Noncommutative geometric regularization

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Studies in string theory and in quantum gravity suggest the existence of a finite lower bound to the possible resolution of lengths which, quantum theoretically, takes the form of a minimal uncertainty in positions Δx_0. A finite minimal uncertainty in momenta Δp_0 has been motivated from the absence of plane waves on generic curved spaces. Both effects can be described as small noncommutative geometric features of space-time. In a path integral approach to the formulation of field theories on noncommutative geometries, we can now generally prove IR regularization for the case of noncommutative geometries which imply minimal uncertainties Δp_0 in momenta. [S0556-2821(96)02420-4]

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I. INTRODUCTION

As has long been known, the resolution of very small scales requires high energetic test particles which, through their gravitational effect, will eventually significantly disturb the space-time structure which was probed. This problem has been approached from several directions and studies in string theory and quantum gravity suggest that, quantum theoretically, a lower bound to the resolution of distances could take the form of a finite minimal position uncertainty Δx_0 of the order of the Planck length of $\approx 10^{-35}m$, see [1–4]. On the other hand, on large scales, there is no notion of plane waves or momentum eigenvectors on generic curved spaces. It has, therefore, been suggested that quantum theoretically there could then exist lower bounds Δp_0 to the possible determination of momentum [7–9].

Independent of the suggested mechanisms for the origins of minimal uncertainties both types of effects, i.e., a Δx_0 or a Δp_0, can be described as small noncommutative geometric corrections to space-time and/or energy-momentum space [5–12].

Intuitively, the presence of finite minimal uncertainties $\Delta x_0, \Delta p_0$ should have UV and IR regularizing effect in field theory. This would imply that minimal uncertainties may also formally be used as UV and/or IR regulators. The example of Euclidean ϕ^4 theory on a restricted class of such noncommutative geometries has been studied in detail and both UV and IR regularizations have been shown for this case [7–10].

Our aim here is to prove the IR regularity of Euclidean propagators $1/(\mathbf{p}^2+m^2c^2)$ for all noncommutative geometries with a minimal uncertainty in momentum Δp_0, both for $m>0$ and for $m=0$.

II. NONCOMMUTATIVE GEOMETRIES WITH MINIMAL UNCERTAINTIES

We consider the possibility of small "noncommutative geometric" corrections to the canonical commutation relations in the associative Heisenberg algebra $\mathcal{A}$ generated by the $\mathbf{x}_i, \mathbf{p}_j$, see [5–10]:

$$[\mathbf{x}_i,\mathbf{p}_j]=i\hbar(\delta_{ij}+\alpha_{ijkl}\mathbf{x}_k\mathbf{x}_l+\beta_{ijkl}\mathbf{p}_k\mathbf{p}_l+\cdots), \quad (1)$$

and also

$$[\mathbf{x}_i,\mathbf{x}_j]\neq 0, \quad [\mathbf{p}_i,\mathbf{p}_j]\neq 0 \quad (2)$$

with the involution $\mathbf{x}_i^*=\mathbf{x}_i, \mathbf{p}_i^*=\mathbf{p}_i$.

A priori, we formulate field theories on generic noncommutative background "geometries" $\mathcal{A}$ which may or may not have certain symmetries, similar to the case of curved background geometries. Nontrivial examples of non-Lorentz symmetric noncommutative background geometries have been studied in [5–10]. Lorentz symmetric examples of suitable noncommutative background geometries were found in [11].

The correction terms necessarily imply new physical features, since unitary transformations generally preserve the commutation relations. Here, for appropriate small α,β one obtains ordinary quantum-mechanical behavior at medium scales while the presence of small α and β imply modified IR and UV behavior, respectively.

The uncertainty relations, holding in all * representations of the commutation relations on some dense domain $D\subset H$ in a Hilbert space H, are of the form $\Delta A \Delta B \geq (1/2)|\langle[A,B]\rangle|$ so that $[\mathbf{x}_i,\mathbf{x}_j]\neq 0$, yields $\Delta x_i \Delta x_j \geq 0$. The noncommutativity implies that the $\mathbf{x}_i$ (as well as the $\mathbf{p}_i$) can no longer be simultaneously diagonalized. Because of Eq. (1) and the corresponding uncertainty relations, there can appear the even more drastic effect that the $\mathbf{x}_i$ (as well as the $\mathbf{p}_i$) may also not be separately diagonalizable.

Already in one dimension the uncertainty relation (assuming small positive α,β with $\alpha\beta<1/\hbar^2$ and neglecting higher order corrections)

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1+\alpha(\Delta x)^2+\alpha\langle\mathbf{x}\rangle^2+\beta(\Delta p)^2+\beta\langle\mathbf{p}\rangle^2] \quad (3)$$

implies nonzero minimal uncertainties in $\mathbf{x}$ as well as in $\mathbf{p}$ measurements: $\Delta x_0=(1/\beta\hbar^2-\alpha)^{-1/2}$, $\Delta p_0=(1/\alpha\hbar^2$

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$-\beta)^{-1/2}$. For $\alpha=0$ and a small β we cover the example of an ultraviolet-modified uncertainty relation that has been discussed in string theory and quantum gravity, for a review see [4]. For all physical domains D, i.e., for all * representations of the commutation relations, there are now no physical states in the minimal uncertainty ''gap''

$$\nexists|\psi\rangle\in D: \quad 0\leqslant(\Delta x)_{|\psi\rangle}<\Delta x_0, \tag{4}$$

$$\nexists|\psi\rangle\in D: \quad 0\leqslant(\Delta p)_{|\psi\rangle}<\Delta p_0, \tag{5}$$

where $|\psi\rangle$ generally stands for a normalized element of D. Thus, unlike on ordinary geometry, there now do not exist sequences $\{|\psi_n\rangle\}$ of physical states which would approximate point localizations in position or momentum space, i.e., for which the uncertainty would decrease to zero: $\nexists|\psi_n\rangle\in D:\lim_{n\to\infty}(\Delta x)_{|\psi_n\rangle}=0$.

Technically, the new infrared and ultraviolet behavior has important consequences for the representation theory. For example, a finite minimal uncertainty Δx_0 in positions implies that the commutation relations do no longer find a spectral representation of $\mathbf{x}$, so that one has to resort to other Hilbert space representations.

The interplay between the functional analysis of the position and the momentum operators was first studied in [5,6]. In fact, we are giving up (essential) self-adjointness of the $\mathbf{x}$ and $\mathbf{p}$ operators, to retain only their symmetry. While giving up essential self-adjointness is necessary for the description of the new short distance behavior, the symmetry is sufficient to guarantee that all physical expectation values are real and also that uncertainties can be calculated applying the usual definition of the standard deviation: $\Delta x=\langle\psi|(\mathbf{x}-\langle\psi|\mathbf{x}|\psi\rangle)^2|\psi\rangle^{1/2}$. Nevertheless, this is a nontrivial step which goes beyond the conventional quantum-mechanical treatment, and it also goes beyond Connes' ''dictionary'' [14] of how to treat ''real variables'' on noncommutative geometries.

The key observation is that although self-adjoint extensions and (discrete) diagonalizations of $\mathbf{x}$ or $\mathbf{p}$ exist in H, under the circumstances described, these diagonalizations are not in any common domain, i.e., not in any physical domain, of $\mathbf{x}$ and $\mathbf{p}$ [5,6]. Instead, there is now the finite uncertainty ''gap'' separating the physical states from formal $\mathbf{x}$ or $\mathbf{p}$ eigenstates. For details and proofs, see [5,6,10–12].

The physical states of maximal localization in the presence of minimal uncertainties have, in the meanwhile, been extensively studied, first in the special case $\alpha=0$, $\beta>0$, see [11] and recently also in the general (though one-dimensional) case $\alpha,\beta>0$, see [12]. Explicitly, the physical states $|\phi_\xi^{mlx}\rangle$, $|\phi_\pi^{mlp}\rangle$ which realize the maximal localization in positions or momenta obey

$$\Delta x_{|\phi_\xi^{mlx}\rangle}=\Delta x_0, \quad \langle\phi_\xi^{mlx}|\mathbf{x}|\phi_\xi^{mlx}\rangle=\xi,$$
$$\langle\phi_\xi^{mlx}|\mathbf{p}|\phi_\xi^{mlx}\rangle=0, \tag{6}$$

and similarly for $|\phi_\pi^{mlp}\rangle$. The projection $\langle\phi_\xi^{mlx}|\psi\rangle$ is then the probability amplitude for finding the particle maximally localized around ξ. For $\alpha,\beta\to0$, one recovers the position and the momentum eigenvectors. Studies in n dimensions are in progress. We here only state one key result, the mutual projection of maximal localization states:

$$\langle\psi_{\xi'}^{mlx}|\psi_\xi^{mlx}\rangle=\frac{1}{\pi}\left[\frac{\xi-\xi'}{2\hbar\sqrt{\beta}}-\left(\frac{\xi-\xi'}{2\hbar\sqrt{\beta}}\right)^3\right]^{-1}\sin\left(\frac{\xi-\xi'}{2\hbar\sqrt{\beta}}\pi\right). \tag{7}$$

It is the generalization of the Dirac δ function which, on ordinary geometry, would be obtained from projecting maximal localization states, i.e., then from projecting position eigenstates onto another: $\langle x|x'\rangle=\delta(x-x')$. The nonmultiplicativity of δ distributions is related to the appearance of ultraviolet divergencies, whereas the behavior of Eq. (7) (note that the singularities of its first factor are canceled by zeros of the sine) suggests UV regularity in field theory.

Concerning the infrared we remark that because of the correction terms, the momenta $\mathbf{p}_i$ no longer generate translations on flat space. Under certain conditions, the $\mathbf{p}_i$ do generate translations of normal coordinate frames on curved spaces. As was shown in [9], translations on curved space, and therefore the momenta defined to generate these translations, generally do not commute and lead to commutation relations of the type of Eqs. (1) and (2) (with $[\mathbf{x}_i,\mathbf{x}_j]=0$). This then allows us to explicitly investigate the relation between the absence of plane waves (i.e., of $\mathbf{p}$ eigenstates) and the presence of a minimal uncertainty in momentum.

While the possible origins of minimal uncertainties need further investigation we will in the following focus only on the field theoretical consequences of minimal uncertainties.

III. PATH INTEGRATION

We adopt the ansatz for the formulation of field theories on noncommutative geometries given in [6–10]. The partition function of charged Euclidean ϕ^4 theory in natural units $c=1=\hbar$ (with μ the mass and N a constant),

$$Z[J]\equiv N\int D\phi\exp\left(\int d^4x\,\phi^*(\partial_i\partial_i-\mu^2)\phi-\frac{\lambda}{4!}(\phi\phi)^*\phi\phi+\phi^*J+J^*\phi\right), \tag{8}$$

we write in the form

$$Z[J]=N\int_D D\phi\exp\left[-\mathrm{tr}\left(\frac{l^2}{\hbar^2}(\mathbf{p}^2+m^2c^2)\cdot|\phi\rangle\langle\phi|-\frac{\lambda l^4}{4!}|\phi^*\phi\rangle\langle\phi^*\phi|+|\phi\rangle\langle J|+|J\rangle\langle\phi|\right)\right]. \tag{9}$$

In order to make the units transparent, we reintroduced c and $\hbar$, and we introduced an arbitrary positive length l to render the fields unitless (l could trivially be reabsorbed in the fields).

Equation (8) is recovered from Eq. (9) by assuming the ordinary relations $[\mathbf{x}_i,\mathbf{p}_j]=i\hbar\delta_{ij}$ in $\mathcal{A}$ and by choosing the spectral representation of the $\mathbf{x}_i$. We then have as usual $\phi(x):=\langle x|\phi\rangle$ with the scalar product $\langle\phi|\psi\rangle=$

$\int d^4x\,\phi^*(x)\,\psi(x)$. The trace reads $\mathrm{tr}(q)=\int d^4x\langle x|q|x\rangle$, and the operators act as $\mathbf{x}_i\cdot\phi(x)=x_i\phi(x)$, $\mathbf{p}_i\cdot\phi(x)=-i\hbar\partial_{x_i}\phi(x)$.

The pointwise multiplication "$*$,"

$$(\phi_1 * \phi_2)(x)=\phi_1(x)\,\phi_2(x),$$

i.e.,

$$\langle x|\phi_1 * \phi_2\rangle=\langle x|\phi_1\rangle\langle x|\phi_2\rangle \tag{10}$$

which expresses point interaction, is (and can also on noncommutative geometries be kept) commutative for bosons. Since fields are in a representation of $\mathcal{A}$, similar to quantum-mechanical states, we here formally extended Dirac's bra-ket notation for states to fields. In Eq. (9) this yields a convenient notation for the functional analytic structure of the action functional, but of course the quantum-mechanical interpretation does not simply extend, see [15]. The space D of fields that is formally to be summed over can be taken to be the dense domain S_∞ in the Hilbert space H of square integrable fields.

Generally, the unitary transformations that map from one Hilbert basis to another have trivial determinant, so that no anomalies are introduced into the field theory and changes of basis can be performed arbitrarily, in the action functional, in the Feynman rules or in the end results of the calculation of n-point functions.

Let us now assume that the commutation relations, i.e., $\mathcal{A}$ are represented on a dense domain D spanned by a Hilbert basis of vectors $\{|n\rangle\}$, where n may be discrete, as in the case of a Bargmann Fock representation, or continuous as in the case of position or momentum representations, or generally, a mixture of both. For simplicity we use the notation for n discrete. The identity operator on H can then be written $1=\Sigma_n|n\rangle\langle n|$ and fields and operators are expanded as

$$\phi_n=\langle n|\phi\rangle$$

and

$$(\mathbf{p}^2+m^2c^2)_{nn'}=\langle n|\mathbf{p}^2+m^2c^2|n'\rangle. \tag{11}$$

The pointwise multiplication, which expresses local interaction, has the general form for arbitrary choice of Hilbert basis $|n\rangle$:

$$*=\sum_{n_i} L_{n_1,n_2,n_3}|n_1\rangle\otimes\langle n_2|\otimes\langle n_3|. \tag{12}$$

Equation (10) is recovered for the choice of Hilbert basis $|n\rangle=|x\rangle$ with $L_{x,x',x''}=\delta(x-x')\,\delta(x-x'')$.

We remark that in presence of a minimal uncertainty Δx_0 in positions, $*$ can naturally be generalized to include slightly nonlocal ultraviolet regularizing corrections. Strict observational locality is preserved as long as the generalized $*$ is chosen not more nonlocal than the intrinsic position uncertainty Δx_0 of the underlying geometry. While this ultraviolet structure is studied in [10,13], we will in the following focus on the propagator and the infrared structure.

In the Hilbert basis $\{|n\rangle\}$ the partition function reads, summing over repeated indices,

$$Z[J]=N\int_D D\phi\,\exp\biggl(-\frac{l^2}{\hbar^2}\phi^*_{n_1}(\mathbf{p}^2+m^2c^2)_{n_1n_2}\phi_{n_2} - \frac{\lambda l^4}{4!}L^*_{n_1n_2n_3}L_{n_1n_4n_5}\phi^*_{n_2}\phi^*_{n_3}\phi_{n_4}\phi_{n_5}+\phi^*_nJ_n+J^*_n\phi_n\biggr). \tag{13}$$

Pulling the interaction term in front of the path integral, completing the squares, and carrying out the Gaussian integrals yields

$$Z[J]=N'\exp\biggl(-\frac{\lambda l^4}{4!}L^*_{n_1n_2n_3}L_{n_1n_4n_5}\frac{\partial}{\partial J_{n_2}}\frac{\partial}{\partial J_{n_3}}\frac{\partial}{\partial J^*_{n_4}}\frac{\partial}{\partial J^*_{n_5}}\biggr) \times\exp\biggl[-\frac{\hbar^2}{l^2}J^*_n(\mathbf{p}^2+m^2c^2)^{-1}_{nn'}J_{n'}\biggr]. \tag{14}$$

The inversion of $(\mathbf{p}^2+m^2c^2)$ is nontrivial and involves a self-adjoint extension in which it can be diagonalized and inverted. This will be investigated below. We obtain the Feynman rules:

$$\Delta_{n_1n_2}=\left(\frac{-\hbar^2}{l^2(\mathbf{p}^2+m^2c^2)}\right)_{n_1n_2}$$

and

$$\Gamma_{n_1n_2n_3n_4}=-\frac{\lambda l^4}{4!}L^*_{n'n_1n_2}L_{n'n_3n_4}. \tag{15}$$

Note that since each vertex attaches to four propagators, l drops out of the Feynman rules, as it should.

Recall that the usual formulation of the partition function in Eq. (8) implies that $\mathbf{p}^2$ can be represented as the Laplacian on a spectral representation of the $\mathbf{x}_i$. In this case, the $\mathbf{p}_i$ are represented as $-i\hbar\partial_i$, which implies that $[\mathbf{x}_i,\mathbf{p}_j]=i\hbar\delta_{ij}$. It is crucial that in our formulation of partition functions in abstract form, such as in Eq. (9), the commutation relations of the underlying algebra $\mathcal{A}$ are not implicitly fixed and can be generalized to the form of Eqs. (1) and (2).

The generalized $\mathcal{A}$ can be represented on an arbitrary dense domain in a Hilbert space H with a Hilbert basis $\{|n\rangle\}$. The Feynman rules are then obtained straightforwardly through Eqs. (9) and (11)–(15). Therefore, the formalism allows us to explicitly check noncommutative geometries $\mathcal{A}$ on UV and IR regularization.

IV. IR REGULARIZATION

On ordinary geometry, a finite mass term in the propagator $(\mathbf{p}^2+m^2c^2)^{-1}$ ensures that, as an operator, it is bounded. However, for $m=0$ the operator $1/\mathbf{p}^2$ is unbounded, causing infrared divergencies. Indeed, on geometries that imply a minimal uncertainty Δp_0, even the massless propagator $1/\mathbf{p}^2$ is as well behaved as if it contained a mass term.

To be precise, we intend to show that for all noncommutative geometric algebras $\mathcal{A}$ of the type of Eqs. (1) and (2), which imply a minimal uncertainty Δp_0, the following propositions hold for $m>0$ *and* for $m=0$.

(A) The operator $(\mathbf{p}^2+m^2c^2):=(\Sigma_i \mathbf{p}_i\mathbf{p}_i+m^2c^2)$ has exactly one self-adjoint extension $(\mathbf{p}^2+m^2c^2)_F$, which is contained in its form domain.

(B) The operator $(\mathbf{p}^2+m^2c^2)_F$ has a unique inverse (the free propagator) which is self-adjoint and defined on the entire Hilbert space H.

(C) The propagator $(\mathbf{p}^2+m^2c^2)_F^{-1}$ is infrared regular, i.e., it is a bounded operator (implying also that its matrix elements are bounded) with bound $\|(\mathbf{p}^2+m^2c^2)_F^{-1}\| \leqslant [n(\Delta p_0)^2+m^2c^2]^{-1}$.

(D) Also propagators that are the inverse to arbitrary other self-adjoint extensions of $(\mathbf{p}^2+m^2c^2)$ (for finite deficiency indices) are IR regular; i.e., they are bounded self-adjoint operators on H. To see this, let $\mathcal{A}$ be represented on a dense domain $D\subset H$ in a Hilbert space H. By assumption, the momenta $\mathbf{p}_i$ exhibit a minimal uncertainty $\Delta p_0>0$, i.e, for all *normalized* vectors $|\phi\rangle\in D$ there holds $\Delta p_{i|\phi\rangle}\geqslant\Delta p_0$ (with $i=1,\ldots,n$), so that

$$\langle\phi|\mathbf{p}_i^2|\phi\rangle=\langle\phi|\mathbf{p}_i|\phi\rangle^2+(\Delta p_{i|\phi\rangle})^2\geqslant(\Delta p_0)^2 \tag{16}$$

and by linearity, for vectors of arbitrary norm:

$$\langle\phi|\mathbf{p}^2|\phi\rangle\geqslant n\|\phi\|^2(\Delta p_0)^2. \tag{17}$$

Thus, the operator $(\mathbf{p}^2+m^2c^2)$ is a densely defined symmetric *positive definite* operator (now even for $m=0$), and therefore has, by a theorem of Friedrich, see for example [18–22], a unique self-adjoint extension within its form domain. It has the same lower bound as the original operator. Explicitly, the Friedrich extension $(\mathbf{p}^2+m^2c^2)_F$ of $(\mathbf{p}^2+m^2c^2)$ has the domain $D_F=D_{(\mathbf{p}^2+m^2c^2)^*}\cap H'$, which is the intersection of the domain $D_{(\mathbf{p}^2+m^2c^2)^*}$ of the adjoint $(\mathbf{p}^2+m^2c^2)^*$ with the Hilbert space H' obtained by completion of D with respect to the norm $\|\phi\|':=\langle\phi|\mathbf{p}^2+m^2c^2|\phi\rangle^{1/2}$ induced by the quadratic form which is defined through the positive definite operator $(\mathbf{p}^2+m^2c^2)$. The range of $(\mathbf{p}^2+m^2c^2)_F$ is $R[(\mathbf{p}^2+m^2c^2)_F]=H$, the inverse $(\mathbf{p}^2+m^2c^2)_F^{-1}$ exists, has the domain $D_{(\mathbf{p}^2+m^2c^2)_F^{-1}}=R[(\mathbf{p}^2+m^2c^2)_F]=H$, and is a self-adjoint bounded operator:

$$\|(\mathbf{p}^2+m^2c^2)_F^{-1}\|\leqslant\frac{1}{n(\Delta p_0)^2+m^2c^2}. \tag{18}$$

For a constructive proof of the properties of the Friedrich extension, see for example [21]. To see the invertibility note that, since $(\mathbf{p}^2+m^2c^2)_F$ has the same bound as $(\mathbf{p}^2+m^2c^2)$, i.e., $\forall|\phi\rangle\in D_F:\langle\phi|(\mathbf{p}^2+m^2c^2)_F|\phi\rangle\geqslant m^2c^2+n\|\phi\|^2(\Delta p_0)^2$, its kernel is empty: $(\mathbf{p}^2+m^2c^2)_F|\phi\rangle=0\Rightarrow 0=\langle\phi|(\mathbf{p}^2+m^2c^2)_F|\phi\rangle\geqslant m^2c^2+n\|\phi\|^2(\Delta p_0)^2$. Because of the Cauchy-Schwarz inequality the matrix elements of $(\mathbf{p}^2+m^2c^2)_F^{-1}$ are also bounded:

$$\begin{aligned}\forall|\phi\rangle,|\psi\rangle\in H:\ &|\langle\phi|(\mathbf{p}^2+m^2c^2)_F^{-1}|\psi\rangle|\\ &\leqslant\|\phi\|\,\|\psi\|\,\|(\mathbf{p}^2+m^2c^2)_F^{-1}\|\\ &\leqslant\|\phi\|\,\|\psi\|[n(\Delta p_0)^2+m^2c^2]^{-1}.\end{aligned} \tag{19}$$

So far we have shown (A)–(C), i.e., that there exists a canonical inverse $(\mathbf{p}^2+m^2c^2)_F^{-1}$ and that, as a propagator, it does not lead to infrared problems since it is bounded also in the case $m=0$.

To see (D) we consider the bi-adjoint $(\mathbf{p}^2+m^2c^2)^{**}$, which is symmetric and closed, as is every bi-adjoint of a densely defined symmetric operator. Because of the existence of one self-adjoint extension, $(\mathbf{p}^2+m^2c^2)_F$, the deficiency indices (r,r) are equal. Note that the deficiency indices are by definition the dimensions of the eigenspaces of $(\mathbf{p}^2+m^2c^2)^*$ to the eigenvalues $\pm i$.

We recall that in ordinary geometry the deficiency indices are (0,0), implying that $(\mathbf{p}^2+m^2c^2)_F$ is the only self-adjoint extension. The deficiency indices can now be nonzero, examples of which are known, see [5,6,11,12]. There then exists a whole family of further self-adjoint extensions $(\mathbf{p}^2+m^2c^2)_f$ (labeled by f), and a corresponding family of "nonstandard propagators" $(\mathbf{p}^2+m^2c^2)_f^{-1}$ [for invertible $(\mathbf{p}^2+m^2c^2)_f$] which, in explicit representations, differ by their boundary conditions.

A priori, we do not want to exclude these nonstandard propagators (although we exclude as unphysical the case of infinite deficiency indices in which case the propagator would require an infinite set of boundary conditions).

Indeed, also the nonstandard propagators are IR regular. To see this, we note first that also $(\mathbf{p}^2+m^2c^2)^{**}$ is semibound from below by $[n(\Delta p_0)^2+m^2c^2]$ since $(\mathbf{p}^2+m^2c^2)_F$, which is an extension of $(\mathbf{p}^2+m^2c^2)^{**}$, has this property. As seen by the v. Neumann method, the unitary extension of the isometric Cayley transform only involves a finite-dimensional mapping of the deficiency spaces and thus all self-adjoint extensions of a closed symmetric operator have the same essential spectrum, see for example Thm.8.18 in [18]. Indeed, since $(\mathbf{p}^2+m^2c^2)^{**}$ is closed, symmetric, and bounded from below, the now interesting part of the spectrum $\sigma((\mathbf{p}^2+m^2c^2)_f)\cap[-\infty,n(\Delta p_0)^2+m^2c^2]$ of its self-adjoint extensions consists of isolated eigenvalues only, of total multiplicity $\leqslant r$, see Cor.2 of Thm8.18 in [18]. Thus, for all invertible self-adjoint extensions there exist $\epsilon>0$ such that the spectrum is empty in the finite interval $[-\epsilon,\epsilon]$, i.e., in the neighborhood of zero. We can, therefore, conclude the boundedness of the spectra of the inverses of arbitrary invertible self-adjoint extensions of $(\mathbf{p}^2+m^2c^2)$. To be precise, for all invertible self-adjoint extensions, zero is a regular point $0\in\rho((\mathbf{p}^2+m^2c^2)_f)$, since it is not in the spectrum. For self-adjoint operators A, it is generally true that, see for example [18] (Thm5.24), [19] (Thms129.1,2), or [20–22]:

$$\begin{aligned}&z\in\rho(A)\Leftrightarrow\exists c>0,\forall v\in D(A):\\ &\quad\|(z-A)\cdot v\|\geqslant c\|v\|(\text{and}\Rightarrow\|1/(z-A)\|\leqslant c^{-1})\\ &\quad\Leftrightarrow R(z-A)=H.\end{aligned} \tag{20}$$

Here, therefore, we have:

$$\exists\epsilon>0,\forall v\in D_f:\ \|(\mathbf{p}^2+m^2c^2)_f\cdot v\|\geqslant\epsilon\|v\|$$

and

$$\mathrm{R}((\mathbf{p}^2+m^2c^2)_f)=H. \tag{21}$$

Thus, the corresponding propagators are bounded $||(\mathbf{p}^2+m^2c^2)_f^{-1}||\leq 1/\epsilon$ and are defined on the entire Hilbert space $D_{(\mathbf{p}^2+m^2c^2)_f^{-1}}=R((\mathbf{p}^2+m^2c^2)_f)=H$. The propagators are also self-adjoint, as the inverses of self-adjoint operators generally are.

V. OUTLOOK

Concerning the structure in the ultraviolet, the same arguments prove of course that for example a background Coulomb potential $A_\mu(\mathbf{x}):=[q/\sqrt{(\Sigma_i \mathbf{x}_i^2)_F},0,0,0]$ [the square root is well defined since $(\Sigma_i \mathbf{x}_i^2)_F$ is positive definite] is bounded in the presence of a minimal uncertainty Δx_0 in positions. Given a representation of the algebra $\mathcal{A}$, the propagator $\Delta=(\{[\mathbf{p}_i+eA_i(\mathbf{x})]^2+m^2c^2\}_F)^{-1}$ can be calculated straightforwardly. Investigations into the ''local'' gauge principle on geometries with minimal uncertainties should eventually allow one to study also dynamical gauge fields and to check for UV regularization. We note, however, that in the presence of a minimal uncertainty Δx_0 in positions, the very notion of ''local'' gauging will need to be redefined. In the simpler ϕ^4 theory, explicit loop integrations on certain examples of geometries with a minimal uncertainty in positions and momenta have been carried out in [7,10] and UV regularization has been shown for these cases. We remark that for UV regularization the structure of the pointwise multiplication ''$*$,'' which describes local interaction, is crucial. Because of the absence of a position representation, $*$ is nonunique in the case of $\Delta x_0>0$. Crucially, an interaction is now observationally local if any formal nonlocality of $*$ is not larger than the scale of the nonlocality Δx_0 inherent in the underlying space. Thus, intuitively, UV regularity and strict observational locality become more compatible than on ordinary geometry. There exist ''quasiposition representations'' [10–12], built on maximal localization states, which can be used to establish the locality and causality properties of pointwise multiplications. A detailed study on the special case $\Delta x_0>0,\Delta p_0=0$ (which allows a convenient momentum space representation) is in preparation [13].

We remark that an alternative approach with a similar motivation, but based on the canonical formulation of field theory is given in [16], see also [17].

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