No hair for spherical black holes: Charged and nonminimally coupled scalar field with self-interaction

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We prove three theorems in general relativity which rule out classical scalar hair of static, spherically symmetric, possibly electrically charged black holes. We first generalize Bekenstein's no-hair theorem for a multiplet of minimally coupled real scalar fields with not necessarily quadratic action to the case of a charged black hole. We then use a conformal map of the geometry to convert the problem of a charged (or neutral) black hole with hair in the form of a neutral self-interacting scalar field nonminimally coupled to gravity to the preceding problem, thus establishing a no-hair theorem for the cases with a nonminimal coupling parameter $\xi<0$ or $\xi\geqslant 1/2$. The proof also makes use of a causality requirement on the field configuration. Finally, from the required behavior of the fields at the horizon and infinity we exclude hair of a charged black hole in the form of a charged self-interacting scalar field nonminimally coupled to gravity for any ξ. [S0556-2821(96)00420-1]

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I. INTRODUCTION

''Black holes have no hair'' was introduced by Wheeler in the early 1970s as a principle predicting the simplicity of the stationary black hole family. The proliferation in the 1990s of stationary black hole solutions with ''hair'' of various sorts [1] may give the impression that the principle has fallen by the wayside. However, this is emphatically not the case for scalar field hair, possibly accompanied by Abelian gauge fields. The only exceptions known to ''black holes have no hair'' in this department are the Bronnikov-Melnikov-Bocharova-Bekenstein (BMBB) spherical extremal black hole with electric charge and a scalar field nonminimally coupled to gravity in conformally invariant fashion [2,3], its magnetic monopole extension [4], and the Achucarro-Gregory-Kuijken (AGK) black hole [5], a charged black hole transfixed by a Higgs local cosmic string. Even these examples are not contrary to the spirit of the no-hair conjecture: the first seems to be unstable [6], the second is too similar to the first to escape its fate, while the third is not asymptotically flat. What is the evidence for ''no scalar hair'' for black holes?

The first no-scalar hair theorems applied to the common massless scalar field [7,8] and to the neutral Klein-Gordon field [8,9]. The latter theorem's proof is also found to work for the neutral scalar field with a monotonically increasing self-interacting potential. Little progress was made in extending these theorems during the 1970s and 80s. A notable exception was the Adler-Pearson theorem [10] which excludes charged Higgs hair for a charged black hole. This theorem has, however, occasionally been regarded as flawed [11]. Lately theorems by Heusler [12], Sudarsky [13], and Bekenstein [14] have become available which exclude electrically neutral black holes with hair as minimally coupled scalar fields endowed with positive semidefinite self-interaction potentials of otherwise arbitrary shape. The last mentioned theorem applies also to fields whose Lagrangians are not necessarily quadratic in the gradients of the fields.

Whereas simple scalar fields are covered by all these theorems, various complications such as charge of the field and the hole, nonminimal coupling to gravity, etc., are not. Early works in this more challenging direction are the papers by Xanthopoulos and Zannias [15] and Zannias [16] which establish the uniqueness of the BMBB black hole among the asymptotically flat static solutions of the Einstein and *conformal* scalar field equations, and the recent theorem by Saa [17] which excludes, for spherical black holes, a broader, but still limited, class of nonminimally coupled neutral scalar hair (see Sec. IV A).

In the present work we consider whether a charged black hole may possess hair in the form of a scalar field with self-interaction and with nonminimal coupling to gravity and gauge covariant coupling to the electromagnetic field. The motivation for looking at nonminimal gravitational coupling is supplied by the existence of the BMBB black hole solution with nonminimally coupled scalar hair. The motivation for considering coupling of the scalar to the electromagnetic field comes from the existence of the AGK black hole. Since nonminimal gravitational coupling entails not necessarily positive field energy, one loses one of the earlier tools for proving no hair theorems [14]. Our assumption of spherical symmetry simplifies things enough to allow us to prove several useful theorems.

In Sec. II we formulate the equations of the scalar field coupled nonminimally to gravity and gauge covariantly to the Maxwell field, write down the energy-momentum tensor, and discuss restrictions on it from regularity of the horizon and causality requirements. The last, in particular, do not seem to have been taken advantage of by previous workers. Section III generalizes a theorem by one of us [14] which excludes hair in the form of a multiplet of mutually interacting real scalar fields with possibly nonquadratic kinetic action. The theorem is here extended to an electrically charged

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black hole, still under the assumption of positivity of energy of the fields. The extended theorem provides one of the tools for proving, in Sec. IV, a theorem ruling out, for an electrically charged or neutral black hole, hair in the form of a neutral scalar field with standard quadratic kinetic action, a positive semidefinite self-interacting potential, and nonminimal coupling to gravity. The theorem is proved for the ranges $\xi<0$ or $\xi\geqslant\frac{1}{2}$ of the nonminimal coupling parameter; its proof uses a conformal map to convert the problem to the one dealt with by the theorem of Sec. III. Also central to its proof are the causality restrictions on the energy-momentum tensor. In Sec. V a theorem is proved which rules out, for an electrically charged black hole, hair in the form of a charged scalar field with standard kinetic action, a positive semidefinite self-interaction potential, and nonminimal coupling to gravity with any ξ. The proof, which is given separately for nonextremal and extremal black holes, centers on the behavior of the various fields near the horizon and at infinity. Section VI summarizes our findings and speculates on their implications.

II. BASIC EQUATIONS AND PHYSICAL RESTRICTIONS

Here we derive the energy-momentum tensor from the Maxwell-charged scalar action with self-interaction and nonminimal coupling to gravity. Then we derive the field equations for the scalar and the Maxwell fields. Throughout we use units with $c=1$.

A. The energy-momentum tensor

We assume the existence of an asymptotically flat joint solution of the Einstein, scalar field, and Maxwell equations, having the character of a static, spherically symmetric, charged black hole spacetime. The symmetries entitle us to write the metric outside the horizon as

$$ds^2=-e^{\nu}dt^2+e^{\lambda}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{2.1}$$

with $\nu(r)$ and $\lambda(r)$ both non-negative and obeying $\nu(r),\lambda(r)\sim O(r^{-1})$ as $r\to\infty$ because of asymptotic flatness. The event horizon is at $r=r_H$ where $e^{-\lambda(r_H)}=0$ (see Sec. V A below). In case there are several such zeroes, the horizon corresponds to the outer one. Anticipating the results of Secs. V B and V E, we note that near the event horizon of a black hole of nonextremal or extremal kind

$$e^{\nu}\sim e^{-\lambda}\sim\begin{cases} r-r_H & \text{nonextremal,}\\ (r-r_H)^2 & \text{extremal.}\end{cases} \tag{2.2}$$

These results apply whatever the matter content of the spacetime.

The action of a charged scalar field with nonminimal coupling to gravity, gauge covariant coupling to electromagnetism [or any U(1) gauge field], and with a general self-interaction potential is

$$S_{SM\xi}=-\frac{1}{2}\int\left(D_\alpha\psi(D^\alpha\psi)^*+\xi R\psi\psi^*+V(\psi\psi^*)\right.$$
$$\left.+\frac{1}{8\pi}F^{\alpha\beta}F_{\alpha\beta}\right)\sqrt{-g}d^4x, \tag{2.3}$$

where ψ is the complex scalar field, A_μ the Maxwell vector potential, $D_\mu=\partial_\mu-iqA_\mu$ the gauge covariant derivative (q is the charge), $F_{\nu\mu}=A_{\mu,\nu}-A_{\nu,\mu}$ the Faraday field tensor, $V=V(|\psi|^2)$ the self-interaction potential, R the scalar curvature, and ξ the strength of the nonminimal coupling to gravity. We assume throughout that V is everywhere regular (V and its first derivative bounded for finite argument) as well as positive semidefinite.

The energy-momentum tensor that follows from the action is

$$T_{\mu\nu}=\frac{1}{2}D_\mu\psi(D_\nu\psi)^*+\frac{1}{2}(D_\mu\psi)^*D_\nu\psi-\frac{1}{2}D_\alpha\psi(D^\alpha\psi)^*g_{\mu\nu}$$
$$-\xi(\psi^*\psi)_{,\mu;\nu}+\xi\Box(\psi^*\psi)g_{\mu\nu}+\xi(\psi^*\psi)G_{\mu\nu}$$
$$-\frac{1}{2}Vg_{\mu\nu}+T^{(\mathrm{em})}_{\mu\nu}, \tag{2.4}$$

where $G_{\mu\nu}$ denotes the Einstein tensor and

$$T^{(\mathrm{em})}_{\mu\nu}=\frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha-\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{2.5}$$

Here and elsewhere $\Box$ is the d'Alembertian. By virtue of the symmetries, the components of the electromagnetic energy-momentum tensor satisfy [18]

$$T^{(\mathrm{em})r}{}_r=T^{(\mathrm{em})t}{}_t=-T^{(\mathrm{em})\theta}{}_\theta=-T^{(\mathrm{em})\phi}{}_\phi=-\frac{Q^2}{8\pi r^4}, \tag{2.6}$$

where $Q(r)$ is the electric charge enclosed by the sphere of radius r. Eliminating R from Eq. (2.4) by means of the trace of Einstein's equations, $R=-8\pi GT$, we obtain

$$T^\nu_\mu(1-8\pi G\xi\psi^*\psi)=\frac{1}{2}D_\mu\psi(D^\nu\psi)^*+\frac{1}{2}(D_\mu\psi)^*D^\nu\psi$$
$$-\frac{1}{2}D_\alpha\psi(D^\alpha\psi)^*\delta^\nu_\mu-\xi(\psi^*\psi)^{;\nu}_{,\mu}$$
$$+\xi\Box(\psi^*\psi)\delta^\nu_\mu-\frac{1}{2}V\delta^\nu_\mu+T^{(\mathrm{em})\nu}{}_\mu. \tag{2.7}$$

In light of the angular and temporal symmetries, F_{tr} is the only nonvanishing field component. Thus only A_r and A_t need be nonvanishing. Then the gauge transformation $A_\mu\to A_\mu+\Lambda_{,\mu}$ with $\Lambda=-\int A_r dr$ removes A_r. The new A_t must have the form $f(r)+g(t)$ in order to give a stationary F_{tr}. A further gauge transformation with $\Lambda=-\int g(t)dt$ makes A_t static. In this second gauge any temporal variation of the phase of ψ must be linear in t in order that both the current and charge density be time independent. More precisely, the phase must be $\varphi(r)-\omega t$ with ω a real constant. A last gauge transformation with $\Lambda=-\omega t/q$ reduces ψ to the form $\psi=a(r)e^{i\varphi(r)}$, while merely adding a constant to A_t.

Now on the one hand, the radial current component is

$$J_r=\imath(\psi^*\partial_r\psi-\psi\partial_r\psi^*)=-2a^2\partial_r\varphi. \tag{2.8}$$

On the other hand, conservation of charge together with the symmetries implies that

$$J_r e^{-\lambda}\sqrt{-g} = \text{const.} \tag{2.9}$$

The constant here must vanish; otherwise charge would leak out continually to infinity. It follows from Eq. (2.8) that φ cannot depend on r, except possibly for jumps at nodes where $a=0$. However, an arbitrary jump of φ through a node causes an unacceptable discontinuity in $\psi,_r$ unless the jump is by an odd multiple of π in which case its effect on $\psi,_r$ is cancelled by the change in sign of $a,_r$ at the node. Accordingly, in all that follows we regard φ as strictly constant (and thus irrelevant) at the cost of allowing $a(r)$ to change sign. Thus in our problem ψ, henceforth denoted by a, is real while A_μ reduces to a static temporal component.

In what follows it will be convenient to look at the differences

$$T_t^t - T_\phi^\phi = \xi e^{-\lambda}\frac{(2/r-\nu')aa,_r}{1-8\pi G\xi a^2} - \frac{Q^2/(4\pi r^4)+q^2e^{-\nu}A_t^2a^2}{1-8\pi G\xi a^2}, \tag{2.10}$$

$$T_t^t - T_r^r = e^{-\lambda}\frac{(2\xi-1)a,_r^2 - \xi(\nu+\lambda)'aa,_r + 2\xi aa,_{rr}}{1-8\pi G\xi a^2} - \frac{q^2e^{-\nu}A_t^2a^2}{1-8\pi G\xi a^2} \tag{2.11}$$

(here and henceforth $' \equiv \partial/\partial r$), as well as at the negative of the energy density

$$T_t^t = e^{-\lambda}\frac{(2\xi-1/2)a,_r{}^2 + 2\xi aa,_{rr} + \xi(4/r-\lambda')aa,_r}{1-8\pi G\xi a^2} - \frac{1}{2}\frac{V+Q^2/(4\pi r^4)+q^2A_t^2a^2e^{-\nu}}{1-8\pi G\xi a^2}. \tag{2.12}$$

B. The scalar-Maxwell field equations

The field equation for a that follows from Eq. (2.3) is

$$D_\mu D^\mu a - (\xi R + \dot{V})a = 0, \tag{2.13}$$

where $\dot{V} \equiv \partial V(a^2)/\partial a^2$. After substitution of the metric functions and simplification we get

$$a,_{rr} + (1/2)(4/r+\nu'-\lambda')a,_r - (\xi R + \dot{V} - q^2e^{-\nu}A_t^2)e^\lambda a = 0. \tag{2.14}$$

Finally the temporal component of the Maxwell equations,

$$F^{\mu\nu}{}_{;\mu} = 4\pi J^\nu, \tag{2.15}$$

takes the form

$$A_t,_{rr} + (1/2)(4/r-\nu'-\lambda')A_t,_r - 4\pi q^2a^2e^\lambda A_t = 0. \tag{2.16}$$

Note that when $q=0$ the equations for A_t and for the scalar field decouple so that we can consider the two fields separately.

C. Finiteness of T_μ^ν and the causality restriction

There are two types of restrictions on the total energy-momentum tensor which must be obeyed everywhere in the black hole exterior and horizon in order for a solution to be physically acceptable: boundedness of the mixed components T_μ^ν, and the causality restriction.

Staticity and spherical symmetry imply that the only nonvanishing mixed components of T_μ^ν are T_t^t, T_r^r, and $T_\theta^\theta = T_\phi^\phi$. Thus $T_{\alpha\beta}T^{\alpha\beta} = (T_t^t)^2 + (T_r^r)^2 + (T_\theta^\theta)^2 + (T_\phi^\phi)^2$. But $T_{\alpha\beta}T^{\alpha\beta}$ is a physical invariant and must thus be bounded everywhere, including at the horizon: any divergence would imply divergence of the curvature invariant $G_{\mu\nu}G^{\mu\nu}$. It follows that the *mixed* components T_t^t, T_r^r, and $T_\theta^\theta = T_\phi^\phi$ are all bounded everywhere including at the horizon (this is no longer true for a component like T_{rr}).

Along with finiteness of T_μ^ν in general we should cite here an important result to be obtained in Secs. V B and V E; it has also been noticed by Achucarro, Gregory, and Kuijken [5] and by Nún̄ez, Quevedo, and Sudarsky [19]. For any static and spherical black hole and for any matter content,

$$T_t^t(r_H) = T_r^r(r_H). \tag{2.17}$$

Now consider the Poynting vector according to a physical observer with four-velocity u^ν

$$j^\mu = -T_\mu^\nu u^\mu \tag{2.18}$$

with $u^\mu u_\mu = -1$, and the associated energy density

$$\varepsilon \equiv T_{\mu\nu}u^\mu u^\nu. \tag{2.19}$$

If $\varepsilon > 0$ then j^μ should be a nonspacelike four-vector, for in this case j^μ defines a future-directed four-velocity (with positive time component $\propto \varepsilon$), and on grounds of causality this ''velocity of transfer of energy'' should not be superluminal. If $\varepsilon < 0$ the Poynting vector points into the past. We still expect that j^μ should be nonspacelike because the flow of negative energy can be interpreted as flow of positive energy in the opposite space direction from that demarcarted by j^μ. In other words j^μ should, in this case, point into the past lightcone. Hence, for any observer we must have

$$T_\mu^\nu u_\nu T_\sigma^\mu u^\sigma \leqslant 0. \tag{2.20}$$

Now suppose that our observer moves in any way in the equatorial plane $\theta = 0$ in the hole's exterior, Eq. (2.20) becomes

$$u^t u_t (T_t^t)^2 + u^r u_r (T_r^r)^2 + u^\phi u_\phi (T_\phi^\phi)^2 \leqslant 0. \tag{2.21}$$

Substituting here $u^t u_t$ from the normalization of the velocity

$$u^t u_t + u^r u_r + u^\phi u_\phi = -1 \tag{2.22}$$

and rearranging the inequality gives

$$(T_t^t)^2 \geqslant \frac{u_r u^r (T_r^r)^2 + u_\phi u^\phi (T_\phi^\phi)^2}{1 + u^r u_r + u^\phi u_\phi}. \tag{2.23}$$

In light of the positivity of $u^r u_r$ and $u^\phi u_\phi$, it follows that inequality (2.21) or (2.20) can be true for any velocity only if

$$|T^{\theta}_{\theta}|=|T^{\phi}_{\phi}|\leq|T^{t}_{t}|\geq|T^{r}_{r}|. \tag{2.24}$$

The energy conditions (2.24) have been discussed by Hawking and Ellis [20] who, however, considered them only for the positive energy density case. When dealing with nonminimal coupling to gravity, negative energy density is not excluded. In the Appendix we prove that either the energy conditions (2.24), or the causality condition (2.20) for all observers, are equivalent to consensus of all observers as to the sign of the energy density. From all this it is clear that the energy conditions (2.24) are a must for a nonpathological solution of the field equations, and henceforth we assume them to hold.

The energy conditions (2.24) can also be stated as

$$\mathrm{sgn}(T^{t}_{t})=\mathrm{sgn}(T^{t}_{t}-T^{r}_{r})=\mathrm{sgn}(T^{t}_{t}-T^{\phi}_{\phi}). \tag{2.25}$$

We stress that no assumption is made here about the sign of the energy density $-T^{t}_{t}$, so that these inequalities are more broadly valid than the *weak* energy condition $T^{r}_{r}-T^{t}_{t}\geq 0$ which is sometimes invoked.

III. MINIMALLY COUPLED NEUTRAL SCALAR FIELD WITH NONQUADRATIC ACTION

There exists a no-hair theorem for black holes which rules out hair in the form of a minimally coupled (to gravity), real multiplet scalar field for any asymptotically flat, static, spherically symmetric neutral black hole [14]. The field is assumed to bear positive energy, but its field Lagrangian need not be quadratic in the field derivatives. Here we generalize that theorem to charged black holes [21], not only for its intrinsic interest, but for use in our later theorems for nonminimally coupled fields.

Consider the action for real scalar fields, $\psi,\chi,\ldots,$ accompanied by an electromagnetic field

$$S_{\psi,\chi,\ldots}=-\int\left(\mathcal{E}(\mathcal{I},\mathcal{J},\mathcal{K},\ldots,\psi,\chi,\ldots)+\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\right)\times\sqrt{-g}d^{4}x. \tag{3.1}$$

Here $\mathcal{E}$ is a function (which for static fields turns out to be identical to the energy density), and $\mathcal{I}\equiv g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}$, $\mathcal{J}\equiv g^{\alpha\beta}\chi_{,\alpha}\chi_{,\beta}$, and $\mathcal{K}\equiv g^{\alpha\beta}\chi_{,\alpha}\psi_{,\beta}$ are examples of the invariants that can be formed from first derivatives of the scalar fields. We do not assume that the kinetic part of the scalar's Lagrangian density can be separated out, nor that it is a quadratic form in first derivatives.

Assume the existence of a spherically symmetric static black hole solution with the said scalar fields as hair. Because the scalar fields are assumed decoupled from the electromagnetic field, the energy-momentum tensor of the scalar fields is conserved separately. From the radial component of the conservation law $T^{(\mathrm{sc})\nu}{}_{\mu;\nu}=0$ together with the result $T^{(\mathrm{sc})t}{}_{t}=T^{(\mathrm{sc})\phi}{}_{\phi}$ which follows from the form of $S_{\psi,\chi,\ldots}$ and the symmetries, one obtains, as in [14], the results

$$T^{(\mathrm{sc})r}{}_{r}=-\frac{e^{-\nu/2}}{r^{2}}\int_{r_H}^{r}(r^{2}e^{\nu/2})'\rho\,dr, \tag{3.2}$$

$$(T^{(\mathrm{sc})r}{}_{r})'=-\frac{e^{-\nu/2}}{r^{2}}(r^{2}e^{\nu/2})'(\rho+T^{(\mathrm{sc})r}{}_{r}). \tag{3.3}$$

Here $\rho=\mathcal{E}=-T^{(\mathrm{sc})t}{}_{t}$ is the (assumed positive) energy density of the scalar fields. In order for the mass to be finite, we shall require that asymptotically $\rho=O(r^{-3})$. Now the positivity of ρ, the relation (2.6) and the causality restriction (2.25) for the overall T^{ν}_{μ} tell us that $\rho+T^{(\mathrm{sc})r}{}_{r}>0$. Since e^{ν} vanishes at $r=r_H$ (see Sec. V A below) and must be positive for $r>r_H$, $r^{2}e^{\nu/2}$ must grow with r at least sufficiently near the horizon. It is then immediately obvious from Eqs. (3.2) and (3.3) that sufficiently near the horizon $T^{(\mathrm{sc})r}{}_{r}$ and $(T^{(\mathrm{sc})r}{}_{r})'$ are both negative.

Asymptotic flatness considerations together with Eqs. (3.2) and (3.3) tell us that as $r\to\infty$, $T^{(\mathrm{sc})r}{}_{r}=O(r^{-2})$ and $(T^{(\mathrm{sc})r}{}_{r})'<0$. From these follows that asymptotically $T^{(\mathrm{sc})r}{}_{r}>0$ so that $T^{(\mathrm{sc})r}{}_{r}$ must switch sign at some finite point $r=r_c$. Hence we infer that in some intermediate interval $[r_a,r_b]$, $(T^{(\mathrm{sc})r}{}_{r})'>0$ and the point where $T^{(\mathrm{sc})r}{}_{r}$ changes sign is $r_c\in[r_a,r_b]$. $T^{(\mathrm{sc})r}{}_{r}$ is positive in $[r_c,r_b]$.

Now it turns out that this last conclusion clashes with Einstein's equations. The relevant one are

$$e^{-\lambda}\left(\frac{1}{r^{2}}-\frac{\lambda'}{r}\right)-\frac{1}{r^{2}}=8\pi GT^{t}_{t}=-8\pi G\left(\rho+\frac{Q^{2}}{8\pi r^{4}}\right),$$

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^{2}}\right)-\frac{1}{r^{2}}=8\pi GT^{r}_{r}=8\pi G\left(T^{(\mathrm{sc})r}{}_{r}-\frac{Q^{2}}{8\pi r^{4}}\right), \tag{3.4}$$

where Q, a constant in the present section as well as in Sec. IV, is the total charge of the black hole. Sometimes the difference of these comes in handy:

$$e^{-\lambda}(\nu'+\lambda')=-8\pi G(T^{t}_{t}-T^{r}_{r})r. \tag{3.5}$$

Now by integrating Eq. (3.4) out from the horizon radius, r_H, and solving for e^{λ} we obtain

$$e^{-\lambda}=1-\frac{r_H}{r}-\frac{8\pi G}{r}\int_{r_H}^{r}\left(\rho+\frac{Q^{2}}{8\pi r^{4}}\right)r^{2}dr, \tag{3.6}$$

where the integration constant has been set so that $e^{-\lambda}$ vanishes at r_H. It follows from Eq. (3.6) that $e^{\lambda}\geq 1$ throughout the black hole exterior.

Consider now the second Einstein equation. Since $e^{-\lambda(r_H)}=0$, but $\nu'(r)$ may diverge positively at $r=r_H$ (see Sec. V A below), we can write it as

$$8\pi GT^{(\mathrm{sc})r}{}_{r}(r_H)-\frac{GQ^{2}}{r_H{}^{4}}\geq-\frac{1}{r_H{}^{2}}. \tag{3.7}$$

Because in the proximity of the horizon $T^{(\mathrm{sc})r}{}_{r}<0$, we can infer that

$$GQ^{2}\leq r_H{}^{2} \tag{3.8}$$

so that trivially

$$\frac{-GQ^{2}}{2r^{3}}+\frac{1}{2r}>0. \tag{3.9}$$

Now rewriting Eq. (3.4) we infer from Eq. (3.9) that, in a region where ${T^{(\mathrm{sc})}}^r_r>0$,

$$\frac{e^{-\nu/2}}{2}(r^2e^{\nu/2})'=\left(4\pi G r {T^{(\mathrm{sc})}}^r_r-\frac{GQ^2}{2r^3}+\frac{1}{2r}\right)e^{\lambda}+\frac{3}{2r}>0. \tag{3.10}$$

We found that in $[r_c,r_b]$, ${T^{(\mathrm{sc})}}^r_r>0$. Thus $(e^{-\nu/2}/2)(r^2e^{\nu/2})'>0$ there. According to Eq. (3.3) this means that $({T^{(\mathrm{sc})}}^r_r)'<0$ throughout $[r_c,r_b]$. However, we determined that $({T^{(\mathrm{sc})}}^r_r)'>0$ throughout the encompassing interval $[r_a,r_b]$. Thus there is a contradiction. The only way to resolve it is to accept that the the scalar field component must be constant throughout the black hole exterior, taking values such that all components of ${T^{(\mathrm{sc})}}^\mu_\nu$ vanish identically. Such values must exist in order that the trivial solution of the scalar field equation be possible in free empty space. It is this solution which served implicitly as an asymptotic boundary condition in our argument.

Thus *the unique asymptotically flat, static, spherically symmetric black hole solution of the action (3.1) is the Reissner-Nordström black hole with no scalar hair.*

IV. NONMINIMALLY COUPLED ($\xi<0$ OR $\xi\geqslant\frac{1}{2}$) NEUTRAL SCALAR FIELD WITH SELF-INTERACTION

A. Case $\xi<0$

We now consider hair described by action (2.3) with $q=0$ and a potential restricted by $V\geqslant 0$ for a black hole which may or may not be charged. In order to prove that the field can only be in a trivial configuration, we shall use a conformal map to show that in a new metric the action is equivalent to that considered in Sec. III. This approach has also been used by Saa [17], who also started from the theorem discussed in Sec. III in its neutral black hole version [14].

With the proposed solution for the black hole with nonminimally coupled and neutral scalar field hair, $\{a,g_{\mu\nu}\}$, we construct the map

$$g_{\mu\nu}\rightarrow\overline{g_{\mu\nu}}\equiv g_{\mu\nu}\Omega,$$
$$\Omega\equiv 1-8\pi G\xi a^2. \tag{4.1}$$

For $\overline{g_{\mu\nu}}$ to be nondegenerate and of like signature to $g_{\mu\nu}$, Ω must be strictly positive and bounded in $r\in[r_H,\infty)$. Obviously $\Omega>0$ here. Saa leaves pending the question of boundedness of the conformal factor. We now *prove* that Ω is bounded in the black hole exterior both for $\xi<0$ and for $\frac{1}{2}\leqslant\xi$.

Ω can blow up only where a does so. However, at such a point r_c, $a,_r$ and $a,_{rr}$ would be even more singular than a. Specifically, $a,_r^2/a^2$ and $a,_{rr}/a$ should both behave like $(r-r_c)^{-2}$. We see from Eqs. (2.10) and (2.12) that if $r_c>r_H$, the physical components of the energy-momentum tensor definitely diverge at r_c, which is unphysical (Sec. II C). If $r_c=r_H$, the factor $e^{-\lambda}$, ameliorates the divergence. According to Eq. (2.2) for a nonextremal black hole $e^{-\lambda}\sim(r-r_H)$; this is not enough to cancel the divergence. By contrast, for an extremal black hole $e^{-\lambda}\sim(r-r_H)^2$ so it would seem that the divergence is quenched. But since $a^2\rightarrow\infty$ for r approaching r_H from the right, it is evident that $aa,_r$ and $aa,_{rr}>0$ near the singularity. We thus see from Eq. (2.11) in its variant form (4.3) that for $r\rightarrow r_H$, $T_t^t-T_r^r$ has a *negative* definite limit for either $\xi<0$ or $\xi\geqslant 1/2$. However, Eq. (2.17) tells us that $T_t^t-T_r^r$ must vanish at a regular spherical horizon. Thus for all black holes and for $\xi<0$ or $\xi\geqslant\frac{1}{2}$, Ω can blow up in $r\in[r_H,\infty)$ only for physically unacceptable solutions. Additionally, if a were to diverge for $r\rightarrow\infty$, the various $T_\mu{}^\nu$ would blow asymptotically. We conclude that $\Omega<\infty$ everywhere outside the black hole both for $\xi<0$ and for $\xi\geqslant 1/2$. And since Ω cannot vanish for $\xi<0$ we see that the map is regular for physically acceptable black holes and $\xi<0$.

Under the map the action (2.3) together with the Hilbert-Einstein action is transformed into

$$S=\frac{1}{16\pi G}\int\overline{R}\sqrt{-\overline{g}}d^4x-\frac{1}{2}\int\Bigg((1+f)\overline{g^{\alpha\beta}}a,_\alpha a,_\beta+\overline{V}$$
$$+\frac{1}{8\pi}\overline{g^{\alpha\gamma}}\overline{g^{\beta\delta}}F_{\alpha\beta}F_{\gamma\delta}\Bigg)\sqrt{-\overline{g}}d^4x,$$
$$f\equiv 48\pi G\xi^2a^2(1-8\pi G\xi a^2)^{-2},$$
$$\overline{V}\equiv V(1-8\pi G\xi a^2)^{-2}. \tag{4.2}$$

The transformed action is of the form (3.1). It is easily checked that in the static situation the field bears positive energy with respect to $\overline{g_{\mu\nu}}$, not least because of the assumed positivity of $V(a^2)$. It is also easily seen that the map leaves the mixed components T_μ^ν unaffected. Hence the finiteness of these, and the causality sign relations (2.25), can be used in the new geometry.

There is one little complication. We know that Ω goes to some finite positive value at infinity (an oscillatory behavior is excluded by the argument to be given presently that Ω determines the effective gravitational constant, which cannot oscillate spatially in a physical solution). Since this value is not necessarily unity, the asymptotically Minkowskian metric $g_{\mu\nu}$ will be mapped into a not necessarily asymptotically Minkowskian $\overline{g_{\mu\nu}}$. But $\overline{g_{\mu\nu}}$ *is* asymptotically flat. One need only redefine globally the units of length and time to make it of standard Minkowski form at infinity. With this proviso we may apply the theorem of Sec. III to show that a must be constant.

Thus *there exists no static spherically symmetric neutral or charged black hole endowed with nontrivial hair in the form of a neutral scalar field nonminimally coupled to gravity with $\xi<0$ and with a non-negative self-interaction potential.*

B. Case $\xi\geqslant\frac{1}{2}$ with $Q\neq 0$

Before starting that proof, we comment on the asymptotic value of Ω. This is determined by the value of a for which $a,_r\rightarrow 0$ and $a,_{rr}\rightarrow 0$ as $r\rightarrow\infty$ according to the scalar equation, Eq. (2.14). Since $q=0$ here and $R\rightarrow 0$ asymptotically, $a(\infty)$ is determined by $\dot{V}(a(\infty)^2)=0$. Further, in order for the energy density $-T_t^t$ to vanish in the same limit (asymptotic flatness), we need, according to Eq. (2.12), that $V(a^2)$

itself vanish where $\dot{V}(a^2)$ vanishes. In addition, this common zero of V and $\dot{V}$ must be such as to make $\Omega > 0$. For otherwise the *effective* gravitational constant would be negative far away from the black hole. One way to see this is to imagine adding to the background of the black hole solution with energy-momentum tensor given by Eq. (2.4) a small positive mass. In Eqs. (2.10) and (2.12) the additional energy-momentum tensor would appear as contributions to the numerators, with everything divided by Ω. In a region where $\Omega < 0$ that mass would thus contribute to the gravitational field as if it were negative. This contribution will repel a second particle of the same kind (treated as a test particle). Thus positive masses would repel each other gravitationally and the effective gravitational constant $G_{\rm eff} = G(1-8\pi G\xi a^2)^{-1}$ would be negative. This is certainly unphysical if the region is far from the black hole (it could be our neighborhood). We conclude that a physically reasonable black hole solution must have $\Omega > 0$ asymptotically, which requires that both $V(a^2)$ and $\dot{V}(a^2)$ have at least one common root $a^2 < (8\pi G\xi)^{-1}$.

We now proceed to prove by contradiction that Ω cannot vanish in $[r_H, \infty)$. Suppose that there is a nontrivial physically reasonable neutral black hole solution, for which Ω vanishes at some point $r = \tilde{r}$ (if there are several points $\tilde{r}$, we focus on the *rightmost* one). It is obvious from Eq. (2.10) that $a_{,r} \neq 0$ and $2/r - \nu' \neq 0$ at $\tilde{r}$ for if either vanished, $T_t^t - T_\phi^\phi$ would necessarily diverge there contrary to the requirements in Sec. II C. In fact $a^2{}_{,r} < 0$ at $r = \tilde{r}$ because Ω must be positive as $r \to \infty$.

Now a^2 cannot have a minimum. For at such point $r = \hat{r}$, $a_{,r} = 0$, and $aa_{,rr} > 0$. Obviously $\hat{r} \neq \tilde{r}$ because we found $a_{,r} \neq 0$ at the latter. But then according to Eqs. (2.10) and (2.11), $T_t^t - T_\phi^\phi$ and $T_t^t - T_r^r$ will have opposite signs at $r = \hat{r}$ (we assume $\xi > \frac{1}{2}$). But this contradicts the causality restriction (2.25). Thus in our solution $a(r)^2$ must be monotonically decreasing.

It follows that near infinity we must have $aa_{,r} < 0$ and $aa_{,rr} > 0$. From asymptotic flatness it follows that $\nu' \sim 1/r^2$ for sufficiently large r. Hence by Eq. (2.10), $T_t^t - T_\phi^\phi < 0$ asymptotically. By causality [Eq. (2.25)] $T_t^t - T_r^r$ must then be negative for large r. This condition together with Eq. (3.5) tells us that asymptotically $(\nu + \lambda)' > 0$. Substituting all these in Eq. (2.11) we find that $T_t^t - T_r^r > 0$ for large r. But this contradicts our previous conclusion. Our supposition that Ω vanishes somewhere is thus rebutted, at least for $\xi \geq \frac{1}{2}$, $q = 0$, and $Q \neq 0$.

Recalling from Sec. IV A that Ω cannot blow up in the black hole exterior, we see that the map used there is equally valid in the present case. Thus by the same logic as used in Sec. IV A, hair is excluded in the present case.

C. Case $\xi \geq \frac{1}{2}$ with $Q = 0$

The vanishing of Q compromises our proof in Sec. IV B that a^2 has no minimum. We thus adopt here a new strategy unrelated to the map (4.1). Again the proof of this claim proceeds by contradiction. We assume there is a nontrivial physically reasonable neutral black hole solution with $Q = 0$.

Let us first eliminate $\nu' + \lambda'$ from Eq. (2.11) with the help of Eq. (3.5) to get

$$T_t^t - T_r^r = \frac{e^{-\lambda}[(2\xi - 1)a_{,r}^2 + 2\xi aa_{,rr}]}{1 - 8\pi G\xi a^2 - 8\pi G\xi raa_{,r}}. \tag{4.3}$$

Obviously as $r \to \infty$, $aa(r)_{,r}$ must fall off faster than r^{-1}, so that the denominator here is asymptotically positive by the positivity of the asymptotic gravitational constant.

Now suppose that asymptotically a^2 decreases, which means that $aa_{,r} < 0$ and $aa_{,rr} > 0$. It is then plain from Eq. (2.10) that $T_t^t - T_\phi^\phi < 0$ while from Eq. (4.3) it is clear that $T_t^t - T_r^r > 0$. This would violate causality, and must be excluded. Thus suppose the opposite, that a^2 increases asymptotically so that $a^2{}_{,r} > 0$ while $a^2{}_{,rr} < 0$. Rewriting Eq. (4.3) in the form

$$T_t^t - T_r^r = \frac{e^{-\lambda}[\xi a^2{}_{,rr} - a_{,r}^2]}{1 - 8\pi G\xi a^2 - 4\pi G\xi ra^2{}_{,r}}, \tag{4.4}$$

we see that now $T_t^t - T_\phi^\phi > 0$ while $T_t^t - T_r^r < 0$. This new possibility is thus also ruled out by causality. Likewise, were a^2 to oscillate indefinitely as $r \to \infty$, a similar clash would ensue over part of each cycle. We must thus conclude that a is strictly constant for r greater than some finite but large r_*. A Taylor expansion of $a(r)$ about a point to the right of r_* must obviously sum up to the asymptotic value $a(\infty)$. Now the differential equation for a, Eq. (2.14), has singular points only at $r = r_H$ and $r = \infty$ (R must be bounded in the black hole exterior while we assume that V is a regular function). Thus the series must converge to the correct $a(r)$ all the way to the horizon and $a \equiv$ const so that there is no hair.

Summarizing this and the last section, *there exists no static spherically symmetric neutral or charged black hole endowed with nontrivial hair in the form of a neutral scalar field nonminimally coupled to gravity with* $\xi \geq 1/2$ *and with a non-negative and regular self-interaction potential.*

V. NONMINIMALLY COUPLED (ANY ξ) CHARGED SCALAR FIELD WITH SELF-INTERACTION

Next we consider charged scalar hair, possibly nonminimally coupled to gravity (any ξ) and with a positive semidefinite self-interaction potential assumed to be a regular function of its argument a^2. We shall here invoke a new strategy, namely looking at the analytic behavior of various quantities in the horizon's vicinity, as dictated by the very nature of the horizon. The following two subsections contain general conclusions about the horizon and its neighborhood which are independent of the matter content of the black hole exterior, first in general and then for nonextremal black holes. These are extended to extremal black holes in Sec. V F. In this section $Q(r)$ denotes the charge of black hole plus scalar field up to radial coordinate r.

A. General properties of a spherical static event horizon

We return to Eq. (3.6) written as

$$e^{-\lambda} = 1 - \frac{r_H}{r} + \frac{8\pi G}{r}\int_{r_H}^{r} T_t^t r^2 dr. \tag{5.1}$$

As anticipated already, the point $r=r_H$ where $e^{-\lambda}$ vanishes is to be interpreted as the location of the horizon. To see why define a family of spherical hypersurfaces by the conditions $\{\forall t; f(r)=\text{ const}\}$ with f monotonic. Each value of the constant labels a different surface. The normal to each such hypersurface is

$$n_\mu = \frac{\partial f}{\partial x^\mu} = (0,1,0,0) f'. \tag{5.2}$$

Hence

$$n_\mu n^\mu = e^{-\lambda}(f')^2 \tag{5.3}$$

which vanishes only for $r=r_H$. This must thus be the location of the horizon which is defined as a null surface (hence null normal).

Proceeding with the argument, assume that e^ν vanishes at some point $\overline{r}$. Then $\nu\to-\infty$ and $\nu'\to\infty$ as $r\to\overline{r}$ from the right. It is then obvious from Eq. (3.4) that $e^{-\lambda}$ must vanish as $r\to\overline{r}$ since T_r^r must be bounded. But since $e^{-\lambda}$ vanishes only for $r=r_H$, we see that $\overline{r}=r_H$: $e^{-\lambda}$ vanishes wherever e^ν vanishes. The converse is also true: the horizon $r=r_H$ must always be an infinite redshift surface with $e^\nu=0$. For if e^ν were positive at $r=r_H$, then according to the metric Eq. (2.1) the t direction would be timelike there, while the θ and ϕ directions would be, as always, spacelike. But since the horizon is a null surface, it must have a null tangent direction, and this must obviously be the t direction. Thus it is inconsistent to assume that $e^\nu\neq 0$ at $r=r_H$.

B. Matter independent characterization of nonextremal event horizon

Since T_t^t must be bounded on the horizon, we may write the first approximation (in Taylor's sense) for $e^{-\lambda}$ near the horizon as

$$e^{-\lambda} = L(r-r_H) + O[(r-r_H)^2], \quad L \equiv \frac{1+8\pi G T_t^t(r_H) r_H^2}{r_H}. \tag{5.4}$$

Since $e^{-\lambda}$ must be non-negative outside the horizon, we learn that $L>0$, that is, *at every static spherically symmetric event horizon*

$$-(8\pi G r_H^2)^{-1} \leq T_t^t(r_H). \tag{5.5}$$

Note that the energy density at the horizon, if positive, is limited by the very condition of regularity at the horizon. The inequality is saturated for the extremal black hole; we consider this case in Sec. V E below.

Under the assumption of asymptotic flatness, we can integrate Eq. (3.5) to get

$$\nu+\lambda = 8\pi G \int_r^\infty r'(T_t^t - T_r^r) e^\lambda dr'. \tag{5.6}$$

Here the T_μ^ν are finite everywhere, including at the horizon, and T_t^t must vanish asymptotically faster than $1/r^3$ in order for e^λ not to diverge at infinity [see Eq. (5.1)]. In view of Eq. (2.24) the difference $T_t^t - T_r^r$ vanishes at least as fast as $1/r^3$. We thus conclude that $\nu+\lambda$ is regular everywhere, except possibly on the horizon.

Now in view of Eq. (5.4) we get from Eq. (5.6)

$$\nu+\lambda = \text{const} - 8\pi G r_H^2 \frac{T_t^t(r_H) - T_r^r(r_H)}{1+8\pi G T_t^t(r_H) r_H^2} \ln(r-r_H) + O(r-r_H). \tag{5.7}$$

But Eq. (5.4) informs us that

$$\lambda = \text{const} - \ln(r-r_H) + O(r-r_H). \tag{5.8}$$

Thus

$$\nu = \text{const} + \beta \ln(r-r_H) + O(r-r_H),$$
$$\beta \equiv \frac{1+8\pi G T_r^r(r_H) r_H^2}{1+8\pi G T_t^t(r_H) r_H^2}. \tag{5.9}$$

The value of β is restricted by the requirement that the scalar curvature

$$R = e^{-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 + \frac{2}{r}(\nu'-\lambda') - \frac{1}{2}\nu'\lambda' + \frac{2}{r^2}\right) - \frac{2}{r^2} \tag{5.10}$$

be bounded on the horizon (this is the same as boundedness of T). If we substitute here Eqs. (5.8) and (5.9) we get

$$R = -\frac{2}{r_H^2} + L(r-r_H)\left(\frac{1}{2}\frac{\beta(\beta-1)}{(r-r_H)^2} + \frac{2}{r_H}\frac{\beta+1}{(r-r_H)} + \frac{2}{r_H^2}\right). \tag{5.11}$$

Obviously the terms in Eq. (5.11) that diverge at the fastest rate must cancel. Since we are considering a nonextremal black hole, $L>0$, so we are left with the condition

$$\beta(\beta-1) = 0. \tag{5.12}$$

The alternative $\beta=0$ is excluded by the requirement (Sec. V A) that $e^\nu=0$ at the horizon. Thus necessarily $\beta=1$. We thus recover Eq. (2.17). In addition, we learn that

$$e^\nu = N(r-r_H) + O((r-r_H)^2), \tag{5.13}$$

where N denotes a positive constant.

C. A_t is bounded on the horizon

Our choice of gauge in Sec. II A does not guarantee that $A_t(\infty)=0$. For that same gauge transformation with $\Lambda=\text{const}\times t$ which we used to make ψ static adds a constant to A_t, and so may make $A_t(\infty)\neq 0$. To show this does not happen in a physically acceptable solution, we assume otherwise and exhibit a contradiction. Thus suppose $A_t(\infty)\neq 0$ with $A_{t,r}$ and $A_{t,rr}$ vanishing asymptotically. Then it follows from Maxwell's Eq. (2.16) that $a(\infty)=0$ so that all derivatives of a vanish asymptotically. Putting this fact together with the requirement from asymptotic flatness that $T_t^t\to 0$ into Eq. (2.12), we see that the potential must satisfy $V(0)=0$. But the potential is positive semidefinite so that we must also require that $\dot{V}(0)=0$.

Turn now to the scalar equation Eq. (2.14) and realize that because of the asymptotic vanishing of R, the equation must reduce as $r\rightarrow\infty$ to

$$a,_{rr}+2r^{-1}a,_{r}-q^2A_t(\infty)^2a=0 \tag{5.14}$$

with general solution

$$a=Kr^{-1}\sin[qA_t(\infty)r+\chi], \tag{5.15}$$

where K and χ are integration constants. Although this a falls off asymptotically, it does so too slowly. The electric charge density it implies,

$$\rho\propto q^2A_ta^2\propto r^{-2}\sin^2[qA_t(\infty)r+\chi] \tag{5.16}$$

leads to a total charge which diverges asymptotically as r. The implication that the black hole is surrounded by a cloud with infinite charge is clearly physically unacceptable. We conclude that the assumption $A_t(\infty)\neq 0$ is incompatible with a physically acceptable solution. We shall thus assume henceforth that $A_t(\infty)=0$.

We shall now prove that $|A_t|$ is a monotonically decreasing function of r. F_{tr} must obviously vanish at spatial infinity. Consider the case that A_t is of one sign throughout and, with no loss of generality, assume that A_t is non-negative. Assume further that A_t has an extremum at some point $r=\hat{r}$ outside of the horizon. But according to Eq. (2.16), at any extremum $\mathrm{sgn}(A_t,_{rr})=\mathrm{sgn}(A_t)$ so that an extremum must be a minimum. On the other hand, since A_t vanishes asymptotically, it cannot have a minimum without also having a maximum. There is thus a contradiction which signals the incorrectness of the assumption that there is an extremum.

When A_t can change sign, assume with no loss of generality that A_t changes from negative to positive with increasing r. In that case A_t would have to attain a positive maximum in order for $A_t\rightarrow 0$ as $r\rightarrow\infty$. But the previous argument shows that A_t is forbidden positive maxima. Thus $|A_t|$ cannot change sign. It follows from the preceding argument that $|A_t|$ must be monotonically decreasing in r.

Introduce now the set of orthonormal differential forms

$$d\hat{t}=-e^{\nu/2}dt,$$

$$d\hat{r}=e^{\lambda/2}dr,$$

$$d\hat{\theta}=r\,d\theta,$$

$$d\hat{\phi}=r\sin\theta d\phi. \tag{5.17}$$

The physical components of the Faraday tensor, $\hat{F}_{\mu\nu}$, are related to the coordinates components by $\hat{F}_{\mu\nu}d\hat{\omega}^{\mu}\wedge d\hat{\omega}^{\nu}=F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$, so that

$$\hat{F}_{\mathrm{tr}}e^{(\nu+\lambda)/2}=F_{\mathrm{tr}}. \tag{5.18}$$

The physical component $\hat{F}_{\mathrm{tr}}$ must be finite. From Eqs. (5.4) and (5.13) we see that $e^{(\nu+\lambda)/2}$ is bounded at the horizon. Thus $A_t,_r=-F_{\mathrm{tr}}$ must be bounded at the horizon. Integrating it once we obtain

$$A_t=\text{ const}-\text{const}\times(r-r_H), \quad r\rightarrow r_H. \tag{5.19}$$

This completes our proof.

D. Proof for nonextremal black hole

First consider the Maxwell equation (2.16). We know that $A_t,_r$ must be bounded on the horizon, so that even if $A_t,_{rr}$ diverges there, it can only do so slower than $(r-r_H)^{-1}$. Now since e^{λ} diverges as $(r-r_H)^{-1}$ while $(\nu+\lambda)'$ remains bounded, a must vanish on the horizon; otherwise, the last term in the equation would blow up without being balanced.

We now look at the scalar equation (2.14). If the potential is regular as assumed, $\dot{V}$ has to be bounded as $a\rightarrow 0$ at the horizon. The curvature R is likewise bounded by assumption of a regular horizon. Therefore, according to Eqs. (5.13) and (5.19), the last term of the equation is dominated by the factor proportional to q^2. It follows from Eq. (5.4) that near the horizon the scalar equation has the limiting form

$$a,_{rr}+(r-r_H)^{-1}a,_r+(LN)^{-1}q^2A_t(r_H)^2(r-r_H)^{-2}a=0. \tag{5.20}$$

The two solutions of this Euler equation are $(r-r_H)^{\pm\imath\alpha}$ with $\alpha\equiv qA_t(r_H)(NL)^{-1/2}$. Combining them we get the general solution

$$a(r)=B\sin\Phi, \quad \Phi\equiv\alpha\ln[(r-r_H)/D] \tag{5.21}$$

with B and D arbitrary constants.

Obviously for no choice of the constants does $a(r)$ vanish for $r\rightarrow 0$ as required. Not only that, but when we substitute this $a(r)$ into the expressions (2.10), (2.11), and (2.12) for the components of T_{μ}^{ν} every derivative of a brings out a factor $(r-r_H)^{-1}$, so that the expressions are singular at the horizon. For instance, from Eqs. (2.10), (2.11), (5.4), and (5.9) we get, near the horizon,

$$T_r^r-T_\phi^\phi=-\frac{LB^2}{r-r_H}\frac{\alpha\sin\Phi(\xi\cos\Phi+\alpha\sin\Phi)+O(r-r_H)}{1-8\pi G\xi B^2\sin^2\Phi}. \tag{5.22}$$

Obviously $T_r^r-T_\phi^\phi$ cannot remain bounded on the horizon as required. Thus the solution with regular horizon we have been assuming is untenable.

In conclusion *there exists no nonextremal static and spherical charged black hole endowed with hair in the form of a charged scalar field, whether minimally or nonminimally coupled to gravity, and with a regular positive semidefinite self-interaction potential.*

E. Matter independent characterization of extremal event horizon

When inequality (5.5) is saturated, namely when

$$T_t^t(r_H)=-(8\pi Gr_H^2)^{-1}, \tag{5.23}$$

we must continue the expansion of $e^{-\lambda}$ to second order:

$$e^{-\lambda}=\mathcal{L}(r-r_H)^2+O((r-r_H)^3);$$

$$\mathcal{L}\equiv\frac{4\pi Gr_H^3T_t^{t\prime}(r_H)-1}{r_H^2}. \tag{5.24}$$

Because $e^{-\lambda}$ has to be positive for $r>r_H$, $\mathcal{L}>0$ or

$$T_t^{t\prime}(r_H)>(4\pi G r_H^3)^{-1}. \tag{5.25}$$

Of course Eq. (2.17) must still hold since the saturated case is a special member of the black hole family which can be reached continuously from the main branch. We note that Eqs. (5.23), (5.24) and (2.17) are all satisfied at the extremal Reissner-Nordström and BMBB black hole horizons.

Substituting these results in Einstein's equation (3.4), expanding T_r^r about its value at $r=r_H$, solving for ν', and integrating we have

$$\nu=\text{const}+2\kappa\ln(r-r_H)+O(r-r_H),$$

$$\kappa=\frac{4\pi G r_H^3 T_r^{r\prime}(r_H)-1}{4\pi G r_H^3 T_t^{t\prime}(r_H)-1}. \tag{5.26}$$

We now show that causality restricts the possible values of κ.

Assume that $T_t^{t\prime}(r_H)\neq T_r^{r\prime}(r_H)$. Then in light of Eq. (2.17) we may expand near the horizon

$$T_t^t(r)-T_r^r(r)=(T_t^{t\prime}-T_r^{r\prime})\big|_{r_H}(r-r_H)+O((r-r_H)^2). \tag{5.27}$$

Because $T_t^t(r_H)<0$, $T_t^t(r)$ must be negative in a neighborhood of the horizon. The causality condition (2.25) then tells us that in that same neighborhood, $T_t^t(r)<T_r^r(r)$. Then Eq. (5.27) implies that $T_r^{r\prime}(r_H)>T_t^{t\prime}(r_H)$. In light of Eq. (5.25) this means that $\kappa>1$ in Eq. (5.26).

Thus the assumption $T_t^{t\prime}(r_H)\neq T_r^{r\prime}(r_H)$ implies that e^{ν} vanishes at the horizon faster than $(r-r_H)^2$ [presumably as $(r-r_H)^4$ if the metric coefficients are to avoid branch points at the horizon and if the metric is not to change signature upon traversal of the horizon]. However, there is nothing wrong with the possibility $T_t^{t\prime}(r_H)=T_r^{r\prime}(r_H)$; it would simply mean that the second order term in Eq. (5.27) is not allowed to be positive. In fact $T_t^{t\prime}(r_H)=T_r^{r\prime}(r_H)$, which corresponds to $\kappa=1$, is attained at the extremal Reissner-Nordström and BMBB horizons. In view of all these facts we find it natural to define extremal black holes as those characterized by Eqs. (5.23), (5.24) and (2.17) together with

$$e^{\nu}=\mathcal{N}(r-r_H)^2+O((r-r_H)^3), \tag{5.28}$$

where $\mathcal{N}$ is a positive constant. Higher order black holes with $\kappa=2,3,\ldots$ may not exist, just as third- and higher-order phase transitions do not.

F. Proof for extremal black hole

With the extremal black hole forms of the metric near the horizon, Eqs. (5.24) and (5.28), no change transpires in the conclusions of Sec. V C, namely, the field A_t must be monotonic in r, and from the regularity of the physical components of $F_{\mu\nu}$ one concludes that A_t attains a bounded and nonvanishing value at the horizon. Repeating the argument in Sec. V D with the new forms of the metric coefficients, one concludes that a must vanish at the horizon faster than $(r-r_H)^{1/2}$ in order for the Maxwell equation (2.16) to hold.

With this in mind let us look at the scalar equation (2.14) in the neighborhood of the horizon. Recall that R and $\dot{V}(0)$ must be bounded, so the corresponding terms are negligible compared with the q^2 term. Substituting Eqs. (5.24) and (5.28) and retaining the leading contributions we get

$$a_{,rr}+2(r-r_H)^{-1}a_{,r}+(\mathcal{L}\mathcal{N})^{-1}q^2A_t(r_H)^2(r-r_H)^{-4}a=0 \tag{5.29}$$

which is to be contrasted with Eq. (5.20). In the variable $u=(r-r_H)^{-1}$ we have

$$a_{,uu}+\overline{\alpha}^2a=0,\quad \overline{\alpha}\equiv qA_t(r_H)(\mathcal{L}\mathcal{N})^{-1/2} \tag{5.30}$$

with the general solution

$$a(r)=\mathcal{B}\sin\overline{\Phi},\quad \overline{\Phi}\equiv\overline{\alpha}(r-r_H)^{-1}+\zeta, \tag{5.31}$$

where $\mathcal{B}$ and ζ are integration constants. For no choice of $\mathcal{B}$ and ζ does a vanish for $r\to r_H$ as required. In addition, its very singular derivative leads, for instance, to the expression

$$T_r^r-T_\phi^\phi=\frac{\mathcal{L}\mathcal{B}^2}{(r-r_H)^2}\,\frac{\overline{\alpha}\sin\overline{\Phi}(2\xi\cos\overline{\Phi}-\overline{\alpha}\sin\overline{\Phi})+O((r-r_H)^2)}{1-8\pi G\xi\mathcal{B}^2\sin^2\overline{\Phi}} \tag{5.32}$$

which is incompatible with a regular horizon. Thus the theorem stated at the end of Sec. V D is extended to extremal black holes.

VI. CONCLUSIONS AND SPECULATIONS

We have extended to charged static spherical black holes the exclusion of hair in the form of a neutral scalar multiplet with action which need not be quadratic in the derivatives. From this theorem we have excluded, for charged or neutral static spherical black holes, hair in the form of a neutral scalar field with standard kinetic action, positive semidefinite self-interaction potential, and nonminimal coupling to gravity with $\xi<0$ and $\xi\geq\frac{1}{2}$. Finally, for charged static spherical black holes, we have excluded hair in the form of a charged scalar field with standard kinetic action, regular self-interaction potential, and nonminimal coupling to gravity with any ξ.

Extension of the theorem excluding the neutral scalar field to the full range $0<\xi<\frac{1}{2}$ is blocked by the existence of the BMBB black hole, an extremal spherical black hole solution for the case $\xi=\frac{1}{6}$ with no self-interaction. Xanthopoulos and Zannias [15,16] have shown that there are no more static black holes in this case, even if extremality or spherical symmetry are given up. It may be that $\xi=\frac{1}{6}$ is the unique value for which nonminimally coupled scalar black hole hair appears. In that case it should not be prohibitively difficult to produce a single theorem proving this. But if there exists a whole family of black holes with nonminimally coupled hair within the domain $0<\xi<\frac{1}{2}$, of which the BMBB black hole is just one example, it would seem that at least two theorems involving different approaches would be needed to exclude the unoccupied hair parameter space on both sides of the putative family.

It seems unlikely that slightly aspherical charged black holes with self-interacting neutral or charged scalar hair exist. For one would expect any such family to be governed by a parameter quantifying the departure from spherical symmetry. This parameter should reach the spherical black hole. Yet the spherical example is rigorously ruled out by our theorems. This heuristic argument obviously cannot be applied to very aspherical black holes, or to those which show a topological distinction from the spherical one. Such is the case of the AGK black hole, a charged black hole with minimally coupled self-interacting (Higgs) scalar hair in the form of a local cosmic string which transfixes the black hole. Strictly speaking, our third theorem does not rule out such a solution because of its lack of spherical symmetry and asymptotic flatness. But it is really the distinct topology of the scalar field phase with its multiple connectivity around the string which makes our proof far from relevant.

One can speculate on more complicated situations. Suppose a black hole forms with two local Higgs strings through it. The situation would seem unstable. Strings with the same sense of winding of the phase repel each other, so the two strings will become antiparallel and approach. If the winding numbers were originally equal in absolute value, the strings will annihilate with the Higgs phase topology becoming simple. The configuration will then relax. But by our third theorem the end point cannot be a spherical black hole with Higgs hair. With due caution we infer that the black hole will swallow part of the field and jettison the rest, so that we end up with a Reissner-Nordström hole. By extension we may surmise that if a black hole is transfixed by an even number of unit winding-number strings, it will end up with no scalar field, whereas if it has an odd number, it will end up in the AGK configuration.

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APPENDIX: THE ENERGY CONDITIONS

At a given spacetime event consider the eigenvalue problem

$$T_\mu^\nu w^\mu = \sigma w^\nu. \tag{A1}$$

Because T_μ^ν is a 4×4 matrix, there must be four distinct eigenvectors w^μ. Obviously

$$0 \equiv w_\nu^{(2)} T_\mu^\nu w^{(1)\mu} - w_\nu^{(1)} T_\mu^\nu w^{(2)\mu} = (\sigma^{(1)} - \sigma^{(2)}) w_{(2)\nu} w^{(1)\nu}. \tag{A2}$$

Hence for distinct eigenvalues the eigenvectors are orthogonal with respect to the spacetime metric (for degenerate eigenvalues they can be made orthogonal by the Schmidt procedure). We gloss over the possibility that some eigenvectors may be null (radiative solutions). Thus one must be timelike; call it $w^{(0)\mu}$ and normalize so that $w^{(0)\mu} w_\mu^{(0)} = -1$. The other three must be spacelike; call them $\{w^{(1)\mu}, w^{(2)\mu}, w^{(3)\mu}\}$ and normalize them so that $w^{(1)\mu} w_\mu^{(1)} = +1$, etc.

The four eigenvectors obviously furnish a basis for writing any four-vector, in particular the four-velocity of an observer:

$$u^\mu = c^{(0)} w^{(0)\mu} + \Sigma_i^3 c^{(i)} w^{(i)\mu}, \tag{A3}$$

where obviously

$$(c^{(0)})^2 = \Sigma_i^3 (c^{(i)})^2 + 1 \tag{A4}$$

in order to satisfy $u_\mu u^\mu = -1$. The various choices of $\{c^{(i)}\}$ label all possible observers at a given event.

Now use Eqs. (A1), (A3), and (A4) and the normalizations to reexpress (see Sec. II C)

$$j^\mu j_\mu = -(\sigma^0)^2 - \Sigma_i (c^{(i)})^2 [(\sigma^{(0)})^2 - (\sigma^{(i)})^2] \tag{A5}$$

and

$$\varepsilon \equiv T_{\mu\nu} u^\mu u^\nu = -\sigma^0 - \Sigma_i (c^{(i)})^2 (\sigma^{(0)} - \sigma^{(i)}). \tag{A6}$$

We now see that the energy conditions

$$|\sigma^{(0)}| \geq \{|\sigma^{(i)}|\} \tag{A7}$$

are necessary and sufficient for $j^\mu j_\mu$ to be nonpositive for all observers (all choices of $\{c^{(i)}\}$) and for the energy density ε to be of like sign (that of $-\sigma^{(0)}$) for all observers. Likewise $j^\mu j_\mu \leq 0$ for all observers is a necessary and sufficient condition for the energy conditions to be satisfied. And consensus of all observers as to the sign of the energy density is necessary and sufficient for the energy conditions to be satisfied and the causality condition $j^\mu j_\mu \leq 0$ to hold for all observers.

In the static spherically symmetric situation considered in Sec. II C, the T_μ^ν is diagonal, so that $\sigma^{(0)} = T_t^t$, $\sigma^{(1)} = T_r^r$, $\sigma^{(2)} = \sigma^{(3)} = T_\theta^\theta = T_\phi^\phi$ We thus recover the energy conditions (2.24).

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