PHYSICAL REVIEW D VOLUME 54, NUMBER 8 15 OCTOBER 1996

General matching of two spherically symmetric spacetimes

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(Received 4 April 1996)

In this paper we show some very general results concerning the *general* matching of any two spherically symmetric spacetimes through a timelike hypersurface. We present a set of necessary conditions for this case which are very simple to use. As an important result, these conditions allow us to ascertain, by mere inspection of the conformal diagrams, which matchings are feasible in principle and which are not allowed. We shall illustrate these results by applying them to the general matching of Vaidya's radiating metric and the general flat Robertson-Walker spacetime with a linear equation of state, where all possible models are obtained. These particular models are relevant on their own as they describe interesting physical situations; some of them had not been considered hitherto. [S0556-2821(96)01018-1]

PACS number(s): 04.20.Jb, 04.20.Cv, 04.40.−b, 98.10.+z

I. INTRODUCTION

The matching of two different spacetimes has been often used in order to describe some interesting physical situations, or to construct new models which can throw some light onto theoretical aspects which could not be treated with the original separated models. Thus, for instance, models for collapsing or expanding stars, for the description of local inhomogeneities in a cosmological context, for the collision of gravitational waves, etc., can be studied by means of the junction of two different known and given spacetimes. The purpose of this paper is twofold. We want to show some very general results for the *general* matching of any two spherically symmetric spacetimes through a timelike hypersurface. Furthermore, we shall illustrate these results with the general matching of Vaidya's radiating metric with the Robertson-Walker spacetimes. The resulting particular models will be of great interest on their own, and they will further describe some interesting old and new physical situations.

But first of all, let us make some standard considerations about the matching problem, both from the *practical* and the *theoretical* point of view. Let $\mathcal{V}^+$ and $\mathcal{V}^-$ be two C^3 orientable spacetimes carrying C^2 metrics g^+ and g^-, and each of them with boundary Σ^+ and Σ^-, respectively (see [1] for standard definitions). These boundaries are of course hypersurfaces in $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively. In general relativity, the junction conditions for two such spacetimes with boundary have been extensively considered and studied, even for the general case of matching hypersurfaces with nonconstant signature (see [2–4] and references therein). When Σ^+ and Σ^- are diffeomorphic, these junction conditions have been clearly established. An extensive and full summary of these junction conditions is given in Sec. II.

However, the *practical* problems have their own difficulties and subtleties, which are usually overlooked in the general theoretical approach. Thus, in general, we are given two orientable *full* spacetimes $\mathcal{V}$ and $\bar{\mathcal{V}}$ (instead of two spacetimes with a boundary), and we want to know, first, whether or not they are *matchable*, and second, which are the possible matching hypersurfaces. (Think, for instance, of the following questions: Are Robertson-Walker and Vaidya geometries matchable? Where?) Notice that the matching hypersurfaces must be found along with the resolution of the matching problem itself, which is one of the main differences with the theoretical ''view.'' Furthermore, the correct choice between the two possible directions (relative signs) of the vectors normal to the hypersurfaces is crucial in the practical problems such as those we treat in this paper (this has been recently pointed out also in [5]). We perform a detailed study of these choices and their consequences in Sec. II, and we find some necessary conditions which must be valid in a general matching. One of the main aims of this paper is to study, in some depth, the possibilities that arise in this sense for a general spherically symmetric matching. We devote Sec. III to the application of the above results to the spherically symmetric case, where the considerations made in Sec. II will be immediately and easily applicable. Thus, we obtain a whole set of necessary (but nevertheless, easily verifiable) conditions for this case. These simple conditions will allow us to ascertain, by mere inspection of the conformal diagrams, which matchings are feasible and which are not. We will be able to sketch the conformal diagrams for the whole matched spacetimes at this level and *even before* we write down the matching equations.

We shall then solve the particular but interesting problem of the junction of a Vaidya spacetime with a Robertson-Walker geometry. This subject has been previously treated in the literature. For example, in the pioneering work by Oppenheimer and Snyder [6], the matching of a closed collapsing dust Robertson-Walker model with a Schwarzschild vacuum exterior was solved. Later, the complementary problem (interchanging interior with exterior) was considered by Einstein and Straus [7] in another historically relevant paper in which they studied the influence of the universal expan-

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sion on the gravitational fields around the stars. The generalization from Schwarzschild to Vaidya's metric started with the first attempts to describe the so-called primordial black holes in [8,9]. The general treatement of this problem can be found in [10]. Lake and Hellaby [11] used this same type of matching to study the formation of naked singularities in the collapse of radiating stars, as possible counterexamples of the cosmic censorship conjecture. The necessary and sufficient conditions for the completely general matching of the Vaidya and Robertson-Walker spacetimes were presented in [12]. In this work, the physical interpretation of the results was also given, settling down some apparent problems which had been raised concerning the impossibility of gluing a radiative spacetime such as Vaidya with a Robertson-Walker model (the final comments of Sec. VI are devoted to this subject). Moreover, some interesting examples were fully integrated in [12]. In the present paper, Secs. IV and V are devoted to the treatment and resolution of this problem, which is exhausted for a flat Robertson-Walker model with a linear equation of state $p=\gamma\rho$: all possible and qualitatively different matchings with their corresponding conformal diagrams are explicitly given. Finally, in Sec. VI we analyze and discuss the results, and give some possible physical interpretations for the explicitly found models.

II. GENERAL MATCHING CONDITIONS

The purpose of this section is to present briefly the main results concerning the general junction conditions. For further details, the reader is referred to [2–4]. To that end, and as sketched in the Introduction, let us consider two C^3 orientable spacetimes $\mathcal{V}^{\pm}$, each of them with *timelike* boundary $\Sigma^{\pm}$ and C^2 metric $g^{\pm}$. In order to study their possible matching by means of identification of points on the boundaries, we will assume that there is a C^3 diffeomorphism from Σ^- to Σ^+. Then, we define $\mathcal{V}_4$, the whole spacetime, as the disjoint union of $\mathcal{V}^+$ and $\mathcal{V}^-$ with diffeomorphically related points of Σ^+ and Σ^- identified. Henceforth, the identified images of $\Sigma^{\pm}$ in $\mathcal{V}_4$ will be denoted simply by Σ. The question arises whether or not we can dovetail $\mathcal{V}^{\pm}$ to form $\mathcal{V}_4$ in such a way that there is a Lorentzian geometry in $\mathcal{V}_4$ with well-defined Einstein's equations and with the original metrics $g^{\pm}$ in the images of $\mathcal{V}^{\pm}$ in $\mathcal{V}_4$. As shown in [3], it turns out that this is possible if and only if Σ^+ and Σ^- are isometrical with respect to their first fundamental forms $\overline{g^+}$ and $\overline{g^-}$ inherited from $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively, because in this case there is a natural *continuous* extension g of the metric to the whole $\mathcal{V}_4$. Here "natural" means that g coincides with $g^{\pm}$ in the images of $\mathcal{V}^{\pm}$ in $\mathcal{V}_4$, respectively.

In a practical problem, we are given two embeddings $x^{\mu}_{\pm}=x^{\mu}_{\pm}(\xi^a)$ of Σ, where $x^{\mu}_{\pm}$ are local coordinates for $\mathcal{V}^{\pm}$, respectively, and ξ^a are intrinsic coordinates for Σ ($\mu,\nu,\ldots=0,1,2,3;\quad a,b,\ldots=1,2,3$). The mentioned condition that the first fundamental forms inherited from both embeddings and computed in the coordinate system $\{\xi^a\}$ must agree, then reads

$$\overline{g^+_{ab}}=\overline{g^-_{ab}}, \tag{1}$$

where [2–4]

$$\overline{g^{\pm}_{ab}}\equiv g^{\pm}_{\mu\nu}(x_{\pm}(\xi))\frac{\partial x^{\mu}_{\pm}(\xi)}{\partial\xi^a}\frac{\partial x^{\nu}_{\pm}(\xi)}{\partial\xi^b}.$$

However, as pointed out in [3,4], one should specify how the tangent spaces are to be identified in order to give a global well-defined geometry. To that end we consider

$$\vec{e}^{\,+}_a\equiv\frac{\partial x^{\mu}_{+}(\xi)}{\partial\xi^a}\frac{\partial}{\partial x^{\mu}_{+}}\quad\text{and}\quad\vec{e}^{\,-}_a\equiv\frac{\partial x^{\mu}_{-}(\xi)}{\partial\xi^a}\frac{\partial}{\partial x^{\mu}_{-}}.$$

These are two different sets of three vector fields tangent to Σ (because they are obviously tangent to the images of $\Sigma^{\pm}$ in $\mathcal{V}_4$; remember that these images have been identified). Condition (1) implies that the scalar products in $\mathcal{V}_4$ of $\{\vec{e}^{\,\pm}_a\}$ coincide. Therefore, we should identify these two sets of vectors. However, we need to identify the whole four-dimensional $\pm$-tangent spaces at Σ, and not merely the part tangent to Σ. To do that, let us now consider the spacelike unit vectors $\vec{n}^{\pm}$ normal to $\Sigma^{\pm}$ on $\Sigma^{\pm}$, defined *up to sign* by the conditions

$$n^+_{\mu}e^{+\mu}_a=0,\quad n^+_{\mu}n^{+\mu}=1,$$

and similarly for $\vec{n}^-$. We *must* choose these two normal vectors in such a way that if $\vec{n}^-$ points from $\mathcal{V}^-$ outwards, then $\vec{n}^+$ points inwards over $\mathcal{V}^+$, and vice versa. It is important to remark now that this is not an arbitrary choice since any curve with affine parameter s which crosses Σ at a point p must have a uniquely well-defined tangent vector $d/ds|_p$. Thus, if $d/ds|^+_p$ is the tangent vector of the curve at p as seen from $\mathcal{V}^+$ and is given by

$$\left.\frac{d}{ds}\right|^+_p=A\vec{n}^+|_p+B^a\vec{e}^{\,+}_a|_p,$$

then $\vec{n}^-$ must be chosen in such a way that the tangent vector $d/ds|^-_p$ of the curve at p as seen from $\mathcal{V}^-$ is

$$\left.\frac{d}{ds}\right|^-_p=A\vec{n}^-|_p+B^a\vec{e}^{\,-}_a|_p.$$

This choice fixes two pairs of reasonable signs for the normal vectors and allows for the complete identification of the two bases $\{\vec{n}^+,\vec{e}^{\,+}_a\}$ and $\{\vec{n}^-,\vec{e}^{\,-}_a\}$ for the tangent space of $\mathcal{V}_4$ at Σ. In particular, this identification determines an orientation and/or a time arrow for $\mathcal{V}_4$ whenever such concepts are defined (or fixed by any means) in any of the $\mathcal{V}^+$ or $\mathcal{V}^-$.

Once this identification has been done, we can drop the $\pm$ and write $\{\vec{n},\vec{e}_a\}$ for the basis of the tangent spaces at Σ. In the resulting whole spacetime $\mathcal{V}_4$, as we wished, there exists a unique C^1 atlas $\mathcal{A}$ (compatible with the original C^3 structures in $\mathcal{V}^{\pm}$) and a continuous extension g for the metric of $\mathcal{V}_4$ which coincides with $g^{\pm}$ at the corresponding regions $\mathcal{V}^{\pm}$. Moreover, according to the previous identification of tangent spacetimes, the components n^{μ} and e^{μ}_a in the local charts of $\mathcal{A}$ are well defined and $\mathcal{V}_4$ is globally orientable.

When Eq. (1) holds and given that the extension g of the metric is continuous in $\mathcal{V}_4$, the Einstein equations are well defined in the distributional sense, see [3,4]. Relation (1) is necessary if we want to compute the Riemann tensor distribution and its contractions [4]. In general, these distributions will have two separated parts, one of them called the singular part of the considered tensor distribution. This singular part is proportional to the Dirac one-form distribution δ_μ associated with Σ [3,4], so that it describes an infinite jump at Σ. Of course, in general, we want to avoid these infinities in the curvature and matter tensors because the only physically meaningful discontinuities across a timelike matching hypersurface must be finite. Thus, we need to get rid of the singular part of the Riemann tensor distribution. An easy calculation leads to the conclusion that, for a general timelike matching hypersurface Σ, the vanishing of the singular part for the Riemann tensor distribution is equivalent to the vanishing of the singular part of the Einstein tensor distribution [3,4], and this happens if and only if the *second fundamental forms* of Σ calculated from $\mathcal{V}^+$ and $\mathcal{V}^-$ are identical. The components $\mathcal{K}^{\pm}_{ab}$ of the second fundamental forms of Σ in the natural cobasis $\{d\xi^a\}$ are defined by [3,4]

$$\mathcal{K}^{\pm}_{ab} \equiv -n_\mu e^\nu_a \nabla^{\pm}_\nu e^\mu_b ,$$

where the $\pm$ appears now because, even though the preliminary junction conditions hold, the connection coefficients need *not* be continuous across Σ. In fact, in the practical problems and given that we may only know the $\pm$-local charts $\{x^\mu_\pm\}$ for $\mathcal{V}^\pm$, the formulas we must use to compute $\mathcal{K}^{\pm}_{ab}$ are

$$\mathcal{K}^{\pm}_{ab} \equiv -n^{\pm}_\mu \left(\frac{\partial^2 x^\mu_\pm(\xi)}{\partial \xi^a \partial \xi^b} + \Gamma^{\pm\mu}_{\rho\nu} \frac{\partial x^\rho_\pm(\xi)}{\partial \xi^a} \frac{\partial x^\nu_\pm(\xi)}{\partial \xi^b} \right),$$

from where it is obvious that $\mathcal{K}^{\pm}_{ab} = \mathcal{K}^{\pm}_{ba}$ Therefore, the aforementioned condition for the vanishing of the singular part for the Riemann tensor distribution simply reads

$$\mathcal{K}^{-}_{ab} = \mathcal{K}^{+}_{ab} . \tag{2}$$

Because of the above reasons, two spacetimes $\mathcal{V}^\pm$ are said to be matchable across their common boundary Σ if the *junction conditions* (1) and (2) hold.

As outlined in the Introduction, in practice we will usually have two full space times $\mathcal{V}$ and $\bar{\mathcal{V}}$ and we will want to know if they are matchable. Let us choose a generic timelike hypersurface σ splitting $\mathcal{V}$ into two complementary parts which we call 1 and 2. In the same way, we can define $\bar{\sigma}$, $\bar{1}$, and $\bar{2}$ for $\bar{\mathcal{V}}$. The matching of $\mathcal{V}$ and $\bar{\mathcal{V}}$ can be done in *four* different manners: 1 with $\bar{2}$, 1 with $\bar{1}$, 2 with $\bar{1}$, and 2 with $\bar{2}$ (see [5]). For every different matching we shall denote the overbarred region by $\mathcal{V}^-$ (and then the region without bar is $\mathcal{V}^+$). The whole spacetime $\mathcal{V}_4$ will thus be formed by the disjoint union of these two $\pm$ parts, and Σ will be the image of both σ and $\bar{\sigma}$ in $\mathcal{V}_4$. According to the above criterion for the normal vector of the matching hypersurface, it is easy to see that if Σ matches a part of $\mathcal{V}$ with a part of $\bar{\mathcal{V}}$, say 1 and $\bar{2}$, then this same hypersurface matches 2 with $\bar{1}$ automatically. This is why we call 1-$\bar{2}$ and 2-$\bar{1}$ *complementary matchings*. Therefore, there are only two properly inequivalent matchings, given by 1-$\bar{2}$ and 1-$\bar{1}$.

At this level, our task reduces to finding the matching hypersurfaces σ and $\bar{\sigma}$ defined by $x^\mu = x^\mu(\xi^a)$ and $\bar{x}^\mu = \bar{x}^\mu(\xi^a)$, respectively, which are solutions of the complete set of junction conditions. In general situations this may be rather difficult, but we will now obtain a *necessary condition* that will allow us to ascertain which possible matchings are valid and which are definitely excluded beforehand. To that end, suppose that one of the possible matchings has been performed. Then, it follows that for every point p on Σ there exists a local coordinate system of $\mathcal{V}_4$ in which the metric is C^1 [4,13,14]. These are called *admissible coordinates* [13]. Therefore, every quantity in $\mathcal{V}_4$ constructed with the metric, its first derivatives, and some C^1 tensor fields must be continuous across Σ. In particular, the expansions of the geodesic congruences must be continuous because, in admissible coordinates, the correct identification of tangent spaces of $\mathcal{V}$ and $\bar{\mathcal{V}}$ on Σ implies that the vectors tangent to geodesic curves are C^1 across Σ and, also, because the expansions depend only on the metric and its first derivatives.

As we shall see in the next section, this condition is particularly useful in the spherically symmetric case, where we can apply the above result to the invariantly defined null geodesic congruences. Given that the sign of the expansion remains invariant when we change the parametrization of any future-directed congruence, we can conclude that the sign of the expansion of a congruence of null geodesics moving on $\mathcal{V}_4$ will be continuous across Σ, independently of the parameters used to describe the congruence at both sides of Σ. Furthermore, the expansion is a scalar so that the continuity of its sign is independent of the coordinate systems we use to describe $\mathcal{V}$ and $\bar{\mathcal{V}}$ in a neighborhood of a point p on Σ.

III. THE SPHERICALLY SYMMETRIC CASE

A spacetime is said to be *spherically symmetric* [1,15] if it admits the group of rotations SO(3) as a group of isometries, with the group orbits spacelike two-surfaces. Then, these orbits are necessarily two-surfaces of constant positive curvature, usually called *two-spheres*, and also there exist two-surfaces orthogonal to the orbits [15]. We can choose two angular coordinates $\{x^2, x^3\} \equiv \{\theta, \phi\}$, with ranges $0 \leqslant \theta < \pi$, $0 \leqslant \phi < 2\pi$, describing the orbits, and two other coordinates $\{x^0, x^1\}$ describing the orthogonal surfaces. Each two-sphere is thus marked by constant values of the $\{x^A\} \equiv \{x^0, x^1\}$ coordinates $(A, B, \ldots = 0,1)$. We can also define a positive function $R(x^A)$ in such a way that $4\pi R^2$ is the total area of the corresponding two-sphere. Thus, the line element of a general spherically symmetric spacetime reads

$$ds^2 = g_{BC}(x^A)\, dx^B dx^C + R^2 (d\theta^2 + \sin^2\theta\, d\phi^2),$$

where the two-metric g_{BC} has Lorentzian signature.

For all these spacetimes, there are two preferred congruences of null geodesics defined as those invariant by the group of isometries (and also as the two principal null directions of the type-D Weyl tensor, see [15]). A straightforward calculation for the product of the expansions κ_1 and κ_2 of these two congruences (when affinely parametrized) gives

$$\kappa_1\kappa_2 = -\frac{\chi}{2R^2}, \quad \chi \equiv g^{\mu\nu}\partial_\mu R \partial_\nu R.$$

It is well known that these expansions coincide (up to a factor 1/2) with the traces of the two null second fundamental forms of the two-spheres (see [1] for definitions). Therefore, the two expansions have the same sign where $\chi<0$, which is the region of closed trapped surfaces [1], while at regions with $\chi>0$, the two expansions have different sign. The hypersurface defined by $\chi=0$, where at least one of the expansions is zero, is called the *apparent horizon* (AH) [1]. It is easy to see that

$$m \equiv \frac{R}{2}(1-\chi) \tag{3}$$

is the usual mass function introduced by Hernández and Misner [16–18].

Let us then consider the question of the general matching of two spherically symmetric spacetimes $\mathcal{V}$ and $\overline{\mathcal{V}}$. We assume that both spacetimes are time oriented and we want to know which, among the four different possible matchings, are permitted in principle. Applying the previous general results about the necessary continuity across Σ of the sign of the expansions for the two preferred null geodesic congruences, we have that, for every point p in Σ, the signs of $\chi|_p$ and $\overline{\chi}|_p$ must be the same. Furthermore, in the case $\chi|_p>0$ both expansions have opposite signs in $\mathcal{V}$, and the same for $\overline{\mathcal{V}}$. Obviously, if $\mathrm{sgn}(\chi)=\mathrm{sgn}(\overline{\chi})=+1$ in some region of Σ, *only* one of the possible inequivalent matchings (and, of course, its complementary) is allowed with regard to this region. This matching should be given by gluing the part of $\mathcal{V}$ into which the future-directed expanding congruence enters, with the part of $\overline{\mathcal{V}}$ towards which the future-directed contracting congruence points in. The other inequivalent matching is impossible in this case. The case with $\mathrm{sgn}(\chi)=\mathrm{sgn}(\overline{\chi})=-1$ in some region of Σ is a little bit different. Now, both expansions κ_1 and κ_2 have the same sign and the same happens for both $\overline{\kappa}_1$ and $\overline{\kappa}_2$. All of them can be assumed to be positive by appropriate choice of the time orientations. Then, all four matchings are allowed at this level in such regions. Similar reasonings apply to the case in which $\chi=\overline{\chi}=0$ in some region of Σ. In this case, one of the expansions vanishes there. The other may be either zero or different from zero. In the first case the situation is of the same kind that the one with $\mathrm{sgn}(\chi)=\mathrm{sgn}(\overline{\chi})=-1$, while the second case is similar to that with $\mathrm{sgn}(\chi)=\mathrm{sgn}(\overline{\chi})=+1$. Summarizing, we have proven the following.

Theorem 1. *If there is at least one point p in Σ where $\chi|_p>0$ (or where $\chi|_p=0$ but with one of the expansions nonzero), then only one matching (and its complementary) is possible in principle.*

Of course, even this *a priori* possible matching may turn out to be impossible by several reasons *a posteriori*. First, because the possible matching around one of the regions with $\chi\geqslant 0$ induces a global identification of the $\pm$-tangent spaces which may be inadequate for the discussed continuity of the sign of the expansions in other regions. And second, because one still has to check whether or not the full set of junction conditions holds.

IV. A RELEVANT APPLICATION: MATCHING OF THE VAIDYA AND ROBERTSON-WALKER SPACETIMES

In this section we apply the above results to the matching of Vaidya's radiating metric with the Robertson-Walker (RW) spacetimes. We proceed as follows: First we recall some properties of the Vaidya and RW spacetimes and we draw their respective Penrose conformal diagrams; then we make use of theorem 1 in order to pick out the possible candidates for matching hypersurfaces in both spacetimes. We will then describe the Penrose diagrams that can be obtained in this way for the complete manifold $\mathcal{V}_4$, *before* the resolution of the matching conditions is carried out in the next section.

A. Vaidya's solution

Locally, Vaidya's metric [19] can be described in radiative coordinates [20] as

$$ds_V^2 = -\overline{\chi}du^2 + 2\varepsilon\, du\, dR + R^2(d\overline{\theta}^2 + \sin^2\overline{\theta}\, d\overline{\phi}^2), \tag{4}$$

where the mass function (3) depends only on u [so that $\overline{\chi}=1-2\overline{m}(u)/R$] and $\varepsilon^2=1$. The range of the coordinates is

$$-\infty<u<\infty, \quad 0\leqslant R<\infty, \quad 0\leqslant\theta<\pi, \quad 0\leqslant\phi<2\pi.$$

The energy-momentum tensor for the metric (4) is of pure radiation type

$$\overline{T}_{\mu\nu} = \frac{2\varepsilon}{R^2}\frac{d\overline{m}(u)}{du} l_\mu l_\nu, \quad l_\mu dx^\mu = -du, \quad l_\mu l^\mu = 0\,. \tag{5}$$

This energy-momentum tensor has only finite jumps whenever $\overline{m}(u)$ is a continuous function. Moreover, the weak energy conditions demand that $\varepsilon\, d\overline{m}(u)/du \geqslant 0$. We shall assume from now on that u grows towards the future. Then, $\overline{m}$ must be a nonincreasing (nondecreasing) function of u for $\varepsilon=-1$ $(\varepsilon=1)$ and the incoherent radially directed radiation described by Eq. (5) is outgoing (ingoing). Here, outgoing means going towards bigger values of R, and similarly for ingoing. The apparent horizon (AH) for the Vaidya metric (4) is located at the hypersurface $R-2\overline{m}(u)=0$. Taking into account the weak energy conditions it is easily seen that the AH is a spacelike hypersurface in general, but it becomes null for values of u such that $d\overline{m}(u)/du=0$.

Let us consider a general timelike hypersurface $\overline{\sigma}$ preserving the spherical symmetry of the spacetime and with intrinsic coordinates $\{\xi^a\}=\{\tau,\vartheta,\varphi\}$, where τ is a future-directed time coordinate defined only on the hypersurface. The general parametric equations of $\overline{\sigma}$ are: $u=u(\tau)$, $R=R(\tau)$, $\overline{\theta}=\vartheta$, $\overline{\phi}=\varphi$. This hypersurface is timelike if and only if $\overline{\chi}\dot{u}-2\varepsilon\dot{R}>0$, where overdots stand for derivatives with respect to τ and we have taken into account that $\dot{u}>0$. The unit normal vector of σ is

$$\vec{\overline{n}} = \frac{\overline{\epsilon}}{\sqrt{\dot{u}(\overline{\chi}\dot{u}-2\varepsilon\dot{R})}}\left[\varepsilon\dot{u}\frac{\partial}{\partial u} + (\overline{\chi}\dot{u}-\varepsilon\dot{R})\frac{\partial}{\partial R}\right], \quad \overline{\epsilon}^2=1.$$

The future-directed vector fields tangent to the two invariant null geodesic congruences are given by $\vec{l}$ of Eq. (5) and

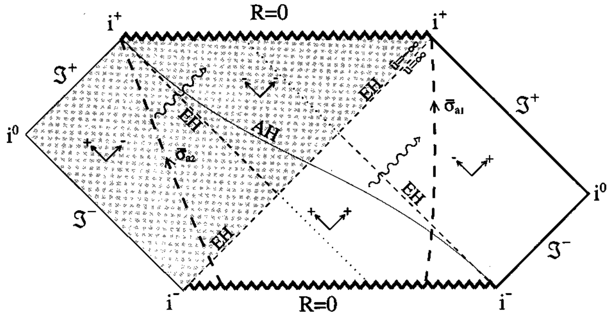

FIG. 1. Penrose diagram for the extended Vaidya spacetime when $\overline{m(u)}$ is a positive function and the waved arrows represent the flow of radiation. For every region we show the light cones and their respective signs of expansions. The dashed lines marked with $\overline{\sigma_{a1}}$ and $\overline{\sigma_{a2}}$ represent two candidates for matching hypersurfaces.

$$\vec{k}=\frac{\partial}{\partial u}+\varepsilon\frac{\overline{\chi}}{2}\frac{\partial}{\partial R},$$

where we have set $l_\mu k^\mu=-1$. A trivial calculation gives

$$\mathrm{sgn}(k_\mu \overline{n^\mu})=-\ \mathrm{sgn}(l_\mu \overline{n^\mu})=\varepsilon\overline{\epsilon}.$$

These relations will be very useful in order to use the criteria defined in Sec. II. Thus, if we want that both $\vec{\overline{n}}$ and $\vec{l}$ point towards the same region, we will have to choose $\overline{\epsilon}\varepsilon=-1$, and analogously for the other cases.

As is well known, Vaidya's metric is extendible in general [21–23]. We can consider two different possibilities. First, Penrose conformal diagram for the line element (4) when $\varepsilon=-1$ and $\overline{m(u)}$ is a nonincreasing and positive function of u is shown in the nonshadowed region of Fig. 1. The ingoing radial null geodesics (coming from the past null infinity $\mathcal{J}^-$ or the spacelike singularity $R=0$) are incomplete since they reach the event horizon $R=2\overline{m(u\to\infty)}$ for a finite value of their affine parameters. The shadowed region of Fig. 1 extends maximally the original Vaidya's metric. This shadowed region is by itself the Penrose conformal diagram for the line element (4) when $\varepsilon=+1$ and with a mass $\widehat{\overline{m}}(\hat{u})$ which is a nondecreasing function of the new null time $\hat{u}$ [23]. With such a mass function the energy conditions are satisfied in the new shadowed region as well. The mass function must be continuous at the event horizon in order to have, at worst, finite jumps of the curvature tensor there.[1] Thus, we have $\overline{m(u\to\infty)}=\widehat{\overline{m}}(\hat{u}\to-\infty)$.

The second possibility is shown in Fig. 2, where we give the Penrose conformal diagram for the line element (4) with $\varepsilon=-1$ and for a mass function which vanishes from some instant $u=0$ on. We could have extended $\overline{m(u)}$ beyond $u=0$ with negative values since the only requirement is that $\overline{m(u)}$ must be a C^0 function. Despite this, we have chosen a continuation with $\overline{m(u>0)}=0$. It is easy to see that the spacetime thus obtained is inextendible. We have to emphasize the fact that, in this case, the differential equations for the ingoing radial null geodesics have a critical point at $R=0,u=0$ because the derivatives of both R and u with respect to the affine parameter vanish there. If

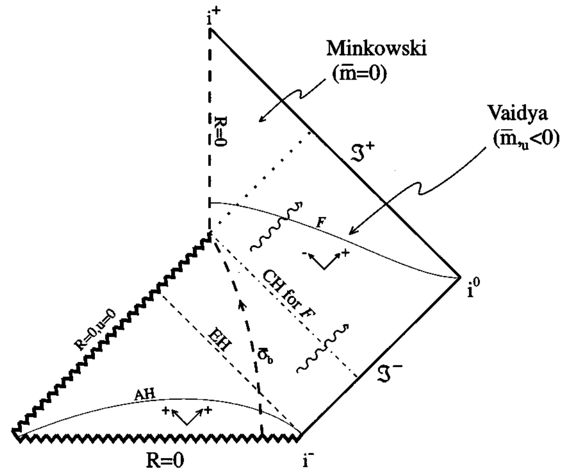

FIG. 2. The Penrose diagram for Vaidya's spacetime when $\overline{m(u)}$ vanishes from some instant $u=0$ on. A generic spacelike hypersurface F has the past Cauchy horizon in the null hypersurface marked as ''CH for F.''

$$\lim_{u\to 0^-}\frac{\overline{m(u)}}{u}\leqslant\frac{1}{16}, \tag{6}$$

then $u=R=0$ has a node (or col-node) structure [24]. The case sketched in Fig. 2 is precisely the linear one: $\overline{m(u)}=\alpha u$ where $0<\alpha=\mathrm{const}\ <1/16$. In particular, the time reversal of Fig. 2 (shown in Fig. 3) has a locally naked singularity in $u=R=0$ (see [25,26]). Obviously, we can interpret Fig. 3 as a Vaidya's spacetime ($\varepsilon=+1$) with a vanishing mass function that begins to grow after a certain $u=0$ and satisfying the time reversal of Eq. (6). In both Figs.

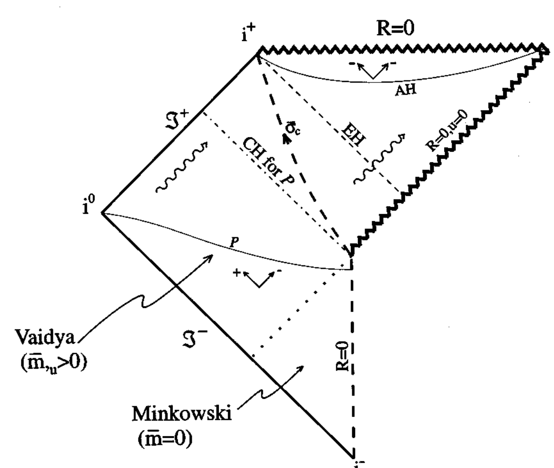

FIG. 3. The Penrose diagram for the time reversal of Fig. 2. A generic spacelike hypersurface P has the future Cauchy horizon at the null hypersurface marked as ''CH for P.''

[1]This condition can be easily verified by matching the two Vaidya's solutions through a null ($u=\mathrm{const}$) hypersurface (see [4,23]).

2 and 3, the region with $\overline{m}=0$ is, obviously, part of a Minkowski flat spacetime. Despite the fact that these diagrams are qualitatively the same for other $\overline{m(u)}$, there are still cases with a different causal structure [depending on whether Eq. (6) is accomplished or not] [25]. These cases are not drawn here because it is easy to see, according to the necessary condition, that any candidate for matching hypersurface does not have a corresponding candidate on the Robertson-Walker spacetime.

B. The Robertson-Walker metrics

We can describe the Robertson-Walker line element in its isotropic form as

$$ds^2_{RW}=-dt^2+\frac{a^2(t)}{b^2(r)}[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)], \quad (7)$$

where $b(r)\equiv 1+kr^2/4$ and $k=0,\pm 1$. As is evident, the $t=$const hypersurfaces are spacelike while the $r=$const ones are timelike. The mass function (3) for the RW spacetime is

$$m(t,r)=\frac{ar^3(a,_t^2+k)}{2b^3}. \quad (8)$$

The energy-momentum tensor of Eq. (7) takes the form of a perfect fluid with respect to the unit velocity vector $\vec{v}=\partial/\partial t$:

$$T_{\mu\nu}=(\rho+p)v_\mu v_\nu+pg_{\mu\nu},$$

where $\rho(t)$ and $p(t)$ are the energy density and the isotropic pressure relative to $\vec{v}$, respectively. Friedmann's equations are given by

$$\rho=\frac{3(a,_t^2+k)}{a^2}, \quad \rho,_t+3(\rho+p)\frac{a,_t}{a}=0.$$

Combining the first of these equations with Eq. (8), we obtain

$$m(t,r)=\frac{\rho}{6}\left(\frac{ra}{b}\right)^3$$

from where we learn that the m will be positive if and only if so is ρ (notice that $\rho\geqslant 0$ for $k=0,1$, in general).

Let us take a general timelike hypersurface σ preserving the spherical symmetry of Eq. $\underline{(7)}$. This hypersurface is assumed to be diffeomorphic to $\overline{\sigma}$, and thus the intrinsic coordinates are chosen to be the same $\{\xi^a\}=\{\tau,\vartheta,\varphi\}$. The general parametric equations for σ are: $t=t(\tau)$, $r=r(\tau)$, $\theta=\vartheta$, $\phi=\varphi$. This hypersurface is timelike whenever

$$\frac{\dot{r}^2}{\dot{t}^2}=\left(\frac{dr}{dt}\right)^2<\frac{b^2}{a^2}. \quad (9)$$

The corresponding unit vector normal to σ is simply

$$\vec{n}=\frac{\epsilon}{a\sqrt{b^2\dot{t}^2-a^2\dot{r}^2}}\left(a^2\dot{r}\frac{\partial}{\partial t}+b^2\dot{t}\frac{\partial}{\partial r}\right),$$

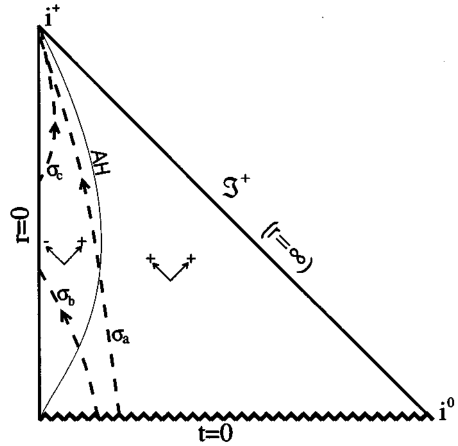

FIG. 4. Conformal diagram for a flat Robertson-Walker spacetime with equation of state $p=\gamma\rho$ when $-1/3<\gamma<1$. The dashed lines represent the valid candidates for matching hypersurfaces. The apparent horizon (AH) drawn corresponds to the case $-1/3<\gamma<1/3$; if $\gamma=1/3$ the AH would be a null hypersurface going from the lower left corner to the future null infinity $\mathcal{J}^+$; finally, if $1/3<\gamma<1$ the AH would be a spacelike hypersurface from the lower left corner to the spacelike infinity i^0.

where $\epsilon^2=1$. Now, the meaning of ϵ is very clear given that $\partial/\partial r$ is a spacelike vector field everywhere. Thus, for instance, if t is future directed (so that $\dot{t}>0$) and $\epsilon=1$ (-1), then $\vec{n}$ points towards greater (smaller) values of r, and so $\vec{n}$ points into the same direction as the vector field tangent to the radial future-directed outgoing (ingoing) null geodesic congruence. The other case ($\dot{t}<0$) is analogous.

Let us consider the interesting particular case of a flat RW metric ($k=0$) with a linear equation of state $p=\gamma\rho$, γ being a constant which we assume to lie in the range $-1<\gamma<1$ (so that the dominant energy conditions hold [1]). Friedmann's equations can be easily solved to give

$$a(t)=Ct^{2/3(1+\gamma)}, \quad (10)$$

where C is a positive integration constant and we have set $a(0)=0$. The apparent horizon $r=2m$ for these particular models is the hypersurface $ra,_t=1$, that is to say

$$\text{AH:}ra,_t=1\Leftrightarrow r-\frac{3\,(1+\gamma)}{2\,C}t^{(1+3\gamma)/3(1+\gamma)}=0. \quad (11)$$

It is easy to check with the help of Eq. (9) that this apparent horizon is timelike if $-1<\gamma<1/3$, lightlike if $\gamma=1/3$, and spacelike if $1/3<\gamma<1$. The typical Penrose diagram for these spacetimes when $-1/3<\gamma<1$ is sketched in Fig. 4, while $\gamma=-1/3$ and $-1<\gamma<-1/3$ are sketched in Figs. 5 and 6, respectively. We shall make use of these particular models later.

C. Making use of theorem 1

Let us focus on the cases $-1/3<\gamma<1/3$. In Figs. 1–4 we have drawn some lines corresponding to some possible matching hypersurfaces for the matching of RW and Vaidya spacetimes. Thus, for example, the lines σ_a, σ_b, and σ_c in

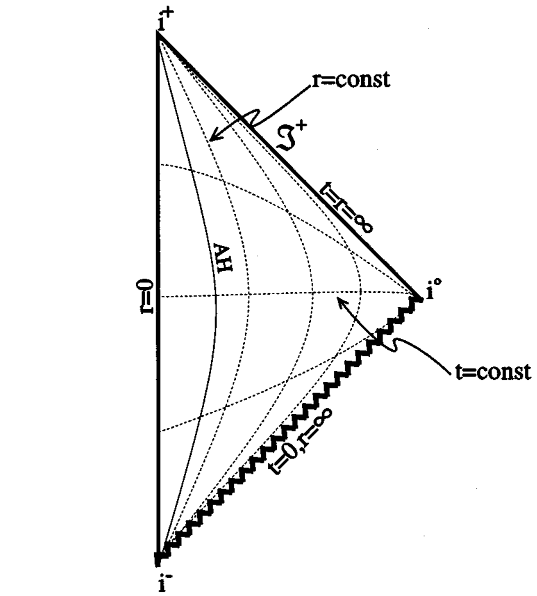

FIG. 5. Conformal diagram for the $\gamma=-1/3$ case. We do not draw the possible candidates for matching hypersurface because they have the same qualitative behavior as those in Fig. 4.

Fig. 4 represent three possible typical matching hypersurfaces σ. As we can see, σ_a starts at $t=0$ and moves first through a $\chi<0$ region up to a certain instant $t=t_{\rm AH}$ in which it crosses the apparent horizon. The expansions for the ingoing and outgoing null geodesic congruences are both positive up to $t=t_{\rm AH}$, and then the matching hypersurface enters a

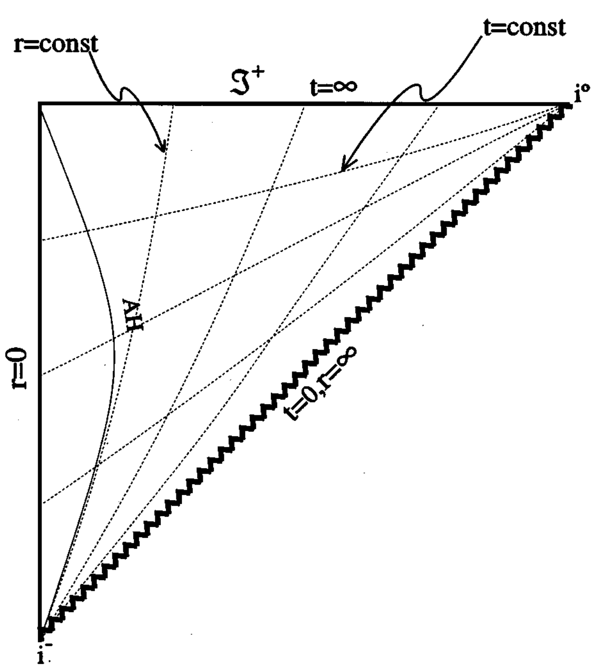

FIG. 6. Conformal diagram for the $-1<\gamma<-1/3$ case. We do not draw the candidates for matching hypersurface for the same reason as in Fig. 5.

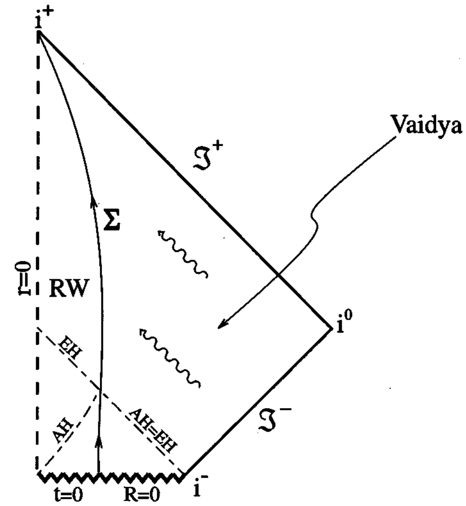

FIG. 7. The Penrose diagram for the matching of the Vaidya and Robertson-Walker spacetimes when σ_a is identified with $\overline{\sigma_{a2}}$.

$\chi>0$ region, where, of course, the expansions have different sign, and moves asymptotically towards i^+. The hypersurface σ_b is similar to σ_a, the only difference is that σ_b reaches $r=0$ at a finite t instead of moving towards i^+. Finally, the hypersurface σ_c is quite different from the previous ones. Now, σ_c starts at $r=0$ with $t>0$ and reaches i^+ asymptotically. The hypersurface σ_c always lies in a $\chi>0$ region.

The question is now to try and find the corresponding possible hypersurfaces in the Vaidya metrics, that is, in Figs. 1–3. By making use of theorem 1 and the considerations preceding it regarding the expansions κ_1 and κ_2, we can easily identify these candidates. For example, in Fig. 1 we have drawn two different and inequivalent possibilities which can be, in principle, good matching partners for σ_a. They are labeled as $\overline{\sigma_{a1}}$ and $\overline{\sigma_{a2}}$. Obviously, the reasoning presented in Sec. III implies also that the matching with the hypersurfaces σ_a and $\overline{\sigma_{a1}}$ must join the region to the left of σ_a in Fig. 4 with the region to the right of $\overline{\sigma_{a1}}$ in Fig. 1 (or the corresponding complementaries, giving the complementary matching). Similarly, the matching with the hypersurfaces σ_a and $\overline{\sigma_{a2}}$ has to be that gluing the part to the left of σ_a in Fig. 4 with the region to the left of $\overline{\sigma_{a1}}$ in Fig. 1 (and its complementary). In this case, the complete Penrose diagram, once the matching has been performed, is presented in Fig. 7 (and the corresponding complementary in Fig. 8). We shall see in the next section that the first possibility with σ_a and $\overline{\sigma_{a1}}$ is forbidden by the complete set of matching equations.

Analogously, the hypersurfaces $\overline{\sigma_b}$ in Fig. 2 and $\overline{\sigma_c}$ in Fig. 3 are good matching partners for σ_b and σ_c, respectively. In the first case we can join the part to the left of σ_b in Fig. 4 with the part to the right of $\overline{\sigma_b}$ in Fig. 2. The resulting conformal diagram is presented in Fig. 9 (and its complementary in Fig. 10). In the second case we can match the part to the left of σ_c in Fig. 4 with the part to the left of $\overline{\sigma_c}$ in Fig. 2. The whole conformal diagram is presented in Fig. 11 (and its complementary in Fig. 12). In the next section we justify all these conformal diagrams by solving com-

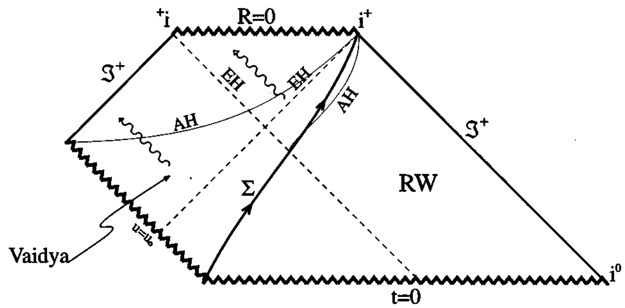

FIG. 8. Conformal diagram for the complementary matching of Fig. 7. Integration of the matching conditions shows that Σ starts from a partly spacelike and partly null singularity.

pletely the full set of matching conditions in some pertinent cases.

V. INTEGRATION OF THE MATCHING CONDITIONS

We are going to write down and solve the general matching equations for the Vaidya and RW spacetimes. As we shall presently see, these equations will depend explicitly on the product $\overline{\epsilon}\epsilon$. The full set of matching conditions is given by Eqs. (1) and (2), which in the case under consideration become, after a little computation [12,27],

$$R \overset{\Sigma}{=} a\frac{r}{b}, \tag{12}$$

$$\overline{m} \overset{\Sigma}{=} \frac{a r^3 (a_{,t}^2+k)}{2\, b^3}, \tag{13}$$

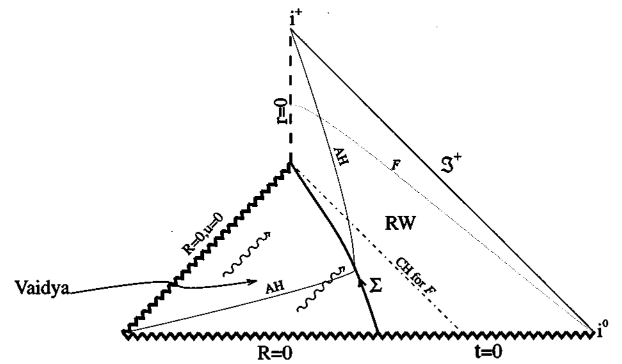

FIG. 10. Conformal diagram for the complementary matching of Fig. 9 .

$$\dot{u} \overset{\Sigma}{=} \varepsilon\frac{a\dot{r}+\varepsilon\,\overline{\epsilon}\epsilon b\dot{t}}{b^2(r/b)_{,r}-\varepsilon\,\overline{\epsilon}\epsilon r a_{,t}}, \tag{14}$$

$$\dot{r} \overset{\Sigma}{=} \epsilon\overline{\epsilon}\epsilon\frac{b}{a}\frac{p}{\rho}\dot{t}, \tag{15}$$

where $\overset{\Sigma}{=}$ means that both sides of the equality must be evaluated on Σ. Condition (9) on the matching hypersurface can be written now with the help of Eq. (15) as [12]

$$\frac{p^2}{\rho^2}<1,$$

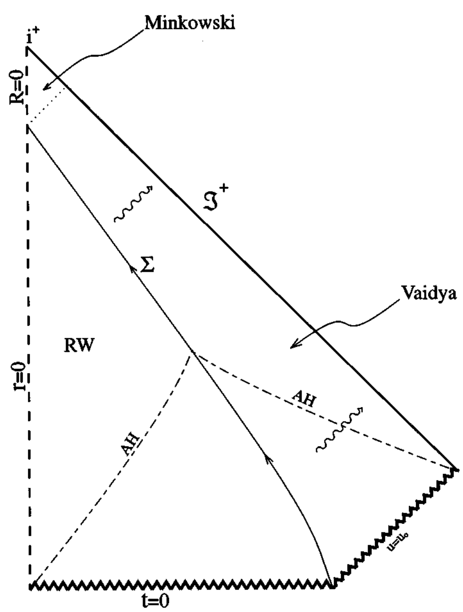

FIG. 9. Penrose diagram for the matching of the Vaidya and Robertson-Walker spacetimes when σ_b and $\overline{\sigma_b}$ are identified. As for the previous case Σ starts from a partly spacelike and partly null singularity.

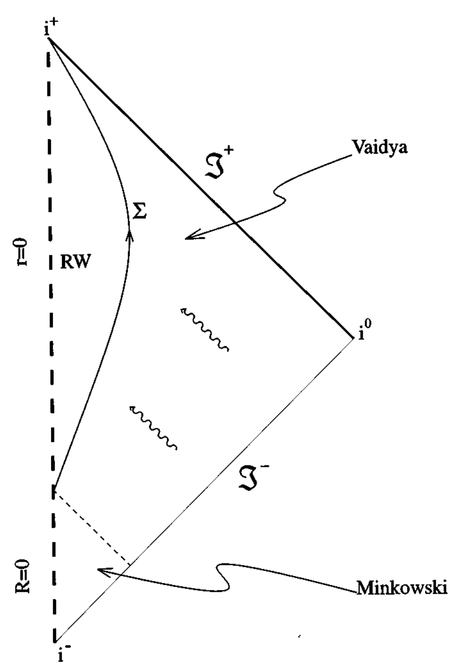

FIG. 11. The identification of σ_c and $\overline{\sigma_c}$ leads to this conformal diagram.

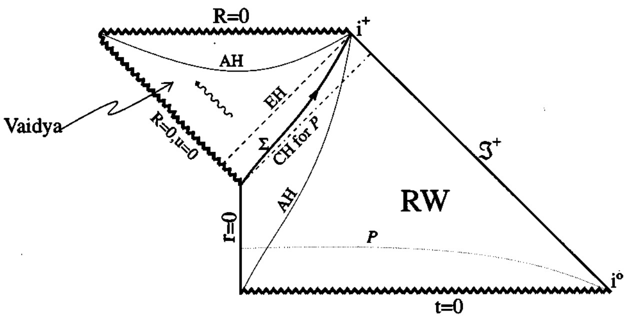

FIG. 12. Complementary Penrose diagram of Fig. 11.

so that this will hold whenever the dominant energy conditions are satisfied. In the case $p=\gamma\rho$, the above inequality is equivalent to simply $\gamma^2<1$.

Assuming that we know explicitly the RW models [i.e, $a(t)$], the general procedure to solve the matching conditions (12)–(15) is the following [12,27]. (i) Since $\dot{t}>0$ (for future-directed t), we can always choose $t(\tau)=\tau$. Then, Eq. (15) becomes a simple ordinary differential equation for the unknown $r(\tau)$, which will have solution if the right-hand side satisfies Lipschitz's conditions. The solution will depend on one arbitrary constant r_0, and thus we obtain a one-parameter family of possible hypersurfaces Σ. (ii) By substituting the solution for $r(\tau,r_0)$ into Eq. (14), we obtain another ordinary differential equation for the unknown $u(\tau)$. Solving this equation, we obtain $u(\tau,r_0,u_0)$, where u_0 is a new arbitrary constant. It should be noticed, however, that in this case u_0 is an additive constant. (iii) From Eq. (12) we immediately find $R(\tau,r_0)$, which together with $u(\tau,r_0,u_0)$ provides the form of the hypersurfaces Σ as seen from the Vaidya spacetime. (iv) Finally, we get $\overline{m}(\tau,r_0)$ from Eq. (13). This gives us, in combination with $u(\tau,r_0,u_0)$, the mass function $\overline{m}(u)$ for Vaidya's metric explicitly.

In the particular case $k=0$, $p=\gamma\rho$ of the previous section, and according to Eq. (10), Eqs. (14) and (15) read, respectively,

$$\dot{u}\stackrel{\Sigma}{=}\frac{\gamma+1}{\overline{\epsilon\epsilon}-\varepsilon a_{,t}r},\qquad \dot{r}\stackrel{\Sigma}{=}\varepsilon\overline{\epsilon\epsilon}\frac{\gamma}{a}=\varepsilon\overline{\epsilon\epsilon}\frac{\gamma}{C}\tau^{-2/3(1+\gamma)}. \tag{16}$$

The second of these equations is easily integrable, and the solutions are

$$r(\tau,r_0)=\varepsilon\overline{\epsilon\epsilon}\frac{3\gamma}{C}\left(\frac{1+\gamma}{1+3\gamma}\right)\tau^{1+3\gamma/3(1+\gamma)}+r_0 \tag{17}$$

for $\gamma\neq-1/3$, while for $\gamma=-1/3$ we have $r(\tau,r_0)=-(\epsilon\varepsilon\bar{\epsilon}/3C)\ln\tau+r_0$, where r_0 is an integration constant. These formulas define a one-parameter family of matching hypersurfaces in the RW models. We should now integrate the first of the Eqs. (16), but this is not necessary in order to know the main characteristics of the solution. Since the left-hand side of this relation is bigger than zero ($\dot{u}>0$) and also $\gamma+1>0$, the matching is only possible in those regions with

$$\overline{\epsilon\epsilon}-\varepsilon a_{,t}(\tau)r(\tau,r_0)>0. \tag{18}$$

From this inequality and the AH hypersurface given in Eq. (11), we learn that if $\varepsilon\overline{\epsilon\epsilon}=1$ we will not be able to describe the matching after crossing the apparent horizon (see cases A and C below). This is partly because of the fact that the coordinate u is only valid to describe a part of the spacetime (for example, the nonshadowed region of Fig. 1) because when $\overline{\epsilon\epsilon}-\varepsilon a_{,t}(\tau)r(\tau,r_0)\to 0$ then $u\to\pm\infty$ and $r(\tau,r_0)$ just reaches the AH.

On the other hand, Eq. (13) comes from the matching condition $\overline{m}(u)\stackrel{\Sigma}{=}m(t,r)$. The derivative on Σ of this equation leads us to

$$\overline{m}_{,u}\dot{u}\stackrel{\Sigma}{=}m_{,t}+m_{,r}\dot{r}.$$

Taking into account Eq. (14) and the above equation we can see that, in order to have $\varepsilon\, d\overline{m}(u)/du\geqslant 0$, a necessary condition is that $\gamma\geqslant 0$. Thus, by making Eq. (18) explicit we finally get

$$\gamma\geqslant 0,\quad \overline{\epsilon\epsilon}-\varepsilon\frac{2Cr_0(1+3\gamma)}{3(1+\gamma)^2}\tau^{-(1+3\gamma)/3(1+\gamma)}>0. \tag{19}$$

From the last relation we immediately see that the case with $\varepsilon=1$ and $\overline{\epsilon\epsilon}=-1$ is not possible. The other cases are:

Case A: $\varepsilon=+1$ and then, from Eq. (19), $\overline{\epsilon\epsilon}=+1$. Examination of Eq. (19) tells us that two different subcases may still appear, depending on whether $r_0\geqslant 0$ or not. When $r_0\geqslant 0$, we can define a lower bound for τ given by

$$\tau_H=\left(\frac{2Cr_0(1+3\gamma)}{3(1+\gamma)^2}\right)^{3(1+\gamma)/(1+3\gamma)} \tag{20}$$

which is the limit value of τ when Σ approaches the apparent horizon in the RW metric. The ranges of τ, $r(\tau,r_0)$ and $u(\tau,r_0,u_0)$ are, respectively

$$\tau_H<\tau<+\infty,$$
$$r(\tau_H,r_0)<r(\tau,r_0)<+\infty,$$
$$-\infty<u(\tau,r_0,u_0)<+\infty. \tag{21}$$

This behavior for $u(\tau,r_0,u_0)$ follows from the expression for $\dot{u}>0$ in Eq. (16), which implies that $u(\tau_H,r_0,u_0)\to-\infty$.

On the other hand, if $r_0<0$, there is another lower bound τ_m for τ, defined by $r(\tau_m,r_0)=0$. Because of Eq. (17), we have

$$\tau_m=\left(\frac{(-r_0)C(1+3\gamma)}{3\gamma(1+\gamma)}\right)^{3(1+\gamma)/(1+3\gamma)}.$$

The mass function vanishes on Σ at τ_m, and thus we must have Minkowski's spacetime previously. The corresponding ranges for τ, $r(\tau,r_0)$ and $u(\tau,r_0,u_0)$ are, respectively,

$$\tau_m\leqslant\tau<\infty,\quad 0\leqslant r(\tau,r_0)<\infty,\quad u_m<u(\tau,r_0,u_0)<+\infty$$

for some finite value u_m.

Case B: $\varepsilon=-1$ with $\overline{\epsilon\epsilon}=+1$. From Eq. (17) it follows that $r_0\geqslant 0$. Then, Eq. (19) is always trivially satisfied. Now, we have an upper bound for τ given by τ_M such that

$r(\tau_M, r_0)=0$, from where we again get a vanishing mass function on Σ at τ_M. From Eq. (17) we have

$$\tau_M=\left(\frac{Cr_0(1+3\gamma)}{3\gamma(1+\gamma)}\right)^{3(1+\gamma)/(1+3\gamma)}$$

and the ranges for τ, $r(\tau,r_0)$, and $u(\tau,r_0,u_0)$ are

$$0\leqslant\tau\leqslant\tau_M,\quad r_0\geqslant r(\tau,r_0)\geqslant 0,\quad u_0\geqslant u(\tau,r_0,u_0)\geqslant u_M.$$

Notice that at $\tau=0(\overset{\Sigma}{\Leftrightarrow}u=u_0)$ the mass function becomes infinite on Σ, then we have $\overline{m}(u_0)=\infty$ and, therefore, Vaidya's hypersurface $u=u_0$ is singular. In this case B, Σ crosses the apparent horizon of the RW metric at the value

$$\tau_H'=\left(\frac{2Cr_0(1+3\gamma)}{3(1+\gamma)(1+5\gamma)}\right)^{3(1+\gamma)/(1+3\gamma)}$$

which is within the allowed range of τ.

Case C: $\varepsilon=-1$ with $\epsilon\overline{\epsilon}=-1$. From Eq. (19) we must have $r_0\geqslant 0$. Now, there appears an upper bound for τ given by the limit value of τ when Σ approaches the RW apparent horizon. This value is exactly τ_H given in Eq. (20). The corresponding ranges of τ, $r(\tau,r_0)$, and $u(\tau,r_0,u_0)$ are now

$$0\leqslant\tau<\tau_H,\quad r_0\leqslant r(\tau,r_0)<r(\tau_H,r_0),$$
$$u_0\leqslant u(\tau,r_0,u_0)<+\infty. \tag{22}$$

Let us interpret the solutions found. In both cases A and C, we have $\varepsilon\epsilon\overline{\epsilon}=+1$, and therefore the solution (17) has the same functional form. The only difference appears in the valid intervals for τ. In fact, the ranges (21) for case A when $r_0\geqslant 0$ and the ranges (22) for case C are obviously two parts of a unique solution since the valid intervals are perfectly complementary. The valid interval for case C describes the part of $\overline{\sigma}_{a2}$ in Fig. 1 which goes from $R=0$ up to the event horizon (EH), while the valid interval for case A describes the part of the same $\overline{\sigma}_{a2}$ in Fig. 1 which goes from the same point at EH up to i^+. Obviously, this same ''double'' description holds for the partner hypersurface σ_a in Fig. 4. The whole matched spacetime is sketched in Fig. 7 (and Fig. 8 for the complementary). For case B, we saw that Σ always crosses the AH hypersurface and also that r and the mass function go to zero for a finite value of τ. Thus, it describes appropriately the hypersurface σ_b shown in Fig. 4, and its partner $\overline{\sigma}_b$ in Fig. 2. The whole matched spacetime and its complementary are sketched in Figs. 9 and 10. The same type of reasoning shows that case A with $r_0<0$ describes σ_c in Fig. 4, and its partner $\overline{\sigma}_c$ in Fig. 3, appropriately. The whole matched spacetimes are shown in Figs. 11 and 12. Finally, it is obvious that the hypersurface $\overline{\sigma}_{a1}$ in Fig. 1 does not correspond to any solution of the matching equations.

VI. DISCUSSION

By making use of theorem 1 and related criteria on the continuity of the sign of the expansions for the invariant null geodesic congruences, we have been able to choose some possible matchings of the Vaidya and RW spacetimes *before* solving the full set of matching equations. The subsequent integration of these matching equations and the requirement of weak energy conditions have distinguished between those which are actually viable from those which are not. We may wonder, however, whether or not we have found all possible matchings for these spacetimes, since we have assumed that both u and t are future-directed throughout. In this sense, the change of time direction for t may supply some extra freedom that could produce new results. A closer examination of the conformal diagrams sketched in Figs. 1–6 proves that, in fact, the global spacetimes we can obtain in this manner are just the time reversals of those already obtained in Figs. 7–12. This is because of the *qualitatively* symmetric nature of future and past in Vaidya's spacetime of Fig. 1, and also to the fact that Fig. 3 is the time reversal of Fig. 2.

The diagrams found can certainly represent interesting physical situations. For example, the time reversal of Fig. 7 is a typical diagram for a radiating star collapsing to form a black hole. The matching hypersurface contracts continuously, crosses the event horizon entering into a region of closed trapped surfaces, and eventually reaches the spacelike singularity. This is a radiating generalization of the classical collapsing star model of Oppenheimer and Snyder [6], which has a $k=+1$ RW dust interior with Schwarzschild exterior. The collapsing time interval in [6] corresponds to the contracting phase of the closed RW spacetime. We present the time reversal of Fig. 7 as a model for a flat ($k=0$) ever-contracting RW star with an equation of state of the type $p=\gamma\rho$ for $\gamma\geqslant 0$ and radiation coming out of the star.

Regarding Fig. 8, Refs. [8–10] deal with radiative voids in expanding RW universes, which are called *primordial inhomogeneities* in those papers. The extension of this type of models through the EH, providing its maximal Penrose diagram, corresponds to the matching of Fig. 8. As far as we know, this full conformal diagram had not been given previously. A relevant property of this model is the presence of a null initial singularity in the past of the radiating void. Thus, the structure of the whole initial singularity is partly spacelike and partly null.

Let us consider now the Figs. 9 and 10. For instance, the Penrose diagram in Fig. 9 can be interpreted as a radiating star (or Universe) that starts in a spacelike singularity and evolves in a collapsing and radiating way until it loses *all* its mass, ending in a flat spacetime state [12]. On the other hand, the time reversal of its complementary matching shown in Fig. 10 describes a flat RW contracting universe in which a radiating Vaidya's void appears. This induces the creation of a locally naked singularity. To our knowledge this model (and its time reversal) is presented here for the first time.

Finally, the diagrams in Figs. 11 and 12 may have greater interest. The time reversal of Fig. 11 represents a *completely evaporating star*, that is to say, a collapsing and radiating star which radiates all its energy away. After the complete evaporation, the manifold becomes flat Minkowski spacetime. We have seen that the mass function in Vaidya's spacetime must be continuous in order to avoid infinite jumps on the energy-momentum tensor. Therefore, we do not need to continue the metric with negative masses once the mass becomes zero. This is not in accordance with a claim in [11]. In Fig. 12 we sketch the complementary matching of Fig. 11. It represents an expanding RW universe generating a radiating void, with the appearance of a locally naked singularity. This model, as well as some of the previous ones, violate the uniqueness of the Cauchy initial value problem for the RW solution, as was pointed out by Ellis in a note added in proof in paper [10]. This is natural and logical, however, because the Cauchy problem needs a differentiability for the metric functions which does not hold here at the matching hypersurface Σ. In [10] this type of models is called a *nonprimordial inhomogeneity*.

As a closing remark, we would like to emphasize here that, in all the above models, the matching hypersurface is not comoving with matter in the usual perfect-fluid interpretation of the RW spacetime. Thus, the matching hypersurface Σ moves with respect to the preferred observer defined by the perfect-fluid velocity vector $\vec{v}=\partial/\partial t$. This is why we can match a radiative metric (e.g., Vaidya) with one usually considered a nonradiative one (e.g., RW). Of course, the nonradiative character of RW metrics depends on the observer and, as a matter of fact, any observer *comoving with* Σ on Σ will certainly see radiation crossing this hypersurface. A correct and full interpretation of these results was given in [12] and, in general, the physical interpretation of any possible matching for spherically symmetric spacetimes was also given in [27].

ACKNOWLEDGMENTS

This work was partially supported by the Ministerio de Educación y Ciencia under Project No. PB93-1050 and the Direcció General de Recerca under Project No. 1995SGR00066.

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