(b)} + T^{(\xi)}_{(b)} \widetilde{D}^{(\xi)}_0(p) T^{(\xi)*}_{(d)} \\ & + T^{(\xi)}_{(c)} \widetilde{\Delta}^{(\xi)}_0(p) T^{(\xi)*}_{(c)} + T^{(\xi)}_{(d)} \widetilde{D}^{(\xi)}_0(p) T^{(\xi)*}_{(d)}],\end{aligned} \tag{7.3}$$

where the tilde acting on the tree-level propagators simply projects out the corresponding absorptive parts. Such a projection can effectively be obtained by applying the Cutkosky rules. More explicitly, we have

$$\widetilde{\Delta}_{0H}(p_H) = 2\pi \delta_+(p_H^2 - M_H^2), \tag{7.4}$$

$$\widetilde{D}^{(\xi)}_0(p) = 2\pi \delta_+(p^2 - \xi M_W^2), \tag{7.5}$$

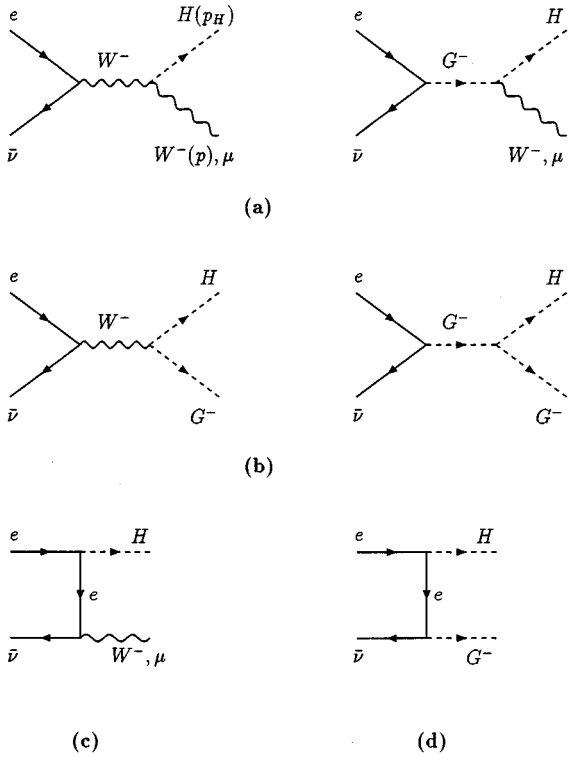

FIG. 6. Graphs contributing to the amplitudes $T^{(\xi)}_{(a)\mu}$, $T^{(\xi)}_{(b)}$, $T_{(c)}$, and $T_{(d)}$.

$$\begin{aligned}\widetilde{\Delta}^{(\xi)}_{0\mu\nu}(p) &= 2\pi \left[Q_{\mu\nu}(p)\, \delta_+(p^2 - M_W^2) \right. \\ &\quad \left. - \frac{p_\mu p_\nu}{M_W^2} \delta_+(p^2 - \xi M_W^2) \right] \\ &= \widetilde{U}_{\mu\nu}(p) - \frac{p_\mu p_\nu}{M_W^2} \widetilde{D}^{(\xi)}_0(p),\end{aligned} \tag{7.6}$$

where the W-boson polarization tensor $Q_{\mu\nu}(p)$ is given in Eq. (6.8) and $\delta_+(p^2 - M^2) = \delta(p^2 - M^2)\theta(p^0)$. After identifying the PT piece, $T_P = -ig_w S_R/(2M_W)$, which is obtained from Eq. (7.2) each time the $p^\mu p^\nu$-dependent part of $\widetilde{\Delta}^{(\xi)}_{0\mu\nu}$ gets contracted with $T^{(\xi)}_{(c)}$, we observe that the imaginary propagatorlike part may be decomposed as

$$\mathrm{Im}\hat{\mathcal{M}}_1 = \mathrm{Im}\hat{\mathcal{M}}_1^{(\mathrm{phys})} + \delta\hat{\mathcal{M}}_1, \tag{7.7}$$

where

$$\begin{aligned}\mathrm{Im}\hat{\mathcal{M}}_1^{(\mathrm{phys})} &= \widetilde{\Delta}_{0H}(p_H)(2\pi)\delta_+(p^2 - M_W^2) \\ &\quad \times \left(T^{(\xi)}_{(a)\mu} Q^{\mu\nu}(p) T^{(\xi)*}_{(a)\nu} + T_P \frac{p^\nu}{M_W} T^{(\xi)*}_{(a)\nu} \right. \\ &\quad \left. + T^{(\xi)}_{(a)\lambda} \frac{p^\lambda}{M_W} T_P^* + T_P T_P^* \right)\end{aligned} \tag{7.8}$$

and

$$\delta\hat{\mathcal{M}}_1 = -\widetilde{\Delta}_{0H}(p_H)\widetilde{D}_0^{(\xi)}(p)\Bigg(T^{(\xi)}_{(a)\lambda}\frac{p^\lambda p^\nu}{M_W^2}T^{(\xi)*}_{(a)\nu} - T^{(\xi)}_{(b)}T^{(\xi)*}_{(b)} + T_P\frac{p^\nu}{M_W}T^{(\xi)*}_{(a)\nu} + T^{(\xi)}_{(a)\lambda}\frac{p^\lambda}{M_W}T^*_P + T_P T^*_P\Bigg). \tag{7.9}$$

In the first term $\mathrm{Im}\hat{\mathcal{M}}_1^{(\mathrm{phys})}$ we have collected all contributions originating from the physical poles at $p_H^2 = M_H^2$ and $p^2 = M_W^2$, whereas all those occurring at $p^2 = \xi M_W^2$ and are proportional to $\widetilde{D}_0^{(\xi)}(p)$ are included in $\delta\hat{\mathcal{M}}_1$.

The first important observation is that $\delta\hat{\mathcal{M}}_1 = 0$, which can be shown with the help of the EWI in Eq. (7.1). So, the full exploitation of this WI gives rise to a propagatorlike imaginary part where all unphysical thresholds have been canceled. In addition, with the help of the same WI, we obtain, for $\mathrm{Im}\hat{\mathcal{M}}_1^{(\mathrm{phys})}$,

$$\mathrm{Im}\hat{\mathcal{M}}_1 = \mathrm{Im}\hat{\mathcal{M}}_1^{(\mathrm{phys})} = \frac{1}{2}\int dX_{\mathrm{LIPS}}(-T^{(\xi)}_{(a)}T^{(\xi)*}_{(a)} + T^{(\xi)}_{(b)}T^{(\xi)*}_{(b)}). \tag{7.10}$$

We must now demonstrate that the final dependence on ξ cancels in the above equation. Note that even though we use the on-shell conditions $p^2 = M_W^2$ and $p_H^2 = M_H^2$, the amplitudes T in the last equation are *not* really ''on shell,'' because they are *not* contracted by the corresponding polarization vectors; therefore, the ξ cancellation is not immediate. To verify the cancellation, we must employ the identity of Eq. (6.9) to decompose the internal tree-level W propagators, and the WI's, which relate the tree-level vertices involved: i.e.,

$$q^\nu \Gamma^{HW^+W^-}_{0\mu\nu} = -M_W\Gamma^{HW^+G^-}_{0\mu} + \frac{ig_w}{2}M_W p_\mu,$$

$$q^\nu \Gamma^{HG^+W^-}_{0\nu} = -M_W\Gamma^{HG^+G^-}_{0} - \frac{ig_w}{2}M_W^2. \tag{7.11}$$

Thus, the final expression can be cast into the form

$$\mathrm{Im}\hat{\mathcal{M}}_1 = \frac{1}{2}\int dX_{\mathrm{LIPS}}(-T^{(\infty)}_{(a_1)}T^{(\infty)*}_{(a_1)} + T^{(\infty)}_{(b_1)}T^{(\infty)*}_{(b_1)}), \tag{7.12}$$

where the index a_1 (b_1) denotes the first graph in Fig. 6(a) 6(b), and the superscript ''∞'' means that the internal tree-level W propagators are in the unitary gauge.

This is precisely what one would obtain from the straightforward computation of the imaginary part of the one-loop PT WW self-energy, presented in [8]. The expression for the GFP-independent propagatorlike part of $\hat{\mathcal{M}}$, $\hat{\mathcal{M}}_1$, in terms of the PT WW self-energy, $\hat{\Pi}_{\mu\nu}(q)$, is given by

$$\hat{\mathcal{M}}_1 = V_{L\sigma}U^{\sigma\mu}(q)\hat{\Pi}_{\mu\nu}(q)U^{\nu\rho}(q)V_{L\rho}. \tag{7.13}$$

The Higgs-dependent part of $\hat{\Pi}_{\mu\nu}$, call it $\hat{\Pi}^{(HW)}_{\mu\nu}$ is given by [29]

$$\hat{\Pi}^{(HW)}_{\mu\nu}(q) = \pi\alpha_w\int\frac{d^nk}{i(2\pi)^n}I(q,k) \times[(2k+q)_\mu(2k+q)_\nu - 4M_W^2 g_{\mu\nu}], \tag{7.14}$$

where $\alpha_w = g_w^2/(4\pi)$ is the $\mathrm{SU}(2)_L$ fine structure constant and

$$I(q,k) = \frac{1}{(k^2 - M_W^2)[(k+q)^2 - M_H^2]}. \tag{7.15}$$

It is now easy to see that the imaginary part of $\hat{\Pi}^{(HW)}_{\mu\nu}$ is indeed equal to Eq. (7.12). This can be verified by an explicit application of the Cutkosky rules on the expression in the RHS of Eq. (7.14). Actually, this amounts to determining where the logarithmic terms, which are obtained after the integration over the virtual momenta, turn negative. One could then compare that result with the result we will obtain after integrating Eq. (7.1) over the phase-space integral given above. To that end, we must make use of the fact that the typical integral over the Feynman parameter x

$$\mathrm{Im}\left[\int\frac{d^nk}{i(2\pi)^n}I(q,k)\right] = -\frac{1}{16\pi^2}\mathrm{Im}\left\{\int_0^1 dx\,\ln[M_H^2 x + M_W^2(1-x) - q^2x(1-x)]\right\} = \frac{\theta[q^2-(M_W+M_H)^2]}{8\pi q^2}\lambda^{1/2}(q^2,M_H^2,M_W^2) = \frac{1}{2}\int dX_{\mathrm{LIPS}}. \tag{7.16}$$

The above relation gives an explicit connection between Cutkosky rules and the two-body LIPS given in Eq. (3.5). As has been discussed in Sec. II, the analytic continuation of the logarithmic function in the RHS of Eq. (7.16) is uniquely determined via the prescription $s \to s + i\varepsilon$.

It is important to emphasize the conclusions of this section: We have proceeded in two different ways. First, we have calculated the propagatorlike imaginary part by applying the Cutkosky rule, and exploiting the tree-level EWI's. Then, we have computed the imaginary part of the one-loop PT W self-energy, obtained by the usual S-matrix PT rules. The two analytic results have turned out to be identical. We can, therefore, conclude that the PT Green's functions, contrary to their conventional counterparts, satisfy individually the OT. We consider that a crucial point for the success of our resummation algorithm. In addition, the above analysis demonstrates that one can work freely on either side of the OT and arrive at a unique result, just by following the same rules, i.e., by fully exploiting the EWI's of the theory.

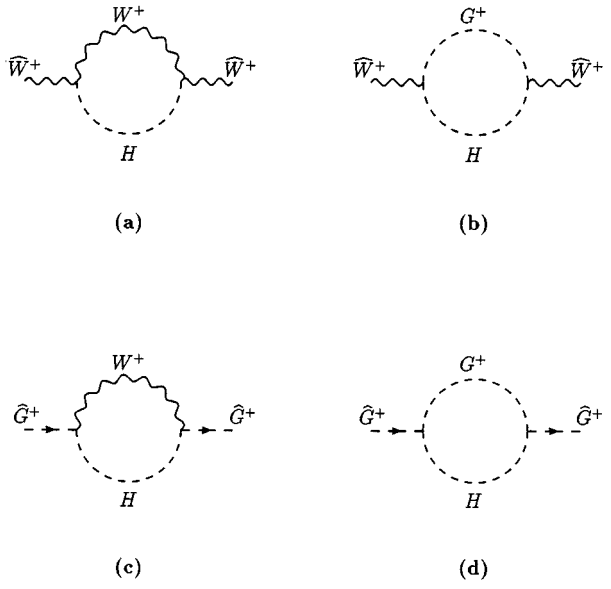

FIG. 7. *WH* contributions to $\Pi_{\mu\nu}^{\hat{W}^+x\hat{W}^+}$ [(a),(b)] and $\Pi_{\mu\nu}^{\hat{G}^+\hat{G}^+}$ [(c),(d)].

VIII. THE BACKGROUND FIELD GAUGE

The formulation of non-Abelian gauge field theories in the framework of the BFG endows the n-point functions obtained from the generating functional with a number of characteristic properties. Most remarkably, the BFG n-point functions satisfy the *tree-level* WI's, to all orders in perturbation theory. This fact is to be contrasted with the Slavnov-Taylor identities of the conventional covariant formulation, where the tree-level WI are spoiled by the appearance of ''ghost'' Green's function, as soon as quantum corrections are introduced. On the other hand, the BFG n-point functions display, in general, a residual dependence on the quantum GFP ξ_Q, which is used to ''gauge fix'' the gauge fields inside the quantum loops. As we will show in this section, the functional dependence of the BFG two-point functions on ξ_Q is such that it leads to the appearance of *unphysical thresholds*, at $q^2=\xi_Q M^2$.

What is rather striking in this context is the following observation. Consider a BFG two-point function computed at one loop at some arbitrary ξ_Q. Let us then separate it into two parts: the part that has only physical thresholds (at $q^2=M^2$) and the part that has unphysical thresholds (at $q^2=\xi_Q M^2$). Interestingly enough, one finds that each part satisfies *separately* the correct tree-level WI.

Defining I_Q as

$$I_Q(q,k)=\frac{1}{(k^2-\xi_Q M_W^2)[(k+q)^2-M_H^2]} \tag{8.1}$$

and using the identity

$$\frac{1-\xi_Q}{(k^2-M_W^2)(k^2-\xi_Q M_W^2)}=\frac{1}{M_W^2}\left[\frac{1}{k^2-M_W^2}-\frac{1}{k^2-\xi_Q M_W^2}\right], \tag{8.2}$$

we have, for the Feynman diagrams (a) and (b) in Fig. 7 (loop integration, $\int d^nk/i(2\pi)^n$, implied),

$$\text{(a)}=g_w^2 M_W^2\left[\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{M_W^2}\right)I(q,k)-\frac{k_\mu k_\nu}{M_W^2}I_Q(q,k)\right],$$

$$\text{(b)}=\frac{g_w^2}{4}(2k+q)_\mu(2k+q)_\nu I_Q(q,k), \tag{8.3}$$

from which follows that

$$\Pi_{\mu\nu}^{(HW)}(q)=g_w^2 M_W^2\left[\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{M_W^2}\right)I(q,k)+\frac{1}{4M_W^2}\times[(2k+q)_\mu(2k+q)_\nu-4k_\mu k_\nu]I_Q(q,k)\right]$$

$$=\overline{\Pi}_{\mu\nu}(q)+\Pi_{\mu\nu}^Q(q), \tag{8.4}$$

where $\overline{\Pi}_{\mu\nu}$ contains only physical thresholds, at $q^2=(M_W+M_H)^2$, whereas $\Pi_{\mu\nu}^Q$ contains unphysical thresholds at $q^2=(\sqrt{\xi_Q}M_W+M_H)^2$. Similarly, from Figs. 7(c) and 7(d), we calculate

$$\text{(c)}=g_w^2 q^\rho q^\sigma\left[\left(-g_{\rho\sigma}+\frac{k_\rho k_\sigma}{M_W^2}\right)I(q,k)-\frac{k_\rho k_\sigma}{M_W^2}I_Q(q,k)\right], \tag{8.5}$$

$$\text{(d)}=\frac{g_w^2}{4M_W^2}(M_H^2-\xi_Q M_W^2)^2 I_Q(q,k), \tag{8.6}$$

and so

$$\Omega^{(HW)}(q)=g_w^2\left[\frac{(qk)^2}{M_W^2}-q^2\right]I(q,k)+g_w^2\left[\frac{(M_H^2-\xi_Q M_W^2)^2}{4M_W^2}-\frac{(qk)^2}{M_W^2}\right]I_Q(q,k)$$

$$=\overline{\Omega}(q)+\Omega^Q(q). \tag{8.7}$$

It is elementary to check that up to irrelevant tadpole terms, the following WI's hold:

$$q^\mu q^\nu\overline{\Pi}_{\mu\nu}(q)-M_W^2\overline{\Omega}(q)=0 \tag{8.8}$$

and

$$q^\mu q^\nu\Pi_{\mu\nu}^Q(q)-M_W^2\Omega^Q(q)=0. \tag{8.9}$$

It is worth noticing that the tree-level WI's, Eqs. (8.8) and (8.9), are *individually* satisfied by the contributions having physical and gauge-dependent unphysical thresholds, respectively. This property is not an accidental feature of the specific example considered above, but, as we will argue in a moment, it must be valid for any individual contribution to an *analytic* two-point correlation function. On the other hand, it is obvious that neither $\overline{\Pi}$ nor Π^Q can be obtained from a specific choice of the ξ_Q value. An exception to this is the value $\xi_Q=1$. In this gauge, the physical and unphysical sectors are not distinguishable. If we impose the constraint of the absence of unphysical thresholds in the BFG, a property which is always preserved within the PT framework [4], then the two-point correlation functions of the PT and

the BFG for $\xi_Q=1$ have to coincide at one loop. This feature should also hold true for all n-point functions at one loop.

In the following, we argue that the reason which forces $\overline{\Pi}_{\mu\nu}(q)$ and $\Pi^{Q}_{\mu\nu}(q)$ to satisfy individually the same treelike WI's as those of the full $\Pi_{\mu\nu}(q)$, is the analyticity of $\Pi_{\mu\nu}(q)$. In fact, it is sufficient to show that $\mathrm{Im}\Pi_{\mu\nu}(q)=\mathrm{Im}\overline{\Pi}_{\mu\nu}(q)\neq 0$ for a finite domain of q^2 (for $\xi_Q\neq 1$). Then, Eq. (8.8) will be valid for the finite kinematic region and will also hold true for any q^2, since $\mathrm{Im}\overline{\Pi}_{\mu\nu}$ is analytic. That $\mathrm{Re}\overline{\Pi}_{\mu\nu}$ will also satisfy Eq. (8.8) is guaranteed through a DR. Finally, it is evident that $\Pi^{Q}_{\mu\nu}(q)=\Pi_{\mu\nu}(q)-\overline{\Pi}_{\mu\nu}(q)$ will obey the same WI (8.9).

To give a specific example, let us consider the absorptive part of the WW self-energy in the BFG at one loop, in which only the $W\gamma$ contributions are considered. It is clear that, for the finite domain $M_W^2\leq q^2<\min[\xi_Q M_W^2,(M_W+M_Z)^2]$ ($\xi_Q\neq 1$), $\mathrm{Im}\Pi_{\mu\nu}(q)=\mathrm{Im}\Pi^{(\gamma W)}_{\mu\nu}(q)$. The latter leads to the fact that $\overline{\Pi}^{(\gamma W)}_{\mu\nu}(q)$ satisfies Eq. (8.8) independently, for any q^2. Similar arguments can carry over to the other distinct threshold contributions.

IX. ISSUES OF UNIQUENESS

In this section, we will address issues related to the uniqueness of the PT rearrangement. We know that the PT rearrangement gives rise to effective self-energies ($\hat{\Pi}$), vertices ($\hat{\Gamma}$), and box graphs ($\hat{B}$), endowed with several characteristic properties. The question naturally arises whether these effective Green's functions are unique. By ''unique'' we mean, whether after the PT rearrangement has been completed, one could still define new Green's functions, by moving GFP-independent terms around, in such a way as (i) the new Green's functions have the same properties with the old ones, and (ii) the above reshuffling does not change the unique value of the S matrix, order by order, in perturbation theory.

In what follows, we will show a ''mild'' version of uniqueness, namely, that the one-loop PT effective Green's functions are unique, provided the following is true.

(i) The PT procedure can be generalized to higher orders in perturbation theory, as described in [4]. In particular, we assume that effective GFP-independent Green's functions can be constructed, satisfying the simple QED-like WI known from the one-loop explicit constructions, and that the effective self-energies so constructed can be Dyson resummed. Regarding the last point, the resummation algorithm proposed in [4] not only is inextricably connected to the fact that the PT self-energies do not shift the position of the pole [4], but has already passed another nontrivial consistency check [30]; still, one has not conclusively shown its validity for the most general of cases.

(ii) The renormalization has been successfully carried out, giving rise to UV-finite effective PT Green's functions. This assumption is crucial, and is the main reason why we characterize the uniqueness proved here as ''mild.'' Things may be different if one attempts the aforementioned reshuffling *before* renormalization, but this will not concern us in the present work.

It is known [7] that the PT self-energy in QCD, $\hat{\Pi}(q^2)$ (the lower and upper indices T and R are dropped for convenience), captures the running of the coupling, exactly as happens in QED. To be specific, setting

$$\hat{d}_1(q^2)=[q^2+\hat{\Pi}_1(q^2)]^{-1}, \tag{9.1}$$

at one loop, then the combination

$$\hat{D}_1(q^2)=g^2\hat{d}_1(q^2) \tag{9.2}$$

obeys the renormalization group equation (RGE)

$$\left(\mu\frac{\partial}{\partial\mu}+g\beta_1\frac{\partial}{\partial g}\right)\hat{D}_1(q^2)=0\,, \tag{9.3}$$

where $\beta_1=-b_1\alpha_s/(4\pi)$. The reason for this is exactly the same as in QED, namely, the fact that the PT vertex and quark self-energy satisfy an Abelian, tree-level-type WI, i.e.,

$$q^\mu\hat{\Gamma}_\mu=\hat{\Sigma}(p+q)-\hat{\Sigma}(p) \tag{9.4}$$

or equivalently, $\hat{Z}_g=\hat{Z}_A^{-1/2}$ where $\hat{Z}_g$, $\hat{Z}_A$ are the gluon-field and strong-coupling-constant renormalizations, respectively.

Let us now assume that the PT rearrangement, as described in [4], works to higher orders in perturbation theory. In particular, let us assume that Eq. (9.3) holds to all orders of perturbation, i.e., for

$$\beta=-\left[b_1\left(\frac{\alpha_s}{4\pi}\right)+b_2\left(\frac{\alpha_s}{4\pi}\right)^2+\cdots+b_n\left(\frac{\alpha_s}{4\pi}\right)^n+\cdots\right], \tag{9.5}$$

and

$$\hat{\Pi}(q^2)=\hat{\Pi}_1(q^2)+\hat{\Pi}_2(q^2)+\cdots+\hat{\Pi}_n(q^2)+\cdots, \tag{9.6}$$

where $\hat{\Pi}_n$ are one particle irreducible of n-loop order and independent of the GFP. Note that the coefficients b_n in Eq. (9.5) are renormalization prescription dependent, for $n>2$. The first three coefficients for quarkless QCD are

$$b_1=\frac{11}{3}c_A,\quad b_2=\frac{34}{3}c_A^2,\quad b_3=\frac{2857}{54}c_A^3, \tag{9.7}$$

and have been evaluated in Refs. [31], [32], and [33], respectively. The values of b_1 and b_2 quoted above are renormalization scheme independent, whereas b_3 has been evaluated within the minimal subtraction (MS) scheme [34].

Substituting Eqs. (9.5) and (9.6) into Eq. (9.3), and equating powers of g^2, it is easy to obtain

$$\mu\frac{\partial\hat{\Pi}_n(q^2)}{\partial\mu}=2\beta_n q^2+2\sum_{k=1}^{n-1}(1-k)\beta_{n-k}\hat{\Pi}_k(q^2), \tag{9.8}$$

with $\beta_n=-b_n(a_s/4\pi)^n$. Note that Eq. (9.8) is identical to the one obtained for the photon vacuum polarization in QED [35]. As happens in the QED case, for $n=1,2$, the dependence of $\hat{\Pi}_n$ on the renormalization point μ is logarithmic, whereas for $n>2$, higher powers of logarithms start appearing.

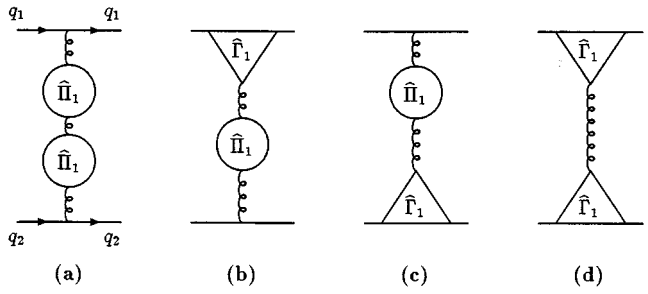

FIG. 8. PT resummation at two loops in QCD.

Let us now assume that we were to change by hand the value of $\hat{\Pi}_1$, $\hat{\Gamma}_1$, and $\hat{B}_1$, in such a way as to not change the value of the S matrix at one loop. So, we make the replacements

$$\hat{\Pi}_1 \to \widetilde{\Pi}_1 \equiv \hat{\Pi}_1 + f_1, \quad \hat{\Gamma}_1 \to \widetilde{\Gamma}_1 \equiv \hat{\Gamma}_1 + u_1,$$
$$\hat{B}_1 \to \widetilde{B}_1 \equiv \hat{B}_1 + h_1, \tag{9.9}$$

where f_1, u_1, and h_1 are, in principle, arbitrary *functions* of q^2, subject to the constraint

$$f_1 + 2q^2 u_1 + q^4 h_1 = 0, \tag{9.10}$$

which guarantees that the value of the S matrix does not change at one loop, after the substitution given in Eq. (9.9).

The functions f_1, u_1, and h_1 do not depend on the gauge-fixing parameter, and are UV and IR finite. Therefore, they do not depend on the renormalization point μ, viz.,

$$\frac{\partial f_1}{\partial \mu} = \frac{\partial u_1}{\partial \mu} = \frac{\partial h_1}{\partial \mu} = 0 .$$

In the case of QCD, the only physical choice for f_1 would be $f_1 = Cq^2$, where C is a numerical constant, since the only available mass scale is q^2. In other words, since f does not depend on μ, we cannot have ratios of momenta q^2/μ^2. At the same time, one does not want to use the mass of the external fermions, since that would convert $\hat{\Pi}_1$ to a process-dependent quantity. Moreover, the RGE in Eq. (9.8) would then be modified by the μ dependence of the running quark masses. For the sake of argument, let us, however, assume that one uses a ''universal'' mass scale M_u, such as the Planck mass, or some combination involving the sum of all quark masses. So, f_1 may contain ratios of q^2/M_u^2. For example, f_1 could be of the form $f_1 = q^2 \exp(-q^2/M_u^2)$. However, it is important to emphasize that M_u should *not* depend on μ, i.e., $\partial M_u/\partial \mu = 0$.

Returning to the uniqueness issue, since the PT self-energies can be Dyson summed [4], one should impose the same property on their new counterparts. Therefore, following the method developed in [4], a string of the form $\hat{\Pi}_1 (1/q^2) \hat{\Pi}_1$ must be converted to $\widetilde{\Pi}_1 (1/q^2) \widetilde{\Pi}_1$. To accomplish this, one must provide the appropriate combinations involving the functions f_1, u_1, and h_1, just as we had to provide the missing pinch parts in going from $\Pi_1 (1/q^2) \Pi_1$ to $\hat{\Pi}_1 (1/q^2) \hat{\Pi}_1$ (see [4]). To see this in detail, we return to the diagrams of Fig. 8, and assume that the PT rearrangement has already been completed. So, now all bubbles and vertices in these graphs refer to the PT objects. The relevant equations are

$$\widetilde{\Pi}_1 \widetilde{\Pi}_1 = (\hat{\Pi}_1 + f_1)(\hat{\Pi}_1 + f_1) = \hat{\Pi}_1 \hat{\Pi}_1 + 2\hat{\Pi}_1 f_1 + f_1^2, \tag{9.11}$$

$$\widetilde{\Pi}_1 \widetilde{\Gamma}_1 = (\hat{\Pi}_1 + f_1)(\hat{\Gamma}_1 + u_1) = \hat{\Pi}_1 \hat{\Gamma}_1 + f_1 \hat{\Gamma}_1 + u_1 \hat{\Pi}_1 + f_1 u_1, \tag{9.12}$$

$$\widetilde{\Gamma}_1 \widetilde{\Gamma}_1 = (\hat{\Gamma}_1 + u_1)(\hat{\Gamma}_1 + u_1) = \hat{\Gamma}_1 \hat{\Gamma}_1 + 2u_1 \hat{\Gamma}_1 + u_1^2. \tag{9.13}$$

Hereafter, the explicit q^2 dependence of the functions $\widetilde{\Pi}$, $\hat{\Pi}$, $\widetilde{\Gamma}$, etc., will not be displayed for brevity. Omitting a common factor of $(1/q^2)^3$, we obtain, for the aforementioned diagrams,

$$\hat{\Pi}_1 \hat{\Pi}_1 + 2q^2 \hat{\Pi}_1 \hat{\Gamma}_1 + q^4 \hat{\Gamma}_1 \hat{\Gamma}_1 = \widetilde{\Pi}_1 \widetilde{\Pi}_1 + 2q^2 \widetilde{\Pi}_1 \widetilde{\Gamma}_1 + q^4 \widetilde{\Gamma}_1 \widetilde{\Gamma}_1 - R, \tag{9.14}$$

with

$$R = (f_1 + q^2 u_1)[2\hat{\Pi}_1 + 2\hat{\Gamma}_1 + (f_1 + q^2 u_1)]. \tag{9.15}$$

At one loop, the new effective charge $\widetilde{D}_1$ satisfies the correct RGE. In particular, since $\partial f/\partial \mu = 0$ by assumption, we have that

$$\mu \frac{\partial \widetilde{\Pi}_1}{\partial \mu} = \mu \frac{\partial (\hat{\Pi}_1 + f_1)}{\partial \mu} = 2\beta_1 q^2, \tag{9.16}$$

which is what Eq. (9.8) yields for $n = 1$.

According to the method in [4], the propagatorlike parts of R must be allotted to Π_2. The second term in Eq. (9.15) is process dependent, since it is proportional to $\hat{\Gamma}_1$. This term should be given to the two-loop vertex or box graphs. In any case, as we will see, this will make no difference in our analysis. But Π_2 has already been converted into $\hat{\Pi}_2$, because we assumed that the PT procedure has been completed. Therefore, $\widetilde{\Pi}_2$ must be defined as

$$\widetilde{\Pi}_2 = \hat{\Pi}_2 + R_2^p, \tag{9.17}$$

where R_2^p is the propagatorlike part of R_2. After all appropriate powers of $1/q^2$ have been restored, R_2^p is given by

$$R_2^p = \frac{2}{q^2}(f_1 + q^2 u_1)\hat{\Pi}_1 + \cdots, \tag{9.18}$$

where the ellipses denote the optional inclusion of the third term in Eq. (9.15), which is irrelevant for what follows, because it is μ independent.

It is now clear that $\widetilde{\Pi}_2$ fails to satisfy the correct RGE, since its μ dependence is not in compliance with the result deduced from Eq. (9.8) for $n = 2$. In particular, we have

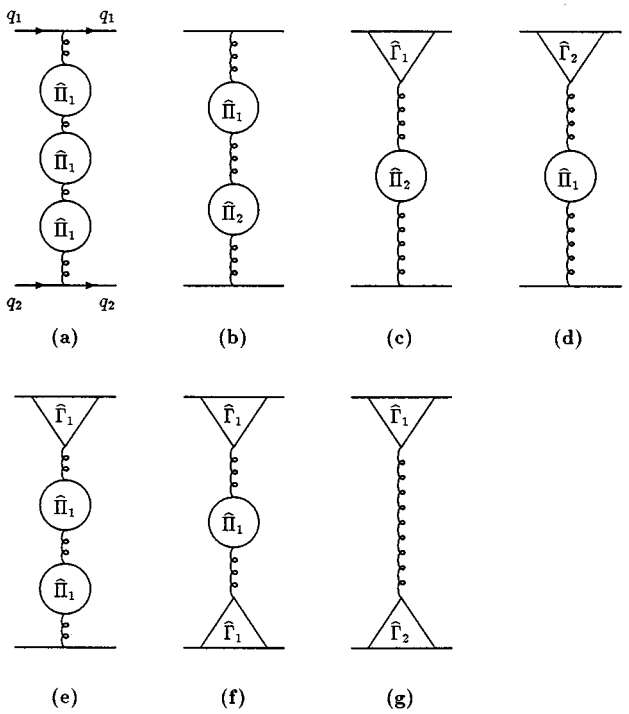

FIG. 9. PT resummation at three loops in QCD.

$$\mu\frac{\partial\widetilde{\Pi}_2}{\partial\mu}=\mu\frac{\partial}{\partial\mu}\left[\hat{\Pi}_2+\frac{2}{q^2}(f_1+q^2u_1)\hat{\Pi}_1\right]$$
$$=2\beta_2q^2+4\beta_1(f_1+q^2u_1)\neq 2\beta_2q^2. \quad (9.19)$$

So, in order to reconcile Dyson summation and the correct RGE behavior to the next order, we must impose the additional constraint that

$$f_1+q^2u_1=0\,. \quad (9.20)$$

Combining this together with Eq. (9.10), we find that $h_1=-u_1/q^4$. Thus, the entire expression for R in Eq. (9.15) vanishes, and Eq. (9.14) becomes

$$\hat{\Pi}_1\hat{\Pi}_1+2q^2\hat{\Pi}_1\hat{\Gamma}_1+q^4\hat{\Gamma}_1\hat{\Gamma}_1=\widetilde{\Pi}_1\widetilde{\Pi}_1+2q^2\widetilde{\Pi}_1\widetilde{\Gamma}_1+q^4\widetilde{\Gamma}_1\widetilde{\Gamma}_1. \quad (9.21)$$

It appears at this point that we have succeeded in implementing the substitution given in Eq. (9.9), without compromising any of the PT properties, at the seemingly modest expense of imposing on f_1 and u_1 the additional constraint given in Eq. (9.20). However, as we will see in a moment, Eq. (9.20) is very crucial, because it actually guarantees the uniqueness of our gauge-invariant resummation method [4], at one loop.

To make this explicit, we proceed to the next order in perturbation theory. The situation may be slightly more cumbersome calculationally, but the conceptual issues are the same. By converting the old strings into new strings, we pick up additional terms, which, when allotted to $\widetilde{\Pi}_3$, will invalidate the RGE that $\widetilde{\Pi}_3$ is expected to satisfy, i.e., Eq. (9.8) for $n=3$, unless a further constraint is imposed on f_1. To determine that constraint, we focus on the three-loop diagrams shown in Fig. 9.

Again, in order to be as general as possible, we assume that one can reshuffle the second order PT Green's functions, without affecting the value of the S matrix to that order. In other words, we allow the additional substitutions

$$\hat{\Pi}_2\to\widetilde{\Pi}_2\equiv\hat{\Pi}_2+f_2\,,$$
$$\hat{\Gamma}_2\to\widetilde{\Gamma}_2\equiv\hat{\Gamma}_2+u_2\,,$$
$$\hat{B}_2\to\widetilde{B}_2\equiv\hat{B}_2+h_2\,, \quad (9.22)$$

with

$$f_2+2q^2u_2+q^4h_2=0. \quad (9.23)$$

Of course, the proof becomes easier if we assume $f_2=u_2=h_2=0$, but we do not have to. We will need the following algebraic relations:

$$\widetilde{\Pi}_1^3=(\hat{\Pi}_1+f_1)^3=\hat{\Pi}_1^3+3\hat{\Pi}_1^2f_1+3\hat{\Pi}_1f_1^2+f_1^3, \quad (9.24)$$

$$\widetilde{\Pi}_1\widetilde{\Pi}_2=(\hat{\Pi}_1+f_1)(\hat{\Pi}_2+f_2)=\hat{\Pi}_1\hat{\Pi}_2+\hat{\Pi}_1f_2+\hat{\Pi}_2f_1+f_1f_2, \quad (9.25)$$

$$\widetilde{\Pi}_1\widetilde{\Gamma}_2=(\hat{\Pi}_1+f_1)(\hat{\Gamma}_2+u_2)=\hat{\Pi}_1\hat{\Gamma}_2+\hat{\Pi}_1u_2+f_1\hat{\Gamma}_2+f_1u_2, \quad (9.26)$$

$$\widetilde{\Pi}_2\widetilde{\Gamma}_1=(\hat{\Pi}_2+f_2)(\hat{\Gamma}_1+u_1)=\hat{\Pi}_2\hat{\Gamma}_1+\hat{\Pi}_2u_1+f_2\hat{\Gamma}_1+f_2u_1, \quad (9.27)$$

$$\widetilde{\Pi}_1^2\widetilde{\Gamma}_1=(\hat{\Pi}_1+f_1)^2(\hat{\Gamma}_1+u_1)=\hat{\Pi}_1^2\hat{\Gamma}_1+u_1\hat{\Pi}_1^2+2f_1u_1\hat{\Pi}_1$$
$$+2f_1\hat{\Pi}_1\hat{\Gamma}_1+f_1^2\hat{\Gamma}_1+f_1^2u_1, \quad (9.28)$$

$$\widetilde{\Pi}_1\widetilde{\Gamma}_1^2=(\hat{\Pi}_1+f_1)(\hat{\Gamma}_1+u_1)^2$$
$$=\hat{\Pi}_1\hat{\Gamma}_1^2+u_1^2\hat{\Pi}_1+2u_1\hat{\Pi}_1\hat{\Gamma}_1+f_1\hat{\Gamma}_1^2$$
$$+2f_1u_1\hat{\Gamma}_1+f_1u_1^2, \quad (9.29)$$

$$\widetilde{\Gamma}_1\widetilde{\Gamma}_2=(\hat{\Gamma}_1+u_1)(\hat{\Gamma}_2+u_2)=\hat{\Gamma}_1\hat{\Gamma}_2+u_2\hat{\Gamma}_1+u_1\hat{\Gamma}_2+u_1u_2. \quad (9.30)$$

Using the above formulas, the crucial constraint of Eq. (9.20), and remembering that the graphs of the Figs. 9(b)–9(e) and 9(g) must be multiplied by a factor of 2, which takes account of the symmetric (mirror-image) graphs, we have that the original set of graphs, call $\hat{\mathcal{A}}$ [we factor out a factor $(1/q^2)^4$]

$$\hat{\mathcal{A}}=\hat{\Pi}_1^3+2q^2(\hat{\Pi}_1\hat{\Pi}_2+\hat{\Pi}_1^2\hat{\Gamma}_1)+q^4(2\hat{\Pi}_1\hat{\Gamma}_2+2\hat{\Pi}_2\hat{\Gamma}_1+\hat{\Pi}_1\hat{\Gamma}_1^2)$$
$$+2q^6\hat{\Gamma}_1\hat{\Gamma}_2 \quad (9.31)$$

and the new one, say $\widetilde{\mathcal{A}}$, which is obtained by replacing all quantities with carets in Eq. (9.31) by ''tilded'' ones, are related by

$$\hat{\mathcal{A}}=\widetilde{\mathcal{A}}-R_3\,, \quad (9.32)$$

where R_3 is given by

$$R_3 = f_1\hat{\Pi}_1^2 + 2q^2(f_2+q^2u_2)\hat{\Pi}_1 + 2q^2 f_1\hat{\Pi}_1\hat{\Gamma}_1 + q^4 f_1\hat{\Gamma}_1^2 + 2q^4(f_2+q^2u_2)\hat{\Gamma}_1. \tag{9.33}$$

Clearly, the first two terms in Eq. (9.33) must be allotted to $\hat{\Pi}_3$, thus converting it to $\widetilde{\Pi}_3$. The rest of the terms cannot be absorbed by $\widetilde{\Pi}_3$, since they are explicitly process dependent, because they contain $\hat{\Gamma}_1$. Therefore, the remaining terms must be distributed among the two-loop vertex and/or box graphs. So, after all powers of $1/q^2$ are restored, the propagatorlike part R_3^p of R_3 reads

$$R_3^p = \frac{f_1}{q^4}\hat{\Pi}_1^2 + \frac{2}{q^2}(f_2+q^2u_2)\hat{\Pi}_1, \tag{9.34}$$

and so

$$\widetilde{\Pi}_3 = \hat{\Pi}_3 + R_3^p. \tag{9.35}$$

It is now important to observe that, because of the particular structure of R_3^p, the RGE satisfied by $\widetilde{\Pi}_3$ will be modified. Indeed, from Eq. (9.8), we derive, for $n=3$,

$$\mu\frac{\partial\hat{\Pi}_3}{\partial\mu} = 2\beta_3 q^2 - 2\beta_1\hat{\Pi}_2 \tag{9.36}$$

and after the substitution $\hat{\Pi}_i \to \widetilde{\Pi}_i$, we must have

$$\mu\frac{\partial\widetilde{\Pi}_3}{\partial\mu} = 2\beta_3 q^2 - 2\beta_1\widetilde{\Pi}_2. \tag{9.37}$$

Subtracting the last two equations by parts, we obtain

$$\mu\frac{\partial}{\partial\mu}(\widetilde{\Pi}_3 - \hat{\Pi}_3) = -2\beta_1(\widetilde{\Pi}_2 - \hat{\Pi}_2) = -2\beta_1 f_2. \tag{9.38}$$

Instead, from Eqs. (9.34) and (9.35), we find

$$\mu\frac{\partial}{\partial\mu}(\widetilde{\Pi}_3 - \hat{\Pi}_3) = \mu\frac{\partial R_3^p}{\partial\mu} = \frac{4f_1}{q^2}\beta_1\hat{\Pi}_1 + 4\beta_1(f_2+q^2u_2). \tag{9.39}$$

Given the fact that $\hat{\Pi}_1$ depends explicitly on μ, in order to reconcile Eqs. (9.38) and (9.39) one must necessarily choose $f_1=0$. Thus, the only possible solution for the set of substitutions described in Eq. (9.9) is the *trivial* one, i.e., $f_1=u_1=h_1=0$, which proves the uniqueness of the PT resummation approach to one loop, after renormalization.

After setting $f_1=0$, we must impose the additional constraint $3f_2+2q^2u_2=0$, in order that Eqs. (9.38) and (9.39) become equal. Evidently, the same arguments presented above must be repeated to the next order, which will finally determine the value of f_2; we will not pursue this issue any further here. Instead, we add some further clarifications regarding the assumptions made in the previous proof of the one-loop uniqueness of the PT resummation formalism. As emphasized at the beginning of this section, we assume that the PT can be extended to higher orders, giving rise to effective Green's function with all the characteristics known from the explicit one-loop analysis. We further assume that the renormalization program has been carried out to all orders. Thus, all Green's functions with carets appearing here are UV finite. So far, the renormalization scheme chosen has been left unspecified. Because of Eq. (9.8), the effect of adopting different renormalization-scheme choices will be to modify the values of b_n, for $n>2$. However, within a specific renormalization scheme, the values of b_n are fixed, and this is what we have implicitly assumed.

The resummation formalism discussed for the case of Yang-Mills theories such as QCD can equally carry over to SSB models such as the SM. In the SM, W and Z bosons are considered to be unstable gauge particles. In the case of the W boson, a RGE similar to Eq. (9.8) will hold for the leading logarithmic part of the transverse W-boson self-energy. Again, one can form the RGE-invariant combination involving the W-boson Green's function

$$g_w^2[q^2 - M_W^2 + \hat{\Pi}_T^W(q^2)]^{-1}.$$

Analogously with Eq. (9.4), one can derive a similar relation between the weak-coupling-constant renormalization $\hat{Z}_{g_w}$ and the wave-function renormalization of the W boson $\hat{Z}_W$, i.e., $\hat{Z}_{g_w} = \hat{Z}_W^{-1/2}$ Hence, one can show the uniqueness of this expression by following a line of arguments similar to the case of QCD. Furthermore, possible modifications of the longitudinal part of the W-boson self-energy, $\hat{\Pi}_L^W$, will result in direct violations of the tree-level WI's, which govern the gauge invariance of the classical action.

X. CONCLUSIONS

We have presented a formalism for resummation of off-shell two-point correlation functions, which relies entirely on arguments of analyticity, unitarity, gauge invariance, and multiplicative renormalization. In addition, several crucial aspects of the GFP-independent resummation approach presented in [4] have been clarified. Specifically, we have shown that unitarity requires the absence of unphysical thresholds for the resummed Green's functions at the quantum loop level. Within the PT resummation approach this property is satisfied, since the effective gauge-invariant Green's functions are directly derived from S-matrix elements, with the only additional inputs being the use of elementary tree-level WI's and analyticity.

This is, however, not true in other approaches. For instance, we have explicitly shown that ξ_Q-dependent unphysical thresholds appear in the BFG, even though the Green's functions obey the same tree-level WI's as the PT Green's functions. For the very specific value of $\xi_Q=1$, the results of BFG and PT coincide to one loop, as this is the only gauge that avoids unphysical propagator poles. The situation may change in higher orders. Furthermore, we have found that the BFG Green's functions can be decomposed into two parts, one containing only physical poles and the other containing ξ_Q-dependent unphysical thresholds, which *separately* satisfy the same WI's as the total BFG Green's functions.

Furthermore, we have addressed issues of gauge invariance by resorting to the BRS symmetries at the one-loop quantum level. We have explicitly demonstrated that the PT two-point correlation function may be obtained from its ab-

sorptive part through a DR. The absorptive part of the PT Green's functions can equally well be calculated from the optical relation of the anti-Hermitian part of the transition amplitude. As a result of this, we have also been able to identify the pinching parts of the PT algorithm, as those terms that quantify the deviation from the intrinsic BRS symmetries. Most importantly, we have been able to show how gauge invariance is restored, within the PT framework, by reinforcing BRS symmetries inside the quantum loops.

In Sec. IX, we have examined the issue of ''uniqueness'' of the gauge-invariant resummation approach proposed in [4]. In the context of QCD, we have focused on the most basic RGE-invariant quantity involving the PT two-point correlation function, namely, the effective (running) strong coupling. By means of a three-loop analysis, we have shown that, at one loop, the PT resummation method gives rise to unique results. We have also briefly outlined how these considerations can be naturally extended to spontaneously broken gauge theories.

Considering the fact that all the basic field-theoretical requirements imposed thus far are preserved within the PT resummation approach that was introduced in [4] and was further analyzed in the present paper, one might be tempted to argue that some deeper underlying principle is in effect, which has yet to be discovered. Here, we wish to point out two possibly relevant directions in such a quest. First, there is a interesting recent result of ''stringy'' origin [36], which seems to single out the one-loop BFG Green's functions for the special value of $\xi_Q=1$, which are, of course, identical to the PT Green's functions. This observation makes the question of whether the correspondence between the PT and the BFG at $\xi_Q=1$ persists beyond one loop even more pressing. Second, one should investigate possible connections between the PT and the Vilkovisky-DeWitt formalism [37]. In particular, the gauge-invariant and GFP-independent Green's functions obtained from the Vilkovisky-DeWitt effective action must be compared with their PT counterparts, establishing the origin and the physical significance of any possible difference between them.

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