Topological and nontopological self-dual Chern-Simons solitons in a gauged O(3) σ model

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We present topological and nontopological self-dual soliton solutions in an O(2) gauged O(3) σ model on $\mathbb{R}_2$ with Chern-Simons rather than Maxwell dynamics. These solutions are not vortices in the usual sense in that the *magnetic flux* is irrelevant to the stability of the topological solitons, which are stabilized by the degree N, but it plays a crucial role in the stabilization of the nontopological solitons. It turns out that *topological* and *nontopological* solitons of arbitrary vorticity N exist. We have studied both types of vortices with $N=1$ and $N=2$, and the nontopological soliton with $N=0$ numerically. We present analytic proofs for the existence of these topological and nontopological solitons. The qualitative features of the gauged O(3) solitons are contrasted with those of the gauged CP^1 solitons. [S0556-2821(96)00120-8]

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I. INTRODUCTION

The problem of gauging σ models is a pertinent one in both $(2+1)$- and $(3+1)$-dimensional physics, and was investigated by Faddeev [1] a while ago. In $2+1$ dimensions, it is relevant to the planar physics of superconductivity theories especially at high T_c when nonstandard statistics is exhibited [2] in the CP^1 model. In $3+1$ dimensions it features in the gauging [3,4] of the O(4) σ model, namely, of the Skyrme model [5]. The work of Ref. [4], in particular, addresses the problem of the topological stability of the soliton in the gauged Skyrme model. In this work we shall require that the gauged σ model support topologically stable solitons.

Recently, there have been several works dealing with the problem of gauging σ models defined on $\mathbb{R}_2$ in such a way that the gauged model can support solitons. These were reported in the Refs. [6–9] chronologically.

In [6,7] the solitons in question were the static solutions to a Chern-Simons theory in $(2+1)$-dimensional Minkowski space featuring a CP^1 field. Implicit in this work [6,7] was the analysis of a gauged CP^1 model on $\mathbb{R}_2$ with Maxwell dynamics, in exact analogy with the role played by the usual Abelian Higgs model [10] with Maxwell dynamics in the analysis of the Chern-Simons Higgs solitons introduced by Hong *et al.* [11] and Jackiw and Weinberg [12]. This aspect was highlighted in [8], where both the Chern-Simons and Maxwell CP^1 models were augmented by a Skyrme term, respectively, on $(2+1)$-dimensional Minkowski space and $\mathbb{R}_2$.

The solitons of the gauged CP^1 models of [6–8] are *not self-dual* solutions and their stability is characterized by a topological charge which coincides with the magnetic flux taking on discrete values related to the vorticity of the soliton. Insofar as the stability of the topological solitons is characterized by the magnetic flux, these vortices are similar to the Maxwell [11] and Chern-Simons Higgs models [11,12]. They differ, however, from the latter [10–12], in that they are *not self-dual*. This lack of self-duality can present itself as a disadvantage, at least technically. One reason is that self-dual vortices, such as, for example, those of the Abelian Higgs model [10], do not interact by virtue of the stress tensor vanishing identically, and hence multivortex configurations arbitrarily distributed on the plane can be studied [10] systematically.

In complete contrast to the above, the gauged O(3) model proposed by Schroers [9] can support **self-dual** solutions whose topological charge, however, is unrelated to the *magnetic flux*. It appears therefore that the static solitons of gauged σ models on $\mathbb{R}_2$ are *either* non-self-dual and stabilized by the magnetic flux *or* they can be self-dual, but are stabilized by a topological charge which is unrelated to the magnetic flux. This in turn can also be regarded as a disadvantage since stabilization by magnetic flux was the original motivation for the introduction of gauge fields in soliton theory. In the background of these contrasting features, it is worthwhile making a comparative study of the gauged CP^1 and O(3) models. This is one of the aims of the present work.

It follows from the descriptions given above that the gauged O(3) σ model [9] which supports *self-dual* solitons can be adapted to the construction of self-dual static solitons in a $(2+1)$-dimensional Chern-Simons gauged O(3) model devised in exact analogy with the work of [11,12] in the analogous case of the Higgs models. Furthermore, static nontopological solutions can also be constructed in this case, in exact analogy with the work of [13]. This is the other aim of the present work. The task is undertaken below in the following manner. Section II is devoted to the stability analysis, which is indirectly employed as a means of constructing

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models that support stable solitons. In Sec. II A we reanalyze the CP^1 model [8], gauged with a U(1) Maxwell term, and show that the model which can support self-dual solutions has only the trivial solution. That implies also that the model obtained by replacing the Maxwell term with a Chern-Simons term also cannot have nontrivial self-dual solutions. In Sec. II B we introduce the Chern-Simons gauged O(3) model and establish the topological inequalities and hence the Bogomol'nyi equations. This is followed in Sec. III by the construction of the soliton solutions of this last model. Since we restrict to radially symmetric solutions only, we give our radially symmetric ansatz and state the asymptotic values of our solutions. In the subsequent Secs. III A and III B, we give the asymptotic behaviors of the topological and nontopological solitons, respectively, and discuss the numerical integrations of each case. The analytic proofs of existence of both topological and nontopological solutions in the gauged O(3) model are given, respectively, in Secs. IV A and IV B, and a summary of the results is presented in Sec. V.

II. STABILITY ANALYSIS AND MODELS

When gauging σ models such that the ensuing gauged model supports topologically stable lump solutions, say, in two static dimensions, the qualitative results strongly depend on whether the σ model in question is a complex or real model. Specifically, consider the gauging of the CP^1 model according to the prescriptions given in Ref. [8] and the O(3) model according to the prescriptions given in Ref. [9]. In the first case, it turns out that the U(1) gauged CP^1 model does not possess self-dual solutions, but the non-self-dual solitons are stabilized by the magnetic flux which is physically a useful feature of the model. In the second case, on the other hand, the solitons are *not* stabilized by the magnetic flux, but they are self-dual, which is technically a very useful feature of the model. Note that in the case of the gauged Higgs models [10,14,15], the vortices are both self-dual and are stabilized by the magnetic flux. Here, for the σ models, the situation is that either the topological charge coincides with the magnetic charge or the solitons are self-dual, but not both. It is therefore interesting to consider both cases and to compare them. This is the aim of the present section.

A. U(1) gauged CP^1 model

Here we restrict ourselves to U(1) Maxwell, rather than Chern-Simons dynamics. This is because the topological inequalities which establish the lower bound on the energy of the soliton for the Chern-Simons gauged model are essentially the same as those which occur in the stabilization of the Maxwell gauged model. This is true for the Maxwell [10] and Chern-Simons [11,12] Higgs models, the Maxwell [14] and Chern-Simons [15] *generalized* Higgs models, and the Maxwell [8] and Chern-Simons [7] CP^1 models.

Our aim in this subsection is to construct *another* U(1) gauged CP^1 model, different from the one introduced in Ref. [8], whose energy functional can be absolutely minimized by a set of Bogomol'nyi equations. It will turn out, however, that these Bogomol'nyi equations do not support soliton solutions and hence we must conclude that gauging the CP^1 model does not lead to a model that supports self-dual solitons. We shall arrive at this result by studying a Maxwell gauged model, but since this is a negative result, it implies that the corresponding Chern-Simons term will also not support self-dual solitons and hence it is sufficient to consider the Maxwell case only.

As in previous examples [14,8], we shall first establish the topological inequalities and then find the model whose dynamics is stabilized by these inequalities. We start from the inequalities

$$|D_i z - i\varepsilon_{ij}\sigma_3 D_j z|^2 \geqslant 0, \tag{1}$$

$$(\tfrac{1}{2} F_{ij} - \varepsilon_{ij}\sqrt{V})^2 \geqslant 0, \tag{2}$$

in which $z=(z_1,z_2)$ is the CP^1 field subject to the constraint $z^\dagger z = 1$ and the U(1) curvature $F_{ij} = -iD_{[i}D_{j]}$ is defined in terms of the covariant derivative $D_i = \partial_i + iA_i$, with $i=1,2$.

Developing the inequalities (1) and (2), and adding them, we find the topological inequality

$$[(\tfrac{1}{4}F_{ij}^2 + 2V) + \tfrac{1}{2} D_i z^\dagger D_i z] \geqslant \partial_i \Omega_i, \tag{3}$$

$$\Omega_i = \frac{i}{2}\varepsilon_{ij}[vA_j + z^\dagger(\sigma_3 D_j z)], \tag{4}$$

provided that the function $\sqrt{V}$ in Eq. (2) is chosen to be $\sqrt{V} = \frac{1}{4}(v - z^\dagger\sigma_3 z)$. Here v is a dimensionless constant. Then the topological charge, which is the "surface" integral of the density Ω_i, Eq. (4), is the magnetic flux arising from the first term in Eq. (4), while the integral of the second term in Eq. (4) vanishes as a result of the usual finite-energy decay conditions. The energy integral which is bounded from below by this topological (magnetic) charge is the integral of the static Hamiltonian density

$$\mathcal{H}_0 = \tfrac{1}{4}F_{ij}^2 + \tfrac{1}{2} D_i z^\dagger D_i z + \tfrac{1}{8}(v - z^\dagger\sigma_3 z)^2. \tag{5}$$

The static Hamiltonian density (5) pertains to a U(1) gauged CP^1 model which differs from the one considered in Ref. [8] in that it is described by a different potential, even when v is set equal to 1. While in the model proposed in Ref. [8] the topological inequalities could not be saturated by construction, here the energy integral of Eq. (5) can be minimized absolutely by saturating the topological inequalities (1) and (2). This leads to the pair of Bogomol'nyi equations

$$D_i z = i\varepsilon_{ij}\sigma_3 D_j z, \tag{6}$$

$$F_{ij} = \tfrac{1}{2}\varepsilon_{ij}(v - z^\dagger\sigma_3 z), \tag{7}$$

whose radially symmetric restriction, obtained by using the radially symmetric ansatz for the fields with vorticity n,

$$z_1 = \cos\frac{f(r)}{2}e^{-in\theta}, \quad z_2 = \sin\frac{f(r)}{2},$$

$$A_i = \frac{a(r)-n}{r}\varepsilon_{ij}\hat{x}_j, \tag{8}$$

yields the *three* equations

$$\frac{1}{2}f'\sin f+\frac{a}{r}(1+\cos f)=0, \tag{9}$$

$$\frac{1}{2}f'\sin f+\frac{a-n}{r}(1-\cos f)=0, \tag{10}$$

$$a'=-\frac{r}{2}(\upsilon-\cos f). \tag{11}$$

There are three equations (9), (10), and (11) for two functions $f(r)$ and $a(r)$, and hence the system is overdetermined. The three equations (9), (10), and (11) reduce to the pair of algebraic equations

$$r^2\cos^3 f(\upsilon-\cos f)=n^2(1-\cos^2 f), \tag{12}$$

$$2a\cos f=n(\cos f-1), \tag{13}$$

to which we have *not* found solutions satisfying *finite-energy* asymptotic conditions. We must conclude therefore that when the CP^1 model is gauged with a U(1) field so as to support soliton solutions, the field configuration corresponding to these solutions *cannot* saturate the topological lower bound on the energy. The same conclusion follows when we replace the Maxwell dynamics in Eq. (5) by Chern-Simons dynamics. There exist, of course, non-self-dual solutions to the model (5), which is closely related to the U(1) gauged CP^1 model given in Ref. [8]. The radially symmetric restriction of Eq. (5) is readily calculated by use of Eq. (8), and the corresponding one-dimensional static energy functional is

$$H_0=\frac{1}{2r}a'^2+\frac{r}{4}f'^2+\frac{1}{r}\left[a^2+\frac{1}{2}n(n-2a)(1-\cos f)\right]$$
$$+\lambda(1-\cos f)^2, \tag{14}$$

in which the coupling constant λ multiplying the symmetry breaking potential is introduced by way of emphasizing that the solutions we seek are non-self-dual, with $\lambda=1$, in which case the topological lower bound can actually be saturated, but only for a trivial field configuaration as the solution. The system (13) differs from the one studied in Ref. [8] only in the potential function and the fact that in the present model we have not included any Skyrme terms. We do not integrate the Euler-Lagrange equations pertaining to Eq. (14) numerically since that was done in detail Ref. [8].

It is because of the absence of self-dual solutions in the gauged CP^1 model that it is interesting to study the U(1) Maxwell gauged O(3) model introduced in Ref. [9], since the latter does support self-dual solutions saturating the topological lower bound. The next natural step then is to find and study the corresponding U(1) Chern-Simons gauged O(3) model. The corresponding analysis for the latter is given in the next subsection.

B. Chern-Simons O(3) model

We start this subsection with the identification of the topological charge of the putative soliton and the corresponding topological current. This applies both to the Maxwell gauged [9] and the Chern-Simons gauged O(3) models. Since the topological charge density of the ungauged O(3) model is not a total divergence like that of the ungauged CP^1 model and, instead, it is only locally a total divergence, we expect [9] that it will be essentially the density for the degree of the map

$$\varrho_0=\varepsilon_{ij}\varepsilon^{abc}\partial_i\phi^a\partial_j\phi^b\phi^c. \tag{15}$$

Now the volume integral of Eq. (15) is a gauge-*variant* quantity so that it cannot as it stands supply a lower bound on the volume integral of the energy density or the static Hamiltonian, since the latter is by construction a gauge-*invariant* quantity. To modify Eq. (15) suitably, we consider its gauge-covariant extension

$$\varrho_1=\varepsilon_{ij}\varepsilon^{abc}D_i\phi^a D_j\phi^b\phi^c, \tag{16}$$

in which the covariant derivative $D_i\phi^a=(D_i\phi^\alpha,D_i\phi^3)$, with $\alpha=1,2$, is defined by

$$D_\mu\phi^\alpha=\partial_\mu\phi^\alpha+A_\mu\varepsilon^{\alpha\beta}\phi^\beta,\quad D_\mu\phi^3=\partial_\mu\phi^3, \tag{17}$$

resulting in the U(1) curvature $D_{[\mu}D_{\nu]}\phi^\alpha=\varepsilon^{\alpha\beta}\phi^\beta F_{\mu\nu}$.

It can now be verified that ϱ_0 and ϱ_1 are related as

$$\varrho=\varrho_1+\varepsilon_{ij}\phi^3F_{ij}=\varrho_0+2\varepsilon_{ij}\partial_i(\phi^3A_j), \tag{18}$$

which is a definition of the *gauge-invariant* topological charge density ϱ. The topological charge then will be the degree of the map N, namely, the volume integral of ϱ_0, Eq. (15), provided that the volume integral of the gauge-*variant* total divergence term on the right-hand side of Eq. (18) vanishes. This requirement will have to be verified to hold true for any solution which is a candidate to be a soliton.

Having defined the topological charge density ϱ, Eq. (18), we identify it with the zero-component of the topological current j^μ, given by

$$j^\mu=\varepsilon^{\mu\rho\sigma}(\varepsilon^{abc}D_\rho\phi^a D_\sigma\phi^b\phi^c+\phi^3F_{\rho\sigma}). \tag{19}$$

The divergence of this current is readily calculated to be

$$\partial_\mu j^\mu=3\varepsilon^{\mu\rho\sigma}\varepsilon^{\alpha\beta}D_\rho\phi^\alpha D_\sigma\phi^\beta\partial_\mu\phi^3, \tag{20}$$

which in turn can easily be shown to be a locally total divergence. Thus the volume integral of the divergence of the topological current (20) vanishes as required. We note in passing that the definition (19) for the topological current is arbitrary up to the addition of a divergeless density and in the case where the gauge group is Abelian; then, this means we can add the density $\varepsilon^{\mu\rho\sigma}F_{\rho\sigma}$ since the divergence of the latter happens to be the Abelian Bianchi identity. One can then redefine Eq. (19) as

$$\tilde{j}^\mu=j^\mu+\zeta\varepsilon^{\mu\rho\sigma}F_{\rho\sigma}=\varepsilon^{\mu\rho\sigma}[\varepsilon^{abc}D_\rho\phi^a D_\sigma\phi^b\phi^c$$
$$+(\phi^3+\zeta)F_{\rho\sigma}]. \tag{21}$$

Choosing the constant $\zeta=-\upsilon$, and with $\upsilon=1$, then leads to the topological current stated in Ref. [9], involving the factor $(\upsilon-\phi^3)$ which features in the symmetry-breaking potential. This seems to be a coincidence arising from the Abelian nature of the gauge group.

We now turn to the dynamics and propose the Lagrangian on $(2+1)$-dimensional Minkowski space:

$$\mathcal{L}=\frac{\kappa}{2\sqrt{2}}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho+(D_\mu\phi^a)^2-4V(\phi^3), \qquad (22)$$

where κ is a constant with the dimensions of length and the three-component field $\phi^a=(\phi^\alpha,\phi^3)$, with $\alpha=1,2$, is constrained by $\phi^a\phi^a=1$. The Lagrangian (22) is U(1) gauge invariant by virtue of Eq. (17), since the potential function not yet specified is nevertheless allowed to depend only on the U(1)-invariant component ϕ^3 of ϕ^a.

The Hamiltonian density and the Gauss law equation of motion are given, respectively, by

$$\mathcal{H}=\tfrac{1}{2}[(D_0\phi^\alpha)^2+(\partial_0\phi^3)^2+(D_i\phi^\alpha)^2+4V], \qquad (23)$$

$$\frac{\kappa}{2\sqrt{2}}\varepsilon_{ij}F_{ij}=\varepsilon^{\alpha\beta}\phi^\alpha D_0\phi^\beta. \qquad (24)$$

In the static limit $\mathcal{H}$ reduces to

$$\mathcal{H}_0=(\phi^\alpha)^2A_0^2-\tfrac{1}{2}(D_i\phi^a)^2+2V \qquad (25)$$

and the Gauss law equation yields

$$A_0=-\frac{\kappa}{2\sqrt{2}(\phi^\alpha)^2}\varepsilon_{ij}F_{ij}, \qquad (26)$$

so that substituting for A_0 from Eq. (26) into Eq. (25), we end up with the static Hamiltonian density defined on $\mathbb{R}_2$:

$$\mathcal{H}_0=\frac{\kappa^2}{4(\phi^\alpha)^2}F_{ij}^2+\frac{1}{2}(D_i\phi^a)^2+2V. \qquad (27)$$

The choice of the potential function in Eq. (27) is dictated by the requirement that the volume integral of Eq. (27) be bounded from below by a topological charge and be fixed uniquely by that criterion. The lower bound on the volume integral of the density (27) follows from the inequalities

$$\left(\frac{\kappa}{2|\phi^\alpha|}F_{ij}-\varepsilon_{ij}|\phi^\alpha|U\right)^2\geqslant 0, \qquad (28)$$

$$(\varepsilon_{ij}D_i\phi^a-\varepsilon^{abc}D_j\phi^b\phi^c)^2\geqslant 0, \qquad (29)$$

where $|\phi^\alpha|=\sqrt{(\phi^\alpha)^2}$. The sum of Eqs. (28) and (29) implies the (topological) inequality

$$\frac{\kappa^2}{4|\phi^\alpha|^2}F_{ij}^2+2|\phi^\alpha|^2U^2+\frac{1}{2}(D_i\phi^a)^2\geqslant\frac{1}{2}\varrho_0+\varepsilon_{ij}\partial_i(\phi^3A_j)-\frac{1}{2}\varepsilon_{ij}(\phi^3-2\kappa U)F_{ij}. \qquad (30)$$

The left-hand side of Eq. (30) can now be identified as $\mathcal{H}_0$ in Eq. (27) if $|\phi^\alpha|^2U^2$ in Eq. (30) is identified as the potential function V in Eq. (27). What we choose for V then determines the lower bound on $\mathcal{H}_0$. This fixes the potential function uniquely, subject to our requirement that the volume integral of the right-hand side of Eq. (30) reduce to the volume integral of the winding number density ϱ_0. This means that the right-hand side of Eq. (30) must reduce to ϱ_0 plus at most a total divergence term such that the surface integral corresponding to the latter vanishes. This situation obtains when we choose

$$U=\frac{1}{2\kappa}(\phi^3-v). \qquad (31)$$

Here the constant v is related to the constant ζ in Eq. (21) and is analogous to the synonymous constant v of the gauged CP^1 model in Eq. (4). In the present work, we shall restrict the value of the constant v in Eq. (31) equal to *unity*, whence the topological inequality (30) leads to

$$\int d^2x\,\mathcal{H}_0\geqslant\frac{1}{2}\int d^2x\,\varrho=\frac{1}{2}\int d^2x\,\varrho_0+\int dS_i\,\varepsilon_{ij}A_j(\phi^3-1), \qquad (32)$$

with $\mathcal{H}_0$ now given by

$$\mathcal{H}_0=\frac{\kappa^2}{2|\phi^\alpha|^2}F_{ij}^2+(D_i\phi^a)^2+\frac{1}{\kappa^2}(1-\phi^3)^3(1+\phi^3). \qquad (33)$$

Had we not restricted to $v=1$ already in Eq. (32), the potential featured in Eq. (33) would have had the more general form

$$2|\phi^\alpha|^2U^2=\frac{1}{\kappa^2}[1-(\phi^3)^2](v-\phi^3)^2. \qquad (34)$$

The required topological inequality is Eq. (32), *provided* that the surface integral on its right-hand side vanishes. This last requirement is easily satisfied, in the case of topologically stable solutions, by the asymptotic conditions

$$\lim_{|\vec{x}|\to 0}\phi^3=-1,\quad \lim_{|\vec{x}|\to\infty}\phi^3=1, \qquad (35)$$

which guarantee that the volume integral of ϱ yields a nonzero integer winding number. This statement assumes that A_i does not grow too fast at infinity, an assumption which is amply justified as will be seen below when we specialize to the radial field configuration. The conditions (35) pertain to the *topologically stable* solutions of nonzero winding number and also guarantee that the volume integral of Eq. (33), namely, that the energy, be finite. With the particular potential in Eq. (33), however, there is a second set of asymptotic conditions for which the energy integral is also guaranteed to be finite. These are stated as

$$\lim_{|\vec{x}|\to 0}\phi^3=-1,\quad \lim_{|\vec{x}|\to\infty}\phi^3=-1. \qquad (36)$$

The winding number for a field configuration satisfying Eq. (36) *vanishes* and then the lower bound on $\mathcal{H}_0$ follows from Eqs. (32) and (36) to be the magnetic flux. Such solutions, which we expect to find, are the static nontopological solutions analogous to those found in [13] for the Chern-Simons Higgs model.

The topological inequality (32) is saturated when the inequalities (28) and (29) are saturated, yielding the Bogomol'nyi equations

$$F_{ij}=\mp\varepsilon_{ij}(1-\phi^3)^2(1+\phi^3), \qquad (37)$$

$$\varepsilon_{ij}D_i\phi^a = \pm\varepsilon^{abc}D_j\phi^b\phi^c, \tag{38}$$

where we have set $\kappa=1$ and, the lower and upper signs pertain to anti-self-duality and self-duality, respectively.

In terms of the complex-valued functions $\varphi=\ln u$ of the coordinates $\vec{x}$ on $\mathbb{R}_2$, defined by $u=(\phi^1+i\phi^2)/(1+\phi^3)$, the Bogomol'nyi equations (38) reduce to the partial differential equation

$$\Delta(\varphi+\overline{\varphi}) = \frac{16e^{-(\varphi+\overline{\varphi})}}{(1+e^{-(\varphi+\overline{\varphi})})^3}, \tag{39}$$

which we do not study further here but proceed in the next sections to subject Eqs. (37) and (38) to radial symmetry and subsequently to integrate them numerically.

III. SOLITONS

Since we shall be concerned with radially symmetric field configurations only, we proceed to state our radially symmetric ansatz for the fields A_i and ϕ^a:

$$A_i = \frac{a(r)-N}{r}\varepsilon_{ij}\hat{x}_j, \tag{40}$$

$$\phi^\alpha = \sin f(r) n^\alpha, \quad \phi^3 = \cos f(r), \tag{41}$$

where $\hat{x}_i = x_i/r$ and $n^\alpha = (\cos N\theta,\ \sin N\theta)$ are unit vectors, with N defined to be an integer.

The Hamiltonian density of the corresponding one-dimensional subsystem arising from the imposition of radial symmetry on the system (33) is defined by

$$\int d^2x\,\mathcal{H}_0 = 4\pi\int H_0\,dr,$$

which as a function of $f(r)$ and $a(r)$ is expressed as

$$H_0 = \frac{a'^2}{r\sin^2 f} + rf'^2 + \frac{a^2\sin^2 f}{r} + (1-\cos f)^2\sin^2 f. \tag{42}$$

We have set $\kappa=1$ in Eq. (33). The one-dimensional Hamiltonian density (42), which gives the energy density $\mathcal{E}=rH_0$, will be needed below in the numerical computation of the total energies.

The Bogomol'nyi equations (37) and (38) now reduce to the following pair of coupled nonlinear ordinary differential equations:

$$\frac{a'}{r} = \pm(1-\cos f)^2(1+\cos f), \quad f' = \pm\frac{a\sin f}{r}. \tag{43}$$

We notice that the second member of Eqs. (43) coincides with the Bogomol'nyi equation for the ungauged O(3) σ model when $a(r)$ in it is replaced by the integer N. Thus at the origin this equation is the same for both the gauged and the ungauged O(3) models, with $a(0)=N$. This is not surprising since the topological charge, namely, the volume integral of ϱ_0 in Eq. (15), depends on the asymptotic values of the function $f(r)$.

Since it is sufficient to study *either* the self-dual *or* the anti-self-dual case, we shall restrict ourselves in what follows below to the anti-self-dual case, namely, to the equations with the *lower* signs in Eqs. (43).

The *topological* asymptotic conditions (35), which were chosen in anticipation of our restriction to the anti-self-dual case, now read

$$\lim_{r\to 0} f(r) = \pi, \lim_{r\to\infty} f(r) = 0, \tag{44}$$

which for the field configuration (41) imply *vorticity* $=-N$. This is the same as in the usual (ungauged) O(3) model where the radially symmetric anti-self-dual vortices satisfy the asymptotic conditions (44), while the self-dual vortices satisfy instead

$$\lim_{r\to 0} f(r) = 0, \quad \lim_{r\to\infty} f(r) = \pi.$$

We shall henceforth restrict ourselves to the anti-self-dual case only and will denote the vortex number by N, on the understanding that this is read as $|N|$. This applies in particular to Eq. (41).

Concerning the asymptotic conditions (36) of the *nontopological* solitons, there is a further refinement to be taken into account. The condition at the origin in Eq. (36) is designed to ensure that the field ϕ^a is single valued at the origin. This is true for $N>0$, but is too strong a condition for the nontopological soliton with $N=0$. In that case, it is possible to relax this condition, so that the *nontopological* asymptotic conditions now read, for $N>0$,

$$\lim_{r\to 0} f(r) = \pi, \quad \lim_{r\to\infty} f(r) = \pi \tag{45}$$

and, for $N=0$,

$$\lim_{r\to 0} f(r) = f_0, \quad \lim_{r\to\infty} f(r) = \pi, \tag{46}$$

with f_0 constant. For the fields (41) of all vorticities $N>0$, Eqs. (45) imply *zero degree*, i.e., $\int d^2x\,\varrho_0 = 0$.

The asymptotic behavior of the function $a(r)$ in Eq. (40) is of no consequence to the topological stability of the soliton, unlike in the cases of the Higgs models [8,12,13] and of the gauged CP^1 models [7,8]. This is because in the latter systems the topological charge, which is again related to the vorticity, is also proportional to the *magnetic flux*. In the case of the gauged O(3) models, given in [9] and here, the magnetic flux of the solution is not restricted by the requirement of the stability of the soliton. The only constraint on the large-r behavior of $a(r)$ here is the requirement that the surface integral on the right-hand side of Eq. (32) vanish. This means that $a(r)$ should not grow faster than the quantity $(\cos f - 1)$ in that region. Since the magnetic flux is proportional to the quantity

$$-a(\infty)+a(0),$$

we shall seek solutions for which both $a(\infty)$ and $a(0)$ are finite, since it is reasonable that the solutions we seek correspond to finite magnetic flux field configurations. As explained above, we shall take $a(0)=N$, but will take $a(\infty)=\alpha$, where α is a nonzero constant whose sign will depend on whether we are considering the topological or the

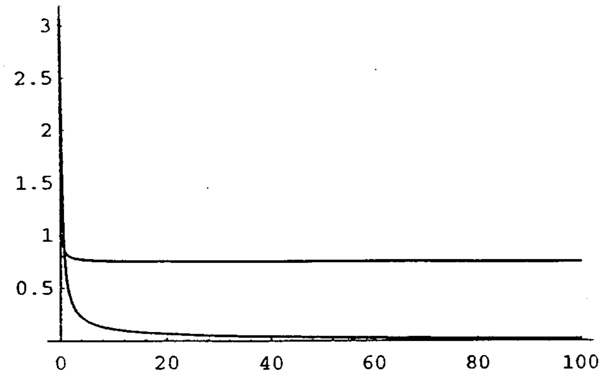

FIG. 1. Profiles of $f(r)$ and $a(r)$ of the *topological* Chern-Simons vortices with $N=1$ and $\alpha=0.75$. The curves are identified by their asymptotic values $f(0)=\pi$, $f(\infty)=0$ and $a(0)=1$, $a(\infty)=0.75$.

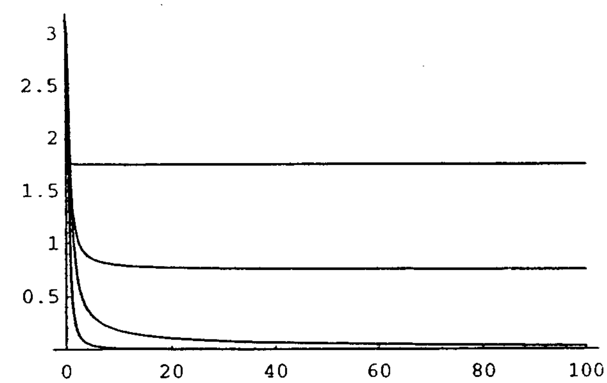

FIG. 2. Profiles of $f(r)$ and $a(r)$ of for the *topological* Chern-Simons vortices of $N=2$ with $\alpha=0.75$ and 1.75, where $f(\infty)=0$ and $a(\infty)=\alpha$. The curves are identified by their asymptotic values with $f(r)$ as in Fig. 1, and $a(0)=2$, $a(\infty)=\alpha$. The faster decaying profile of $f(r)$ corresponds to the larger value of $\alpha=1.75$.

nontopological solutions. We shall see in the following subsections why it is not possible to choose $\alpha=0$, in which case the magnetic flux would have been proportional to the degree N. Corresponding to the asymptotic conditions (44) for the function $f(r)$, we state the asymptotic conditions on the function $a(r)$ as

$$\lim_{r\to 0} a(r)=N, \quad \lim_{r\to\infty} a(r)=\alpha. \tag{47}$$

In the following two subsections, respectively, we will study the topological and nontopological solitons. This involves the numerical integration of the two Bogomol'nyi equations (43), preceded by the asymptotic solutions in the $r\gg 1$ region from which we will learn the restrictions on the possible values of the constant α. Concerning the solutions of Eqs. (43) in the $r\ll 1$ region, it is clear from Eqs. (44) and (45) that there is no distinction between the topological and the $N>0$ nontopological cases so that we give this solution for both these types of solitons forthwith:

$$f(r)=\pi-Ar^N, \quad a(r)=N-\frac{A^2}{N+1}r^{2(N+1)}. \tag{48}$$

For the nontopological soliton with $N=0$, however, we have the asymptotic value $f(0)=f_0$, where f_0 does not have to vanish for the field ϕ^α in Eqs. (41) to be single valued. Indeed, in our numerical integrations below, we have found nontrivial solutions for values of $f_0>0$ and only trivial solutions for $f_0=0$ and $f_0=\pi$. This leads to the following behavior for $N=0$, corresponding to Eqs. (48) of $N>0$, of the function $f(r)$ in the $r\ll 1$ region:

$$f(r)=f_0-\tfrac{1}{4}\sin^3 f_0(1+\cos f_0)r^2,$$

$$a(r)=-\tfrac{1}{2}\sin^2 f_0(1+\cos f_0)r^2. \tag{49}$$

A. Topological solitons

The topological properties of the solitons of our Chern-Simons gauged O(3) model are determined by the asymptotic conditions pertaining to the O(3) field ϕ only, and not the U(1) gauge field. Thus the relevant conditions in the case of the topological solitons are Eqs. (35) and (44) for the full field and its restriction to the radially symmetric configurations, respectively. The asymptotic properties of the U(1) gauge field, on the other hand, are given by Eqs. (48), both for the topological and nontopological cases.

The topological solitons of vorticity N have have degree N given by the two-dimensional ''volume'' integral of the density ϱ_0, Eq. (15). The topological charge, however, is the integral of the gauge-invariant density ϱ, Eq. (18), which, however, in this case reduces to the integral of ϱ since the one-dimensional ''surface'' integral vanishes by virtue of the asymptotic limit (44).

Since we have already solved the Bogomol'nyi equations (43) in the $r\ll 1$ region, there remains only to solve for these in the $r\gg 1$ region, where we find the power behavior

$$f(r)=\frac{C}{r^\alpha}, \tag{50}$$

$$a(r)=\alpha+\frac{C^4}{4(2\alpha-1)r^{2(2\alpha-1)}}. \tag{51}$$

We see from Eq. (51) that the positive constant α must satisfy the condition $\alpha>\frac{1}{2}$, and since the largest value of $a(r)$ is equal to $N=a(0)$, it follows that all integer values of the degree N satisfy this restriction. The value of the constant A in Eqs. (48) is fixed in the numerical integration. We have integrated Eqs. (43) with the lower sign for fields of vorticities $N=1$ and $N=2$. In the $N=1$ case, we have used $\alpha=0.75$, and in the $N=2$ case, $\alpha=0.75$ and 1.75. For the pairs $\{N,\alpha\}$, our numerical integrations yielded the following values of the constant A in Eqs. (48) to be

$$A\{1,0.75\}=6.7800, \quad A\{2,0.75\}=2.3933,$$

$$A\{2,1.75\}=7.0360.$$

The profiles of the functions $f(r)$ and $a(r)$ are exhibited in Figs. 1 and 2, respectively, for $N=1$ and $N=2$, and the

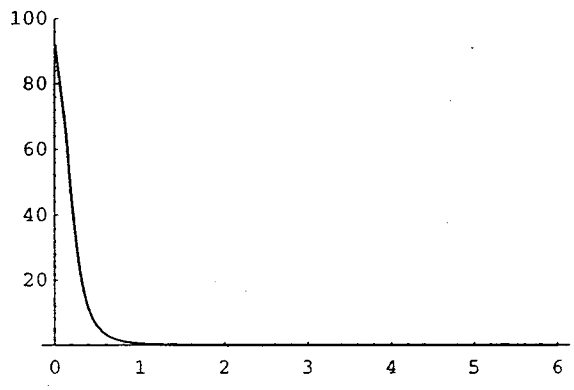

FIG. 3. Profile of the energy density corresponding to the $N=1$ *topological* soliton depicted in Fig. 1.

profiles of the energy densities of these solitons are given in Figs. 3 and 4, respectively.

The total energies $E\{N,\alpha\}$ corresponding to these solutions, labeled again by the pairs $\{N,\alpha\}$, were calculated from the numerical solutions to be

$$E\{1,0.75\}=3.9951, \quad E\{2,0.75\}=7.9992,$$

$$E\{2,1.75\}=7.9999,$$

from which we can confirm that the energy of an $N=2$ soliton is twice as large as the energy of an $N=1$ soliton. This is expected since the solutions in question are *(anti-)self-dual*. Not surprisingly, the numerical accuracy of this statement is better when the *same* value for the parameter α is employed in the numerical integrations of both the $N=1$ and $N=2$ solutions.

B. Nontopological solitons

There are two types of nontopological solutions to the (anti-)self-duality equations (43). The nontopological solitons with $N=0$ and the nontopological vortices with $N>0$, just as in the case of the Chern-Simons Higgs model analyzed in Ref. [13]. In both these cases, the function $f(r)$ tends to π according to Eqs. (45) at large r. In the region with $r\ll 1$, however, the asymptotic value of the $N>0$ vortices is given by Eqs. (45), while that of the $N=0$ soliton by Eqs. (46).

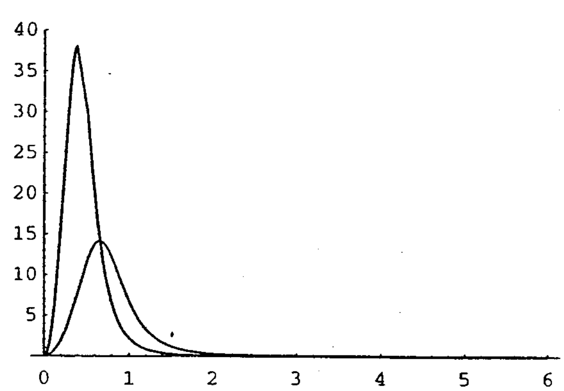

FIG. 4. Profiles of the energy densities corresponding to the $N=2$ *topological* solitons with $\alpha=0.75$ and $\alpha=1.75$, depicted in Fig. 2. The higher peak pertains to the larger $\alpha=1.75$ soliton.

Before we proceed to integrate the self-duality equations (43), it is important to consider the question of the stability of the nontopological solitons, since their stability is not gauarnteed by the topological criteria as for the topological solitons of the previous subsection. Here we notice that the degree of the soliton given by the two-dimensional ''volume'' integral of the density of ϱ_0 in Eq. (32) vanishes for all vorticities $N>0$. The second, one-dimensional ''surface'' integral in Eq. (32) however does *not* vanish in this case, but in fact becomes identical with the magnetic flux of the field configurations. This is because according to Eqs. (44) $\cos f(\infty)=-1$, which causes this second integrand to become equal to the U(1) magnetic flux density. Since this field configuration is a (anti-)self-dual solution, the energy of this solution is equal to the magnetic flux. Of course, the value of this magnetic flux is not quantized, but is determined by the arbitrary value of the parameter $\alpha=a(\infty)$ used in Eqs. (47). We are now in exactly the same situation as that of the nontopological vortices of the Chern-Simons Higgs model discussed in Ref. [13]. As in any Chern-Simons theory, the soliton carrying magnetic flux Φ must carry electric charge

$$Q=-\kappa\Phi, \tag{52}$$

which means that the energy of the nontopological soliton with a given charge Q is

$$E=|\Phi|=\frac{1}{\kappa}|Q|=\mu Q, \tag{53}$$

where μ is the scalar mass in the symmetric vacuum. Thus the argument for the stability of nontopological vortices in this Chern-Simons gauged O(3) model is identical to the corresponding stability analysis for the Chern-Simons gauged Higgs model given Ref. [13], to which we refer the reader for further details. Briefly, the nontopological vortices with $N>0$ are marginally stable, being at the threshold of decay into elementary excitations.

The situation with the $N=0$ nontopological soliton is different. In this case the energy is equal to the magnetic flux coming from the second term of Eq. (32), *plus* the contribution of the integral of the first term ϱ_0. In this case, however, this last integral does not vanish by virtue of the asymptotic condition (46) and contributes a positive amount, resulting in the inequality

$$E\geqslant|\Phi|=\frac{1}{\kappa}|Q|=\mu Q, \tag{54}$$

from which we must conclude that the $N=0$ topological soliton is unstable. In this connection, it would be interesting to consider the solitons of ungauged O(3) models in $2+1$ dimensions and to study their stability, in the same spirit as the corresponding works using O(2) models [16,17]. We intend to return to this question elsewhere.

As in the above subsection, the behaviors of the functions $a(r)$ and $f(r)$ in the $r\ll 1$ region are already known to be Eqs. (48), which hold also for the nonzero vorticity $N>0$ *nontopological* case at hand, while for the $N=0$ these are

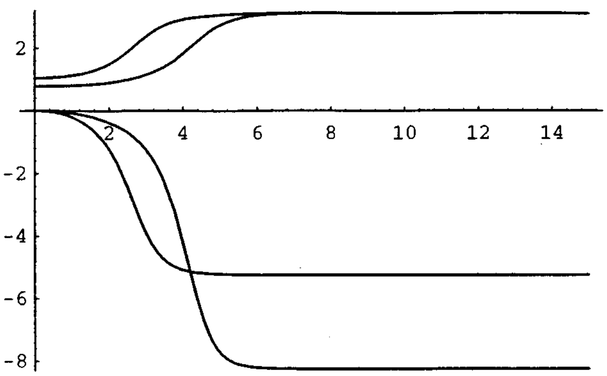

FIG. 5. Profiles of $f(r)$ and $a(r)$ of the *nontopological* Chern-Simons solitons of $N=0$, with $\alpha=-8.2480$ for which $f_0=\pi/4$ and with $\alpha=-5.2447$ for which $f_0=\pi/3$. The maximum of $f(r)$ occurs where $a(r)$ crosses the r axis. The higher $f(r)$ curve pertains to $f_0=\pi/3$.

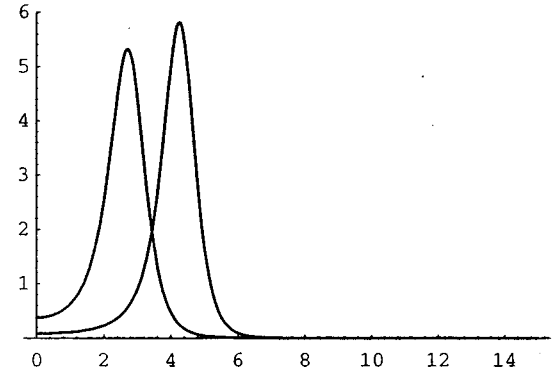

FIG. 6. Profiles of the energy densities corresponding to the $N=0$ *nontopological* solitons. The higher peak pertains to the soliton with $f_0=\pi/4$.

given by Eqs. (49). There remains, therefore, to solve Eqs. (43), with the lower signs, in the $r\gg 1$ region subject to the asymptotic conditions (36) or (45), which are the same for the $N>0$ and $N=0$ cases. We find the following solutions with power behavior:

$$f(r)=\pi+\frac{C}{r^{|\alpha|}}, \tag{55}$$

$$a(r)=\alpha+\frac{C^2}{(|\alpha|-1)r^{2(|\alpha|-1)}}. \tag{56}$$

We see from Eq. (56) that the negative constant α must satisfy the condition $|\alpha|>1$. For a given choice of the parameter α, the value of the parameter A, or equivalently the parameter f_0 in Eqs. (46) and (49) when $N=0$, is fixed by the numerical integration.

For the $N=0$ soliton, we have integrated Eqs. (43) numerically with two values of the parameter α. The value of the parameter f_0 for each pair $\{N,\alpha\}$ is fixed by the numerical integration to be

$$f_0\{0,-8.2480\}=\frac{\pi}{4}, \quad f_0\{0,-5.2447\}=\frac{\pi}{3}.$$

The profiles of the functions $f(r)$ and $a(r)$ for these solutions are given in Fig. 5, and the profiles of the energy densities of these solutions in Fig. 6. The total energies corresponding to these solutions were calculated numerically to be

$$E\{0,-8.2480\}=32.9921, \quad E\{0,-3.707\}=20.9788.$$

For the $N>0$ case, we seek a solution for which the function $f(r)$ increases from its value π at the origin, rises, and then descends asymptotically to its value π at infinity. This implies that the gradient of $f(r)$ changes its sign in this interval, as a consequence of which we see that the value α to which $f(r)$ tends at infinity must have the opposite sign of its value at the origin, and since we shall choose the latter value $a(0)=N$ to be positive, the required value of α is negative in this case.

We have integrated Eqs. (43) for $N=1$ and $N=2$ numerically with two values of the parameter α in each case. The value of the parameter A in Eqs. (48), for each pair $\{N,\alpha\}$, is then fixed by the numerical integration to be

$$A\{1;-3.0849\}=-0.2, \quad A\{1;-3.0343\}=-0.01,$$

$$A\{2;-4.2245\}=-0.1, \quad A\{2;-4.044\}=-0.01.$$

The profiles of the functions $f(r)$ and $a(r)$ for the solutions with $N=1$ and $N=2$ are exhibited, respectively, in Figs. 7 and 8 and the corresponding energy densities in Figs. 9 and 10. Again, the total energies corresponding to each of these solutions were calculated numerically to be

$$E\{1;-4.2245\}=16.3396, \quad E\{1;-4.0438\}=16.1371,$$

$$E\{2;-4.2245\}=24.8978, \quad E\{2;-4.0438\}=24.1754.$$

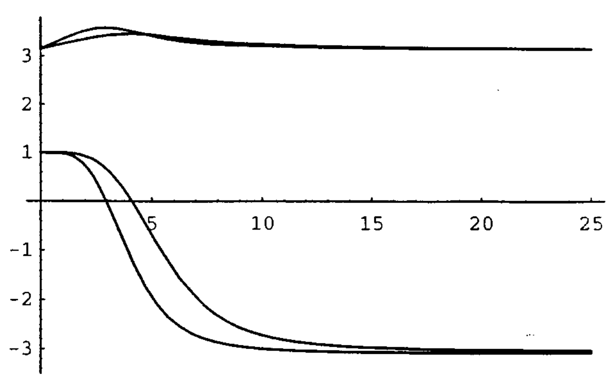

FIG. 7. Profiles of $f(r)$ and $a(r)$ of the *nontopological* Chern-Simons vortices of $N=1$ with $\alpha=-3.0849$ and -3.0343, where $f(\infty)=\pi$ and $a(\infty)=\alpha$. The maximum of $f(r)$ occurs where $a(r)$ crosses the r axis.

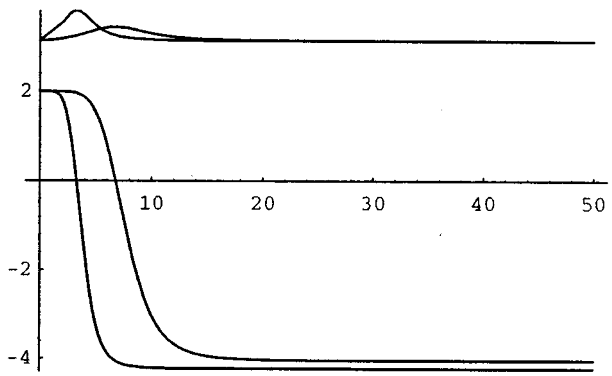

FIG. 8. Profiles of $f(r)$ and $a(r)$ of the *nontopological* Chern-Simons vortices of $N=2$ with $\alpha=-4.2244$ and -4.0438, where $f(\infty)=\pi$ and $a(\infty)=\alpha$. The maximum of $f(r)$ occurs where $a(r)$ crosses the r axis.

IV. RIGOROUS EXISTENCE RESULTS

We establish now the existence of topological and nontopological solitons in the two-dimensional Chern–Simons O(3) σ model.

Theorem 1. The self-dual equations (37) and (38) have two classes of radially symmetric solutions which satisfy the topological and nontopological boundary conditions (35) and (36), respectively. For each integer $N\geqslant 1$ the topological solutions have the flux

$$\Phi=2\pi N,$$

whereas for each $N\geqslant 0$ the nontopological solutions have the flux

$$\Phi=2\pi N+\pi\beta,$$

where β takes its value in the interval

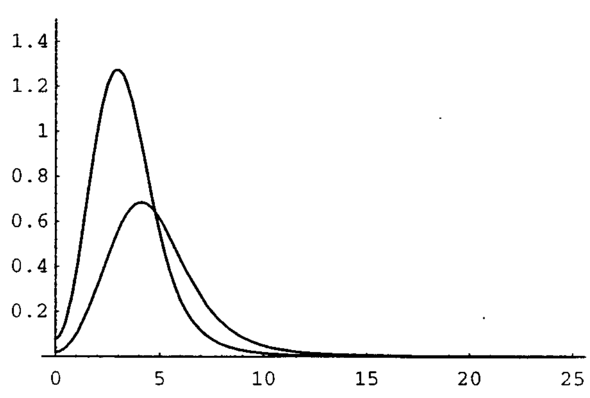

FIG. 9. Profiles of the energy densities corresponding to the $N=1$ *nontopological* solitons with $\alpha=-3.0343$ and $\alpha=-3.0849$, depicted in Fig. 7. The higher peak pertains to the $\alpha=-3.0849$ soliton.

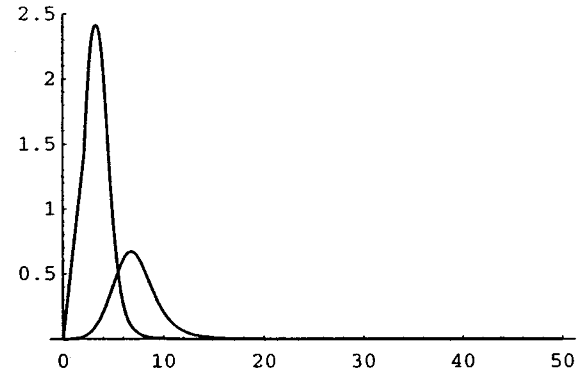

FIG. 10. Profiles of the energy densities corresponding to the $N=2$ *nontopological* solitons with $\alpha=-4.2244$ and $\alpha=-4.0438$, depicted in Fig. 8. The higher peak pertains to the $\alpha=-4.2244$ soliton.

$$\frac{16}{27}+\sqrt{\left(\frac{16}{27}\right)^2+\frac{32}{27}N(N+2)}$$

$$<\beta<4+\sqrt{16+N\left(\frac{27}{2}N+16\right)},$$

$$N\geqslant 0.$$

Remark. This theorem reveals an interesting feature which distinguishes our model from the self-dual Chern-Simons model of [11,12]: In the Chern-Simons case the exponent β, which labels the fractional electric charge and magnetic flux, is shown to be allowed to assume any given value in an explicitly determined interval [in [18] we established the exact result $\beta\in(2N+4,\infty)$]. In the model here, however, the exponent β is *confined* from the above.

With the substitution

$$u_1=\frac{\phi_1}{1+\phi_3},\quad u_2=\frac{\phi_2}{1+\phi_3},\quad u=u_1+iu_2,$$

we see that $\phi(p)=(0,0,-1)$ [or $\phi_3(p)=-1$] implies that p is a pole of integer degree for the complex function u and that $\phi(q)=(0,0,1)$ [or $\phi_3(q)=1$] implies that q is a zero of integer multiplicity for u. Suppose that the origin of $\mathbb{R}^2$ is the only pole of u and that the corresponding degree is $N\geqslant 1$. Then the new variable $v=\ln|u|^2=\varphi+\overline{\varphi}$ transforms the Bogomol'nyi equations into the elliptic equation

$$\Delta v=\frac{16e^{-v}}{(1+e^{-v})^3}-4\pi N\delta(x)\quad\text{in }\mathbb{R}^2. \tag{57}$$

The corresponding boundary conditions to be imposed on v are as follows.

(a) $v(x)\to-\infty$ as $|x|\to\infty$: topological solutions.

(b) $v(x)\to\infty$ as $|x|\to\infty$: nontopological solutions.

For a person not interested in a rigorous mathematical analysis, the rest of this section may be skipped.

Here are our basic existence results for Eq. (57), which give rise to the solutions stated in theorem 1 via standard realizations.

Theorem 2. For any integer $N\geqslant 1$, Eq. (55) has a family of topological solutions satisfying the property that for any $0<\varepsilon<1$ there is a solution to fulfill the asymptotic estimate

$$e^{v(x)}=O(|x|^{-(2N-\varepsilon)}) \quad \text{as} \quad |x|\to\infty.$$

These solutions are all radially symmetric about the origin of $\mathbb{R}^2$ and are strictly increasing with respect to the radial variable $r=|x|$. Further, there also holds the sharp decay estimate $|\nabla v(x)|=O(|x|^{-1})$ as $|x|\to\infty$ for the solutions.

Theorem 3. For given integer $N\geqslant 0$, Eq. (55) has a nontopological solution v satisfying the asymptotic property

$$e^{v(x)}=O(|x|^{\beta}) \quad \text{as} \quad |x|\to\infty,$$

and that in the radial variable $r=|x|$ there holds

$$\lim_{r\to\infty} r v_r(r)=-\beta,$$

where the constant β lies in the interval stated in theorem 1. The solution is radially symmetric about the origin, and there is exactly one point $r_0>0$ so that with the radial variable $r=|x|$ the solution is increasing in the region $r<r_0$, but decreasing in $r>r_0$, and the maximum

$$v(r_0)=\max_{r>0}\{v(r)\}\equiv e^{-a}$$

can be arbitrarily prescribed for the number a in the range $a\geqslant \ln 2$. In other words, we again have a continuous family of distinct solutions to realize the same prescribed ''vortex'' charge and location. Besides, there also holds the decay estimate for the derivatives, $|\nabla v(x)|=O(|x|^{-1})$ as $|x|\to\infty$ for all the solutions.

These results will be established in the next two sections. We only remark here that it is sometimes convenient to use $w=-v$. Then w satisfies the following usual ''vortex'' equation

$$\Delta w=-\frac{16e^{w}}{(1+e^{w})^3}+4\pi N\delta(x). \tag{58}$$

Remark. Theorem 2 says that even in the radial case topological solitons are not unique for given topological charge N which is in contrast to the uniqueness result proven in [18] for the self-dual Chern-Simons vortices. This clearly indicates that the model here possesses some new characteristics. In fact such a conclusion is already further evidenced in theorem 3 above (see also the remark following theorem 1).

A. Topological solitons: Proof of theorem 2

We shall first study topological solutions. Let $w=2N\ln|x|+\eta$. Then Eq. (56) becomes

$$\Delta\eta=-\frac{16|x|^{2N}e^{\eta}}{(1+|x|^{2N}e^{\eta})^3}. \tag{59}$$

Remember that we want to achieve the asymptotic behavior $w(x)\to\infty$ at infinity. Thus it is sufficient to find a solution of Eq. (57) so that $\eta(x)\geqslant -c\ln|x|$ asymptotically for some $c<2N$. For this purpose, we set $r=|x|$ and introduce the new variables $t=\ln r$ and $\zeta=-\eta$. Then, with $\zeta'=d\zeta/dt$, Eq. (57) is turned into

$$\zeta''=\frac{16e^{2t}e^{-4Nt+2\zeta}}{(1+e^{-2Nt+\zeta})^3}\equiv f(t,\zeta). \tag{60}$$

It is important to record the partial derivative

$$\frac{\partial f(t,\zeta)}{\partial\zeta}=\frac{16e^{2t-4Nt+2\zeta}}{(1+e^{-2Nt+\zeta})^4}(2-e^{-2Nt+\zeta}). \tag{61}$$

Consequently, there holds the bound

$$\sup_{\zeta} f(t,\zeta)=\frac{64e^{2t}}{27} \tag{62}$$

and

$$f(t,\cdot)\ \text{is increasing for}\ \zeta\ \text{in the region}\ \zeta<\ln 2+2Nt. \tag{63}$$

We now supplement Eq. (58) with the initial data

$$\zeta(-\infty)=-a,\quad \zeta'(-\infty)=0. \tag{64}$$

As a preparation we establish *Lemma 1. For any $a\in\mathbb{R}$, Eq. (58) subject to the initial condition (62) has a unique global solution in the entire interval $-\infty<t<\infty$.*

Proof. It is standard to consider the equivalent form of the problem in the form of the integral equation

$$\zeta(t)=-a+\int_{-\infty}^{t}(t-s)f(s,\zeta(s))ds,\quad t\in\mathbb{R}. \tag{65}$$

Let T be such that

$$\int_{-\infty}^{T}(T-s)f(s,\zeta(s))ds<1 \tag{66}$$

and that, with $\zeta(s)$ satisfying $|\zeta(s)|<|a|+1$ (say), there holds

$$\int_{-\infty}^{T}(T-s)\left|\left(\frac{\partial}{\partial\zeta}f(s,\zeta)\right)_{\zeta=\zeta(s)}\right|ds<\frac{1}{2}. \tag{67}$$

In view of Eqs. (59) and (60), the existence of such a number T to ensure Eqs. (64) and (65) is clearly guaranteed. Therefore we can use Eq. (63) to define a convergent Picard successive iteration scheme for functions over $(-\infty,T]$. The limit function solves Eq. (63) on $(-\infty,T]$. Using Eq. (60) again we can extend the solution to the entire line $(-\infty,\infty)$. In fact, we can set

$$\zeta_n(t)=-a+\int_{-\infty}^{t}(t-s)f(s,\zeta_{n-1}(s))ds,\quad t\leqslant T,$$

$$\zeta_0(t)=-a,$$

$$n=1,2,\ \dots\ .$$

For any n, there holds by virtue of Eqs. (64) and (65) the uniform bound $\|\zeta_n\|_\infty<|a|+1$ and the recursive bound

$\|\zeta_{n+1}-\zeta_n\|_\infty \leqslant (1/2^n)$. Thus $\{\zeta_n\}$ is convergent uniformly over $(-\infty,T]$ to a solution ζ of Eq. (63), $-\infty<t\leqslant T$. Local uniqueness also follows from the inequality (65).

Thus the lemma is proved.

Using Eq. (60) in Eq. (63), we find that for any $t_0 \leqslant 0$ there holds

$$\zeta(t_0)\leqslant -a+\int_{-\infty}^{0}|s|\frac{128}{27}e^{2s}ds<-a+C_0, \tag{68}$$

where C_0 is a constant independent of a and t_0. On the other hand, with

$$\zeta'(t)=\int_{-\infty}^{t} f(s,\zeta(s))ds$$

and Eq. (60), we have for any $1>\varepsilon>0$ the existence of a $t_0<0$ to ensure that

$$\zeta'(t)\leqslant \frac{\varepsilon}{2},\quad t\leqslant t_0, \tag{69}$$

where t_0 is independent of a.

Because of Eq. (67), we can let $t_1>t_0$ be such that $\zeta'(t)<\varepsilon$ for $t\in(t_0,t_1)$. Then it follows from Eq. (66) that

$$\zeta(t)<-a+C_0+\varepsilon(t-t_0) \tag{70}$$

for $t\in(t_0,t_1)$.

Lemma 2. We can choose suitable $a>0$ *so that Eq. (68) holds for all* $t>t_0$.

Proof. Consider the right-hand side of Eq. (68) first. It is seen that we can let $a>0$ be sufficiently large to make

$$-a+C_0+\varepsilon(t-t_0)<\ln 2+2Nt,\quad t>t_0. \tag{71}$$

In the region where Eq. (68) holds, by virtue of Eqs. (69) and (61), we have

$$\begin{aligned}\zeta(t)&=\zeta(t_0)+\int_{t_0}^{t}(t-s)f(s,\zeta(s))ds\\ &<-a+C_0+\int_{t_0}^{t}(t-s)f(s,-a+C_0+\varepsilon(s-t_0))ds\\ &<-a+C_0+16e^{2(-a+C_0-\varepsilon t_0)}\int_{t_0}^{t}(t-s)e^{-2(2N-1-\varepsilon)s}ds.\end{aligned} \tag{72}$$

Since the integrand on the right-hand side of Eq. (70) for $t>t_0$ is bounded in view of $\varepsilon<1$, we can choose $a>0$ large to make Eq. (68) valid.

The lemma is proved.

Consequently, we have just constructed topological solutions of arbitrary vortex charge N. Our method is essentially a shooting argument starting from $-\infty$.

B. Nontopological solitons: Proof of theorem 2

We next prove the existence of nontopological solutions. The above method does not seem to work out well. Instead, we employ a shooting argument starting from somewhere in the middle and we shoot designated target data at $\pm\infty$.

In this case Eq. (56) is most convenient. Again, we use the variable $t=\ln r$. Then we are to solve the following two-point boundary value problem:

$$\begin{gathered} w''=-\frac{16e^{2t}e^{w}}{(1+e^{w})^3},\\ w'(-\infty)=2N,\quad w(\infty)=-\infty. \end{gathered} \tag{73}$$

The property of a solution of Eq. (71) is transparent from the equation and the boundary asymptotics. In fact, it is seen that $w(\pm\infty)=-\infty$ and that w is concave. Hence there is a unique maximum in the middle. Our shooting argument will start from this *unknown* maximum point, say, t_0. More precisely, to approach Eq. (71), we choose to consider the initial value problem

$$\begin{gathered} w''=-\frac{16e^{2t}e^{w}}{(1+e^{w})^3},\\ w(t_0)=-a,\quad w'(t_0)=0. \end{gathered} \tag{74}$$

It is easy to show that Eq. (72) has a unique global solution for any pair of numbers a and t_0 given. Define a function $h(a,t_0)$ of a,t_0 by

$$h(a,t_0)=\lim_{t\to-\infty}w'(t)=\int_{-\infty}^{t_0}g(s,w(s))ds,$$

$$g(s,w)=\frac{16e^{2t}e^{w}}{(1+e^{w})^3}. \tag{75}$$

As in the last section, we record here some properties of g:

$$\frac{\partial}{\partial w}g(s,w)=16e^{2s+w}\frac{1-2e^{w}}{(1+e^{w})^4},\quad \sup_w g(s,w)=\frac{64}{27}e^{2s}. \tag{76}$$

By the fact that w depends on a,t_0 continuously and that Eqs. (74) imply the uniform convergence of the integral in Eqs. (73), we conclude that $h(a,t_0)$ is a continuous function of the parameters a,t_0.

Lemma 3. For any given $a\geqslant \ln 2$, *there exists at least one* $t_0=t_0(a)$ *so that the function h defined in Eq. (73) satisfies*

$$h(a,t_0(a))=2N. \tag{77}$$

Recall that Eq. (75) is the boundary condition at $t=-\infty$ we would like to achieve in Eq. (71). To prove Eq. (75), it suffices to establish the following two lemmas.

Lemma 4. For any $a\in\mathbb{R}$ *we can find a suitable* t_0 *such that*

$$h(a,t_0)<2N.$$

Proof. From Eq. (72), we have $w''>-16e^{2t+w}$. Set $W=2t+w$. Then,

$$W'' > -16e^{W}. \tag{78}$$

Since $w'(t_0)=0$, we have $w'(t)>0$ for $t<t_0$. Hence $W'(t)>2$ for $t<t_0$. Therefore, multiplying Eqs. (76) by W' and integrating over $-\infty<t<t_0$, we obtain

$$4-[W'(t)]^2 \geqslant 32(e^{W(t)}-e^{2t_0-a}).$$

Returning to the original function w, we have the estimate

$$0<w'(t)<2\sqrt{1+8e^{2t_0-a}}-2\equiv\kappa, \quad t<t_0. \tag{79}$$

In particular, $h(a,t_0)\leqslant 2\sqrt{1+8e^{2t_0-a}}-2$. Consequently, we may choose t_0 sufficiently negative to make $h(a,t_0)<2N$, which proves the lemma.

Lemma 5. For any $a\geqslant \ln 2$, *there is a suitable* t_0 *so that*

$$h(a,t_0)>2N.$$

Proof. First we notice that, in view of Eqs. (74), the function $g(t,\cdot)$ is increasing in the interval $(-\infty,-\ln 2]$. On the other hand, Eq. (77) gives us

$$-\ln 2 \geqslant -a > w(t) > -a-\kappa(t_0-t), \quad t<t_0.$$

Consequently, we are led to the inequality

$$w''(t) < -\frac{16e^{2t-a-\kappa(t_0-t)}}{(1+e^{-a-\kappa(t_0-t)})^3}. \tag{80}$$

Now integrate Eq. (78) over the interval $(-\infty,t_0]$. We obtain the lower estimate

$$\begin{aligned} h(a,t_0)=w'(-\infty) &\geqslant \int_{-\infty}^{t_0}\frac{16e^{2t-a-\kappa(t_0-t)}}{(1+e^{-a-\kappa(t_0-t)})^3}dt \\ &> 2\int_{-\infty}^{t_0} e^{2t-a-\kappa(t_0-t)}dt \\ &= 2e^{-a-\kappa t_0}\int_{-\infty}^{t_0} e^{(2+\kappa)t}dt = \frac{e^{2t_0-a}}{\sqrt{1+8e^{2t_0-a}}}. \end{aligned} \tag{81}$$

Here the last line is derived using the definition of κ given in Eq. (77). The form of the right-hand side of Eq. (79) is crucial for the existence of a sufficiently large t_0 to make $h(a,t_0)>2N$.

The lemma is established.

Finally, the continuity of h and the conclusions in lemmas 4 and 5 ensure the existence of at least one $t_0=t_0(a)$ to satisfy (75). Thus lemma 3 is proved.

In the rest of this section, we always assume that a,t_0 are so chosen that Eq. (75) is valid. We prove that the other boundary condition in Eq. (71), $w(\infty)=-\infty$, is automatically observed. More precisely, we state *Lemma 6. There is a number* $\beta>2$ *so that*

$$\lim_{t\to\infty} w'(t) = -\beta. \tag{82}$$

In particular, $w(t)\to-\infty$ *as* $t\to\infty$.

Proof. Integrating Eq. (72) over $(-\infty,t]$, we write

$$w'(t)=2N-\int_{-\infty}^{t}\frac{16e^{2s}e^{w(s)}}{(1+e^{w(s)})^3}ds. \tag{83}$$

Since $w<0$, we have

$$2N-\int_{-\infty}^{t}16e^{2s}e^{w(s)}ds < w'(t) < 2N-\int_{-\infty}^{t}8e^{2s}e^{w(s)}ds. \tag{84}$$

Therefore the integral

$$\int_{-\infty}^{\infty} e^{2s}e^{w(s)}ds \tag{85}$$

must be convergent. In fact, if it is not, then by Eq. (82) $w'(t)\to-\infty$ as $t\to\infty$, which leads to the convergence of Eq. (83), contradicting the original assumption. By virtue of the convergence of Eqs. (83) and (81), we see that Eq. (80) holds with $\beta>2$ as expected.

Lemma 7. The number β *in lemma 6 actually satisfies*

$$\frac{16}{27}+\sqrt{\left(\frac{16}{27}\right)^2+\frac{32}{27}N(N+2)} < \beta < 4+\sqrt{16+N\left(\frac{27}{2}N+16\right)}. \tag{86}$$

Proof. Using the inequality $w\leqslant -\ln 2$ in Eq. (72), we have the crude bounds for the second derivative of w:

$$-16e^{2t+w}\leqslant w'' \leqslant -\frac{128}{27}e^{2t+w}, \quad -\infty<t<\infty. \tag{87}$$

Since $w'(t)>0$ for $t<t_0$, multiplying Eq. (85) by w', integrating over $(-\infty,t_0]$, and noting that

$$0<\int_{-\infty}^{t_0} e^{2t+w}w'dt = e^{2t_0-a}-2\int_{-\infty}^{t_0}e^{2t+w}dt,$$

we obtain

$$\frac{128}{27}\left(e^{2t_0-a}-2\int_{-\infty}^{t_0}e^{2t+w}dt\right) \leqslant 2N^2 \leqslant 16\left(e^{2t_0-a}-2\int_{-\infty}^{t_0}e^{2t+w}dt\right). \tag{88}$$

Similarly, over the interval $[t_0,\infty)$, multiplying Eq. (85) by $w'<0$, integrating, and noting that lemma 6 implies that $e^{2t+w(t)}\to 0$ as $t\to\infty$, we obtain

$$\frac{128}{27}\left(e^{2t_0-a}+2\int_{t_0}^{\infty}e^{2t+w}dt\right) \leqslant \frac{1}{2}\beta^2 \leqslant 16\left(e^{2t_0-a}+2\int_{t_0}^{\infty}e^{2t+w}dt\right). \tag{89}$$

By virtue of Eqs. (86) and (87), we find

$$\frac{1}{4}\beta^2-\frac{27}{8}N^2\leqslant 16\int_{-\infty}^{\infty}e^{2t+w}dt\leqslant\frac{27}{32}\beta^2-N^2. \tag{90}$$

However, taking $t\to\infty$ in Eq. (82), we have

$$2N+\beta<16\int_{-\infty}^{\infty}e^{2t+w}dt<2(2N+\beta). \tag{91}$$

Inserting Eq. (89) into Eq. (88), we see that the decay exponent β satisfies the two quadratic inequalities

$$\tfrac{27}{32}\beta^2-\beta>N^2+2N, \tag{92}$$

$$2\beta^2-16\beta<27N^2+32N. \tag{93}$$

Solving Eqs. (90) and (91), we arrive immediately at the range (84) and the lemma is proved.

To get nontopological solutions with $N=0$, we study the problem

$$(rw_r)_r=-\frac{16re^w}{(1+e^w)^3}, \tag{94}$$

$$w(0)=w_0,\quad \lim_{r\to 0}rw_r=0.$$

It is easily seen that Eq. (92) has a global solution for any $w_0\in\mathbb{R}$ and that the solution must satisfy $w(r)\to-\infty$ as $r\to\infty$. In fact, we can show that $rw_r(r)\to-\beta$ as $r\to\infty$, with $\beta>2$ lying in the correct interval again.

V. SUMMARY

We have presented above a Chern-Simons gauged O(3) model in $2+1$ dimensions and found (anti-)self-dual solitons. The dynamics of the U(1) gauge field in our model is controlled by a Chern-Simons term only and excludes the Maxwell term. In this sense it is complementary to the model proposed by Schroers in Ref. [9], which features a Maxwell term exclusively. The importance of these models is that thay admit (anti-)self-dual solutions, unlike the Maxwell [8] and Chern-Simon [7] U(1) gauged CP^1 models which we have shown here *do not* admit (anti-)self-dual solutions.

We were able to integrate the relevant Bogomol'nyi equations only asymptotically, and the full integrations were performed numerically. The solitons found are of two categories, *topological* and *nontopological*. We found that *topological* and *nontopological* vortices of arbitrary degree N exist, as well as *nontopological* solitons of degree $N=0$. This contrasts with the restriction on the existence [19] of the (topological) solitons of the Maxwell O(3) model of Ref. [9], where it was found that the $N=1$ soliton did not exist. Like their purely Maxwell gauged counterparts [9], the *topological* vortices are stabilized by the degree N and not by the magnetic flux, which takes on an arbitrary value, unlike the topological vortices of the Maxwell and Chern-Simons Higgs models [10,13]. By contrast, the *nontopological* solitons of vorticity $N>0$, for which the degree $\int d^2x\,\varrho_0=0$ vanishes, are stabilized by the magnetic flux in exactly the same way that the corresponding *nontopological* solitons of the Chern-Simons Higgs model [13] are. The detailed numerical integrations were performed for *topological* and *nontopological* solitons of degrees $N=1$ and $N=2$, and for the *nontopological* $N=0$ soliton. To underpin our results mathematically, analytic proofs for the existence of these solutions were supplied as well.

The qualitative features of our numerical results are exhibited in Figs. 1–10 The profiles of the functions $f(r)$ and $a(r)$, which are of the expected shapes, are given in Figs. 1 and 2 for the topological vortices with $N=1,2$, Fig. 5 for the nontopological soliton with $N=0$, and Figs. 7 and 8 for the nontopological vortices with $N=1,2$. The profiles of the energy densities of the $N=1,2$ topological vortices are given, respectively, in Figs. 3 and 4, those of the $N=0$ solitons in Fig. 6, and for the $N=1,2$ nontopological vortices respectively in Figs. 9 and 10. We note that all these profiles describe ring-shaped energy densities.

To put our results into context, we note that in $(2+1)$-dimensional U(1) gauged models it is possible to describe the dynamics of the U(1) field *either* by a Maxwell term *or* by a Chern-Simons term. In each of these cases, it is possible to establish topological inequalities which confer (topological) stability on the resulting static soliton solutions, while in the Chern-Simons gauged model there occur also nontopological solitons. We ignore here the more general case where both the Maxwell and Chern-Simons terms are present in the Lagrangian, because in that case we were not able to establish the required topological inequalities.

The main physical interest of all these models is as solitons of $(2+1)$-dimensional theories which may be relevant to the theory of superconductivity [10] and especially to anyonic dynamics [2] in that context. We do not discuss here the relative merits of these models from a physical viewpoint, except to remark that in the case where the solitons are (anti-)self-dual it is possible to identify *attractive* and *repulsive* phases both in the Maxwell Higgs [10] and the Chern-Simons Higgs [20] models by allowing the dynamics to deviate from one that allows (anti-)self-dual solutions. We expect that this is the case with the present Chern-Simons O(3) model as also it should be for the Maxwell O(3) model, both of which support (anti-)self-dual solutions. This does not, however, mean that in the absence of (anti-)self-dual solutions it is impossible to find *attractive* and *repulsive* phases, as were shown to exist in the Maxwell CP^1 model by the addition of a suitable Skyrme term [8].

In addition to the above motivations, these $(2+1)$-dimensional gauged σ models can serve as the prototypes of the analogous $(3+1)$-dimensional models. Already a four-dimensional gauged Grassmannian model [21] was proposed which supports instanton solutions [22] on $\mathbb{R}_4$. The latter SU(2) gauged Grassmanian model [21] on $\mathbb{R}_4$ was motivated by the U(1) gauged Grassmanian CP^1 models of Refs. [8,7], which are the simplest examples of a gauged Grassmannian model. In analogy with the U(1) gauged O(3) models of Ref. [9] and the present paper, it would be interesting to find a gauged O(4) σ model, namely, to gauge the usual Skyrme model [5], which can support stable static solitons on $\mathbb{R}_3$. Such an SO(3) gauged model is at present under active investigation and preliminary results [23] have been already obtained. Moreover, it should be possible to gauge an O$(d+1)$ σ model on $\mathbb{R}_d$ with gauge group SO(d) and obtain localized finite action lumps.

Note added. After completing this work, we became aware of two works in which similar results to ours are obtained. One of these, by Ghosh and Ghosh [24], employs the same Chern-Simons gauged O(3) models as ours (33). The other, by Kimm, Lee, and Lee [25] employs a more general model including ours, Eq. (33), featuring the potential (34). Also, we thank R. Jackiw for bringing to our attention the work of Nardelli [26], which also deals with the problem of gauging the O(3) σ model with Chern-Simons dynamics. Our work differs from that of Ref. [26] in that our gauge group is the U(1) as opposed to SO(3) in [26] and, also, in that our solution involves a dynamical U(1) field as opposed to a composite connection field configuration in Ref. [26].

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