Is an effective Lagrangian a convergent series?

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We present some generic arguments demonstrating that an effective Lagrangian $L_{\rm eff}$ which, by definition, contains operators O^n of arbitrary dimensionality in general is not convergent, but rather an asymptotic series. It means that the behavior of the far distant terms has a specific factorial dependence $L_{\rm eff}\sim\Sigma_n(c_nO^n/M^n)$, $c_n\sim n!$, $n\gg1$. We explain the main ideas by using QED as a toy model. However we expect that the obtained results have a much more general origin. We speculate on possible applications of these results to various physical problems with typical energies from 1 GeV to the Planck scale. [S0556-2821(96)05220-4]

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I. INTRODUCTION

Today it is widely believed that all of our present realistic field theories are actually not fundamental, but effective theories. The standard model is presumably what we get when we integrate out modes of very high energy from some unknown theory, and like any other effective field theory, its Lagrangian density contains terms of arbitrary dimensionality, though the terms in the Lagrangian density with dimensionality greater than four are suppressed by negative powers of a very large mass M. Even in QCD, for the calculation of processes at a few GeV we would use an effective field theory with heavier quarks integrated out, and such an effective theory necessarily involves terms in the Lagrangian of unlimited dimensionality.

The basic idea behind effective field theories is that a physical process at energy $E\ll M$ can be described in terms of an expansion in E/M, see recent reviews [1–3]. In this case we can limit ourselves by considering only a few first leading terms and neglect the rest. In this paper we discuss not this standard formulation of the problems, but rather, we are interested in the behavior of the coefficients of the very high dimensional operators in the expansion. We shall demonstrate that these coefficients c_n grow as fast as a factorial $n!$ for sufficiently large n. Thus the series under discussion is not a convergent, but an asymptotic one. Such a behavior raises problems both of a fundamental nature, concerning the status of the expansion and of practical importance, as to whether divergences can be associated with new physical phenomena. It means, first of all, that in order to make sense, such a theory should be defined by some specific prescription, for example, by Borel transformation.

Let us note, that our remarks about the factorial dependence of the series for large $n\gg1$ is an absolutely irrelevant issue for the analysis of standard problems when we are interested only in the low-energy limit. We have nothing new to say about these issues.

However, sometimes we need to know the behavior of a whole series when the distant terms in the series might be important. In this case the analysis of the large order terms in the expansion has some physical meaning.

Such a situation may occur in a variety of different problems as will be discussed in more detail later in the text. Now let us mention that, in general, it occurs when the energy scale E is close to M and/or when two or more intermediate, not well-separated scales, come into the game [4].

This paper is organized in the following way. In the next section we consider our basic QED example, where the factorial behavior of the coefficients in front of the high-dimensional operators is explicitly calculated. After that we argue that this property is a very general phenomenon of the effective field theories.[1]

In conclusion, we make some speculations regarding possible applications of the obtained results to different field theories with very different scales (from QCD problems to the cosmological constant problem).

II. BASIC EXAMPLE: QED

We begin our analysis with the following remark. An effective field theory can be considered as a particular case of the more general idea of the Wilson operator product expansion (OPE). It has been demonstrated recently [6], that the OPE for some specific correlation functions (heavy-light quark system $\bar{Q}q$) in QCD is an *asymptotic*, and not a convergent series. The general arguments of the paper [6] have been explicitly tested in two-dimensional QCD (QCD_2) (where the vacuum structure as well as the spectrum of the theory is known) with the same conclusion concerning the asymptotic nature of OPE [7]. In both cases the arguments

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[1]The generality of this phenomenon can be compared with the well-known property of the large-order behavior in a perturbative series [5]. As is known, a variety of different field theories (gauge theories, in particular) exhibits a factorial growth of the coefficients in the perturbative expansion with respect to a coupling constant. This growth in perturbative expansion is very different from the phenomenon we are discussing, where the factorial behavior is related to high-dimensional operators, and not to coupling constant expansion. However, in spite of the apparent difference of these phenomena, they actually have some common general origin. We shall discuss this connection later.

were based on the dispersion relations and the general properties of the spectrum of the theory. However, the experience with large-order behavior in a perturbative series [5] teaches us that the factorial growth of the coefficients is of a very general nature and it is not a specific property of some Green functions.

Thus we expect that the asymptotic nature of the OPE has a much more general origin and it is not related to the specific correlation functions, for which it was found for the first time [6].

To be more specific and in order to explain what is going on with the effective theory when we integrate out the heavy degrees of freedom, let us consider QED with one heavy electron of mass M. The effective field theory for photons can be obtained by integrating out the fermion degrees of freedom. The most general solution of this problem is not known, however, in the case of a specific (constant) external electric field E the corresponding expression for $L_{\rm eff}$ is known (see, e.g., the textbook [8]). In order to find the OPE coefficients for the high-dimensional operators E^n, one can expand $L_{\rm eff}$ in power of E:

$$L_{\rm eff}=M^4\sum_n c_n\left(\frac{E}{M^2}\right)^n. \tag{1}$$

Of course, the Eq. (1) is not the most general form, because it does not contain all possible operators, in particular those operators which would contain some terms with derivatives $\sim\partial_\mu E$. Our goal now is to demonstrate that we do have already a factorial behavior in this simple case where we select only some specific class of operators, namely those $\sim E^n$.

Our next step is as follows. First of all we shall find an exact formula for the n dependence of the coefficients c_n; secondly, we give a qualitative explanation of why such a factorial behavior takes place. Our argumentation will be so general in form that it will be perfectly clear that this phenomenon is very universal in nature.

The effective Lagrangian for the problem can be written in the following way [8]:

$$L_{\rm eff}=\frac{1}{8\pi^2}\int_0^\infty\frac{ds}{s^2}\left[E\coth(Es)-\frac{1}{s}\right]e^{-isM^2}, \tag{2}$$

where we denote the external field E together with its coupling constant e. We expand this expression in E using the formula

$$\frac{1}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^{k-1}}{k!}, \tag{3}$$

where B_k are Bernoulli numbers. For large k these numbers as is known exhibit factorial growth:

$$B_{2n}=2(-1)^{n+1}(2n)!\sum_{r=1}^{\infty}\frac{1}{(2\pi r)^{2n}}$$
$$\sim 2(-1)^{n+1}(2n)!\frac{1}{(2\pi)^{2n}},\quad n\gg 1. \tag{4}$$

Thus the coefficients c_n in the OPE (1) are factorially divergent for large n:

$$c_{2n}=\frac{1}{8\pi^2}2^{2n}B_{2n}\frac{(2n-3)!}{(2n)!}\sim(2n)!. \tag{5}$$

In particular, for $n=2$ this formula reproduces the well-known Euler-Heisenberg effective Lagrangian $L_{\rm EH}$, which is nothing but the first nontrivial term in the series (1):

$$L_{\rm EH}=\frac{2}{45M^4}\left(\frac{e^2}{4\pi}\right)^2E^4. \tag{6}$$

We have redefined the coupling constant e in this expression to present the formula in a standard way.

Now, how one can understand this factorial behavior (5) in simple terms? We suggest the following almost trivial explanation which, however, is very universal in nature.

Let us look at the function $L_{\rm eff}(z)(1)$ as an analytical function of the complex variable $z=E/M^2$ for which the standard dispersion relations hold. The factorial growth of the coefficients in the real part of $L_{\rm eff}(z)$ implies that the corresponding imaginary part has a very specific behavior $\mathrm{Im}L_{\rm eff}(z)\sim e^{-1/z}$, which follows from the dispersion relations:

$$f(z)\sim\sum_n f_nz^nf_n\sim(a)^nn!\sim\int\frac{dz'}{(z')^{n+2}}\mathrm{Im}f(z')\leftrightarrow\mathrm{Im}f(z')$$
$$\sim e^{-a/z'}. \tag{7}$$

Here we have introduced an arbitrary analytical function $f(z)$ to be more general.

At the same time, an imaginary part of the amplitude, as is known, is related to to a real physical process: the pair creation in the strong external field. We have fairly good physical intuition of what kind of dependence on the field one could expect for such a physical process. Namely, as we shall discuss later, this process can be thought as a penetration through a potential barrier in the quasiclassical approximation. So, from a physical point of view we would expect that the E dependence should have the following form $\mathrm{Im}L_{\rm eff}(E)\sim e^{-1/E}$. As we shall see, this is exactly the case for our QED example (1) and in a full agreement with what the dispersion relations (7) tell us.

Now we would like to present the explicit formula for the probability of pair creation in the constant electric field E. It is given by (see, e.g., [8]):

$$w=-\frac{1}{4\pi^2}\int_0^\infty\frac{ds}{s^2}\left[E\coth(Es)-\frac{1}{s}\right]\mathrm{Im}(e^{-isM^2}). \tag{8}$$

The "only" difference with the formula (2) is the replacement $\mathrm{Re}(e^{-isM^2})\Rightarrow\mathrm{Im}(e^{-isM^2})$. However, this replacement modifies completely the analytical structure. Indeed, the explicit calculation of the coefficients in the power expansion for imaginary part in the formula (8) leads to the following integrals, which are zero $\int dz\sin(z)z^{2n-3}\sim\sin[(n-1)\pi]=0$. Thus the imaginary part is not expandable at $E=0$ in agreement with our arguments about a singular behavior at this point $\sim e^{-1/E}$.

Fortunately, a direct calculation,[2] without using an expansion in power of E can be performed easily with the following final result, explicitly demonstrating the $e^{-1/z}$ structure (see, e.g., [8]):

$$w=\frac{E^2}{4\pi^3}\sum_{n=1}^{n=\infty}\frac{1}{n^2}\exp\left(-\frac{nM^2\pi}{E}\right). \tag{9}$$

A few comments are in order. First, the behavior $w(z)\sim e^{-1/z}$ is exactly what we expected. It can be interpreted as penetration through a potential barrier in the quasiclassical approximation. Indeed, the standard formula for the ionization of a state with bound energy $-V\sim 2M$ and external field E is proportional to

$$\sim\exp\left(-2\int dx\sqrt{2M(V-Ex)}\right)\sim\exp\left(-\frac{\mathrm{const}M^2}{E}\right),$$

which qualitatively explains the exact result (9).

We are not pretending here to have derived a new result in QED. All these classical formulas have been well known for many years. Rather, we wanted to explain, by analyzing this QED example, the main source of the $n!$ dependence in the effective Lagrangian.

The effective Lagrangian, by definition, is a series of operators of arbitrary dimensions constructed from the light fields E. Presumably, this is obtained from some underlying field theory by integrating out the heavy fields of mass M. It is perfectly clear that the probability of the physical creation of the heavy particles with mass M in external field E is strongly suppressed $\sim\exp(-1/E)$. The dispersion relations, thus, imply unambiguously that the coefficients in the real part of the effective Lagrangian are factorially large.

We believe that this simple explanation of formula (5) is so universal in form that it can be applied to almost arbitrary nontrivial effective field theories leading to the same conclusion about factorial behavior. We shall consider another explanation of the same phenomenon later in the text, but now we would like to note that the relation between imaginary and real parts of the amplitudes of course is well known, and heavily used in particle physics.

We would like to come back to formula (5) to explain this factorial behavior in the OPE one more time from an absolutely independent point of view. Again, we use QED as an example to demonstrate an idea, however, as we shall see, the arguments which follow are much more general and universal in nature.

III. SPECULATIONS

As is known, almost all nontrivial field theories exhibit factorial growth of coefficients in the perturbative expansion with respect to a coupling constant[3] [5]. This factorial dependence can be understood as the rapid growth of the number of Feynman graphs.[4]

Now, how one can understand the nature of the Wilson OPE in terms of the Feynman graphs? As is known, the computational recipe of the coefficients in the OPE is simple: it is necessary to separate large and small distance physics. Large distance physics is presented by operators of light fields; the small distance contribution is explicitly calculated from the underlying field theory. Technically, in order to carry out this program, we cut the perturbative graphs in all possible ways over the photon lines (in a general case, a photon field will be replaced by some light degrees of freedom). These lines present the external light fields. They are combined together in a specific way to organize all possible operators. The coefficients in front of these operators can be explicitly calculated and they are determined by the small distance physics.

From this technical explanation of the calculation of the coefficients in the Wilson OPE it should be clear, that if the underlying theory possesses factorial growth in the perturbative expansion, the effective Lagrangian constructed from this theory exhibits the same factorial behavior for the high-dimensional operators. The moral of this argument is very simple: the factorial growth of the perturbative expansion in the underlying theory can not disappear without a trace. It will show up in the coefficients of the high-dimensional operators in the effective Lagrangian obtained from the underlying theory.

Having demonstrated the main result on factorial growth of the coefficients (in an effective Lagrangian) as a universal phenomenon as a consequence of the factorial growth in perturbative series, we would like to discuss some possible applications of this phenomenon.

We start from the QCD (as underlying theory), which is very similar to QED discussed above. The problem in this case can be formulated in the following way (see recent paper [9] on this subject and refences therein). How one can integrate over small distance physics in order to extract the long-distance dynamics? An appropriate way to implement this program is: (a) introduce the collective degrees of freedom, colorless mesons, as the external sources into the underlying lagrangian; (b) integrate over the quarks and gluons with high frequencies by introducing the normalization point μ. The obtained effective Lagrangian is the $1/\mu$ expansion, where operators are expressed in terms of the external fields as well as low-energetic quarks and gluons. Our remark is the coefficients in this expansion grow factorially with the increasing number of meson fields. Let us note that the procedure of obtaining the effective Lagrangian in this case is not much different from the case we discussed previously. The only new element is the introduction of the collective fields, which were not present in our original Lagrangian. However, this does not effect the general arguments on the $n!$ behavior.

Indeed, one can consider the quark-antiquark external lines (instead of the collective meson fields) for the calculation of the OPE coefficients, as discussed in the previous

[2]This integral can be reduced, according to Cauchy's theorem, tthe calculation of the contributions from the poles of the coth$z$ function.

[3]Do not confuse this perturbative expansion with OPE and effective Lagrangian we are dealing with. These series are very different in nature, but they both exhibit a factorial growth.

[4]Here we do not discuss the so-called renormalons, which give the same factorial dependence, but have a very different origin.

section. In this case, all arguments on $n!$ behavior can be applied in a straightforward way. In fact, each extra quark-antiquark external pair raises the dimension of the operator and at the same time it comes with the extra factor α_s. As we learned earlier, the coefficient in front of α_s^n contains $n!$ dependence. Thus the high-dimensional operator with $\sim n$ external fields will be accompanied by the factor α_s^n as well as the factor $\sim n!$.

One more way to understand the same phenomenon is the following. We introduce the collective variables (Goldstone fields) in the course of Ref. [9], where we use the standard form for the interaction:

$$L_{\rm int} \sim \bar{\psi}\gamma_\mu(i\partial_\mu + G_\mu + A_\mu\gamma_5 + \cdots)\psi. \tag{10}$$

In this formula G_μ is the usual gluon field and A_μ is external axial source related to $U^\dagger\partial_\mu U$ with unitary matrix U describing the Goldstone fields. Bearing in mind that the photon-fermion interaction and gluon-fermion interaction are very similar, one can conclude that the effective Lagrangian for the gluon fields derived from Eq. (10) (by integrating over ψ fields) possesses the factorial growth in coefficients in close analogy with QED (1, 5). Moreover, from the similarity of the interaction of gluon field G_μ and axial field A_μ with a fermion ψ, one can conclude that the same factorial growth also is present for the operators constructed from A_μ fields.

In principle, one could imagine that some high-dimensional operators do not contain a factorial dependence. The number of such operators is small (by combinatoric reasons) and they certainly cannot play a dominating role.

Thus, in general, we expect a factorial behavior of the coefficients for the effective QCD Lagrangian, as well as for the chiral Lagrangian, as its particular case. An exact formula for the coefficients depends on the operator under consideration. This is because the different fields (gluons, quarks, mesons), which are constituents of the operator are not equally weighted. However, the precise expression for the coefficients in terms of constituents of these operators is not a relevant issue at the moment.

One more interesting example we would like to mention is the effective field theory of gravity. We refer to the recent review [10] on this subject for a general introduction and references. The only remark we would like to make here is the following. Nowdays it is generally accepted that the Einstein Lagrangian

$$S_{\rm grav} = \int d^4x\sqrt{g}\frac{2}{\kappa^2}R \tag{11}$$

is only the first local term of the expansion of a more complicated theory (string?). Thus general relativity should be considered as an effective field theory with infinitely many terms allowed by general coordinate invariance. As usual, in the effective theory description, only the first term in the expansion plays a role at low energy $E \ll M_{\rm Planck}$. If we were not interested in quantum effects at the Planck scale with $E \simeq M_{\rm Planck}$, Eq. (11) would be the end of the story. However, we intend to discuss physics at the Planck scale; thus we would like to write down the effective Lagrangian in the most general form:

$$S_{\rm eff} = \int d^4x\sqrt{g}\Big[\Lambda + \frac{2}{\kappa^2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \sum_n c_nQ^n\cdots + L_{\rm matter} + L_{\rm dilaton} + L_{\rm inflaton}\cdots + \Big], \tag{12}$$

where the operators Q^n are high-dimensional operators constructed from the relevant fields ($R_{\mu\nu}$, dilaton, inflaton ϕ, gauge fields $F_{\mu\nu}$, etc.). Our remark here is that we believe that the coefficients in the effective Lagrangian, even for pure gravity, exhibit factorial growth. The arguments which support this statement are the same as before: if the underlying theory [in our case it is given by Lagrangian (11)] possesses factorial growth in the perturbative expansion, the effective Lagrangian constructed from this theory exhibits the same factorial behavior for the high-dimensional operators.

As we already mentioned, the factorial behavior of coefficients in the perturbative expansion can be understood as the fast increase in the number of Feynman diagrams. In pure Yang Mills theory we know well that such a growth does take place [5]. We can interpret this growth as a manifestation of the three- and four-gluon vertices, which lead to the factorially divergent number of diagrams. In the case of gravity (11) we expect the same factorial behavior because of the nonlinear nature of the interaction (11) similar to a gauge theory.[5]

IV. CONCLUSION

In this paper we have presented two independent sets of arguments, which support the idea that almost any nontrivial effective Lagrangian obtained by integrating out some heavy fields and/or fast degrees of freedom, is nonconvergent, but an asymptotic series.

The first set of arguments is based on the idea that the imaginary part of the amplitude related to the probability of the physical creation of a heavy particle, is exponentially small $\sim\exp(-1/E)$. The dispersion relations in this case imply unambiguously that the coefficients of the expansion in the real part of the corresponding amplitude exhibit a factorial dependence. Once these coefficients are found to be factorially large, we can forget about the way the result was derived and we can forget about the external auxiliary field E, which we used heavily in our arguments. Coefficients in the OPE do not depend on the applied field E, no matter how small it is.

The second line of reasoning is based on the analysis of the large-order behavior of the perturbative series. As we have argued, if the underlying theory possesses factorial growth of the coefficients of the perturbative series, then the corresponding effective Lagrangian constructed from this theory will exhibit the same factorial behavior for the high-dimensional operators.

We believe that both of these lines of arguments are so general in form that almost all nontrivial effective Lagrangian will demonstrate $n!$ behavior. We believe that this phe-

[5]An explicit calculations based on a simplified version of gravity also support this expectation [11].

nomenon is universal in nature.

Now we would like to discuss some physical consequences, which might result from this phenomenon. As we mentioned in the Introduction, we have nothing new to say in the case of analysis of low-energy phenomena for which the small expansion parameter is $\lambda \equiv E/M \ll 1$. In such a case, the exact formula is approximated perfectly well by the first term of the asymptotic expansion and we can safely forget about all the rest. However, very often the situation is not so fortunate and the expansion parameter $\lambda \sim 1$ (let us say 1/3 or 1/2), like in chiral perturbation theory. In this event people try to improve the situation by considering the next to-leading terms or even next to next to-leading order. If the series were convergent, these efforts would be worthwhile. However, as we argued in this letter, an effective Lagrangian, in general, is represented by an asymptotic, not a convergent series. Thus one may ask the following general questions: (a) How many terms one should keep in the effective Lagrangian for the best approximation of an exact formula for the given parameter λ? (b) What is the fundamental uncertainty (related to our lack of knowledge of the higher-dimensional operators) one should expect for an effective Lagrangian represented as an asymptotic series?

The effective description of QCD, which has been discussed in the previous section is one example where those problems might be a relevant issue.

The factorial behavior in the effective Lagrangian (12) for gravity also might be an interesting observation, which could have some important consequences. Let us recall that the natural scale of the cosmological term Λ is the Planck scale. Indeed, the most popular cosmology today, the inflationary scenario (for a review see [12] and [13]) assumes that our universe passed through an era in which the cosmological term dominated, and it is a total mystery why we should be left in a universe with an almost vanishing vacuum energy. Of course we do not know the answer to this question, but one can argue that the asymptotic nature of the effective Lagrangian of the gravity could have some influence on the vanishing of the vacuum energy provided that the universe has a graceful exit from an inflation epoch [14].

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[1] H. Georgi, Annu. Rev. Nucl. Part. Sci. **43**, 355 (1994).

[2] A. Manohar, Effective Field Theories, Report No. hep-ph/9508245 (unpublished).

[3] D. Kaplan, Effective Field Theories, Report No. hep-ph/9506035 (unpublished).

[4] V. Bhansli and H. Georgi, Running Nonlocal Lagrangians, Report No. hep-ph/9205242, 1992 (unpublished).

[5] *Large Order Behavior of Perturbative Theory*, edited by J.C. Gillou and J. Zinn-Justin, Current Physics-Sources and Comments Vol. 7 (North-Holland, Amsterdam, 1990).

[6] M.A. Shifman, in *Proceedings of the Workshop ''Coutinuous Advances in QCD,''* edited by A. Smilga (World Scientific, Singapore, 1994), pg. 249; talk at PASCOS, Johns Hopkins University, Baltimore, March 1995, Report No. TPI-MINN-95/15-T, hep-ph/9505289 (unpublished); talk at XVIII Kazimier Meeting, Ames, Iowa, May 1995, Report No. TPI-MINN-95/32-T, hep-ph/9511469 (unpublished).

[7] A. Zhitnitsky, Phys. Rev. D **53**, 5821 (1996).

[8] C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[9] P.H. Damgaard, H.B. Nielsen, and R. Sollacher, Nucl. Phys. **B414**, 541 (1994).

[10] J.F. Donoghue, Report No. gr-qc/9512024, hep-ph/9512287 (unpublished).

[11] T. Fugleberg and A. Zhitnitsky, Perturbative series at large order in quantum gravity (work in progress).

[12] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990).

[13] R.H. Brandenberger, in *CP Violation and the Limits of the Standard Model*, Proceedings of the Theoretical Advanced Study Institute, Boulder, Colorado, 1994, edited by J. Donoghue (World Scientific, Singapore, 1995).

[14] R. Brandenberger and A. Zhitnitsky, Report No. hep-ph 9604407, 1996 (unpublished).