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Gravitational vacuum polarization. II. Energy conditions in the Boulware vacuum

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Building on techniques developed in the preceding paper, I investigate the various pointwise and averaged energy conditions for the quantum stress-energy tensor corresponding to a conformally coupled massless scalar field in the Boulware vacuum. I work in the test-field limit, restrict attention to the Schwarzschild geometry, and invoke a mixture of analytical and numerical techniques. In contradistinction to the case of the Hartle-Hawking vacuum, wherein violations of the energy conditions were confined to the region between the event horizon and the unstable photon orbit, I show that in the Boulware vacuum (1) all standard (pointwise and averaged) energy conditions are violated throughout the exterior region, all the way from spatial infinity down to the event horizon, and (2) outside the event horizon the standard pointwise energy conditions are violated in a maximal manner: They are violated at all points and for all null or timelike vectors. (The region inside the event horizon is considerably messier and of dubious physical relevance. Nevertheless, the standard pointwise energy conditions seem to be violated even inside the event horizon.) I argue that this is highly suggestive evidence, pointing to the fact that general self-consistent solutions of semiclassical quantum gravity might *not* satisfy the energy conditions and may in fact for certain quantum fields and certain quantum states violate *all* the energy conditions. [S0556-2821(96)04418-9]

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I. INTRODUCTION

Investigations of the gravitationally induced vacuum polarization produced when a quantum field theory is constructed on a curved background spacetime are a topic of considerable current interest [1–8]. A key aspect of these investigations is the manner in which the various energy conditions are affected by this gravitational vacuum polarization. This general topic is of critical importance to attempts to generalize the classical singularity theorems [9], classical positive mass theorems [10], and classical laws of black hole dynamics to semiclassical quantum gravity [2,3].

It is perhaps a little sobering to realize that none of the currently known versions of these classical theorems survive the introduction of even semiclassical quantum gravity, let alone full-fledged quantum gravity [2,3].

In this paper I shall use techniques developed in the preceding paper [1] to explore these issues in a little more detail: I restrict attention to the conformally coupled massless scalar field on a Schwarzschild spacetime, in the Boulware vacuum. I shall continue to work in the test-field limit.

For this geometry and vacuum state one has both (1) a useful analytic approximation to the gravitational polarization, obtained by combining the Page approximation for the Hartle-Hawking vacuum [11] with the Brown-Ottewill approximation [12] for the difference between the Hartle-Hawking and Boulware vacua, and (2) numerical estimates of the vacuum polarization, these estimates being obtained by combining the numerical calculations of Howard [13] and Howard and Candelas [14] (who calculate the stress-energy tensor in the Hartle-Hawking vacuum), with the further numerical calculations of Jensen, McLaughlin, and Ottewill [15] (who numerically calculate the difference between the Hartle-Hawking and Boulware vacua). Further refinements using the numerical data of Anderson, Hiscock, and Samuel [16–18] are certainly possible, but this avenue has not yet been explored.

For the Hartle-Hawking vacuum I found that the various energy conditions were violated in a nested set of onionlike layers located between the event horizon and the unstable photon orbit [1]. Furthermore, many of the energy conditions continued to be violated inside the event horizon.

For the Boulware vacuum the situation is even easier to describe.

(1) All the standard pointwise energy conditions and standard averaged energy conditions are violated throughout the entire region exterior to the event horizon, all the way from spatial infinity down to the event horizon.

(2) Outside the horizon, the standard pointwise energy conditions are violated in a maximal manner: They are violated at all points and for all null or timelike vectors.

(3) The standard pointwise energy conditions seem to be violated even inside the event horizon.

This includes the obvious (pointwise) null, weak, strong, and dominant energy conditions (NEC, WEC, SEC, and DEC), the averaged null, weak, and strong energy conditions (ANEC, AWEC, and ASEC), more exotic energy conditions such as the partial null energy condition (PNEC), the asymptotic null energy condition (Scri-NEC), and the averaged asymptotic null energy condition (Scri-ANEC), as well as various one-sided integral averages and averages constructed by allowing arbitrary positive weighting along the curve of interest. (For definitions of the basic energy conditions see [9] or [2]. For definitions of some of the more exotic energy conditions see [1].)

Energy conditions *not* completely pinned down by this type of analysis include the ''quantum inequalities'' of Ford and Roman [6–8] and versions of the ANEC in which one is

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willing to countenance some form of bounded negativity for the ANEC integral.

The situation inside the event horizon is rather more complicated. Calculations based on the analytic approximation indicate that some types of energy condition become satisfied sufficiently close to the singularity. On the other hand, it should be borne in mind that the stress tensor for the Boulware vacuum is singular at the event horizon—thus there are good reasons for not taking the Boulware vacuum state too seriously inside the event horizon.

The key results underlying this wholesale violation of the energy conditions are the following.

Theorem 1 (NEC-WEC-SEC-DEC violation). *For a conformally coupled massless scalar field on a Schwarzschild spacetime in the Boulware vacuum state: For* any *point p anywhere outside the event horizon* there exist *null and timelike vectors such that*

$$\langle B|T^{\mu\nu}|B\rangle k_\mu k_\nu \leqslant 0,$$
$$\langle B|T^{\mu\nu}|B\rangle V_\mu V_\nu \leqslant 0,$$
$$\langle B|\overline{T^{\mu\nu}}|B\rangle V_\mu V_\nu \leqslant 0. \quad (1)$$

(Here $\overline{T}$ is the trace-reversed stress-energy tensor.) Thus the NEC, WEC, SEC, and DEC are all violated outside the event horizon.

If one is willing (with suitable caveats to be described below) to accept the analytic approximation to the stress-energy tensor as a reliable guide, then the above result may be extended to the entire spacetime.

Outside the horizon, one can in fact prove a much stronger result.

Theorem 2 (total external NEC violation). *For a conformally coupled massless scalar field on a Schwarzschild spacetime in the Boulware vacuum state: For* any *point p outside the event horizon and* any *null vector k*,

$$\langle B|T^{\mu\nu}|B\rangle k_\mu k_\nu \leqslant 0. \quad (2)$$

The equality is in fact achieved only at spatial infinity.

Similar results can be proved for the other standard pointwise energy conditions.

Even if one is willing (with suitable caveats) to accept the analytic approximation to the stress-energy tensor as a reliable guide, then this second theorem does *not* extend to the entire spacetime.

II. VACUUM POLARIZATION IN SCHWARZSCHILD SPACETIME: BOULWARE VACUUM

By spherical symmetry one knows that

$$\langle B|T^{\hat{\mu}}{}_{\hat{\nu}}|B\rangle \equiv \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & +p & 0 \\ 0 & 0 & 0 & +p \end{bmatrix}, \quad (3)$$

where ρ, τ, and p are functions of r, M, and $\hbar$. [Note that I set $G\equiv 1$ and choose to work in a local Lorentz basis attached to the fiducial static observers (FIDO's).]

A subtlety arises when working in a local Lorentz basis and looking at the two-index-down (or two-index-up) versions of the stress energy. Outside the horizon one has

$$g^{\hat{\mu}\hat{\nu}}|_{\text{outside}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}. \quad (4)$$

Consequently,

$$\langle B|T^{\hat{\mu}\hat{\nu}}|B\rangle|_{\text{outside}} = \begin{bmatrix} +\rho & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & +p & 0 \\ 0 & 0 & 0 & +p \end{bmatrix}. \quad (5)$$

Inside the horizon, on the other hand, it is the radial direction that is timelike, and so

$$g^{\hat{\mu}\hat{\nu}}|_{\text{inside}} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}. \quad (6)$$

Consequently, one has the potentially confusing result that

$$\langle B|T^{\hat{\mu}\hat{\nu}}|B\rangle|_{\text{inside}} = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & +\tau & 0 & 0 \\ 0 & 0 & +p & 0 \\ 0 & 0 & 0 & +p \end{bmatrix}. \quad (7)$$

Thus, inside the horizon, one should interpret τ as the energy density and ρ as the tension (this tension now acting in the spacelike t direction).

Recall that if one wishes the density as measured by a freely falling observer to remain finite as one crosses the event horizon, then one needs $\rho=\tau$ at $r=2M$. This condition is most definitely not satisfied by the Boulware vacuum and leads to discontinuous behavior of the energy conditions as one crosses the horizon. This is not particularly surprising since the Boulware stress energy is itself singular at the event horizon. I shall put aside for now the worry that the Boulware vacuum might be completely nonsensical inside the event horizon and do as much as is currently possible by using the analytic approximation in that region.

To start the actual analysis I require explicit analytic (though approximate) formulas for the stress-energy tensor. By combining Page's analytic approximation [11] for the Hartle-Hawking vacuum state with the Brown-Ottewill [12] analytic approximation for the difference between the Boulware and Hartle-Hawking vacua one obtains a simple rational polynomial approximation to the stress-energy tensor:

$$\rho(r)=-3p_\infty(2M/r)^6\frac{[40-72(2M/r)+33(2M/r)^2]}{(1-2M/r)^2}, \tag{8}$$

$$\tau(r)=-p_\infty(2M/r)^6\frac{[8-24(2M/r)+15(2M/r)^2]}{(1-2M/r)^2}, \tag{9}$$

$$p(r)=-p_\infty(2M/r)^6\frac{[4-3(2M/r)]^2}{(1-2M/r)^2}. \tag{10}$$

Here I have defined a constant

$$p_\infty\equiv\frac{\hbar}{90(16\pi)^2(2M)^4}. \tag{11}$$

In the Hartle-Hawking vacuum p_∞ can be interpreted as the pressure at spatial infinity.

To get these expressions I have explicitly expanded the functions given in Refs. [11,13] as polynomials in $2M/r$ to obtain the formulas for the Hartle-Hawking vacuum given in [1], the results being checked against Elster [19] and the spin-zero case of Brown-Ottewill [12]. [Combine Eqs. (3.11) and (3.12) on p. 2517.]

Next the spin-zero Brown-Ottewill analytic approximation for the difference term is evaluated [12]:

$$\langle B|T^{\hat\mu}{}_{\hat\nu}|B\rangle-\langle H|T^{\hat\mu}{}_{\hat\nu}|H\rangle$$

$$=+p_\infty\frac{1}{(1-2M/r)^2}\begin{bmatrix}+3&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}. \tag{12}$$

[Combine Eqs. (3.16) and (3.17) on p. 2517. Set $T=0$ for the Boulware vacuum and note that the coefficients of their $V^{\mu\nu}$ term cancel for spin zero.]

The history of this last expression is interesting: This expression was first written down by Christensen and Fulling who conjectured that this result was *exact* [20, Eq. (6.29) p. 2101]. Later on, Brown and Ottewill effectively derived this result as an *approximation* in their analytic approximation scheme. Finally, Jensen, McLaughlin, and Ottewill showed that this result is not in fact exact, but is nevertheless a good approximation [15].

As a consistency check, the trace of the stress energy tensor is given by

$$\langle B|T^{\hat\mu}{}_{\hat\mu}|B\rangle\equiv-\rho-\tau+2p\equiv96p_\infty(2M/r)^6. \tag{13}$$

This result, because it is simply a restatement of the conformal anomaly, is known to be exact.

I intend to use this analytic approximation, subject to suitable caveats, over the entire maximally extended Kruskal-Szekeres manifold (that is, over the entire spacetime of an eternal black hole).

Outside the event horizon, the explicit numerical computations of Jensen, McLaughlin, and Ottewill [15] show that this analytic approximation is in good qualitative agreement with the numerically determined stress energy tensor.

Several general results can immediately be extracted from this analytic approximation.

The quantity ρ is negative over the entire range $r\in[0,\infty]$. Outside the event horizon you should interpret this as the energy density, inside the event horizon it is the tension.

The quantity τ is negative in the range $r\in[0,1.7753M)\cup(4.2274M,\infty]$ and positive in the range $r\in(1.7753M,4.2274M)$. Outside the event horizon you should interpret this as the radial tension; inside the event horizon it is the energy density.

The transverse pressure p is negative over almost the entire range $r\in[0,\infty]$, with the exception of $r=3M/2$ where it is zero.

The numerical data [15] are limited to the region outside the event horizon $r\in[2M,\infty]$. By visual inspection of the graphs one draws the general conclusions that the density ρ is negative over the entire range $r\in[2M,\infty]$, the radial tension τ is negative in the range $r\in(2.3M,\infty]$, and positive in the range $r\in[2M,2.3M)$, and the transverse pressure p is negative over the entire range $r\in[2M,\infty]$.

While the analytic approximation and numeric data disagree on where the bumps and zero crossings are located, there is good overall agreement as to the qualitative shape of these curves. In this paper I will need to use only relatively crude aspects of the numeric data and eyeball inspection is quite sufficient. For instance from [15, Fig. 4, p. 3005] it is clear that $-\tau<\rho$ outside the horizon, hence, $\rho+\tau<0$. Furthermore, $p<-\tau$ outside the horizon, hence $\tau+p<0$. Finally $\rho<p$ outside the horizon, hence $\rho-p<0$.

Slightly more subtle are the following relationships, also derivable by visual inspection: $|\tau|<-\rho$ outside the horizon, hence $\rho+|\tau|<0$, and whence $\rho\pm\tau<0$. Also $|\tau|<-p$ outside the horizon, hence $p+|\tau|<0$, and whence $p\pm\tau<0$.

This will be sufficient for current purposes.

III. POINTWISE ENERGY CONDITIONS

A. Outside the horizon

Outside the event horizon, the NEC reduces to the pair of constraints

$$\rho(r)-\tau(r)\geqslant0?,\quad \rho(r)+p(r)\geqslant0?. \tag{14}$$

But we have already seen that both ρ and p are individually negative everywhere outside the event horizon, the numeric data and the analytic approximation agreeing on this point.

Therefore the NEC is definitely violated everywhere outside the event horizon. This automatically implies that all the other pointwise energy conditions (WEC, SEC, and DEC) are violated outside the event horizon.

For future use I point out that (defining $z=2M/r$ to reduce clutter) the analytic approximation gives

$$\rho(z)-\tau(z)=-4p_\infty z^6\frac{[28-48z+21z^2]}{(1-z)^2}, \tag{15}$$

$$\rho(z)+p(z)=-4p_\infty z^6\frac{[34-60z+27z^2]}{(1-z)^2}. \tag{16}$$

It is easy to verify that both of these expressions are strictly negative outside the event horizon and indeed are strictly negative throughout the spacetime.

This completes the proof of theorem 1.

B. Inside the horizon

Inside the event horizon, the radial coordinate becomes timelike, and the roles played by $\rho(r)$ and $\tau(r)$ are interchanged. The NEC reduces to the pair of constraints

$$\tau(r)-\rho(r)\geqslant 0?, \quad \tau(r)+p(r)\geqslant 0?. \tag{17}$$

We now only have the analytic approximation available. We have already seen above that $\tau-\rho>0$, and so this condition is not going to help us. On the other hand,

$$\tau(z)+p(z)=-24p_\infty z^6, \tag{18}$$

which is blatantly negative inside the event horizon (and indeed throughout the spacetime).

Therefore, *assuming the analytic approximation is not wildly inaccurate*, the NEC is violated everywhere inside the event horizon, and consequently all the other pointwise energy conditions (WEC, SEC, and DEC) are violated as well.

In summary, all the pointwise energy conditions are violated throughout the entire Schwarzschild spacetime. Outside the event horizon we have both numeric data and analytic approximations which agree on this point. Inside the event horizon we have only the analytic approximation.

So far, what I have shown is at that at each point in the spacetime there is at least one null-timelike vector along which the pointwise energy conditions are violated. But it is possible to derive much stronger results.

IV. TOTAL EXTERNAL NEC VIOLATION

A. Outside the horizon

Consider a generic null vector inclined at an angle ψ away from the radial direction. Then, without loss of generality, in an orthonormal frame attached to the (t,r,θ,ϕ) coordinate system,

$$k^{\hat{\mu}}\propto(\pm 1,\cos\psi,0,\sin\psi). \tag{19}$$

Ignoring the (presently irrelevant) overall normalization of the null vector, one is interested in calculating

$$\langle B|T_{\mu\nu}|B\rangle k^\mu k^\nu \propto(\rho-\tau\cos^2\psi+p\sin^2\psi)$$
$$=([\rho-\tau]+[\tau+p]\sin^2\psi). \tag{20}$$

I intend to show that this quantity is negative for all values of ψ and r.

We have already seen that the analytic approximation implies that $\rho-\tau$ is negative outside the event horizon (and in fact is negative throughout the spacetime). We have also seen that this observation can be extended to the numerical data by inspection of the graphs plotted in [15].

For the analytic approximation we have also seen that $\tau(z)+p(z)$ is blatantly negative outside the event horizon (and indeed throughout the spacetime). Furthermore, we have already seen that this observation can be extended to the numerical data by inspection of the graphs plotted in [15].

We are now done. (Both terms in square brackets are strictly negative.) What we have shown is that for any point p outside the event horizon and any null vector k the inner product $\langle T_{\mu\nu}\rangle k^\mu k^\nu$ is strictly negative. This completes the proof of theorem 2.

B. Inside the horizon

Inside the event horizon there are additional technical complications. One should now consider a generic null vector inclined at an angle $\tilde{\psi}$ away from the t direction (which is now spacelike). Then without loss of generality, in an orthonormal frame attached to the (t,r,θ,ϕ) coordinate system,

$$k^{\hat{\mu}}\propto(\cos\tilde{\psi},\pm 1,0,\sin\tilde{\psi}). \tag{21}$$

One should now consider the quantity

$$\langle B|T_{\mu\nu}|B\rangle k^\mu k^\nu \propto(\tau-\rho\cos^2\tilde{\psi}+p\sin^2\tilde{\psi})$$
$$\propto(-[\rho-\tau]+[\rho+p]\sin^2\tilde{\psi}). \tag{22}$$

We are now limited to the analytic approximation, and (subject to suitable caveats) have already seen that inside the horizon $\rho-\tau$ and $\rho+p$ are both everywhere negative.

Because of the relative minus sign the critical issue is now the relative magnitude of the terms $|\rho-\tau|$ and $|\rho+p|$. The quantity $\langle T_{\mu\nu}\rangle k^\mu k^\nu$ can be made positive by choosing

$$\sin^2\tilde{\psi}<\sin^2[\tilde{\psi}_{\rm crit}(z)]\equiv\frac{[28-48z+21z^2]}{[34-60z+27z^2]}. \tag{23}$$

Just inside the horizon, $z=1^-$, one has $\tilde{\psi}_{\rm crit}(z=1^-)=(\pi/2)^-$, with all the stress-energy components diverging as one actually hits the horizon. As one approaches the singularity for z large one has $\tilde{\psi}_{\rm crit}(z)\to\sin^{-1}(7/9)\approx 62°$.

In summary, inside the event horizon the analytic approximation suggests that certain null directions allow one to have $\langle T_{\mu\nu}\rangle k^\mu k^\nu>0$. This is the situation for which I introduced the notion of the partial null energy condition (PNEC) in Ref. [1].

In some sense this is the mirror image of the situation in the Hartle-Hawking vacuum. In that vacuum I found that the NEC was satisfied at large radius and that at small radius it was possible to find certain directions such that the PNEC was violated. Here in the Boulware vacuum I find that the NEC is violated at large radius, and that at small radius it is possible to find certain directions such that the PNEC is satisfied.

C. Total external WEC-DEC violation

It is now a simple exercise to extend this type of analysis to generic timelike vectors. Outside the horizon one can take

$$V^{\hat{\mu}}=\gamma(\pm 1,\beta\cos\psi,0,\beta\sin\psi). \tag{24}$$

The quantity of interest is now

$$\langle B|T_{\mu\nu}|B\rangle V^{\mu}V^{\nu} = \gamma^2(\rho - \beta^2\tau\cos^2\psi + \beta^2 p \sin^2\psi)$$
$$= \gamma^2([\rho - \beta^2\tau] + \beta^2[\tau + p]\sin^2\psi). \tag{25}$$

Both of the quantities in square brackets are everywhere negative outside the event horizon. Note that $\rho - \beta^2\tau < \rho + \beta^2|\tau| < \rho + |\tau| < \mathrm{Max}(\rho + \tau, \rho - \tau) < 0$. Consequently on can prove the following.

Theorem 3 (total external WEC violation). *For a conformally coupled massless scalar field on a Schwarzschild spacetime in the Boulware vacuum state: For* any *point p outside the event horizon and* any *timelike vector V*,

$$\langle B|T^{\mu\nu}|B\rangle V_{\mu}V_{\nu} \le 0. \tag{26}$$

The equality is in fact achieved only at spatial infinity.

Observe that while the violations of the ordinary NEC immediately imply violations of the ordinary WEC, there is something extra to be proved here when one wants to discuss the wholesale, everywhere in the phase space, violations addressed in this paper.

The total WEC violation theorem now immediately implies the following theorem.

Theorem 4 (total external DEC violation). *For a conformally coupled massless scalar field on a Schwarzschild spacetime in the Boulware vacuum state: For* any *point p outside the event horizon and* any *timelike vector V the dominant energy condition is violated.*

Turning attention to the region inside the event horizon one can take

$$V^{\hat{\mu}} = \gamma(\beta\cos\tilde{\psi}, \pm 1, 0, \beta\sin\tilde{\psi}). \tag{27}$$

The quantity of interest is

$$\langle B|T_{\mu\nu}|B\rangle V^{\mu}V^{\nu} = \gamma^2(\tau - \beta^2\rho\cos^2\tilde{\psi} + \beta^2 p\sin^2\tilde{\psi})$$
$$= \gamma^2([\tau - \beta^2\rho] + \beta^2[\rho + p]\sin^2\tilde{\psi}). \tag{28}$$

While $\rho + p$ is everywhere negative it is relatively easy to drive the total positive: For instance, take $\beta = 0$ and $r \in (1.7753M, 2M)$. (We have already seen that τ is positive in this range.) Alternatively one can take $\beta \approx 1$ and recover the NEC discussion.

In summary, inside the event horizon there are certainly some points and some timelike vectors for which the analytic approximation suggests $\langle T_{\mu\nu}\rangle V^{\mu}V^{\nu} > 0$.

D. Total external SEC violation

Finally we turn to the issue of wholesale violations of the SEC. If the SEC were to hold, one would wish to prove

$$\langle B|\overline{T^{\mu\nu}}|B\rangle V_{\mu}V_{\nu} \ge 0?. \tag{29}$$

Here $\overline{T}$ is the trace-reversed stress tensor:

$$\overline{T^{\mu\nu}} = T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} T. \tag{30}$$

(The easiest way to remember exactly what the SEC is means is to think of it as the trace-reversed WEC.) Thus outside the horizon we can repeat the analysis used for the WEC by making the substitutions

$$\rho \to \overline{\rho} = (\rho - \tau + 2p)/2,$$
$$\tau \to \overline{\tau} = (\tau - \rho + 2p)/2,$$
$$p \to \overline{p} = (\rho + \tau)/2. \tag{31}$$

For instance, outside the horizon the quantity of interest becomes

$$\langle B|\overline{T_{\mu\nu}}|B\rangle V^{\mu}V^{\nu} = \gamma^2([\overline{\rho} - \beta^2\overline{\tau}] + \beta^2[\overline{\tau} + \overline{p}]\sin^2\psi)$$
$$= \gamma^2\{(1+\beta^2)[\rho - \tau]/2 + (1-\beta^2)[p] + \beta^2[\tau + p]\sin^2\psi\}. \tag{32}$$

You will by now be unsurprised at the refrain: Each quantity in square brackets is everywhere negative outside the event horizon, the analytic approximation and the numerical data agreeing on this point. Thus we can prove the theorem.

Theorem 5 (total external SEC violation). *For a conformally coupled massless scalar field on a Schwarzschild spacetime in the Boulware vacuum state: For* any *point p outside the event horizon and* any *timelike vector V*,

$$\langle B|\overline{T^{\mu\nu}}|B\rangle V_{\mu}V_{\nu} \le 0. \tag{33}$$

The equality is in fact achieved only at spatial infinity.

If we now look at the region inside the horizon, the relevant quantity is

$$\langle B|\overline{T_{\mu\nu}}|B\rangle V^{\mu}V^{\nu} = \gamma^2(\overline{\tau} - \beta^2\overline{\rho}\cos^2\tilde{\psi} + \beta^2\overline{p}\sin^2\tilde{\psi})$$
$$= \gamma^2([\overline{\tau} - \beta^2\overline{\rho}] + \beta^2[\overline{\rho} + \overline{p}]\sin^2\tilde{\psi})$$
$$= \gamma^2\{-(1+\beta^2)[\rho - \tau]/2 - (1-\beta^2)[p] + \beta^2[\rho + p]\sin^2\tilde{\psi}\}. \tag{34}$$

In the last line $\rho - \tau$, p, and $\rho + p$ are individually negative, but the relative minus signs make it easy to drive this quantity positive. Note that if one takes $\beta \to 1$, one recovers the NEC discussion.

In summary, inside the event horizon there are certainly some points and some timelike vectors for which the analytic approximation suggests $\langle \overline{T_{\mu\nu}}\rangle V^{\mu}V^{\nu} > 0$.

V. EXOTIC ENERGY CONDITIONS

The total NEC violation theorem proved above is, at least in the region outside the event horizon, strong enough to destroy all conditional energy conditions such as the PNEC and Scri-NEC introduced in [1].

Furthermore, it is strong enough to destroy any and all energy conditions constructed by averaging the quantity $\langle T_{\mu\nu}\rangle k^{\mu}k^{\nu}$ any sort of null curve and demanding positivity for the resulting integral.

The ANEC is violated along all null geodesics that do not cross the event horizon, and in fact is also violated along all such nongeodesic null curves.

Smearing the ANEC by averaging in transverse directions [4] will not help. The smeared ANEC will still be violated along any and all null curves that avoid the event horizon.

Semilocal versions of the ANEC, obtained by inserting any arbitrary postive weighting function $f(\lambda)$ into the ANEC integral, and demanding that the integral remain positive, are also destroyed by this result.

Similarly, the ''total WEC violation theorem'' and ''total SEC violation theorem'' guarantee that the averaged weak energy condition (AWEC) and averaged strong energy condition (ASEC) and their variants are also guaranteed to be violated for all timelike curves, geodesic or not, with arbitrary weighting functions, provided only they avoid the region behind the event horizon.

On the other hand, the ''quantum inequalities'' of Ford and Roman [6–8] are *not* necessarily violated by these results. Extending the analysis of Ford and Roman, it seems that for timelike curves in a nonflat spacetime the generalized quantum inequalities would take the generic form

$$\int_\gamma f(\tau)\langle T^{\mu\nu}\rangle V_\mu V_\nu d\tau \geqslant -|Q[f,g]|. \tag{35}$$

Here $f(\tau)$ is some specific weighting function, and $Q[f,g]$ is some functional of the weighting function and the spacetime metric. The ''quantum inequality'' states that this f-weighted AWEC is not allowed to become excessively negative. (But because it *is* allowed to become negative, the quantum inequalities are compatible with the results of this paper.)

It would clearly be of interest to consider more general weighting functions than the specific choice made by Ford and Roman, and would also be very interesting to see to what extent one can obtain singularity theorems or positive mass theorems based on such generalized quantum inequalities.

VI. DISCUSSION

In the preceding paper [1] I have studied the Hartle-Hawking vacuum state, discovering a complicated layering of energy-condition violations confined to the region between the unstable photon orbit and the event horizon.

The situation in the Boulware vacuum is more dramatic.

(1) All standard (pointwise and averaged) energy conditions are violated throughout the entire region exterior to the event horizon.

(2) Outside the event horizon, the standard pointwise energy conditions are violated in a maximal manner: They are violated at all points and for all null or timelike vectors.

(3) The standard pointwise energy conditions seem to be violated even inside the event horizon.

It should be borne in mind that these are test-field limit calculations, which gives us a (mild) excuse to not worry too much. Furthermore, the Boulware vacuum is ill behaved on the event horizon itself, and for this reason it might be thought to be an ''unphysical'' quantum state, giving a further excuse for not worrying.

But this is not the whole story: The Boulware vacuum is believed to be a good approximation to the quantum-mechanical vacuum surrounding a large condensed object such as a star or planet that has not been allowed to collapse past its Schwarzschild radius. Because the mode sums and subtractions used in calculating $\langle T\rangle$ are purely local, both the analytic approximations and the numerical calculations should be perfectly adequate for describing the vacuum polarization outside the central body itself.

(There is a potential subtlety here: Outside the central object the modes are simply given by the solutions to the Regge-Wheeler equation, and so are determined in a purely local manner. On the other hand, properly determining the overall normalization of each mode depends on an integral over an entire Cauchy surface. This is where nonlocal effects might sneak in. It is thus conceivable, though maybe unlikely, that the vacuum polarization outside a star or planet could depend on details of its interior composition. On the other hand, one still expects the analytic approximation discussed in this paper to be a rather good approximation outside the central body, and the analytic approximation is blatantly local.)

The analysis of this paper suggests that the entire region outside the central body should violate all the standard energy conditions. These violations will be tiny to be sure, but they will be there in the test-field limit. It is of considerable interest to provide even a single quantum state that leads to such wholesale violations of the energy conditions.

Now it is conceivable that this effect would go away if one were able to find a fully self-consistent solution to the field equations of semiclassical quantum gravity—this is a very interesting question well beyond the scope of this paper.

However, some initial steps towards self-consistency can be made by taking a perturbative point of view. Consider perturbative self-consistency in the sense of Flanagan and Wald [4], and view the spacetime as being described by a class of metrics $g_{\mu\nu}(x,\epsilon)$, where ϵ is to be thought of as a perturbation parameter. [The relevant expansion parameter is in fact $\epsilon=\hbar/M^2=(m_P/M)^2$, and is simply the square of the ratio of the mass of the central body to the Planck mass.]

We want to take $\epsilon=0$ to correspond to the Schwarzschild metric, calculate the gravitational polarization in the Boulware metric (which by definition is of order ϵ), and feed this back in to get the first-order shifted metric $g_{\mu\nu}(x,\epsilon)=g_{\mu\nu}(x,0)+\epsilon\Delta g_{\mu\nu}(x)+O(\epsilon^2)$. Then this first-order shifted metric has a vacuum polarization which is equal to that of the Schwarzschild geometry, up to first order in ϵ, and thus provides a first-order self-consistent solution of semiclassical quantum gravity. Given the smallness of ϵ for heavy objects we should expect the first-order self-consistent solution to be extremely close to the exact solution.

A technical difficulty is that this self-consistent solution breaks down at $r=2M$, the location of the event horizon in zeroth order, which is a much more confusing place at first order. Be that as it may, one may think of a star or planet and chop the almost-Schwarzschild geometry off at some suitably large radius, matching it to some stellar or planetary interior.

Outside the star or planet the entire analysis of this paper should hold, at least qualitatively, and we would then have a self-consistent solution of semiclassical quantum gravity that violates all of the energy conditions. In the notation of Flanagan and Wald [4], $I^{(1)}$, the first-order perturbation of the

ANEC integral, will be negative for geodesics that remain in this region. Because I was able to prove that ANEC was violated at this order for all null curves, it is clear that the transverse averaging advocated by Flanagan and Wald will not help the situation.

Thus I claim that the results of this paper are highly suggestive evidence pointing to the fact that general self-consistent solutions of semiclassical quantum gravity will *not* satisfy the energy conditions, and may in fact for certain quantum fields and certain quantum states violate *all* the standard energy conditions.

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