Area spectrum of the Schwarzschild black hole

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We consider a Hamiltonian theory of spherically symmetric vacuum Einstein gravity under Kruskal-like boundary conditions in variables associated with the Einstein-Rosen wormhole throat. The configuration variable in the reduced classical theory is the radius of the throat, in a foliation that is frozen at the left-hand side infinity but asymptotically Minkowski at the right-hand side infinity, and such that the proper time at the throat agrees with the right-hand side Minkowski time. The classical Hamiltonian is numerically equal to the Schwarzschild mass. Within a class of Hamiltonian quantizations, we show that the spectrum of the Hamiltonian operator is discrete and bounded below, and can be made positive definite. The large eigenvalues behave asymptotically as $\sqrt{2k}$, where k is an integer. The resulting area spectrum agrees with that proposed by Bekenstein and others. Analogous results hold in the presence of a negative cosmological constant and electric charge. The classical input that led to the quantum results is discussed. [S0556-2821(96)04720-0]

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I. INTRODUCTION

One of the most intriguing problems in black hole thermodynamics is the statistical mechanical interpretation of black hole entropy. One surmises that black hole entropy should reflect an outside observer's ignorance about the quantum-mechanical microstates of the hole, but it has proved very difficult to characterize what exactly these quantum-mechanical microstates might be. There are hints that the relevant degrees of freedom may live on the horizon of the hole [1–4], in situations where the horizon can be meaningfully defined. There is also evidence that black hole entropy describes the entanglement between the degrees of freedom in the interior and in the exterior of the hole [5–8]. Recent results from string theory [9] suggest that black hole entropy can be recovered by counting the quantum microstates even in situations where the definition of these states presupposes no black hole geometry. For reviews, see Refs. [9–11].

Even prior to Hawking's prediction of black hole radiation [12], the anticipated connection between black holes and thermodynamics led Bekenstein [13] to propose that the horizon area of a black hole is quantized in integer multiples of a fundamental scale, presumably of the order of the square of the Planck length $l_{\text{Planck}} = \sqrt{\hbar G c^{-3}}$:

$$A = \alpha k l_{\text{Planck}}^2, \qquad (1.1)$$

where k ranges over the positive integers and α is a pure number of order 1. This proposal has since been revived on

various grounds; see Refs. [14–31], and references therein. Although the horizon of a (classical) black hole is a nonlocal object, its total area is completely determined by the irreducible mass [32], and one can therefore alternatively view the rule (1.1) as a proposal for the spectrum of the quantum irreducible mass operator.[1] As the irreducible mass can classically be read off from the asymptotic falloff of the black hole gravitational field, one expects such an operator to be sensibly definable even in a quantum theory that only refers to observations made at an asymptotically flat infinity. In particular, for a Schwarzschild hole, the irreducible mass coincides with the Schwarzschild mass.

The implications of the area spectrum (1.1) for macroscopic physics were recently elaborated on by Bekenstein and Mukhanov [26]. Consider, for concreteness, a Schwarzschild hole. The area is given in terms of the Schwarzschild mass M by

$$A = 16\pi \left(\frac{l_{\text{Planck}}}{m_{\text{Planck}}} \right)^2 M^2, \qquad (1.2)$$

where $m_{\text{Planck}} = \sqrt{\hbar c G^{-1}}$ is the Planck mass. Now, Eqs. (1.1) and (1.2) imply that M can only take discrete values. When the black hole evaporates, it can thus only make transitions between the mass eigenstates corresponding to these discrete values. As a consequence, the radiation comes out in multiples of a fundamental frequency, which is of the same order as the maximum of Hawking's blackbody spectrum, and the corresponding wavelength is of the order of the Schwarzschild radius of the hole. This means that the radiation will differ from the blackbody spectrum in a way that is, in prin-

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ciple, macroscopically observable. For example, if M is of the order of ten solar masses, or 2×10^{31} kg, then the fundamental frequency is of the order of 0.1 kHz, which is roughly the resolving power of an ordinary portable radio receiver. The discussion can also be generalized to accommodate a nonvanishing angular momentum [13].

Arguments presented in favor of the area spectrum (1.1) include quantizing the angular momentum of a rotating black hole [13], information theoretic considerations [14,18,19,26], string theoretic arguments [15,16], periodicity of Euclidean or Lorentzian time [15,17,27,29–31], a treatment of the event horizon as a membrane with certain quantum mechanical properties [22,24], and a Hamiltonian quantization of a dust collapse [25]. Recently, a membrane model for the horizon [33] recovered an area spectrum that is finer than Eq. (1.1), and a calculation within a loop representation of quantum gravity [34] recovered an area spectrum that effectively reproduces the Planckian spectrum for black hole radiation. The purpose of the present paper is to give a derivation of (effectively) the spectrum (1.1) from a Hamiltonian quantum theory of spherically symmetric Einstein gravity, with judiciously chosen dynamical degrees of freedom.

By Birkhoff's theorem [35], the local properties of spherically symmetric vacuum Einstein geometries are completely characterized by a single parameter, the Schwarzschild mass. In a classical Hamiltonian theory of such spacetimes, the true dynamical degrees of freedom are thus expected to contain information only about the Schwarzschild mass and the embedding of the spacelike hypersurfaces in the spacetime. It was demonstrated in Refs. [36–39] that this is indeed the case, under certain types of boundary conditions that specify the (possibly asymptotic) embedding of the ends of the spacelike hypersurfaces in the spacetime. The variables of the reduced theory then consist of a single canonical pair: The coordinate can be taken to be the Schwarzschild mass, and its conjugate momentum carries the information about the evolution of the (asymptotic) ends of the spacelike hypersurfaces in the spacetime. The theory is thus no longer a field theory, but a theory of finitely many degrees of freedom. This means, in particular, that quantization of the theory can be addressed within ordinary, finite-dimensional quantum mechanics.

Our Hamiltonian theory of spherically symmetric vacuum spacetimes will be built on two major assumptions. First, we shall adopt for the spacetime foliation the boundary conditions of Ref. [38]. This implies that the classical solutions have a positive value of the Schwarzschild mass, and that the spacelike hypersurfaces extend on the Kruskal manifold from the left-hand side spacelike infinity to the right-hand side spacelike infinity, crossing the horizons in arbitrary ways. We shall, however, specialize to the case where the evolution of the hypersurfaces at the left-hand side infinity is frozen, and the evolution at the right-hand side infinity proceeds at unit rate with respect to the right-hand side asymptotic Minkowski time. This means that, apart from constraints, the Hamiltonian will consist of a contribution from the right-hand side infinity only, and the value of the Hamiltonian is equal to the Schwarzschild mass. The physical reason for this choice is that while our theory will remain that of vacuum spacetimes, we expect these conditions to correspond to physics accessible to an inertial observer at one

spacelike infinity, at rest with respect to the hole: The proper time of such an observer is our asymptotic Minkowski time, and the Arnowitt-Deser-Misner (ADM) mass observed is the Schwarzschild mass.

Second, we shall adopt as our reduced dynamical variables a canonical pair that is intimately related to the dynamical aspects of the Kruskal manifold. Our configuration variable a can be envisaged as the radius of the Einstein-Rosen wormhole throat [32], in a spacetime foliation in which the proper time at the throat increases at the same rate as the asymptotic Minkowski time at the right-hand side infinity. An example of a foliation satisfying these conditions can be constructed by taking the Novikov coordinates [32] and deforming them near the left-hand side infinity to conform to our boundary conditions there.

The resulting classical theory has two properties whose physical interest should be emphasized. First, every classical solution is bounded, in the sense that the variable a starts from zero, increases to the maximum value $2M$, where M is the Schwarzschild mass, and then collapses back to zero. This evolution corresponds to the wormhole throat starting from the white hole singularity, expanding to the bifurcation two-sphere, and then collapsing to the black hole singularity. The spacetime dynamics in these variables is therefore, in a certain sense, confined to the interior regions of the Kruskal manifold. This property reflects the physics observed by an inertial observer at asymptotic infinity, as such an observer sees her exterior region of the Kruskal manifold as static. Second, as the proper time on the timelike geodesics that form the throat trajectory increases at the same rate as the asymptotic right-hand side Minkowski time, one may regard our foliation as a preferred one, by the principle of equivalence, for relating the experiences of an inertial observer at the asymptotic infinity to the experiences of an inertial observer at the throat. Note, however, that as the total proper time from the initial singularity to the final singularity along the throat trajectory is finite, our choice of the foliation implies that the throat reaches the white and black hole singularities at finite values of the asymptotic right-hand side Minkowski time. As the asymptotic right-hand side Minkowski time evolves at unit rate with respect to our parameter time, this means that the classical theory is incomplete: The classical solutions cannot be extended to arbitrarily large values of the parameter time, neither to the past nor to the future.

We quantize the theory by Hamiltonian methods, treating a as a configuration variable, specifying a class of ''reasonable'' inner products, and promoting the classical Hamiltonian into a self-adjoint Hamiltonian operator. For certain choices of the inner product the Hamiltonian operator turns out to be essentially self-adjoint, whereas in the remaining cases the class of self-adjoint extensions is parametrized by $U(1)$ and associated with a boundary condition at $a = 0$. We find that the spectrum of the Hamiltonian is discrete and bounded below in all the cases. When the Hamiltonian is essentially self-adjoint, the spectrum is strictly positive, and in the remaining cases there always exist self-adjoint extensions for which the spectrum is strictly positive. A WKB estimate for the large eigenvalues of the Hamiltonian yields,

via Eq. (1.2), the result that the large area eigenvalues are asymptotically given by[2]

$$A \sim 32\pi k l_{\text{Planck}}^2 + \text{const} + o(1), \qquad (1.3)$$

where k is an integer and $o(1)$ denotes a term that vanishes asymptotically at large A. The additive constant depends on the choice of the inner product and, when the Hamiltonian is not essentially self-adjoint, also on the choice of the self-adjoint extension. With two particular choices for the inner product, we can verify (and improve on) the accuracy of this WKB result rigorously; with another two particular choices, we can rigorously verify the accuracy of the leading order term. We can, therefore, view our theory as producing, from a Hamiltonian quantum theory constructed from first principles, the area spectrum (1.1) with $\alpha = 32\pi$.

We shall argue that the discreteness of the quantum spectrum is related to the classical incompleteness of the theory. As the variable a classically reaches the singularity at $a = 0$ within finite parameter time, both in the past and in the future, the classical theory can be thought of as particle motion on the positive half-line in a confining potential. Whenever the self-adjoint Hamiltonian operator is constructed so that the possible ''quantum potential'' part does not become significant, general theorems guarantee that the spectrum of the Hamiltonian will be discrete [40,41]. In physical terms, wave packets following classical trajectories will be reflected quantum mechanically from the origin, and the quantum dynamics will in this sense have a quasiperiodic character. In contrast, if the spacetime foliation were chosen so that it would take an infinite amount of parameter time for the variable a to reach the singularity, the classical theory could be thought of as particle motion on the full real line in a potential that is confining on the right but not on the left. In such potentials, the spectrum of a self-adjoint Hamiltonian operator generically has a continuous part, corresponding physically to the fact that wave packets can travel arbitrarily far to the left without being reflected. We shall present a simple example of each of these two types of foliation, handpicked so that the Hamiltonian operator becomes easily tractable. From the first example we can reproduce the area spectrum (1.1) with an arbitrary value of the constant α; in the second example, the spectrum of the Hamiltonian operator will be continuous and consist of the full non-negative half-line.

In addition to the above results for the vacuum theory, we shall also briefly investigate the inclusion of a fixed electric charge and a negative cosmological constant. With the analogous choices for the boundary conditions, the phase space coordinates, and the Hamiltonian quantum theory, we again show that the spectrum of the Hamiltonian operator is discrete and bounded below. The distribution of the large eigenvalues could presumably be analyzed by a suitable generalization of our vacuum techniques; however, we shall not pursue this issue here.

The rest of the paper is as follows. In Sec. II we derive the reduced Hamiltonian theory in our phase space variables, starting from Kuchař's reduced phase space variables [38]

and performing the appropriate canonical transformation. The theory is quantized in Sec. III, with considerable parts of the technical analysis deferred to three appendixes. In Sec. IV we discuss the inclusion of the electric charge and a negative cosmological constant. Section V contains a brief summary and a discussion.

For the remainder of this paper we shall work in natural units $\hbar = c = G = 1$.

II. CLASSICAL WORMHOLE THROAT THEORY

In this section we present a classical Hamiltonian theory of the Schwarzschild black hole in terms of reduced phase space variables that are associated with a wormhole throat, in a sense to be made more precise below. We first briefly recall, in Sec. II A, Kuchař's Hamiltonian reduction of spherically symmetric vacuum geometries [38]. In Sec. II B we derive the throat theory from Kuchař's reduced theory via a suitable canonical transformation.

A. Kuchař reduction

We start from the general spherically symmetric Arnowitt-Deser-Misner (ADM) line element

$$ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 d\Omega^2, \qquad (2.1)$$

where $d\Omega^2$ is the metric on the unit two-sphere, and N, N^r, Λ, and R are functions of t and r only. We adopt the falloff conditions of Ref. [38]. These conditions render the spacetime asymptotically flat both at $r \to \infty$ and $r \to -\infty$, and they make $|r|$ coincide asymptotically with the spacelike radial proper distance coordinate in Minkowski space. Each classical solution consists of some portion of the Kruskal manifold [32], such that the constant t hypersurfaces extend from the left-hand side spacelike infinity to the right-hand side spacelike infinity, crossing the horizons in arbitrary ways. In particular, the Schwarzschild mass is positive for every classical solution. The falloff conditions also guarantee that the four-momentum at the infinities has no spatial component: The black hole is at rest with respect to the left and right asymptotic Minkowski frames.

We fix the asymptotic values of N at $r \to \pm\infty$ to be prescribed t-dependent quantities, denoted by $N_\pm(t)$. The Hamiltonian form of the Einstein action, with appropriate boundary terms, reads then

$$S = \int dt \int_{-\infty}^{\infty} dr (P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H} - N^r \mathcal{H}_r)$$

$$- \int dt (N_+ M_+ + N_- M_-), \qquad (2.2)$$

where $\mathcal{H}$ and $\mathcal{H}_r$ are, respectively, the super-Hamiltonian constraint and the radial supermomentum constraint. The quantities $M_\pm(t)$ are determined by the asymptotic falloff of the configuration variables, and on a classical solution they both are equal to the Schwarzschild mass. We refer to Ref. [38] for the details. Note that when varying the action (2.2), $N_\pm(t)$ are considered fixed.

Through a judiciously chosen canonical transformation, followed by elimination of the constraints by Hamiltonian reduction, Kuchař [38] brings the action (2.2) to the unconstrained Hamiltonian form

$$S = \int dt [\mathbf{p}\dot{\mathbf{m}} - (N_+ + N_-)\mathbf{m}], \tag{2.3}$$

where the independent variables $\mathbf{m}$ and $\mathbf{p}$ are functions of t only. The degrees of freedom constitute thus the single canonical pair $(\mathbf{m}, \mathbf{p})$: The theory is no longer that of fields, but that of finitely many degrees of freedom. The variables take the values $\mathbf{m} > 0$ and $-\infty < \mathbf{p} < \infty$, and their equations of motion are

$$\dot{\mathbf{m}} = 0, \tag{2.4a}$$

$$\dot{\mathbf{p}} = -N_+ - N_-. \tag{2.4b}$$

For our purposes, it will not be necessary to recall the details of the derivation of the action (2.3), but what will be important is the interpretation of the reduced theory in terms of the spacetime geometry. On each classical solution, the time-independent value of $\mathbf{m}$ is simply the value of the Schwarzschild mass. By Birkhoff's theorem, $\mathbf{m}$ thus carries all the information about the local geometry of the classical solutions. The variable $\mathbf{p}$, on the other hand, is equal to the difference of the asymptotic Killing times between the left and right infinities on a constant t hypersurface, in the convention where the Killing time at the right (left) infinity increases towards the future (past). The two terms in the evolution equation (2.4b) arise, respectively, from the two infinities. Thus, $\mathbf{p}$ contains no information about the local geometry, but instead it carries the information about the anchoring of the spacelike hypersurfaces at the two infinities. Note that $\mathbf{p}$, as the difference of the asymptotic Killing times, is invariant under the global isometries that correspond to translations in the Killing time.

B. Hamiltonian throat theory

We shall now make two restrictions on the reduced theory (2.3). First, we specialize to $N_+ = 1$ and $N_- = 0$. This means that the parameter time t coincides with the asymptotic right-hand side Minkowski time, up to an additive constant, whereas at the left-hand side infinity the hypersurfaces remain frozen at the same value of the asymptotic Minkowski time for all t. The action reads then

$$S = \int dt (\mathbf{p}\dot{\mathbf{m}} - \mathbf{m}). \tag{2.5}$$

The fact that the Hamiltonian now equals simply $\mathbf{m}$ reproduces the familiar identification of the Schwarzschild mass as the ADM energy, from the viewpoint of asymptotic Minkowski time evolution at *one* asymptotically flat infinity. Our choice as to which of the two infinities has been taken to evolve is of course merely a convention, but the choice of completely freezing the evolution at the other infinity arises from the requirement that our theory describe physics accessible to observers at just one infinity. We shall return to this issue in Sec. V.

Second, for reasons that will become transparent below, we confine the variables by hand to the range $|\mathbf{p}| < \pi\mathbf{m}$. This means that in each classical solution, the asymptotic right-hand side Minkowski time only takes values within an interval of length $2\pi\mathbf{m}$, centered around a value that is diagonally opposite to the nonevolving left end of the hypersurfaces in the Kruskal diagram. In terms of the parameter time t, each classical solution is then defined only for an interval $-\pi\mathbf{m} < t - t_0 < \pi\mathbf{m}$, where $t = t_0$ is the hypersurface whose two asymptotic ends are diagonally opposite.

Consider now the transformation from the pair $(\mathbf{m}, \mathbf{p})$ to the new pair (a, p_a) defined by the equations

$$|\mathbf{p}| = \int_a^{2\mathbf{m}} \frac{db}{\sqrt{2\mathbf{m}b^{-1} - 1}}$$
$$= \sqrt{2\mathbf{m}a - a^2} + \mathbf{m}\arcsin(1 - a/\mathbf{m}) + \tfrac{1}{2}\pi\mathbf{m}, \tag{2.6a}$$

$$p_a = \operatorname{sgn}(\mathbf{p})\sqrt{2\mathbf{m}a - a^2}. \tag{2.6b}$$

The ranges of the variables are $a > 0$ and $-\infty < p_a < \infty$. The transformation is well defined, one to one, and canonical. The new action reads

$$S = \int dt(p_a \dot{a} - H), \tag{2.7}$$

where the Hamiltonian is given by

$$H = \frac{1}{2}\left(\frac{p_a^2}{a} + a\right). \tag{2.8}$$

The classical solutions are easily written out in the new variables. The value of H on a classical solution is just $\mathbf{m}$, and by writing the canonical momentum p_a in Eq. (2.8) in terms of a and $\dot{a}$, one recovers the equation of motion for a in the form

$$\dot{a}^2 = \frac{2\mathbf{m}}{a} - 1. \tag{2.9}$$

Hence the configuration variable a starts from zero at $t = t_0 - \pi\mathbf{m}$, reaches the maximum value $2\mathbf{m}$ at $t = t_0$, and collapses back to zero at $t = t_0 + \pi\mathbf{m}$.

The interest in the variables (a, p_a) is that they have an appealing geometrical interpretation in terms of the dynamics of a wormhole throat in the black hole spacetime. To see this, we recall that the derivation of the reduced action (2.3) from the original geometrodynamical action (2.2) in Ref. [38] relied on the properties of the spacelike hypersurfaces only through their asymptotic behavior, but otherwise left these hypersurfaces completely arbitrary. We can thus exercise this freedom and seek an interpretation of the variables (a, p_a) in terms of a suitably chosen foliation.

The crucial observation is now that Eq. (2.9) is identical to the equation of a radial timelike geodesic through the bifurcation two-sphere of a Kruskal manifold of mass $\mathbf{m}$, *provided* one identifies a as the curvature radius of the two-sphere and the overdot as the proper time derivative. This is easily seen, for example, from the expression of the interior Schwarzschild metric in the Schwarzschild coordinates. With

these identifications, Eqs. (2.6) show that $-\mathbf{p}$ becomes identified with the proper time elapsed along this geodesic from the bifurcation two-sphere, with positive (negative) values of $-\mathbf{p}$ yielding the part of the geodesic that is in the black (white) hole interior. Thus, if there exists a foliation consistent with the falloff conditions of Ref. [38], intersecting a timelike geodesic through the bifurcation two-sphere so that $-\mathbf{p}$ agrees with the proper time on the geodesic in this fashion, then the quantity a defined by Eq. (2.6a) is the two-sphere radius along this geodesic.

It is easy to see that foliations of this kind do exist. Let us briefly discuss an example that is closely related to the Novikov coordinates [32]. Recall that the geometric idea behind the Novikov coordinates consists of fixing a spacelike hypersurface of constant Killing time through the Kruskal manifold, and releasing from this hypersurface a family of freely falling test particles with a vanishing initial three-velocity in the Schwarzschild coordinates. The coordinates (τ, R^*) are then defined so that they follow these test particles: The trajectories are the lines of constant R^*, and on each trajectory τ is equal to the proper time. The initial hypersurface is $\tau = 0$, with $R^* > 0$ and $R^* < 0$ giving, respectively, the halves living in the right and left exterior regions. Now, to arrive at a foliation satisfying our requirements, we first deform the Novikov coordinates near the left-hand side infinity to accommodate the condition $N_- = 0$, and we then redefine R^* near the infinities in a τ-independent fashion so as to conform to the radial falloff assumed in Ref. [38] (the right-hand side infinity will then have, in the notation of Ref. [38], a falloff with $\epsilon = 1$). The distinguished geodesic through the bifurcation two-sphere is given by $R^* = 0$, and the coordinate τ agrees by construction both with the proper time along this geodesic and with the asymptotic Minkowski time at the right-hand side infinity.

Our interpretation of a gives now a geometrical reason for the restriction $|\mathbf{p}| < \pi \mathbf{m}$, which we above introduced by hand. As a radial timelike geodesic from the initial singularity to the final singularity through the bifurcation two-sphere has the finite total proper time $2\pi \mathbf{m}$, foliations satisfying our conditions do not cover all of the spacetime. The foliations only exist for the duration of $2\pi \mathbf{m}$ in the asymptotic right-hand side Minkowski time.[3]

We summarize. Fix a radial timelike geodesic through the bifurcation two-sphere, and choose any foliation, consistent with our falloff conditions, such that the proper time along the geodesic and the asymptotic right-hand side Minkowski time agree on the constant t hypersurfaces. Then, the variable a equals the radius of the two-sphere on the distinguished geodesic. In particular, if the foliation is chosen so that on each constant t hypersurface the radius of the two-sphere attains its minimum value on the distinguished geodesic, then this geodesic and the ones obtained from it by spherical symmetry form the trajectory of the Einstein-Rosen wormhole throat. This is the case for example in foliations obtained by deforming the Novikov coordinates near the left-

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[3]Note that although the Novikov coordinates (τ, R^*) are globally well defined on the Kruskal manifold, the hypersurfaces of constant τ extend from one infinity to the other only for $|\tau| < \pi \mathbf{m}$. For larger values of $|\tau|$, these hypersurfaces hit a singularity.

hand side infinity in the manner discussed above. We shall therefore, with a minor abuse of terminology, refer to a as the radius of the wormhole throat, and to the theory given by Eqs. (2.7) and (2.8) as the Hamiltonian throat theory.

It is important to note that the configuration variable a is bounded on each classical trajectory, reaching the maximum value $2\mathbf{m}$ as the wormhole throat crosses the bifurcation two-sphere. This means, in a certain sense, that the spacetime dynamics in terms of our configuration variable a is confined ''inside'' the hole. This is physically appealing from the viewpoint of an observer at infinity: Such an observer sees the exterior region of the spacetime as static.

III. THROAT QUANTIZATION

In this section we shall quantize the Hamiltonian throat theory of Sec. II. We saw above that the classical Hamiltonian is numerically equal to the Schwarzschild mass, and that this Hamiltonian arises as the energy with respect to the Minkowski time evolution at one asymptotically flat infinity. The quantum Hamiltonian operator can therefore be viewed as the energy operator with respect to an asymptotic Minkowski frame in which the hole is at rest. In particular, the spectrum of the Hamiltonian operator becomes the ADM mass spectrum of the hole. Our main aim will be a qualitative analysis of this spectrum.

We take the states of the quantum theory to be described by functions of the configuration variable a. The Hilbert space is $\mathsf{H} := L^2(\mathbb{R}^+; \mu da)$, with the inner product

$$(\psi_1, \psi_2) = \int_0^\infty \mu da \, \overline{\psi_1} \psi_2, \qquad (3.1)$$

where $\mu(a)$ is some smooth positive weight function. To obtain the Hamiltonian operator $\hat{H}$, we make in the classical Hamiltonian (2.8) the substitution $p_a \to -id/(da)$ and adopt a symmetric ordering with respect to the inner product (3.1). The result is

$$\hat{H} = \frac{1}{2}\left[-\frac{1}{\mu}\frac{d}{da}\left(\frac{\mu}{a}\frac{d}{da} \right) + a \right]. \qquad (3.2)$$

For technical reasons, it will be useful to work with an isomorphic theory in which the inner product and the kinetic term of the Hamiltonian take a more conventional form. To achieve this, we write $a = x^{2/3}$, $\mu = \frac{3}{2}x^{1/3}\nu^2$, and $\psi = \nu^{-1}\chi$. The theory above is then mapped to the theory whose Hilbert space is $\mathsf{H}_0 := L^2(\mathbb{R}^+; dx)$, with the inner product

$$(\chi_1, \chi_2)_0 = \int_0^\infty dx \, \overline{\chi_1} \chi_2. \qquad (3.3)$$

The new Hamiltonian operator is

$$\hat{H}_0 = \frac{9}{8}\left[-\frac{d^2}{dx^2} + \frac{4x^{2/3}}{9} + \frac{\nu''}{\nu} \right], \qquad (3.4)$$

where $' = d/dx$. By construction, $\hat{H}_0$ is a symmetric operator in H_0. Note that if we had retained dimensions, the ''quantum potential'' term $\frac{9}{8}\nu''/\nu$ would be proportional to $\hbar^2$.

To completely specify the quantum theory, we need to make $\hat{H}_0$ into a self-adjoint operator on H_0. The possible ways of doing this depend on the quantum potential term $\frac{9}{8} \nu''/\nu$. For concreteness, we shall from now on take $\mu = a^s$, where s is a real parameter. The qualitative results would, however, be analogous for any sufficiently similar μ with a power-law asymptotic behavior at $a \to 0$ and $a \to \infty$.

With $\mu = a^s$, $\hat{H}_0$ takes the form

$$\hat{H}_0 = \frac{9}{8}\left[-\frac{d^2}{dx^2} + \frac{4x^{2/3}}{9} + \frac{r(r-1)}{x^2} \right], \qquad (3.5)$$

where

$$r = (2s-1)/6 \quad \text{for } s \geq 2, \qquad (3.6a)$$

$$r = (7-2s)/6 \quad \text{for } s < 2 . \qquad (3.6b)$$

We could have replaced Eq. (3.6) by (say) Eq. (3.6a) for all s, with Eq. (3.5) still holding. However, as Eq. (3.5) is invariant under $r \to 1 - r$, it will be sufficient to analyze $\hat{H}_0$ for $r \geq 1/2$; this range for r is recovered through the definition Eq. (3.6).

It is easy to see that $\hat{H}_0$ has self-adjoint extensions for any r. In the terminology of Ref. [41], infinity is a limit point case, whereas zero is a limit point case for $r \geq 3/2$ and limit circle case otherwise (Ref. [41], theorems X.8 and X.10, and problem 7). For $r \geq 3/2$, $\hat{H}_0$ is therefore essentially self-adjoint. For $1/2 \leq r < 3/2$, on the other hand, the self-adjoint extensions of $\hat{H}_0$ are characterized by a boundary condition at zero and parametrized by $U(1)$.

We now wish to extract qualitative information about the spectrum of the self-adjoint extensions of $\hat{H}_0$.

A first observation is that the essential spectrum of every self-adjoint extension of $\hat{H}_0$ is empty (Ref. [40], theorems XIII.7.4, XIII.7.16, and XIII.7.17). This means that the spectrum is discrete: The spectrum consists of eigenvalues corresponding to genuine, normalizable eigenstates, and the eigenvalues have disjoint neighborhoods.

Second, we shall show in Appendix A that every self-adjoint extension of $\hat{H}_0$ is bounded below: The system has a ground state. For $r \geq 3/2$, the ground state energy is always positive. For $1/2 \leq r < 3/2$, the ground state energy depends on the self-adjoint extension, and the situation is more versatile. On the one hand, there is a certain (open) set among the self-adjoint extensions within which the ground state energy is positive. On the other hand, there exist extensions whose ground state energy is arbitrarily negative.

Third, we shall show in Appendix B that a WKB analysis yields for the squares of the large eigenenergies the asymptotic estimate

$$E_{\text{WKB}}^2 \sim 2k + \text{const} + o(1), \qquad (3.7)$$

where k is an integer and $o(1)$ denotes a term that vanishes asymptotically at large E. The constant depends on r and, for $1/2 < r < 3/2$, also on the self-adjoint extension, in a way discussed in Appendix B. We shall also be able to rigorously verify the accuracy of this WKB result in the special case $r = 5/6$, and the accuracy of its leading order term in the special case $r = 1$.

These properties of the spectrum have consequences of direct physical interest. At the low end of the spectrum, the fact that the Hamiltonian is bounded below indicates stability: One cannot extract from the system an infinite amount of energy. At the high end of the spectrum, the asymptotic distribution of the large eigenenergies yields for the black hole area the eigenvalues (1.3): This agrees with Bekenstein's area spectrum (1.1), with $\alpha = 32\pi$. We shall discuss the physical implications of these results further in Sec. V.

IV. THROAT THEORY WITH CHARGE AND A NEGATIVE COSMOLOGICAL CONSTANT

In this section we shall outline how the throat theory can be generalized to accommodate electric charge and a negative cosmological constant. The classical black hole solutions are in this case given (locally) by the Reissner–Nordström–anti–de Sitter metric [35]

$$ds^2 = -\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} + \frac{R^2}{\ell^2} \right) dT^2$$
$$+ \frac{dR^2}{1 - 2M/R + Q^2/R^2 + R^2/\ell^2} + R^2 d\Omega^2, \quad (4.1a)$$

with the electromagnetic potential one-form

$$A = \frac{Q}{R} dT. \qquad (4.1b)$$

The parameters M and Q are referred to as the mass and the (electric) charge, and the cosmological constant has been written in terms of the positive parameter ℓ as $-3\ell^{-2}$. For the global structure of the spacetime, see Refs. [42,43]. We shall understand the case of a vanishing cosmological constant as the limit $\ell \to \infty$, and in this case the above solution reduces to the Reissner-Nordström solution.

The Hamiltonian structure of the spherically symmetric Einstein-Maxwell system with a cosmological constant was analyzed by a technique related to Ashtekar's variables in Refs. [44,45]. An analysis via a Kuchař-type canonical transformation and Hamiltonian reduction, both with and without a negative cosmological constant, was given in Ref. [46]. Although the focus of Ref. [46] was on thermodynamically motivated boundary conditions, which confine the constant t hypersurfaces to one exterior region of the spacetime, the discussion therein generalizes without essential difficulty to boundary conditions that allow the constant t hypersurfaces to extend from a left-hand side spacelike infinity to the corresponding right-hand side spacelike infinity, crossing the event horizons in arbitrary ways but crossing no inner horizons. The new technical issues arise mainly from the fact that with a negative cosmological constant, the left and right infinities are asymptotically anti–de Sitter rather than asymptotically flat. The new physical issues arise mainly in the choice of the electromagnetic boundary conditions at the infinities.

We shall here concentrate on the theory where the electric charge is fixed at the infinities. In the reduced theory, the charge then becomes an entirely nondynamical, external parameter, which we denote by $\mathbf{q}$. On a classical solution, $\mathbf{q}$ is equal to Q in Eq. (4.1).

The action of the reduced theory takes the form (2.3). On the classical solutions, $\mathbf{m}$ is equal to the mass parameter M of Eq. (4.1a). The canonical conjugate $\mathbf{p}$ again equals the difference in the asymptotic Killing times between the left and right ends of a constant t hypersurface. The range of $\mathbf{m}$ is $\mathbf{m} > \mathbf{m}_{\text{crit}}$, where the critical value $\mathbf{m}_{\text{crit}}(\mathbf{q}, \ell)$ is positive for $\mathbf{q} \neq 0$ and vanishes for $\mathbf{q} = 0$: This restriction arises from the requirement that the classical solutions have a nondegenerate event horizon [42,43,46]. The range of $\mathbf{p}$ is $-\infty < \mathbf{p} < \infty$. The quantities $N_{\pm}$ determine the evolution of the ends of the hypersurface in the asymptotic Killing time. For $\ell \to \infty$, $N_{\pm}$ are simply the asymptotic values of the lapse, whereas for $0 < \ell < \infty$ they are related to the lapse by a factor that diverges at the infinities. Note that in the special case of $\mathbf{q} = 0$ and $0 < \ell < \infty$, we get a theory of vacuum spacetimes with a negative cosmological constant.

Mimicking Sec. II, we freeze the evolution of the hypersurfaces at the left-hand side infinity by setting $N_- = 0$, and fix the evolution at the right-hand side infinity to proceed at unit rate with respect to the Killing time by setting $N_+ = 1$. The action is given by Eq. (2.5). The value of the Hamiltonian on a classical solution is then equal to the mass parameter. For $\ell \to \infty$ this reproduces the identification of the mass as the ADM energy from the viewpoint of asymptotic Minkowski time evolution at one infinity, just as in the uncharged case in Sec. II. For $0 < \ell < \infty$, we similarly recover the interpretation of the mass parameter as the ADM-type energy from the viewpoint of asymptotic anti–de Sitter Killing time evolution at one infinity.

In analogy with Eq. (2.6), we introduce the new variables (a, p_a) via the transformation

$$|\mathbf{p}| = \int_a^{a_+} \frac{db}{\sqrt{2\mathbf{m}b^{-1} - 1 - \mathbf{q}^2 b^{-2} - b^2 \ell^{-2}}}, \quad (4.2a)$$

$$p_a = \text{sgn}(\mathbf{p}) \sqrt{2\mathbf{m}a - a^2 - \mathbf{q}^2 - a^4 \ell^{-2}}, \quad (4.2b)$$

where $a_+(\mathbf{m}, \mathbf{q}, \ell)$ is the unique positive zero of the right-hand side in Eq. (4.2b) for $\mathbf{q} = 0$, and the larger of the two positive zeros for $\mathbf{q} \neq 0$. On a classical solution, a_+ is the radius of the event horizon. To make this transformation well defined, we again need to restrict by hand the range of $\mathbf{p}$. For $\mathbf{q} = 0$, the upper limit for $|\mathbf{p}|$ is obtained from Eq. (4.2a) with $a = 0$. For $\mathbf{q} \neq 0$, the upper limit for $|\mathbf{p}|$ is obtained from Eq. (4.2a) with $a = a_-$, where $a_-(\mathbf{m}, \mathbf{q}, \ell)$ is the smaller of the two positive zeros of the right-hand side in Eq. (4.2b). On a classical solution, a_- is the radius of the inner horizon.

With this restriction on the range of $\mathbf{p}$, the transformation (4.2) is well defined, one to one, and canonical. The new action is given by Eq. (2.7) with the Hamiltonian

$$H = \frac{1}{2} \left(\frac{p_a^2}{a} + a + \frac{\mathbf{q}^2}{a} + \frac{a^3}{\ell^2} \right), \quad (4.3)$$

and the value of this Hamiltonian on a classical solution is just the mass.

As in Sec. II, the theory has an interpretation in terms of a wormhole throat. If there exists a foliation with the appropriate falloff conditions [46], intersecting a timelike geodesic through the event horizon bifurcation two-sphere so that $-\mathbf{p}$ coincides with the proper time from the bifurcation two-

sphere along this geodesic, then the quantity a defined by Eq. (4.2a) is the two-sphere radius on this geodesic. If the foliation is chosen suitably symmetric near the specified geodesic, we can think of a as the radius of the wormhole throat. We shall now examine the existence of such foliations for the different values of the parameters $\mathbf{q}$ and ℓ.

For $\mathbf{q} = 0$ and $0 < \ell < \infty$, the classical solution is the Schwarzschild-anti–de Sitter hole. The Penrose diagram differs from that of the Kruskal manifold only in that the asymptotically flat infinities are replaced by asymptotically anti–de Sitter infinities, represented by vertical lines [42,43]. Foliations of the desired kind clearly exist: The timelike geodesic starts at the initial singularity with $a = 0$, reaches the maximum value of a at the bifurcation two-sphere, and ends at the final singularity with $a = 0$. The situation is thus qualitatively very similar to that with the Schwarzschild hole.

For $\mathbf{q} \neq 0$ and $\ell \to \infty$, the classical solution is the Reissner-Nordström hole with $\mathbf{m} > |\mathbf{q}|$. The Penrose diagram can be found in Refs. [32,47]. Our Kuchař-type Hamiltonian formulation is valid for the part of the spacetime that consists of one pair of spacelike-separated left and right asymptotically flat regions and the connecting region that is bounded in the past and future by the Cauchy horizons. The solutions to the equations of motion obtained from the Hamiltonian (4.3) are periodic oscillations in the interval $a_- \leq a \leq a_+$, but our derivation of this Hamiltonian is only valid on each solution between two successive minima of a. Now, it is clear that foliations of the desired kind exist: The timelike geodesic starts with $a = a_-$ at the past Cauchy horizon bifurcation two-sphere, reaches $a = a_+$ at the event horizon bifurcation two-sphere, and ends with $a = a_-$ at the future Cauchy horizon bifurcation two-sphere.

Finally, for $\mathbf{q} \neq 0$ and $0 < \ell < \infty$, the classical solution is the Reissner-Nordström-anti–de Sitter hole, with $\mathbf{m}$ so large that a nondegenerate event horizon exists. The Penrose diagram is obtained from that of the Reissner-Nordström hole by replacing the asymptotically flat infinities by asymptotically anti–de Sitter infinities [42,43]. Our Kuchař-type Hamiltonian formulation is valid for the part of the spacetime that consists of one pair of spacelike-separated left and right asymptotically anti–de Sitter regions and the connecting region that is bounded in the past and future by the inner horizons. The inner horizons are now not Cauchy horizons, as the asymptotically anti–de Sitter infinities render our part of the spacetime not globally hyperbolic. As with the Reissner-Nordström hole above, the solutions to the equations of motion obtained from the Hamiltonian (4.3) are periodic oscillations in the interval $a_- \leq a \leq a_+$, and our derivation of this Hamiltonian is only valid on each solution between two successive minima of a. Foliations of the desired kind now exist while the timelike geodesic remains sufficiently close to the event horizon bifurcation two-sphere. However, it is seen from the Penrose diagram that as the geodesic progresses towards the past and future inner horizon bifurcation two-spheres, there will occur a critical value of the proper time after which the constant t hypersurfaces would necessarily need to become somewhere timelike. Therefore, the throat interpretation can only be maintained in the full domain of validity of the Hamiltonian (4.3) by ap-

pealing to a foliation where the constant t hypersurfaces need not be everywhere spacelike. We shall return to this issue in Sec. V.

Quantization of the theory proceeds as in Sec. III. The Hamiltonian $\hat{H}_0$, Eq. (3.5), inherits the additional terms

$$\frac{\mathbf{q}^2}{2x^{2/3}} + \frac{x^2}{2\ell^2}. \tag{4.4}$$

The theorems cited in Sec. III show that the additional terms make no difference for the existence and counting of the self-adjoint extensions of $\hat{H}_0$, and they also show that the essential spectrum of any self-adjoint extension of $\hat{H}_0$ is again empty. For $\mathbf{q}=0$, the proof of the lower bound for the spectrum given in Appendix A goes through virtually without change. For $\mathbf{q}\neq0$, the charge term modifies the small x behavior of the wave functions, and the analysis of the self-adjointness boundary condition is more involved; in particular, there is a qualitative difference between the cases $7/6<r<3/2$, $r=7/6$, and $1/2<r<7/6$, arising from whether the next-to-leading term in the counterpart of $v_E(x)$ in Eq. (A2) dominates the leading order term in the counterpart of $u_E(x)$ at small x. However, the modified Bessel function asymptotic behavior (A6) and (A9) still holds, as can be shown by applying the series solution method for u_E and the "second solution" integral formula for v_E (see, for example, Chap. 8 of Ref. [48]). The spectrum of every self-adjoint extension is therefore again bounded below, and certain self-adjoint extensions are strictly positive.

One expects that the asymptotic distribution of the large eigenvalues could be investigated via a suitable generalization of the WKB techniques of Appendix B. We shall, however, not attempt to carry out such an analysis here.

V. DISCUSSION

In this paper we have considered a Hamiltonian theory of spherically symmetric vacuum Einstein gravity under Kruskal-like boundary conditions. The foliation was chosen such that the evolution of the spacelike hypersurfaces is frozen at the left-hand side infinity, but proceeds at unit rate with respect to the asymptotic Minkowski time at the right-hand side infinity. The reduced Hamiltonian theory was written in a set of variables associated with the Einstein-Rosen wormhole throat: The configuration variable is the radius of the throat, in a foliation in which the proper time at the throat agrees with the asymptotic right-hand side Minkowski time. The classical Hamiltonian is numerically equal to the Schwarzschild mass.

We quantized the theory by Hamiltonian methods, taking the wave functions to be functions of the classical configuration variable, and including a general power-law weight factor in the inner product. The classical Hamiltonian was promoted into a self-adjoint Hamiltonian operator. We found that the spectrum of the Hamiltonian operator is discrete and bounded below for all the choices of the weight factor. In the cases where the Hamiltonian operator is essentially self-adjoint, the spectrum is necessarily positive definite; in the remaining cases, self-adjoint extensions with a positive definite spectrum always exist. In all the cases, a WKB estimate gave for the large eigenvalues the asymptotic behavior

$\sqrt{2k}$, where k is an integer, and we were able to rigorously verify the accuracy of this estimate for four particular choices of the weight factor. The resulting spectrum for the area of the black hole agrees with the spectrum (1.1) proposed by Bekenstein and others, with the dimensionless constant α taking the value 32π. We also showed that analogous results can be obtained in the presence of a fixed electric charge and a negative cosmological constant.

It is perhaps worth emphasizing that the basic postulates of our quantum theory consisted of the choice of a Hilbert space and a self-adjoint Hamiltonian operator on it. We did not attempt to define more "elementary" operators, such as those for "position" or "momentum," self-adjoint or otherwise, from which the Hamiltonian operator could be constructed. This issue might merit further study within some geometric or algebraic framework of quantization[4] [49–51].

Even though our theory is that of pure vacuum, our boundary conditions were chosen so as to make the results relevant for physics that is accessible to an inertial observer at a spacelike infinity. Our spacelike hypersurfaces have evolution at only *one* infinity, and there they evolve at unit rate with respect to the asymptotic Minkowski time. Our classical Hamiltonian is therefore the gravitational Hamiltonian with respect to the proper time of an inertial observer at the infinity, at rest with respect to the hole. It is thus reasonable to think of the eigenvalues of the Hamiltonian operator as the possible outcomes that an asymptotic observer would in principle obtain when measuring the ADM mass of the hole. In a given (pure) quantum state, the probability for obtaining a given eigenvalue is determined by the component of the state in the respective eigenspace in the standard way. Although we are here for concreteness using language adapted to a Copenhagen-type interpretation, a translation into interpretations of the many-worlds type could easily be made.

One can also make a case that our throat variable a depicts in a particularly natural way the dynamical aspects of the Kruskal manifold. Classically, the wormhole throat begins life at the white hole singularity, expands to maximum radius at the bifurcation two-sphere, and collapses to the black hole singularity. The dynamics of a is therefore, in a certain sense, confined to the interior regions of the Kruskal manifold, and these are precisely the regions that do not admit a timelike Killing vector. From a physical viewpoint, using a variable with this property is motivated by the fact that an inertial observer at a spacelike infinity sees her exterior region of the Kruskal manifold as static. Further, our foliation made the proper time at the throat increase at the same rate as the asymptotic right-hand side Minkowski time: by the principle of equivalence, one may see this as the preferred condition for relating the experiences of an inertial observer at the asymptotic infinity to the experiences of an inertial observer at the throat. We recall, in contrast, that the reduced phase space variables of Refs. [36–38] reflect more

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[4]For example, note that the kinetic term of the Hamiltonian operator $\hat{H}$, Eq. (3.2), can be written as $\frac{1}{2}\mu^{-1/2}\hat{p}_a a^{-1}\mu\hat{p}_a\mu^{-1/2}$, where $\hat{p}_a$: $\psi\mapsto-iu^{-1/2}(d/da)\mu^{1/2}\psi$ is a symmetric (but not self-adjoint) momentum operator that can be regarded as conjugate to the position operator $\hat{a}$: $\psi\mapsto a\psi$. We thank Thomas Strobl for this observation.

closely the *static* aspects of the Kruskal manifold. Yet another set of variables has been discussed in Refs. [52,53].

With this physical picture, the properties we obtained for the spectrum of the Hamiltonian operator acquire consequences of direct physical interest. At the low end of the spectrum, the fact that the Hamiltonian is bounded below indicates stability: One cannot extract from the system an infinite amount of energy. At the high end of the spectrum, in the semiclassical regime of the theory, the discreteness of the spectrum in accordance with the area quantization rule (1.1) yields the macroscopically observable consequences discussed by Bekenstein and Mukhanov [26]. It should be emphasized, however, that these arguments operate at a somewhat formal level, as our theory does not describe how the quantum black hole would interact with other degrees of freedom, such as departures from spherical symmetry or matter fields.

On the grounds of classical positive energy theorems, one may feel inclined to exclude by fiat quantum theories in which the ground state energy of an isolated self-gravitating system is negative. Among our theories, this would amount to a restriction on the self-adjoint extension in the cases where the Hamiltonian operator is not essentially self-adjoint. However, given the freedom that we have already allowed in the choice of the the inner product, it would be a relatively minor further generalization to add to our Hamiltonian operator the identity operator with some real coefficient, and to take the coefficient as a new parameter in the quantum theory.[5] The classical limit of the quantum theory would still be correct, provided the new parameter is understood to be proportional to Planck's constant. When the Hamiltonian operator is not essentially self-adjoint, any given self-adjoint extension can then be made positive definite by choosing the new parameter sufficiently large. Note, however, that with any fixed value of the new parameter, there still exist self-adjoint extensions whose ground state energy is arbitrarily negative.

In Sec. II, we obtained the classical Hamiltonian throat theory by first going from the geometrodynamical Hamiltonian variables to Kuchař's reduced Hamiltonian theory, and then performing a suitable canonical transformation. The interpretation of our variable a as the radius of the wormhole throat was only introduced after the fact, by appealing to a particular choice of the spacetime foliation. We took the same route in the presence of charge and a negative cosmological constant in Sec. IV. We have not discussed here how to derive a throat theory directly from the unreduced geometrodynamical Hamiltonian theory by introducing a gauge and performing the Hamiltonian reduction, but with the appropriate gauge choice, the resulting theory should by construction be at least locally identical to ours. The only case where one anticipates a difference in the global properties is in the presence of both a nonvanishing charge and a negative cosmological constant. In this case, we saw in Sec. IV that while our throat theory is valid on each classical solution between two successive minima of a, the wormhole throat interpretation could be maintained for all of this interval only by appealing to a foliation that is not everywhere spacelike. A

direct Hamiltonian reduction of the geometrodynamical theory in the corresponding wormhole-type gauge would thus only yield our theory in a more limited domain, valid on each classical solution in a certain interval around a maximum of a.

Our choice of freezing the evolution of the spacelike hypersurfaces at the left-hand side infinity was motivated by the desire to have a theory that would describe physics accessible to observers at just one infinity. For a vanishing charge and cosmological constant, this motivation can be implemented at the very beginning by setting up the Hamiltonian theory not on the Kruskal manifold, but instead on the $\mathbb{R}P^3$ geon [54]. To see this, recall [54] that the $\mathbb{R}P^3$ geon is the quotient of the Kruskal manifold under a freely acting involutive isometry: This isometry consists of a reflection of the Kruskal diagram about the vertical timelike line through the bifurcation point, followed by the antipodal map on the two-sphere. The $\mathbb{R}P^3$ geon has thus only one exterior region, identical to one of the Kruskal exterior regions. Further, the $\mathbb{R}P^3$ geon possesses a distinguished $\mathbb{R}P^2$ of timelike geodesics through the image of the Kruskal bifurcation two-sphere: In the Penrose diagram [54], these geodesics go straight up along the ''boundary'' of the diagram. The existence of the distinguished geodesics reflects the fact that translations in the Killing time on the Kruskal manifold do not descend into globally defined isometries of the $\mathbb{R}P^3$ geon. Now, Kuchař's canonical transformation and Hamiltonian reduction generalize readily to the $\mathbb{R}P^3$ geon [39]. The reduced action is obtained from Eq. (2.3) by setting $N_- = 0$, and the momentum $\mathbf{p}$ is now equal to the difference of the Killing times between the distinguished timelike geodesics and the single spacelike infinity. Setting $N_+ = 1$, we are led to the action (2.5). The variable a defined by Eq. (2.6) is now equal to the curvature radius of the distinguished $\mathbb{R}P^2$ of timelike geodesics.

The above interpretation of our theory in terms of the $\mathbb{R}P^3$ geon generalizes immediately to accommodate a negative cosmological constant. For a nonvanishing charge, on the other hand, there exists again an analogous involutive isometry that can be used to quotient the manifold, but the electric field is invariant under this isometry only up to its sign. This reflects the fact that Gauss's theorem prohibits a regular spacelike hypersurface with just one asymptotic infinity from carrying a nonzero charge. The $\mathbb{R}P^3$ geon interpretation does therefore not extend to the charged case with a conventional implementation of the electromagnetic field.

As we have seen, the central input in our classical theory was to parametrize the geometry in terms of the radius of the wormhole throat in a judiciously chosen foliation: Our variable a is the two-sphere radius on a radial geodesic through the event horizon bifurcation two-sphere, in a foliation such that the proper time along the distinguished geodesic agrees with the asymptotic Killing time at the right-hand side infinity. One possible generalization would be to relax the requirement that the variable ''live'' in the interior regions of the manifold, and use instead timelike geodesics that do not pass through the bifurcation two-sphere. To examine this, let us for concreteness set the charge and the cosmological constant to zero, and let us generalize the canonical transformation (2.6) to

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[5] We thank John Friedman for this observation.

$$|\mathbf{p}| = \int_a^{2\mathbf{m}[1+(R_0^*)^2]} \frac{db}{\sqrt{2\mathbf{m}b^{-1} - [1+(R_0^*)^2]^{-1}}}, \quad (5.1a)$$

$$p_a = \text{sgn}(\mathbf{p})\sqrt{2\mathbf{m}a - a^2[1+(R_0^*)^2]^{-1}}, \quad (5.1b)$$

where R_0^* is a real-valued parameter. The transformation (2.6) is recovered as the special case $R_0^* = 0$. The ranges of the new variables are again $a > 0$ and $-\infty < p_a < \infty$, but the restriction for $\mathbf{p}$ is now $|\mathbf{p}| < \pi[1+(R_0^*)^2]^{3/2}\mathbf{m}$. The action is given by Eq. (2.7) with the Hamiltonian

$$H = \frac{1}{2}\left[\frac{p_a^2}{a} + \frac{a}{1+(R_0^*)^2}\right]. \quad (5.2)$$

On a classical solution, the variable a is now equal to the two-sphere radius on a radial timelike geodesic whose trajectory is given by $R^* = R_0^*$, where R^* is the Novikov space coordinate in the notation of Ref. [32]. Foliations that make this interpretation possible clearly exist. Simple examples are obtained by deforming the Novikov foliation near the left-hand side infinity as in Sec. II, to accommodate the boundary condition $N_- = 0$, and (for $R_0^* \neq 0$) also near the throat, to prohibit the constant t hypersurfaces from reaching the singularity before the geodesic at $R^* = R_0^*$. Upon quantization along the lines of Sec. II, we find that the spectrum depends on the parameter R_0^* only through an overall factor: If the eigenvalues are denoted by $E_k^{(R_0^*)}$, where k ranges over the nonnegative integers, we have

$$E_k^{(R_0^*)} = [1+(R_0^*)^2]^{-3/4}E_k^{(0)}. \quad (5.3)$$

A much wider generalization would be to relax the requirement, which we above motivated by the equivalence principle, that the proper time along the throat trajectory agree with the asymptotic Killing time. If one allows this freedom, it is not difficult to come up with examples of foliations in which the Hamiltonian takes a mathematically simple form. As an illustration, let us exhibit two examples in the case of vanishing charge and cosmological constant.

As the first example, suppose that $\mathbf{p}$ is restricted by hand to have the range $|\mathbf{p}| < \gamma^{-1}\pi\mathbf{m}$, where γ is a positive constant. We perform the canonical transformation

$$\xi = \sqrt{2/\gamma}\,\mathbf{m}\cos\left(\frac{\gamma\mathbf{p}}{2\mathbf{m}}\right), \quad (5.4a)$$

$$p_\xi = \sqrt{2/\gamma}\,\mathbf{m}\sin\left(\frac{\gamma\mathbf{p}}{2\mathbf{m}}\right), \quad (5.4b)$$

where the ranges of the new canonical variables are $\xi > 0$ and $-\infty < p_\xi < \infty$. The Hamiltonian takes the form

$$H = \sqrt{\tfrac{1}{2}\gamma(p_\xi^2 + \xi^2)}. \quad (5.5)$$

We can identify $\sqrt{2\gamma}\xi$ as the two-sphere radius on a radial geodesic through the bifurcation two-sphere in a foliation where the proper time τ along this geodesic is

$$\tau = -\text{sgn}(\mathbf{p})\int_{\sqrt{2\gamma}\xi}^{2\mathbf{m}} \frac{db}{\sqrt{2\mathbf{m}b^{-1}-1}}, \quad (5.6)$$

with ξ given by Eq. (5.4a).

To quantize this theory, we adopt the inner product $(\psi_1, \psi_2) = \int_0^\infty d\xi\,\overline{\psi_1}\psi_2$. We define the Hamiltonian operator $\hat{H}$ by spectral analysis as the positive square root of some positive definite self-adjoint extension of $\gamma\hat{H}_{\text{SHO}}$, where $\hat{H}_{\text{SHO}} := \frac{1}{2}[-(d/d\xi)^2 + \xi^2]$ is the simple harmonic oscillator Hamiltonian operator on the positive half-line. The following statements about $\hat{H}_{\text{SHO}}$ can now be verified: (i) The self-adjoint extensions are specified by the boundary condition $\cos(\theta)\psi - \sin(\theta)d\psi/d\xi = 0$ at the origin, with the parameter θ satisfying $0 \leq \theta < \pi$; (ii) the spectrum of each self-adjoint extension is purely discrete; (iii) the eigenfunctions are parabolic cylinder functions [55], and for $\theta = 0$ and $\theta = \pi/2$ they reduce respectively to the odd and even ordinary harmonic oscillator wave functions; (iv) if ϵ_k denotes the eigenvalues, with k ranging over the nonnegative integers, we have for $\theta = 0$ and $\theta = \pi/2$ the respective exact results $\epsilon_k = 2k + \frac{3}{2}$ and $\epsilon_k = 2k + \frac{1}{2}$, and for other values of θ the asymptotic large k expansion $\epsilon_k \sim 2k + \frac{1}{2} + \pi^{-1}\cot(\theta)k^{-1/2} + o(k^{-1/2})$; (v) the absence of negative eigenvalues is equivalent to the condition that θ not lie in the interval $-2^{-3/2}\pi^{-1}[\Gamma(1/4)]^2 < \tan(\theta) < 0$. The resulting spectrum for $\hat{H}$ therefore agrees asymptotically with the area quantization rule (1.1) for any θ that makes $\hat{H}_{\text{SHO}}$ positive definite. The numerical constant α takes the value $32\pi\gamma$.

As the second example, suppose that $\mathbf{p}$ retains the full range $-\infty < \mathbf{p} < \infty$, and perform the canonical transformation

$$\eta = \frac{\mathbf{m}^2}{\cosh(\mathbf{p}/2\mathbf{m})}, \quad (5.7a)$$

$$p_\eta = \sinh\left(\frac{\mathbf{p}}{2\mathbf{m}}\right), \quad (5.7b)$$

where the ranges of the new canonical variables are $\eta > 0$ and $-\infty < p_\eta < \infty$. The Hamiltonian takes the form

$$H = (\eta^2 p_\eta^2 + \eta^2)^{1/4}. \quad (5.8)$$

We can identify $2\sqrt{\eta}$ as the two-sphere radius on a radial geodesic through the bifurcation two-sphere in a foliation where the proper time τ along this geodesic is

$$\tau = -\text{sgn}(\mathbf{p})\int_{2\sqrt{\eta}}^{2\mathbf{m}} \frac{db}{\sqrt{2\mathbf{m}b^{-1}-1}}, \quad (5.9)$$

with η given by Eq. (5.7a).

In the quantum theory we now adopt the inner product $(\psi_1, \psi_2) = \int_0^\infty \eta^{-1}d\eta\,\overline{\psi_1}\psi_2$. The operator $\hat{H}_{\text{expt}} := -[\eta(d/d\eta)]^2 + \eta^2$ is essentially self-adjoint, its discrete spectrum is empty, and its essential spectrum consists of the non-

negative half-line.[6] We can therefore define the Hamiltonian operator $\hat{H}$ by spectral analysis as $(\hat{H}_{\text{expt}})^{1/4}$. It follows that the spectrum of $\hat{H}$ is now *continuous* and consists of the non-negative half-line.

These examples suggest that the continuity versus discreteness of the spectrum is related to the question of whether the wormhole throat reaches the initial and final singularities within finite parameter time. There are general grounds to expect this to be the case. A classical theory in which the two-sphere radius reaches zero within finite parameter time is singular, in the sense that the classical solutions cannot be continued arbitrarily far into the past and future. If one quantizes such a theory so that the classical Hamiltonian is promoted into a time-independent, self-adjoint Hamiltonian operator, then the unitary evolution generated by the Hamiltonian operator remains well-defined for arbitrarily large times. If one starts with an initial wave function that is a wave packet following some classical trajectory, the quantum time evolution will force the wave packet to be reflected from the classical singularity. The reflection is an entirely quantum-mechanical phenomenon, and the quantum dynamics acquires in this sense a quasiperiodic character. On the other hand, if the classical solutions require an infinite amount of time to reach the singularity, one generically expects [41] that in the quantum theory a wave packet initially following a classical trajectory will just keep following this trajectory, with some spreading, for arbitrarily large times. It is clear that these arguments apply without change also in the presence of a negative cosmological constant. In the presence of a nonvanishing charge, an analogous discussion applies with the singularity replaced by the inner horizon.

One may hold mixed feelings about a wormhole throat quantum theory that introduces a quantum-mechanical bounce at a classical singularity or at an inner horizon. On the one hand, singularities and inner horizons are places where the classical theory behaves poorly, and one anticipates quantum effects to be important. On the other hand, an outright bounce may appear an uncomfortably orderly quantum prediction, given that (semi)classical intuition associates singularities and inner horizons with collapses and instabilities. Related discussion, in this and related contexts, can be found in Refs. [56–62]. While we view the model of the present paper as a useful arena where these issues can be addressed in relatively explicit terms, the model is undoubtedly dynamically too poor to support confident conclusions about the physical reasonableness of a discrete versus continuous black hole spectrum. It would be substantially more interesting if our techniques could be generalized to models containing degrees of freedom that carry Hawking radiation.

ACKNOWLEDGMENTS

APPENDIX A: SEMIBOUNDEDNESS OF $\hat{H}_0$

In this appendix we shall show that every self-adjoint extension of the Hamiltonian $\hat{H}_0$, Eq. (3.5), is bounded below, and that certain self-adjoint extensions are strictly positive. We shall discuss separately the cases $r \geq 3/2$, $1/2 < r < 3/2$, and $r = 1/2$.

1. $r \geq 3/2$

For $r \geq 3/2$, $\hat{H}_0$ is essentially self-adjoint. Let χ be an eigenfunction with energy E. With a suitable choice of the overall numerical factor, χ is real valued and has the small x expansion

$$\chi(x) = x^r[1 + O(x^2)]. \tag{A1}$$

Both χ and χ' are therefore positive for sufficiently small x. If now $E \leq 0$, the eigenvalue equation shows then that χ is increasing for all $x > 0$. This implies that χ cannot be normalizable, which contradicts the assumption that χ is an eigenfunction. Hence the spectrum is strictly positive.

2. $1/2 < r < 3/2$

For $1/2 < r < 3/2$, the self-adjoint extensions of $\hat{H}_0$ form a family characterized by a boundary condition at $x = 0$, and parametrized by $U(1)$. To find these extensions, we note that for any E, the differential equation $\hat{H}_0 \chi = E \chi$ has two linearly independent solutions, denoted by $u_E(x)$ and $v_E(x)$, with the asymptotic small x behavior:

$$u_E(x) = x^r[1 + O(x^2)], \tag{A2a}$$

$$v_E(x) = x^{1-r}[1 + O(x^2)]. \tag{A2b}$$

For real E, both u and v are real valued. Using the techniques of Ref. [41], it is easily shown that the eigenfunctions of a given self-adjoint extension of $\hat{H}_0$ take, up to overall normalization, the form

$$\chi_E = \cos(\theta) u_E + \sin(\theta) v_E, \tag{A3}$$

where $\theta \in [0, \pi)$ is the parameter specifying the self-adjoint extension. The condition (A3) can be written without explicit reference to the solutions (A2) as

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[6]These statements follow in a straightforward manner from Chap. VIII of Ref. [40] after bringing $\hat{H}_{\text{expt}}$ and the inner product to a standard form by the substitution $\eta = e^\zeta$. Note that if we had retained dimensions, this substitution would need to include a dimensional constant.

$$0 = \lim_{x \to 0}\left[(2r-1)\cos(\theta)x^{r-1}\chi - \sin(\theta)x^{2(1-r)}\frac{d(x^{r-1}\chi)}{dx}\right].$$

$$\text{(A4)}$$

We now proceed to obtain a lower bound for the eigenenergies.

Consider first an extension in the range $0 \leq \theta \leq \pi/2$. Comparing the eigenvalue differential equation to the corresponding equation with the term $\frac{1}{2}x^{2/3}$ omitted from $\hat{H}_0$, Eq. (3.5), and the energy set to zero, one sees that the prospective eigenfunctions with $E \leq 0$ are bounded below by the function $\cos(\theta)x^r + \sin(\theta)x^{1-r}$, which does not vanish exponentially at large x. However, as the potential increases without bound as x goes to infinity, every eigenfunction must vanish exponentially at large x. Hence the spectrum is strictly positive.

Consider then an extension in the remaining range $\pi/2 < \theta < \pi$. Let χ be an eigenfunction with energy $E < 0$. Writing $y = (-8E/9)^{1/2}x$, the eigenfunction equation reads

$$0 = \left[-\frac{d^2}{dy^2} + \frac{r(r-1)}{y^2} + 1 + \left(\frac{3y}{8E^2}\right)^{2/3}\right]\chi. \qquad \text{(A5)}$$

The last term in Eq. (A5) is asymptotically small at large negative E, uniformly in the interval $y \in (0,M]$, where M is an arbitrary positive constant. Omitting this last term gives an equation whose linearly independent solutions are $y^{1/2}I_{r-(1/2)}(y)$ and $y^{1/2}I_{(1/2)-r}(y)$, where I is a modified Bessel function [65]. Therefore χ has at large negative E the asymptotic behavior

$$\chi \sim (y/2)^{1/2}[\cos(\theta)\Gamma(r+\tfrac{1}{2})(-2E/9)^{-r/2}I_{r-(1/2)}(y)$$

$$+ \sin(\theta)\Gamma(\tfrac{3}{2}-r)(-2E/9)^{(r-1)/2}I_{(1/2)-r}(y)], \quad \text{(A6)}$$

uniformly for $y \in (0,M]$. The coefficients of the two Bessel functions in Eq. (A6) have been fixed by comparing the small y expansions of Eqs. (A3) and (A6) [65].

By the asymptotic behavior of the Bessel functions at large argument [65], we can now choose M so that $y^{1/2}I_{(1/2)-r}(y)$ is positive and increasing for $y \geq M/2$. For future use, we make this choice so that $M > 1$. Then, the second term in Eq. (A6) dominates the first term at large negative E, uniformly for $M/2 \leq y \leq M$. Therefore there exists a constant $\widetilde{E} < 0$, dependent on r and θ, such that χ and $d\chi/dy$ are positive at $y = M$ whenever $E < \widetilde{E}$. As $|r(r-1)| < 1$, Eq. (A5) then shows, by virtue of the choice $M > 1$, that χ diverges at large y whenever $E < \widetilde{E}$. As χ is by assumption normalizable, we thus see that the eigenenergies are bounded below by $\widetilde{E}$.

Note that the lower bound for the eigenenergies is not uniform in θ. For fixed r and any given E, there exists a unique self-adjoint extension of $\hat{H}_0$ such that E is in the spectrum. This is because the differential equation $\hat{H}_0\chi = E\chi$ has for any E a normalizable solution that is unique up to a multiplicative constant, and matching the small x behavior of this solution to Eq. (A3) uniquely specifies the value of θ. One can thus find extensions with arbitrarily negative ground state energy.

3. $r = 1/2$

For $r = 1/2$, the self-adjoint extensions of $\hat{H}_0$ form again a family parametrized by $U(1)$. The boundary condition characterizing the extensions takes the form (A3), where $\theta \in [0,\pi)$, but now with

$$u_E(x) = x^{1/2}[1 + O(x^2)], \qquad \text{(A7a)}$$

$$v_E(x) = u_E(x)\ln x + O(x^{5/2}). \qquad \text{(A7b)}$$

Condition (A4) is replaced by

$$0 = \lim_{x \to 0}\left\{[\cos(\theta) + \sin(\theta)\ln x]x\frac{d(x^{-1/2}\chi)}{dx} - \sin(\theta)x^{-1/2}\chi\right\}.$$

$$\text{(A8)}$$

For the extension with $\theta = 0$, one sees as above that the spectrum is strictly positive.

Consider then an extension in the range $0 < \theta < \pi$. Let χ be an eigenfunction with energy $E < 0$, and proceed as above. Equation (A6) is replaced by

$$\chi \sim \left(\frac{-8E}{9}\right)^{-1/4}y^{1/2}(\{\cos(\theta) - \sin(\theta)[\gamma + \tfrac{1}{2}\ln(-2E/9)]\}I_0(y)$$

$$- \sin(\theta)K_0(y)), \qquad \text{(A9)}$$

where K is the second modified Bessel function [65] and γ is Euler's constant. At large negative E, the term proportional to $y^{1/2}I_0(y)$ dominates, and one can argue as above. Hence the spectrum is bounded below.

As in the case $1/2 < r < 3/2$, for any given energy E there exists a self-adjoint extension such that E is in the spectrum. One can thus again find extensions with arbitrarily negative ground state energy.

APPENDIX B: LARGE EIGENVALUES OF $\hat{H}_0$

In this appendix we analyze the asymptotic distribution of the large eigenvalues of the self-adjoint extensions of $\hat{H}_0$, Eq. (3.5). The idea is to match a Bessel function approximation at small argument to a WKB approximation in the region of rapid oscillations. We shall again discuss separately the cases $r \geq 3/2$, $1/2 < r < 3/2$, and $r = 1/2$.

1. $r \geq 3/2$

We begin with the case $r \geq 3/2$, where $\hat{H}_0$ is essentially self-adjoint. We shall throughout denote by χ an eigenfunction with energy $E > 0$.

Consider first χ at small argument. Setting $z = (8E/9)^{1/2}x$, the eigenfunction equation reads

$$0 = \left[-\frac{d^2}{dz^2} + \frac{r(r-1)}{z^2} - 1 + \left(\frac{3z}{8E^2}\right)^{2/3}\right]\chi. \qquad \text{(B1)}$$

The last term in Eq. (B1) is asymptotically small at large E, uniformly in the interval $z \in (0,ME^{1/2}]$, where M is an arbitrary positive constant. Omitting this last term gives an equation whose linearly independent solutions are $z^{1/2}J_{r-(1/2)}(z)$ and $z^{1/2}N_{r-(1/2)}(z)$, where J and N are the Bessel functions of the first and second kinds, and only the

former solution is normalizable at small x [65]. The asymptotic large E behavior of χ is, therefore,

$$\chi \stackrel{\sim}{\propto} x^{1/2} J_{r-(1/2)}\left[\frac{(8E)^{1/2}x}{3}\right], \tag{B2}$$

valid uniformly in any bounded region in x. Here, and from now on, we use the symbol $\stackrel{\sim}{\propto}$ to denote the asymptotic form at large E, up to a possibly E-dependent coefficient. Introducing two constants δ_1 and δ_2 that satisfy $0 < \delta_1 < \delta_2$, and using the asymptotic large argument behavior of J [65], we can rewrite Eq. (B2) as

$$\chi \stackrel{\sim}{\propto} \cos\left[\frac{(8E)^{1/2}x}{3} - \frac{\pi r}{2}\right], \tag{B3}$$

valid uniformly for $\delta_1 \leq x \leq \delta_2$.

Consider then the region of rapid oscillations. We take E so large that the eigenvalue equation has two turning points, and we denote the larger turning point by x_0. The region of rapid oscillations at large E is sufficiently far left of x_0 but sufficiently far right of the origin. As χ decays exponentially right of x_0, the WKB approximation to the wave function in the region of rapid oscillations is (see, for example, Ref. [66])

$$\chi_{\mathrm{WKB}} = [p_2(x)]^{-1/2} \cos\left[\int_x^{x_0} dx' p_2(x') - \frac{\pi}{4}\right], \tag{B4}$$

where

$$p_2 = \frac{2}{3}\left[2E - x^{2/3} - \frac{9r(r-1)}{4x^2}\right]^{1/2}. \tag{B5}$$

The evaluation of the integral in Eq. (B4) is discussed in Appendix C. We find

$$\chi_{\mathrm{WKB}} \stackrel{\sim}{\propto} \cos\left[\frac{(8E)^{1/2}x}{3} - \frac{\pi E^2}{2} + \frac{\pi}{4} + O(E^{-1/2})\right], \tag{B6}$$

valid uniformly for $\delta_1 \leq x \leq \delta_2$. Comparing Eqs. (B3) and (B6) yields for the large eigenenergies the WKB estimate

$$E_{\mathrm{WKB}}^2 \sim 2k + r + \tfrac{1}{2} + o(1), \tag{B7}$$

where k is a large integer and $o(1)$ indicates a term that goes to zero at large E.

Note that we have not attempted to control how far the integer k is from the number of the eigenvalue that Eq. (B7) is meant to approximate. To obtain a formula that gives an approximation to the kth eigenenergy in the limit of large k, one may need to add to the right-hand side of Eq. (B7) some even integer.

As the potential term in the Hamiltonian is smooth without oscillations or steplike behavior, and as $|p_2' p_2^{-2}|$ vanishes for large E uniformly in $\delta_1 \leq x \leq \delta_2$, one expects the WKB estimate to the large eigenenergies to be an accurate one. We shall not attempt to investigate the accuracy rigorously for general r, but we shall see below that the accuracy can be verified by independent means in the special case $r = 5/6$, and, to leading order, also in the special case $r = 1$.

2. $1/2 < r < 3/2$

For $1/2 < r < 3/2$, we recall from Appendix A that the self-adjoint extensions of $\hat{H}_0$ are specified by the boundary condition (A3) at small x. Fixing θ and proceeding as above, we see from the small argument behavior of the Bessel functions [65] that Eq. (B2) is replaced by

$$\chi \sim (z/2)^{1/2}[\cos(\theta)\Gamma(r+\tfrac{1}{2})(2E/9)^{-r/2}J_{r-(1/2)}(z)$$
$$+ \sin(\theta)\Gamma(\tfrac{3}{2}-r)(2E/9)^{(r-1)/2}J_{(1/2)-r}(z)]. \tag{B8}$$

When $\theta = 0$, the second term in Eq. (B8) vanishes, and we can proceed as above, with x_0 now denoting the larger one of the two turning points for $r > 1$ and the unique turning point for $r \leq 1$. The WKB estimate for the large eigenenergies is again given by Eq. (B7). When $\theta \neq 0$, on the other hand, the second term in Eq. (B8) dominates the first term at large E, and r in Eq. (B3) is replaced by $1 - r$. The WKB estimate (B7) is therefore replaced by

$$E_{\mathrm{WKB}}^2 \sim 2k - r - \tfrac{1}{2} + o(1). \tag{B9}$$

In the special case $r = 5/6$, we can verify the accuracy of these WKB results rigorously. The eigenfunctions are now $\chi = x^{1/6} U(-\tfrac{1}{2}E^2, \sqrt{2}(x^{2/3} - E))$, where U is the parabolic cylinder function that vanishes at large values of its second argument [55]. The boundary condition (A4) reads

$$0 = \cos(\theta)U(-\tfrac{1}{2}E^2, -\sqrt{2}E) - \sqrt{2}\sin(\theta)U'(-\tfrac{1}{2}E^2, -\sqrt{2}E), \tag{B10}$$

where the prime denotes the derivative of U with respect to its second argument. Using Olver's asymptotic expansions of parabolic cylinder functions [Ref. [67], formula (9.7), and the discussion of the derivative on p. 155], we find

$$E^2 \sim 2k - 2/3 + O(E^{-8/3}) \text{ for } \theta = 0, \tag{B11a}$$

$$E^2 \sim 2k + 2/3 + O(E^{-4/3}) \text{ for } \theta = \pi/2, \tag{B11b}$$

$$E^2 \sim 2k + 2/3 + O(E^{-1/3}) \text{ for } 0 \neq \theta \neq \pi/2. \tag{B11c}$$

This corroborates the WKB results (B7) and (B9) for $r = 5/6$, and gives an improved bound for the error term.

In the special case $r = 1$, the theorem in Sec. 7 of Ref. [68] yields the rigorous asymptotic estimate

$$E^2 \sim 2k + O(\ln E). \tag{B12}$$

[The leading order term of Eq. (B12) also follows from Ref. [40], p. 1614.] This corroborates the leading order term in our WKB results (B7) and (B9) for $r = 1$.

Finally, we note in passing that for $r = 5/6$, the boundary condition (B10) and Olver's expansions [67] yield the asymptotic relation

$$\tan(\theta) \sim -\frac{3^{1/6}[\Gamma(1/3)]^2}{2^{4/3}\pi(-E_0)^{1/3}} + O((-E_0)^{-5/3}) \tag{B13}$$

for the parameter θ and the ground state energy E_0, valid in the limit of large negative E_0.

3. $r = 1/2$

For $r = 1/2$, Eq. (B8) is replaced by

$$\chi \sim \left(\frac{8E}{9}\right)^{-1/4} z^{1/2}(\{\cos(\theta) - \sin(\theta)[\gamma + \tfrac{1}{2}\ln(2E/9)]\}J_0(z)$$

$$+ (\pi/2)\sin(\theta)N_0(z)), \qquad \text{(B14)}$$

where γ is Euler's constant as before. For any θ, the term proportional to $z^{1/2}J_0(z)$ dominates at large energies, and Eq. (B3) holds with $r = 1/2$. The WKB estimate for the eigenenergies is thus given by Eq. (B7) with $r = 1/2$, for any θ.

APPENDIX C: EVALUATION OF THE WKB INTEGRAL

In this appendix we outline the evaluation of the integral in Eq. (B4) at large E. We shall throughout assume that E is so large that a classically allowed domain exists (which is a restriction only for $r > 1$), that x is in the classically allowed domain, and that $x > \delta_1$, where the constant δ_1 was introduced in Appendix B.

Returning to the variable $a = x^{2/3}$ of the main text, and writing $b = (x')^{2/3}$, the integral in the exponent in Eq. (B4) takes the form

$$S := \int_a^{a_0} db \sqrt{2Eb - b^2} \sqrt{1 - \frac{9r(r-1)}{4b^3(2E-b)}}, \qquad \text{(C1)}$$

where $a_0 = x_0^{2/3}$ is the (larger) turning point. At large E, $a_0 \sim 2E + O(E^{-3})$.

We fix a constant $\delta_3 > 0$, take E so large that $|a_0 - 2E| < \delta_3/E$, and restrict a to be in the interval $\delta_1^{2/3} \leq a \leq 2E - \delta_3/E$. We can then replace the upper limit of the integral in Eq. (C1) by $2E - \delta_3/E$, with the error in S being of order $O(E^{-1})$. The second term under the second square root in (C1) is now uniformly of order $O(E^{-1})$. As the first square root is of order $O(E)$, we can expand the second square root in the Taylor series and truncate the series after the third term, with the error in S being of order $O(E^{-1})$. In the truncated integrand, the third term is proportional to $b^{-11/2}(2E-b)^{-3/2}$, which is uniformly of order $O(E^{-3/2})$, and this term can thus be omitted with the consequence of making an error of order $O(E^{-1/2})$ in S. We therefore have

$$S \sim \int_a^{2E - \delta_3/E} db \left[\sqrt{2Eb - b^2} - \frac{9r(r-1)}{8b^2\sqrt{2Eb - b^2}}\right] + O(E^{-1/2}). \qquad \text{(C2)}$$

The integral in Eq. (C2) is elementary. The contribution from the second term turns out to be of order $O(E^{-1/2})$, and in the first term the upper limit can be replaced by $2E$ with the consequence of making an error of order $O(E^{-1})$. We thus obtain

$$S \sim \tfrac{1}{2}(E - a)\sqrt{2Ea - a^2} + \tfrac{1}{2}E^2 \arcsin(1 - a/E) + \tfrac{1}{4}\pi E^2$$

$$+ O(E^{-1/2}). \qquad \text{(C3)}$$

Finally, we restrict a to be in the interval $\delta_1^{2/3} \leq a \leq \delta_2^{2/3}$, where the constant δ_2 was introduced in Appendix B. A large E expansion of Eq. (C3) then gives

$$S \sim -\sqrt{\frac{8Ea^3}{3}} + \frac{\pi E^2}{2} + O(E^{-1/2}), \qquad \text{(C4)}$$

which leads to Eq. (B6).

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