Gravitational radiation from compact binary systems: Gravitational waveforms and energy loss to second post-Newtonian order

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We derive the gravitational waveform and gravitational-wave energy flux generated by a binary star system of compact objects (neutron stars or black holes), accurate through second post-Newtonian order ($O[(v/c)^4]=O[(Gm/rc^2)^2]$) beyond the lowest-order quadrupole approximation. We cast the Einstein equations into the form of a flat-spacetime wave equation together with a harmonic gauge condition, and solve it formally as a retarded integral over the past null cone of the chosen field point. The part of this integral that involves the matter sources and the near-zone gravitational field is evaluated in terms of multipole moments using standard techniques; the remainder of the retarded integral, extending over the radiation zone, is evaluated in a novel way. The result is a manifestly convergent and finite procedure for calculating gravitational radiation to arbitrary orders in a post-Newtonian expansion. Through second post-Newtonian order, the radiation is also shown to propagate toward the observer along true null rays of the asymptotically Schwarzschild spacetime, despite having been derived using flat-spacetime wave equations. The method cures defects that plagued previous ''brute-force'' slow-motion approaches to the generation of gravitational radiation, and yields results that agree perfectly with those recently obtained by a mixed post-Minkowskian post-Newtonian method. We display explicit formulas for the gravitational waveform and the energy flux for two-body systems, both in arbitrary orbits and in circular orbits. In an appendix, we extend the formalism to bodies with finite spatial extent, and derive the spin corrections to the waveform and energy loss. [S0556-2821(96)02820-2]

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I. INTRODUCTION

The generation of gravitational radiation is a longstanding problem that dates back to the first years following the publication of general relativity (GR). In 1916 Einstein calculated the gravitational radiation emitted by a laboratory-scale object using the linearized version of GR [1]. Some of his assumptions were questionable and his answer for the energy flux was off by a factor of 2 (an error pointed out by Eddington [2]). There followed a lengthy debate about whether gravitational waves are real or an artifact of general coordinate invariance, the former interpretation being confirmed by the rigorous, coordinate free theorems of Bondi and his school [3–5] and by the short-wave analysis of Isaacson [6]. Shortly after the discovery of the binary pulsar PSR 1913+16 in 1974, questions were raised about the foundations of the ''quadrupole formula'' for gravitational radiation damping [7] (and in some quarters, even about its quantitative validity [8]). These questions were answered in part by theoretical work designed to shore up the foundations of the quadrupole approximation [9–13], and in part (perhaps mostly) by the agreement between the predictions of the quadrupole formula and the *observed* rate of damping of the pulsar's orbit [14,15].

Because it is a slow-motion system ($v/c\sim10^{-3}$), the binary pulsar is sensitive only to the lowest-order effects of gravitational radiation as predicted by the quadrupole formula. Nevertheless, the first correction terms of order v/c and $(v/c)^2$ to the quadrupole formula were calculated as early as 1976 [16,17]. These are now conventionally called ''post-Newtonian'' (PN) corrections, with each power of v/c corresponding to one-half a post-Newtonian order (1/2PN), in analogy with post-Newtonian corrections to the Newtonian equations of motion [18]. In 1976, the post-Newtonian corrections were of purely academic, rather than observational interest.

Recently, however, the issue of higher post-Newtonian corrections in the theory of gravitational waves has taken on some urgency. The reason is the construction of kilometer-scale, laser interferometric gravitational-wave observatories in the U.S. [Laser Interferometric Gravitational Wave Observatory (LIGO) project] and Europe (VIRGO project), with gravitational-wave searches scheduled to commence around 2000 (see [19] for a review). These broadband antennae will have the capability of detecting and measuring the gravitational waveforms from astronomical sources in a frequency band between about 10 Hz (the seismic noise cutoff) and 500 Hz (the photon counting noise cutoff), with a maximum sensitivity to strain at around 100 Hz of $\Delta l/l\sim10^{-22}$ (rms). The most promising source for detection and study of the gravitational-wave signal is the ''inspiralling compact binary''—a binary system of neutron stars or black holes (or one of each) in the final minutes of a death dance leading to a violent merger. Such is the fate, for example, of the Hulse-Taylor binary pulsar PSR 1913+16 in about 300 M years. Given the expected sensitivity of the ''advanced LIGO'' (around 2001), which could see such sources out to hundreds of megaparsecs, it has been estimated that from 3 to 100 annual inspiral events could be detectable [19–21].

The urgency derives from the realization [22] that extremely accurate theoretical predictions for the orbital evolution, and to a lesser extent, the gravitational waveform, will play a central role in the data analysis from these observatories. That data analysis is likely to involve some form of matched filtering of the noisy detector output against an ensemble of theoretical ''template'' waveforms which depend on the intrinsic parameters of the inspiralling binary, such as the component masses, spins, and so on, and on its inspiral evolution. How accurate must a template be in order to ''match'' the waveform from a given source (where by a match we mean maximizing the signal-to-noise ratio)? In the total accumulated phase of the wave detected in the sensitive bandwidth, the template must match the signal to a fraction of a cycle. For two inspiralling neutron stars, around 16 000 cycles should be detected; this implies a phasing accuracy of 10^{-5} or better. Since $v/c \sim 1/10$ during the late inspiral, this means that correction terms in the phasing at the level of $(v/c)^5$ or higher are needed. More formal analyses confirm this intuition [23–26].

The bottom line is that theorists have been challenged to derive the gravitational waveform and the resulting radiation back reaction on the orbit phasing at least to 2PN, or second post-Newtonian order, $O[(v/c)^4]$, beyond the quadrupole approximation, and probably to 3PN order. Furthermore, because of the extreme complexity of the calculations at such high PN order, independent calculations are called for, in order to inspire confidence in the final formulas. After all, the formulas will ultimately be compared against real data.

This challenge was recently taken up by two teams of workers, one composed of Blanchet, Damour and Iyer (BDI), the other composed of the present authors. The goal was to derive the gravitational waveform and the energy flux for inspiralling compact binaries of arbitrary masses, through 2PN order. Each team adopted a different approach to the calculation, and worked in isolation from the other. Only at the end of the calculation were comparisons made for the key formulas for the waveform and the gravitational energy flux. The results agreed precisely [27].

The BDI approach was based on a mixed post-Newtonian and ''post-Minkowskian'' framework for solving Einstein's equations approximately, developed in a long series of papers by Damour and colleagues [28–33]. The idea is to solve the vacuum Einstein equations in the exterior of the material sources extending out to the radiation zone in an expansion (''post-Minkowskian'') in ''nonlinearity'' (effectively an expansion in powers of Newton's constant G), and to express the asymptotic solutions in terms of a set of formal, time-dependent, symmetric and trace-free (STF) multipole moments [34]. Then, in a near zone within one characteristic wavelength of the radiation, the equations including the material source are solved in a slow-motion approximation (expansion in powers of $1/c$) that yields a set of STF source multipole moments expressed as integrals over the ''effective'' source, including both matter and gravitational field contributions. The solutions involving the two sets of moments are then matched in an intermediate zone, resulting in a connection between the formal radiative moments and the source moments. The matching also provides a natural way, using analytic continuation, to regularize integrals involving the noncompact contributions of gravitational stress energy that might otherwise be divergent.

The approach of this paper is based on a framework developed by Epstein and Wagoner (EW) [16]. Like the BDI approach, it involves rewriting the Einstein equations in their ''relaxed'' form, namely as an inhomogeneous, flat-spacetime wave equation for a field $h^{\alpha\beta}$, whose source consists of both the material stress energy, and a ''gravitational stress energy'' made up of all the terms nonlinear in $h^{\alpha\beta}$. The wave equation is accompanied by a harmonic or de Donder gauge condition on $h^{\alpha\beta}$, which serves to specify a coordinate system, and also imposes equations of motion on the sources. Unlike the BDI approach, a *single* formal solution is written down, valid everywhere in spacetime. This formal solution, based on the flat-spacetime retarded Green's function, is a retarded integral equation for $h^{\alpha\beta}$, which is then iterated in a slow-motion $(v/c<1)$, weak-field $(||h^{\alpha\beta}||<1)$ approximation, that is very similar to the corresponding procedure in electromagnetism. However, because the integrand of this retarded integral is not compact by virtue of the nonlinear field contributions, the original EW formalism quickly runs up against integrals that are not well defined, or worse, are divergent. Although at the lowest quadrupole and first few PN orders, various arguments can be given to justify sweeping such problems under the rug [17], they are not very rigorous, and provide no guarantee that the divergences do not become insurmountable at higher orders. As a consequence, despite efforts to cure the problem, the EW formalism fell into some disfavor as a route to higher orders, although an extension to 3/2PN order was accomplished [35].

One contribution of this paper is a resolution of this problem. The resolution involves taking literally the statement that the solution is a *retarded* integral, i.e., an integral over the *entire* past null cone of the field point. To be sure, that part of the integral that extends over the intersection between the past null cone and the material source and the near zone is still approximated as usual by a slow-motion expansion involving spatial integrals of moments of the source, including the non-compact gravitational contributions, just as in the BDI framework. But instead of cavalierly extending the spatial integrals to infinity as was implicit in the original EW framework, and risking undefined or divergent integrals, we terminate the integrals at the boundary of the near zone, chosen to be at a radius $\mathcal{R}$ given roughly by one wavelength of the gravitational radiation. For the integral over the rest of the past null cone exterior to the near zone (''radiation zone''), we do not make a slow-motion expansion, instead we use a coordinate transformation to convert the integral into a convenient, easy-to-calculate form, that is manifestly convergent, subject only to reasonable assumptions about the past behavior of the source. This transformation was suggested by our earlier work on a nonlinear gravitational-wave phenomenon called the Christodoulou memory [36]. Not only are all integrations now explicitly finite and convergent, we show explicitly that all contributions from the near-zone spatial integrals that grow with $\mathcal{R}$ (and that would have diverged had we let $\mathcal{R}\rightarrow\infty$) are actually *cancelled* by corresponding terms from the radiation-zone integrals. Thus the procedure, as expected, has no dependence on the artificially chosen boundary radius $\mathcal{R}$ of the near zone. In addition, the method can be carried to higher orders in a straightforward, albeit very tedious manner. The result is a manifestly finite, well-defined procedure for calculating gravitational radiation to high, and we suspect all, PN orders.

The result of the calculation is an explicit formula for the gravitational waveform for a two-body system, the transverse-traceless (TT) part of the radiation-zone field, denoted h^{ij}, and representing the deviation of the metric from flat spacetime. In terms of an expansion beyond the quadrupole formula, it has the schematic form

$$h^{ij}=\frac{2G\mu}{Rc^4}\{\tilde{Q}^{ij}[1+O(\epsilon^{1/2})+O(\epsilon)+O(\epsilon^{3/2}) + O(\epsilon^2)\cdots]\}_{\mathrm{TT}}, \tag{1.1}$$

where μ is the reduced mass, and $\tilde{Q}^{ij}$ represents two time derivatives of the mass quadrupole moment tensor (the series actually contains multipole orders beyond quadrupole). The TT projection operation is described below. The expansion parameter ϵ is related to the orbital variables by $\epsilon \sim Gm/rc^2 \sim (v/c)^2$, where r is the distance between the bodies, v is the relative velocity, and $m=m_1+m_2$ is the total mass. The 1/2PN and 1PN terms were derived in [17], the 3/2PN terms in [35]. The contribution of gravitational-wave ''tails,'' caused by backscatter of the outgoing radiation off the background spacetime curvature, at $O(\epsilon^{3/2})$, were derived and studied in [32,37,38].

This paper derives the 2PN terms including 2PN tail contributions; the results are in complete agreement with BDI [39]. We also find that part of the tail terms at 3/2PN and 2PN order serve to guarantee that the outgoing radiation propagates along true null directions of the asymptotic curved spacetime, despite the use of flat spacetime wave equations in the solution. The explicit formula for the general two-body waveform is given below in Eqs. (6.10) and (6.11).

There are also contributions to the waveform due to intrinsic spin of the bodies, which occur at $O(\epsilon^{3/2})$ (spin-orbit) and $O(\epsilon^2)$ (spin-spin); these have been calculated elsewhere [40,41], and are rederived in the EW framework in Appendix F.

Equations of motion for the material sources must also be specified to 2PN order in order to have a consistent solution of Einstein's equations. These have the schematic form

$$d^2\mathbf{x}/dt^2=-(Gm\mathbf{x}/r^3)[1+O(\epsilon)+O(\epsilon^{3/2})+O(\epsilon^2)+\cdots], \tag{1.2}$$

where $\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2$ is the separation vector. The lowest-order contribution is obviously Newtonian. The next term $O(\epsilon)$ is the first post-Newtonian correction, which gives rise to such effects as the advance of the periastron. The term $O(\epsilon^{3/2})$ comes solely from the spin-orbit interaction. The term of $O(\epsilon^2)$ is a *second* post-Newtonian correction to the equation of motion (and also contains spin-spin interactions). The terms in Eq. (1.2) are all non-dissipative, having nothing to do with gravitational radiation reaction. Through 2PN order, these equations are by now standard; see for example [42–44] and Eq. (6.5) below.

Given the gravitational waveform, we can compute the rate energy is carried off by the radiation (schematically $\int \dot{h}\dot{h}\, d\Omega$, the gravitational analog of the Poynting flux). The result has the schematic form

$$dE/dt=(dE/dt)_Q[1+O(\epsilon)+O(\epsilon^{3/2})+O(\epsilon^2)+\cdots]. \tag{1.3}$$

Here $(dE/dt)_Q$ denotes the lowest-order quadrupole contribution, proportional to the square of three time derivatives of the trace-free mass quadrupole moment tensor of the source. The explicit formula for a general two-body system is given below in Eqs. (6.12) and (6.13). For the special case of nonspinning bodies moving on quasicircular orbits (i.e., circular apart from a slow inspiral), the energy flux has the form

$$\frac{dE}{dt}=\frac{32G}{5c^5}\eta^2\left(\frac{Gm}{rc^2}\right)^5\left[1-\frac{Gm}{rc^2}\left(\frac{2927}{336}+\frac{5}{4}\eta\right)+4\pi\left(\frac{Gm}{rc^2}\right)^{3/2} +\left(\frac{Gm}{rc^2}\right)^2\left(\frac{293\,383}{9072}+\frac{380}{9}\eta\right)\right], \tag{1.4}$$

where $\eta=m_1m_2/m^2$. The first term is the quadrupole contribution, the second term is the 1PN contributon [17], the third term, with the coefficient 4π, is the ''tail'' contribution [32,37,38,45], and the fourth term is the 2PN contribution derived here. This new contribution was reported in [27], and was also derived using the BDI approach in [39]. For the contributions of spin-orbit and spin-spin coupling see [40,41,27] and Appendix F.

Similar expressions can be derived for the loss of angular momentum and linear momentum. These losses react back on the orbit to circularize it and cause it to inspiral. The result is that the orbital phase (and consequently the gravitational-wave phase) evolves nonlinearly with time. It is the sensitivity of the broadband LIGO- and VIRGO-type detectors to phase that makes the higher-order contributions to dE/dt so observationally relevant. For example, for an inspiral of two 1.4 $M_\odot$ neutron stars, the 2PN term in Eq. (1.4) contributes about 9 of the 16 000 cycles observable in the bandwidth of the advanced LIGO. More detailed analyses of the effect of the 2PN terms on the matched filtering can be found in [25,46,47]. A ready-to-use set of formulas for the 2PN gravitational waveform template, including the nonlinear evolution of the gravitational-wave frequency (not including spin effects) may be found in [48]. Spin corrections to the waveform templates may be found in Appendix F.

An alternative approach to deriving gravitational waveforms and energy flux for inspiralling compact binaries, in the limit in which one mass is much smaller than the other, is that of black hole perturbation theory. This method provides numerical results that are exact in v/c, as well as analytical results expressed as series in powers of v/c, both for nonrotating and for rotating black holes [37,49–52]. For nonrotating holes, the analytical expansions have been carried to *fourth* PN order [52]. In all cases of overlap, the results agree precisely with our post-Newtonian results, in the limit $\eta\rightarrow 0$.

This paper is an attempt to present, in a relatively complete and self-contained form, the formalism and machinery of our ''improved EW'' approach to higher-order gravitational radiation from binary systems. Indeed, we begin with the raw Einstein equations, and end with a plot of the 2PN waveform. The goal is to provide sufficient detail to allow the reader, using this paper virtually alone, to verify any of the results reported here (we make no statement about the amount of work involved), and to carry the computations to higher PN orders. In Sec. II, we lay out the foundations of gravitational-wave generation, describing the relaxed Einstein equations, the matter sources and the near and radiation zones, and the formal retarded integral solution of the wave equation, including the new treatment of integration over the null cone in the radiation zone. We turn in Sec. III to the weak-field, slow-motion approximation, and write down the matter and field variables to the accuracy needed to find the radiation to 2PN order. The part of the retarded integral for $h^{\alpha\beta}$ that extends over the near zone can be written in terms of a set of ''Epstein-Wagoner'' moments; these are evaluated explicitly in Sec. IV. In Sec. V, we evaluate the contributions to $h^{\alpha\beta}$ from the radiation-zone integrals, showing both the explicit cancellation of those terms in the EW moments that grow with $\mathcal{R}$, and the generation of tail terms. Section VI specializes to two-body systems, and displays the full formulas for the gravitational waveform and energy loss. In Sec. VII, we further specialize to circular orbits. Section VIII makes concluding remarks. A number of technical details are relegated to Appendices.

Our conventions and notation generally follow those of [53,34]. Henceforth we use units in which $G=c=1$. Greek indices run over four spacetime values 0, 1, 2, 3, while Latin indices run over three spatial values 1, 2, 3; commas denote partial derivatives with respect to a chosen coordinate system, while semicolons denote covariant derivatives; repeated indices are summed over; $\eta^{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$; $g\equiv\det(g_{\mu\nu})$; $a^{(ij)}\equiv(a^{ij}+a^{ji})/2$; $a^{[ij]}\equiv(a^{ij}-a^{ji})/2$; ϵ^{ijk} is the totally antisymmetric Levi-Civita symbol ($\epsilon^{123}=+1$). We use a multi-index notation for products of vector components: $x^{ij\cdots k}\equiv x^i x^j\cdots x^k$, with a capital letter superscript denoting a product of that dimensionality: $x^L\equiv x^{i_1}x^{i_2}\cdots x^{i_l}$; angular brackets around indices denote STF products (see Appendix A for definitions). Spatial indices are freely raised and lowered with δ^{ij} and δ_{ij}.

II. FOUNDATIONS OF GRAVITATIONAL-WAVE GENERATION

A. The relaxed Einstein equations

We begin our development of the gravitational-wave generation problem with the Einstein equations

$$R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R=8\pi T^{\alpha\beta}. \tag{2.1}$$

Here $R^{\alpha\beta}$ is the Ricci curvature tensor, $g^{\alpha\beta}$ is the spacetime metric and $T^{\alpha\beta}$ is the stress energy tensor of the matter. Although Eq. (2.1) is a conceptually powerful statement, relating the curvature of spacetime on the left-hand side to the stress energy of matter on the right-hand side, it is not a particularly useful form of the Einstein equations for practical calculations of gravitational-wave generation. For that purpose it is conventional first to define the potential

$$h^{\alpha\beta}\equiv\eta^{\alpha\beta}-(-g)^{1/2}g^{\alpha\beta}, \tag{2.2}$$

(see, e.g., [34]) and to choose a particular coordinate system defined by the de Donder or harmonic gauge condition

$$h^{\alpha\beta}{}_{,\beta}=0. \tag{2.3}$$

The spatial components of $h^{\alpha\beta}$ evaluated far from the source comprise the gravitational waveform and are directly related to the signal which a gravitational-wave detector measures. With these definitions the Einstein equations (2.1) can be recast in the form

$$\Box h^{\alpha\beta}=-16\pi\tau^{\alpha\beta}, \tag{2.4}$$

where $\Box\equiv-\partial^2/\partial t^2+\nabla^2$ is the flat-spacetime wave operator. The source on the right-hand side is given by the ''effective'' stress-energy pseudotensor

$$\tau^{\alpha\beta}=(-g)T^{\alpha\beta}+(16\pi)^{-1}\Lambda^{\alpha\beta}, \tag{2.5}$$

where $\Lambda^{\alpha\beta}$ is the nonlinear ''field'' contribution given by

$$\Lambda^{\alpha\beta}=16\pi(-g)t^{\alpha\beta}_{\rm LL}+(h^{\alpha\mu}{}_{,\nu}h^{\beta\nu}{}_{,\mu}-h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu}), \tag{2.6}$$

and $t^{\alpha\beta}_{\rm LL}$ is the ''Landau-Lifshitz'' pseudotensor, given by

$$\begin{aligned}16\pi(-g)t^{\alpha\beta}_{LL}\equiv\Big\{&g_{\lambda\mu}g^{\nu\rho}h^{\alpha\lambda}{}_{,\nu}h^{\beta\mu}{}_{,\rho}+\frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}h^{\lambda\nu}{}_{,\rho}h^{\rho\mu}{}_{,\nu}\\&-2g_{\mu\nu}g^{\lambda(\alpha}h^{\beta)\nu}{}_{,\rho}h^{\rho\mu}{}_{,\lambda}+\frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu}\\&-g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau}-g_{\rho\sigma}g_{\nu\tau})h^{\nu\tau}{}_{,\lambda}h^{\rho\sigma}{}_{,\mu}\Big\}.\end{aligned} \tag{2.7}$$

By virtue of the gauge condition (2.3), this source term satisfies the conservation law

$$\tau^{\alpha\beta}{}_{,\beta}=0, \tag{2.8}$$

which is equivalent to the equation of motion of the matter $T^{\alpha\beta}{}_{;\beta}=0$.

We emphasize that Eq. (2.4) is not an *approximate*, or *weak-field*, form of the Einstein equations; it is exact, and relies only on the assumption that spacetime can be covered by harmonic coordinates.

The form of Eq. (2.4) is suggestive of the wave equation for the vector potential in electromagnetism (EM). This analogy with EM is at once helpful and deceptive. It is helpful in that it suggests how to proceed to solve the equation, i.e., use a retarded Green function, and an expansion in terms of radiative multipole moments. It further illustrates that, just as the current density in EM is the source for the vector potential, here the stress energy of the matter is a source of the gravitational potential.

However there are several important differences between Eq. (2.4) and its electromagnetic counterpart. First, the ''source'' in Eq. (2.4) also contains a gravitational part that depends explicitly on $h^{\alpha\beta}$, the very quantity for which we are trying to solve. Second, unlike the EM case where the source (the currents) has finite spatial extent (compact support), we can expect $\tau^{\alpha\beta}$, which depends on the fields $h^{\alpha\beta}$, to have infinite spatial extent. Indeed the very outgoing radiation that we hope to detect, will, at some level of approximation, serve as a contribution to the source, thus generating an additional component of the radiation. However, we have found that, for the physical situations of interest, this latter, highly nonlinear effect, often referred to as the Christodoulou memory, is very weak and can be adequately approximated by the methods of this paper [36].

Another complication in Eq. (2.4) is that the second derivative term $h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu}$ in the source really ''belongs'' on the left-hand side with the other second derivative terms in the wave operator. Such a term in a differential equation modifies the propagation characteristics of the field from the flat-spacetime characteristics represented by the d'Alembertian operator. Physically this is a manifestation of the fact that the radiation propagates along null cones of the curved spacetime around the source, which deviate from the flat null cones of the harmonic coordinates. Nevertheless, the techniques to be presented here do recover the leading manifestations of this effect, commonly known as ''tails,'' including modification of the phasing of the solutions from their initial dependance on flat space retarded time to true retarded time of the asymptotic Schwarzschild spacetime of the source.

B. Source, near zone, and radiation zone

We consider a material source consisting of a collection of fluid balls (stars) whose size is typically small compared to their separations. The material will be modeled as perfect fluid, having stress-energy tensor

$$T^{\alpha\beta}\equiv(\rho+p)u^{\alpha}u^{\beta}+pg^{\alpha\beta}, \tag{2.9}$$

where ρ and p are the locally measured energy density and pressure, respectively, and u^{α} is the four-velocity of an element of fluid. We shall assume that the bodies are sufficiently compact that we can ignore all intrinsic multipole moments of the bodies at quadrupole order and beyond. That is, we treat only the bodies' monopole (mass) moments [in an appendix we treat the bodies' dipole (spin) moments]. For inspiralling binaries of compact objects, the effects of rotationally induced and tidally induced quadrupole and higher moments on the orbital evolution or gravitational radiation have been shown, in the case of binary neutron stars, to be negligible until the final coalescence stage, where the post-Newtonian approximation breaks down anyway [54]. For spinning black holes, the effects are small, but can be non-negligible for sufficiently large spin [55]. In the long run, such finite-size effects should (and can) be incorporated into our formalism.

To treat the monopole part of the bodies' mass distributions, we approximate the stress-energy tensor as a distributional tensor representing ''point'' masses, given by

$$T^{\alpha\beta}_{\rm monopole}\equiv\sum_A m_A(-g)^{-1/2}(u^{\alpha}_A u^{\beta}_A/u^0_A)\,\delta^3[\mathbf{x}-\mathbf{x}_A(t)], \tag{2.10}$$

where m_A is the gravitational mass of the Ath body, and u^{α}_A is the four-velocity of its center of mass, $\mathbf{x}_A(t)$. Formally, such a distributional stress-energy tensor is not valid in general relativity. On the other hand, it has been shown in a variety of post-Newtonian contexts to give results that are equivalent to treating the bodies as almost spherical fluid balls, defining a suitable approximate center of mass, and carrying out explicit integrals over the interiors of the balls. The resulting self-field and internal energy effects result in a renormalization of the mass of each body from a ''bare'' mass $\int_A \rho d^3x$ to the gravitational mass m_A. Furthermore, all effects of the internal structure of the bodies are ''effaced,'' so that all aspects of the motion and gravitational radiation are characterized by a single mass m_A for each body (see [35] for demonstration of this effacement in the waveform at 3/2PN order). This is a manifestation of the strong equivalence principle, which is satisfied by general relativity. All these complications, then, can be embodied in the distributional stress-energy tensor of Eq. (2.10), with the caveat that all infinite self-field effects that might result from the use of the δ-function source are to be discarded (self-field effects having already been renormalized into m_A). An alternative viewpoint takes the gravitational field in a zone surrounding each body in a coordinate system that momentarily comoves with the body and notes that it can be characterized by multipole moments that can be identified with the body's asymptotially measured mass and (if desired) higher multipole moments. The fields surrounding each body are then matched to an appropriate interbody gravitational field, with the equations of motion providing consistency conditions for such matching. Apart from tidal effects, the results depend only on the effective masses of the bodies, and all self-field effects are automatically accounted for (see [56,57] for example, for detailed implementations of this approach in various situations).

The effects of spins can be added to the framework in a straightforward way; these are reviewed in Appendix F.

We consider the bodies to comprise a bound system of characteristic size $\mathcal{S}=\max_{\{A,B\}}r_{AB}$, where $r_{AB}=|\mathbf{x}_A-\mathbf{x}_B|$, with a center of mass chosen to be at the origin of coordinates, $\mathbf{X}=0$. The *source zone* then consists of the world tube $\mathcal{T}=\{x^{\alpha}|r<\mathcal{S},-\infty<t<\infty\}$.

The bodies are assumed to move with characteristic velocities $v_A<1$, and for much of their evolution with $v_A\ll 1$. The characteristic reduced wavelength of gravitational radiation, $\lambda\!\!\!^{-}=\lambda/2\pi\sim\mathcal{S}/v\equiv\mathcal{R}$ serves to define the boundary of the *near zone*, defined to be the world tube $\mathcal{D}=\{x^{\alpha}|r<\mathcal{R},-\infty<t<\infty\}$. Within the near zone, the gravitational fields can be treated as almost instantaneous functions of the source variables, i.e., retardation can be ignored or treated as a small perturbation of instantaneous solutions. For most of the evolution, up to the point where the post-Newtonian approximation breaks down, $\mathcal{R}\gg\mathcal{S}$.

The region exterior to the near zone is the *radiation zone*, $r>\mathcal{R}$. In this zone, we evaluate the fully retarded solutions of Eq. (2.4), and focus on the parts that fall off as r^{-1}.

The formal solution to Eq. (2.4) can be written down in terms of the retarded, flat-space Green function:

$$h^{\alpha\beta}(t,\mathbf{x})=4\int\frac{\tau^{\alpha\beta}(t',\mathbf{x}')\,\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^4x'. \tag{2.11}$$

This represents an integration of $\tau^{\alpha\beta}/|\mathbf{x}-\mathbf{x}'|$ over the past harmonic null cone $\mathcal{C}$ emanating from the field point $(t,\mathbf{x})$ (see Fig. 1). This past null cone intersects the world tube $\mathcal{D}$ enclosing the near zone at the three-dimensional hypersurface $\mathcal{N}$. Thus the integral of Eq. (2.11) consists of two pieces, an integration over the hypersurface $\mathcal{N}$, and an inte-

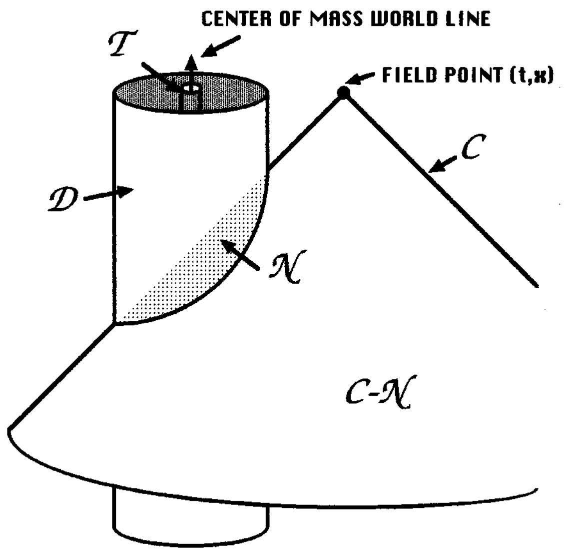

FIG. 1. Past harmonic null cone $\mathcal{C}$ of the field point $(t,\mathbf{x})$ intersects the near zone $\mathcal{D}$ in the hypersurface $\mathcal{N}$.

gration over the rest of the past null cone $\mathcal{C}-\mathcal{N}$. Each of these integrations will be treated differently. We will also treat slightly differently the two cases in which (a) the field point is outside the near zone (Fig. 1), and (b) the field point is within the near zone (Fig. 2). The former case will be relevant for calculating the gravitational-wave signal, while the latter will be important for calculating field contributions to $\tau^{\alpha\beta}$ that must be integrated over the near zone, as well as for calculating fields that enter the equations of motion.

C. Radiation-zone field point, near-zone integration

For a field point in the radiation zone, and integration over the near zone, we first carry out the t' integration in Eq. (2.11), to obtain

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\int_{\mathcal{N}}\frac{\tau^{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \qquad (2.12)$$

Within the near zone, the spatial integration variable $\mathbf{x}'$ satisfies $|\mathbf{x}'|\leqslant\mathcal{R}<r$, where the distance to the field point

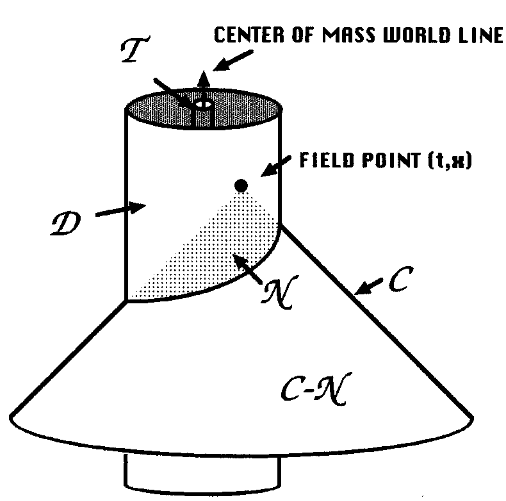

FIG. 2. Same as Fig. 1, for field point inside the near zone.

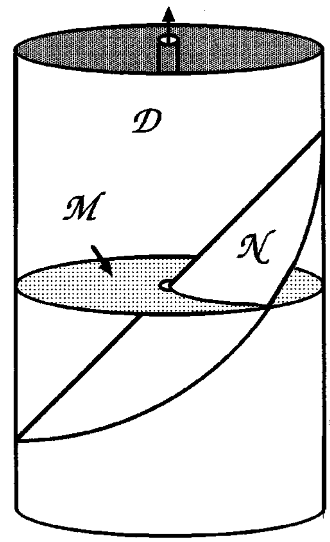

FIG. 3. Taylor expansion of retarded time dependence on $\mathcal{N}$ results in multipole moments integrated over the spatial hypersurface $\mathcal{M}$.

$r=|\mathbf{x}|$. We now expand the x' dependence in the integrand in powers of $|\mathbf{x}'|/r$, using the fact that

$$|\mathbf{x}-\mathbf{x}'|^q=\sum_{m=0}^{\infty}\frac{1}{m!}(-x')^{i_1\cdots i_m}(r^q)_{,i_1\cdots i_m}. \qquad (2.13)$$

We next expand $\tau^{\alpha\beta}$ in a Taylor series about the retarded time $u\equiv t-r$. The integration is now over the hypersurface $\mathcal{M}$, which is the intersection of the near-zone world tube with the constant-time hypersurface $t_{\mathcal{M}}=u=t-r$ (see Fig. 3). Roughly speaking, each term in the Taylor series is smaller than its predecessor by a factor of order $v<1$, thus for any hope of convergence of the series, one must restrict attention to slow-motion sources. We now have an infinite series in x' (expansion of $|\mathbf{x}-\mathbf{x}'|^{-1}$) multiplying a double infinite series (expansion of $|\mathbf{x}-\mathbf{x}'|$ inside the Taylor expansion). Grouping terms with the same powers of x' and carrying out the appropriate combinatorics (including use of ‘‘Faà di Bruno’s formula’’ [58]), it is straightforward to show that

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\left(\frac{1}{r}M^{\alpha\beta k_1\cdots k_q}\right)_{,k_1\cdots k_q}, \qquad (2.14)$$

where

$$M^{\alpha\beta k_1\cdots k_q}(u)\equiv\int_{\mathcal{M}}\tau^{\alpha\beta}(u,\mathbf{x}')x'^{k_1}\cdots x'^{k_q}d^3x'. \qquad (2.15)$$

This general expansion, both in powers of r^{-1} and in retarded-time derivatives of $M^{\alpha\beta k_1\cdots k_q}(u)$ will prove useful in later integrations of field quantities over the far zone.

However, for gravitational-wave detectors, we need only to focus on the spatial components of $h^{\alpha\beta}$, and on the leading component in $1/R$, where R is the distance to the detector. Using the fact that $u_{,i}=-\hat{N}^i$, where $\hat{\mathbf{N}}\equiv\mathbf{x}/R$ denotes the detector direction, we obtain

$$h^{ij}_{\mathcal{N}}(t,\mathbf{x})=\frac{4}{R}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}}\tau^{ij}(u,\mathbf{x}')(\hat{\mathbf{N}}\cdot\mathbf{x}')^m d^3x' + O(R^{-2}). \tag{2.16}$$

Because of the conservation law Eq. (2.8), τ^{ij} satisfies the identities

$$\tau^{ij}=\frac{1}{2}(\tau^{00}x^i x^j)_{,00}+2(\tau^{l(i}x^{j)})_{,l}-\frac{1}{2}(\tau^{kl}x^i x^j)_{,kl}, \tag{2.17a}$$

$$\tau^{ij}x^k=\frac{1}{2}(2\tau^{0(i}x^{j)}x^k-\tau^{0k}x^i x^j)_{,0}+\frac{1}{2}(2\tau^{l(i}x^{j)}x^k-\tau^{kl}x^i x^j)_{,l}. \tag{2.17b}$$

Using these identities in Eq. (2.16) generates the multipole expansion

$$h^{ij}_{\mathcal{N}}(t,\mathbf{x})=\frac{2}{R}\frac{d^2}{dt^2}\sum_{m=0}^{\infty}\hat{N}_{k_1}\cdots\hat{N}_{k_m}I^{ijk_1\cdots k_m}_{\rm EW}(u), \tag{2.18}$$

where the ''Epstein-Wagoner'' (EW) moments are given by

$$I^{ij}_{\rm EW}=\int_{\mathcal{M}}\tau^{00}x^i x^j d^3x+I^{ij}_{\rm EW\,(surf)}, \tag{2.19a}$$

$$I^{ijk}_{\rm EW}=\int_{\mathcal{M}}(2\tau^{0(i}x^{j)}x^k-\tau^{0k}x^i x^j)d^3x+I^{ijk}_{\rm EW\,(surf)}, \tag{2.19b}$$

$$I^{ijk_1\cdots k_m}_{\rm EW}=\frac{2}{m!}\frac{d^{m-2}}{dt^{m-2}}\int_{\mathcal{M}}\tau^{ij}x^{k_1}\cdots x^{k_m}d^3x \quad (m\geqslant 2), \tag{2.19c}$$

where integrating the spatial derivative terms in Eqs. (2.17) by parts generates surface integrals at the two-dimensional coordinate sphere of radius $\mathcal{R}$ bounding the hypersurface $\mathcal{M}$, denoted $\partial\mathcal{M}$, resulting in surface contributions to the first two EW moments given by

$$(d/dt)^2 I^{ij}_{\rm EW(surf)}=\oint_{\partial\mathcal{M}}[4\tau^{l(i}x^{j)}-(\tau^{kl}x^i x^j)_{,k}]\mathcal{R}^2\hat{n}^l d^2\Omega, \tag{2.20a}$$

$$(d/dt)I^{ijk}_{\rm EW\,(surf)}=\oint_{\partial\mathcal{M}}(2\tau^{l(i}x^{j)}x^k-\tau^{kl}x^i x^j)\mathcal{R}^2\hat{n}^l d^2\Omega, \tag{2.20b}$$

where $\hat{n}^l$ denotes an outward radial unit vector, and $d^2\Omega$ denotes solid angle.

One advantage of this multipole expansion is that the field and source variables appearing in the integrand $\tau^{\alpha\beta}$ are evaluated at the single retarded time u; a disadvantage is that because the field contributions to $\tau^{\alpha\beta}$ fall off as some power of r, one can expect to encounter integrals that depend on positive powers of the radius $\mathcal{R}$ of the boundary of integration, especially in some of the higher-order moments. If this boundary is formally taken to ∞ (as was previously done), these integrals would diverge. However, as we shall see, such $\mathcal{R}$-dependent effects are *precisely* canceled by contributions from the integral over the rest of the past null cone, to which we now turn.

D. Radiation-zone field point, radiation-zone integration

The integral over the rest of the past null cone $\mathcal{C}-\mathcal{N}$ can be written in the form

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})=4\int_{-\infty}^{\infty}du'\int_{\mathcal{C}-\mathcal{N}}\frac{\tau^{\alpha\beta}(t',\mathbf{x}')\,\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\times\delta(u'-t'+r')d^4x', \tag{2.21}$$

where we have simply inserted $1=\int du'\,\delta(u'-t'+r')$. We now integrate over t' and r', and note that

$$\int_{-\infty}^{\infty}dt'\int_{\mathcal{R}}^{\infty}dr'\,\delta(u'-t'+r')\,\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)=\begin{cases}\dfrac{|\mathbf{x}-\mathbf{x}'|}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}, & u'<u \quad\text{and}\quad r'>\mathcal{R},\\ 0, \quad u'>u & \text{or}\quad r'<\mathcal{R}.\end{cases} \tag{2.22}$$

The result is

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})=4\int_{-\infty}^{u}du'\oint_{\mathcal{C}-\mathcal{N}}\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d^2\Omega'. \tag{2.23}$$

Note that r' is a function of u' and Ω' via the condition [from the two δ-functions in Eq. (2.22)]: $t-u'=r'+|\mathbf{x}-\mathbf{x}'|$, which gives

$$r'(u',\Omega')=[(t-u')^2-r^2]/[2(t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x})]. \tag{2.24}$$

The integration over solid angle $d^2\Omega'$ for a given value of u', together with the $u'+r'$ ''time'' dependence of $\tau^{\alpha\beta}$, can be seen to represent an integration over the two-dimensional intersection of the past null cone $\mathcal{C}$ with the future null cone $t'=u'+r'$ emanating from the center of mass of the system at $t_{\rm c.m.}=u'$ (Fig. 4). The integration over u' then includes all such future-directed cones, starting from the infinite past, and terminating in the one emanating from the center of mass at time u, which is tangent to the past null cone of the observation point.

However, for $u\geqslant u'\geqslant u-2\mathcal{R}$, the two-dimensional intersections meet the boundary of the near zone, and so the angular integration is not complete. If we choose the field point $\mathbf{x}$ to be in the z direction, so that $\hat{\mathbf{n}}'\cdot\mathbf{x}=r\cos\theta'$, then the condition $r'\geqslant\mathcal{R}$, together with Eq. (2.24) imply that $0\leqslant\phi'\leqslant 2\pi$, $1-\alpha\leqslant\cos\theta'\leqslant 1$, where

$$\alpha=(u-u')(2r-2\mathcal{R}+u-u')/2r\mathcal{R}. \tag{2.25}$$

Note that α ranges from 0 ($u'=u$) to 2 ($u'=u-2\mathcal{R}$). For $u'<u-2\mathcal{R}$, the angular integration covers the full 4π. Thus we write the radiation-zone integral in the form

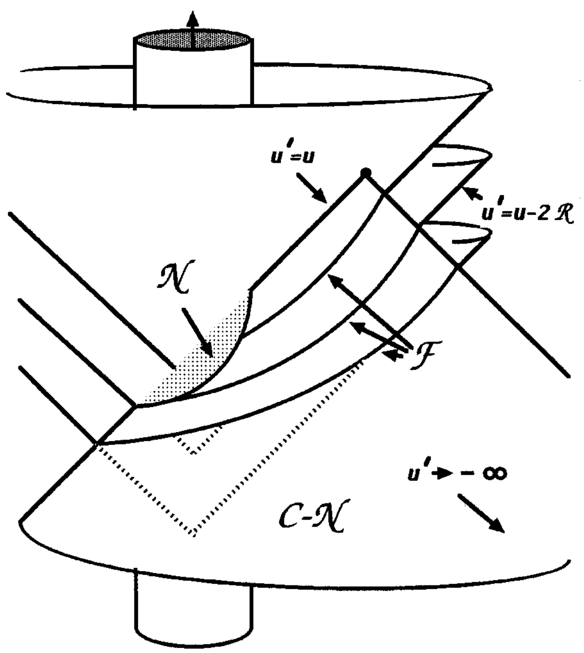

FIG. 4. Two-dimensional hypersurfaces $\mathcal{F}$ formed by intersection of past null cone of field point with future null cones from the origin. The field point is in radiation zone. For u' from $-\infty$ to $u-2\mathcal{R}$, F covers full 4π solid angle around the origin. From $u-2\mathcal{R}$ to u, F terminates at boundary of the near zone $\mathcal{N}$.

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})=4\int_{u-2\mathcal{R}}^{u}du'\int_0^{2\pi}d\phi'\int_{1-\alpha}^{1}\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d\cos\theta' +4\int_{-\infty}^{u-2\mathcal{R}}du'\oint\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d^2\Omega'. \quad (2.26)$$

Note that $\tau^{\alpha\beta}$ contains only field contributions evaluated in the radiation zone; in determining these we will make use of the general expansion (2.14).

To obtain the contribution to the gravitational waveform, we evaluate the spatial components of Eq. (2.26) at distance R and direction $\hat{\mathbf{N}}$ and keep the leading $1/R$ part.

E. Near-zone field point, near-zone integration

In this case, in Eq. (2.12), both $\mathbf{x}$ and $\mathbf{x}'$ are within the near zone, hence $|\mathbf{x}-\mathbf{x}'|\leqslant 2\mathcal{R}$. Consequently, the variation in retarded time can be treated as a small perturbation, since $\tau^{\alpha\beta}$ varies on a time scale $\sim\mathcal{R}$. We therefore expand the retardation in powers of $|\mathbf{x}-\mathbf{x}'|$, to obtain

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}}\tau^{\alpha\beta}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{m-1}d^3x', \quad (2.27)$$

where $\mathcal{M}$ here denotes the intersection of the hypersurface $t=\text{const}$ with the near-zone world tube.

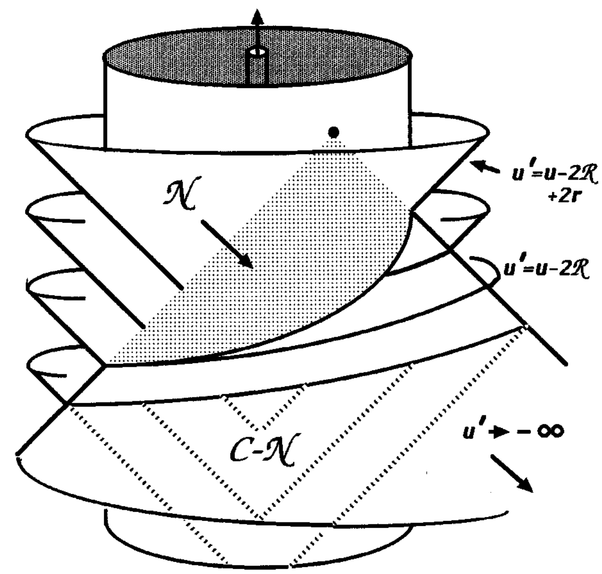

FIG. 5. Same as Fig. 4, for field point in near zone. Integral over u' terminates at $u'=u-2\mathcal{R}+2r$.

F. Near-zone field point, radiation-zone integration

The formulas from Sec. II D, such as Eqs. (2.24) and (2.25), carry over to this case with only one modification. The final future null cone that appears in the integration is the one that intersects the boundary of the near zone and the past null cone of the field point simultaneously at $u'=u-2\mathcal{R}+2r$, rather than $u'=u$ (Fig. 5) (recall that here, $r<\mathcal{R}$). The result is, for a near-zone field point,

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})=4\int_{u-2\mathcal{R}}^{u-2\mathcal{R}+2r}du'\int_0^{2\pi}d\phi'\int_{1-\alpha}^{1}\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d\cos\theta' +4\int_{-\infty}^{u-2\mathcal{R}}du'\oint\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d^2\Omega'. \quad (2.28)$$

G. Gravitational waveform and energy flux

To obtain the gravitational waveform, we combine the two contributions to h^{ij}, Eqs. (2.18) and the leading $1/R$ part of the spatial components of Eq. (2.26), and evaluate the transverse-traceless (TT) part, given by

$$h^{ij}_{\mathrm{TT}}=h^{kl}\left(P^i_kP^j_l-\frac{1}{2}P^{ij}P_{kl}\right), \quad (2.29)$$

where $P^i_k=\delta^i_k-\hat{N}_k\hat{N}^i$.

Note that the two expressions that contribute to h^{kl} in Eq. (2.29) each depend on the radius $\mathcal{R}$ of the near zone. Since $\mathcal{R}$ was an arbitrarily chosen radius, the final physical answer should not depend on it. However, to check that all terms involving $\mathcal{R}$ cancel in the end would be a formidable task. Instead we adopt the following nonrigorous, but reasonable strategy. All terms in the near-zone EW moments and in the radiation-zone integrals that are *independent* of $\mathcal{R}$ are kept.

All terms that *fall off* with $\mathcal{R}$ will be dropped. Close examination shows that, despite our formal choice $\mathcal{R}\sim\lambda\!\!\!^{-}$, nothing in our calculations actually constrains the value of $\mathcal{R}$, apart from the inequality $\mathcal{R}<R$. Thus we are free to make $\mathcal{R}$ sufficiently large, but still less than R, so as to make such terms as small as we wish, whether or not they ultimately cancel. In this regard, it is useful to note that, for a LIGO or VIRGO detector 10 Mpc from a source emitting gravitational waves at $f=100$ Hz, $fR=R/\lambda\sim10^{16}$, and thus many orders of magnitude of $\mathcal{R}$ are available to achieve this suppression. Nevertheless, we believe that all such terms actually cancel. Finally, all terms that *grow* with powers of $\mathcal{R}$ are also kept. In this case we will show explicitly that all terms that vary as positive powers of $\mathcal{R}$ cancel between the near-zone and radiation-zone integrals. This procedure thus isolates the finite terms that arise from convergent integrals, while simultaneously verifying that no truly divergent integrals arise. The result is a well-defined, explicitly finite, method for calculating the gravitational waveform. It is the explicit inclusion of the radiation-zone integral in the formulation of Eq. (2.26) that cures the apparent divergences that plagued the original EW framework.

The energy flux is given by

$$\dot{E}=(R^2/32\pi)\oint \dot{h}^{ij}_{\rm TT}\dot{h}^{ij}_{\rm TT}d^2\Omega. \tag{2.30}$$

III. WEAK-FIELD, SLOW-MOTION APPROXIMATION

A. Iteration of relaxed Einstein equations

We make the standard assumption that, with respect to the orbital motion and mutual gravitational interactions,

$$v_A^2\sim m_A/\mathcal{S}\sim\epsilon\ll1, \tag{3.1}$$

where ϵ will be used as an expansion parameter.

Now, because the field $h^{\alpha\beta}$ appears in the source of the field equation, the usual method of solution is to iterate: substitute $h^{\alpha\beta}=0$ on the right-hand side of Eq. (2.11) and solve for the first-iterated $h_1^{\alpha\beta}$; substitute that into Eq. (2.11) and solve for the second-iterated $h_2^{\alpha\beta}$, and so on [imposing the gauge condition Eq. (2.3) consistently at each order]. The first iterated $h_1^{\alpha\beta}$ is $O(\epsilon)$, and each subsequent iteration improves its accuracy by one order in ϵ. Thus, for example, to obtain a result for the waveform accurate to the order of the quadrupole formula, $h\sim(m/r)\ddot{I}^{ij}\sim(m/r)(v^2+m/\mathcal{S})\sim\epsilon^2$, *two* iterations of Eq. (2.11) are needed. To obtain the first post-Newtonian corrections to the quadrupole approximation, i.e., h to order ϵ^3, $h_3^{\alpha\beta}$, or three iterations, are needed, while to obtain the 2PN contributions (the goal of this paper), the fourth-iterated field is needed. This would be a daunting task, if it was not for the use of the identities, Eqs. (2.17). Consider for example, the quadrupole formula. The source τ^{ij} of the second-iterated field h_2^{ij} contains ρv^iv^j as well as terms of the form $(\nabla h_1^{00})^2$, both of which are $O(\rho\times\epsilon)$. [Note that $(\nabla h)^2\sim h\nabla^2h\sim\rho\epsilon$.] However, the use of the identity Eq. (2.17a) in the near-zone integration converts τ^{ij} into two time derivatives of $\tau^{00}x^ix^j$ (modulo total divergences); because of the slow-motion approximation, two time derivatives increase the order by ϵ, and thus, to sufficient accuracy, only the dominant contribution to τ^{00}, namely ρ, is needed, *without* explicit recourse to the first-iterated $h_1^{\alpha\beta}$. Instead, $h_1^{\alpha\beta}$ is buried implicitly in the equation of motion (2.8) that leads to the identity (2.17a). This circumstance is responsible for the prevalent, but erroneous view that linearized gravity (one iteration) suffices to derive the quadrupole formula. The formula so derived turns out to be ''correct,'' but its foundation is not (see [9] for discussion).

Thus, in practice, in order to evaluate EW moments required for the Nth iterated field, we will only need the $(N-2)$-iterated field contributions to the sources. This is not precisely true for the two EW surface integrals, and formally the full $(N-1)$-iterated field must be used in τ^{ij} there, but with sufficient care, it can be shown without detailed, explicit calculations that the contributions of the $(N-1)$-iterated fields all fall off sufficiently rapidly with $\mathcal{R}$ to have no effect on these surface integrals. Similarly, for the radiation-zone integration, the full $(N-1)$-iterated field must be used in τ^{ij}, Eq. (2.26). However, it will also be possible to show that the contributions of the these fields fall off with $\mathcal{R}$. To obtain the finite contributions and the contributions needed to cancel any divergent terms from the EW moments, only the $N-2$ iterated fields will be needed in practice. Thus to 2PN order (fourth iteration), only second-iterated fields will be needed explicitly in the source terms.

B. Second-iterated fields in source terms

Because the source contributions are integrated over all space, we must evaluate the second-iterated fields $h_2^{\alpha\beta}$ in a form that is valid everywhere (this and the following section follow the approach and notation of BDI; see [33], for example). The first iteration of the field equations (2.4) gives the linearized equations, $\Box h_1^{\alpha\beta}=-16\pi T^{\alpha\beta}$. Since $T^{\alpha\beta}$ has compact support, the solutions are standard Lienard-Wiechert-type retarded functions. The solutions have the leading-order behavior $h^{00}\sim\epsilon$, $h^{0i}\sim\epsilon^{3/2}$, $h^{ij}\sim\epsilon^2$. Taking these orders into account, together with the fact that, because of the slow-motion assumption, $\partial/\partial t\sim\epsilon^{1/2}\partial/\partial x^i$, we can write the second-iterated field equations in the form (we drop the subscripts)

$$\Box h^{00}=-16\pi(-g)T^{00}+\frac{7}{8}h^{00}_{,k}h^{00}_{,k}+O(\rho\epsilon^2), \tag{3.2a}$$

$$\Box h^{0i}=-16\pi(-g)T^{0i}+O(\rho\epsilon^{3/2}), \tag{3.2b}$$

$$\Box h^{ij}=-16\pi(-g)T^{ij}-\frac{1}{4}\left(h^{00}_{,i}h^{00}_{,j}-\frac{1}{2}\delta_{ij}h^{00}_{,k}h^{00}_{,k}\right)+O(\rho\epsilon^2), \tag{3.2c}$$

where we have kept only contributions required to determine h^{00}, h^{0i}, and h^{ij} to the accuracies ϵ^2, $\epsilon^{3/2}$, and ϵ^2, respectively [note that, in identifying orders of source terms with dimension (length)$^{-2}$, we can use $\Box^{-1}\rho\sim\epsilon$]. By defining the densities

$$\sigma\equiv T^{00}+T^{ii}, \tag{3.3a}$$

$$\sigma_i\equiv T^{0i}, \tag{3.3b}$$

$$\sigma_{ij}\equiv T^{ij}, \tag{3.3c}$$

and the retarded potentials

$$V(t,\mathbf{x})\equiv\int_{\mathcal{C}}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\sigma(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}'), \tag{3.4a}$$

$$V_i(t,\mathbf{x})\equiv\int_{\mathcal{C}}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\sigma_i(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}'), \tag{3.4b}$$

$$W_{ij}(t,\mathbf{x})\equiv\int_{\mathcal{C}}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij}+\frac{1}{4\pi}\left(V_{,i}V_{,j}-\frac{1}{2}\delta_{ij}V_{,k}V_{,k}\right)\right]\times(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}'), \tag{3.4c}$$

it is straightforward to solve Eqs. (3.2) to the needed order, with the result

$$h^{00}=4V-4(W-2V^2)+O(\epsilon^3), \tag{3.5a}$$

$$h^{0i}=4V_i+O(\epsilon^{5/2}), \tag{3.5b}$$

$$h^{ij}=4W_{ij}+O(\epsilon^3), \tag{3.5c}$$

where $W=W_{ii}$. It is useful to note that, although these forms of $h^{\alpha\beta}$ are of sufficient accuracy in practice to be used in the effective sources for evaluating the waveform to 2PN order, they are not sufficiently accurate for use in the equations of motion that must also be specified consistently to 2PN order. The 2PN equations of motion require h^{00} to $O(\epsilon^3)$ and h^{0i} to $O(\epsilon^{5/2})$ (h^{ij} is sufficiently accurate as it stands). However, as the 2PN equations of motion are well known, we shall not undertake their derivation here, and will simply use the published equations [44,59] when they are needed.

Because the source of V and V_i has compact support, the integrals (3.4a) and (3.4b) can be evaluated simply for field points within either the near zone or the radiation zone. But because the source of W_{ij} contains both compact and noncompact support pieces, it must be evaluated carefully, with proper attention paid to contributions from the integration over the radiation-zone part of the null cone. The details will depend on the use to which W_{ij} is being put. Evaluation of W_{ij} is discussed in Appendix C.

When we calculate the EW moments, we shall need the field contributions to $\tau^{\alpha\beta}$ evaluated at fixed retarded time u (on the hypersurface $\mathcal{M}$), and for field points with $r<\mathcal{R}$. We therefore expand the retardation $t-|\mathbf{x}-\mathbf{x}'|$ as a perturbation of the potentials V, V_i, and W_{ij} about $t=u$, with $|\mathbf{x}-\mathbf{x}'|$ acting as the expansion parameter [see Eq. (2.27)]. The results are

$$V=U+\frac{1}{2}\partial_t^2X+O(\epsilon^{5/2}), \tag{3.6a}$$

$$V_i=U_i+O(\epsilon^{5/2}), \tag{3.6b}$$

$$W_{ij}=P_{ij}+(W_{ij})_{\mathcal{C}-\mathcal{N}}+O(\epsilon^{5/2}), \tag{3.6c}$$

where the ''instantaneous'' potentials are given by

$$U(u,\mathbf{x})\equiv\int_{\mathcal{M}}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\sigma(u,\mathbf{x}'), \tag{3.7a}$$

$$X(u,\mathbf{x})\equiv\int_{\mathcal{M}}d^3x'|\mathbf{x}-\mathbf{x}'|\sigma(u,\mathbf{x}'), \tag{3.7b}$$

$$U_i(u,\mathbf{x})\equiv\int_{\mathcal{M}}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\sigma_i(u,\mathbf{x}'), \tag{3.7c}$$

$$P_{ij}(u,\mathbf{x})\equiv\int_{\mathcal{M}}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij}+\frac{1}{4\pi}\left(U_{,i}U_{,j}-\frac{1}{2}\delta_{ij}U_{,k}U_{,k}\right)\right]\times(u,\mathbf{x}'). \tag{3.7d}$$

We have used the fact that, by virtue of the conservation of mass and momentum at lowest order, $\partial_t\int\sigma d^3x\sim\epsilon^{5/2}$ and $\partial_t\int\sigma_i d^3x\sim\epsilon^3$. We will drop the contribution from the radiation-zone integral $(W_{ij})_{\mathcal{C}-\mathcal{N}}$, which falls off at least as fast as $\mathcal{R}^{-2}$ (see Appendix C). Note that these potentials satisfy $U_{i,i}=-\dot{U}$, $\nabla^2X=2U$, $P_{ij,j}=-\dot{U}_i$.

C. Near-zone metric, matter stress energy, and effective gravitational source

In order to evaluate the components of the stress-energy tensor $T^{\alpha\beta}$ to the necessary order, we need the components of the near-zone metric to post-Newtonian order. These are given from Eqs. (2.2) and (3.5) by

$$g^{00}=-(1+2V+2V^2)+O(\epsilon^3), \tag{3.8a}$$

$$g^{0i}=-4V_i+O(\epsilon^{5/2}), \tag{3.8b}$$

$$g^{ij}=(1-2V)\delta^{ij}+O(\epsilon^2), \tag{3.8c}$$

$$(-g)=1+4V-8(W-V^2)+O(\epsilon^3). \tag{3.8d}$$

These equations, together with the distributional definition (2.10) of the stress-energy tensor yield, to the requisite order,

$$\sigma=\sum_A m_A\left[1-V+\frac{3}{2}v_A^2+\frac{1}{2}V^2+\frac{1}{2}Vv_A^2+4W+\frac{7}{8}v_A^4-4V_iv_A^i+O(\epsilon^3)\right]\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{3.9a}$$

$$\sigma_i=\sum_A m_Av_A^i\left[1-V+\frac{1}{2}v_A^2+O(\epsilon^2)\right]\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{3.9b}$$

$$\sigma_{ij}=\sum_A m_Av_A^iv_A^j\left[1-V+\frac{1}{2}v_A^2+O(\epsilon^2)\right]\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{3.9c}$$

where the potentials V, V_i, and W are assumed to be evaluated at $\mathbf{x_A}$, excluding contributions of the Ath body itself (to avoid infinite self-field terms). The components of $T^{\alpha\beta}$ can be easily constructed from these expressions.

To the needed order, $\Lambda^{\alpha\beta}$ has the form

$$\Lambda^{00}=-14V_{,k}V_{,k}+16\Big[-V\ddot{V}+V_{,k}\dot{V}_k-2V_k\dot{V}_{,k}+\frac{5}{8}\dot{V}^2+\frac{1}{2}V_{m,k}(V_{m,k}+3V_{k,m})+2W_{,k}V_{,k}-W_{kl}V_{,kl}-\frac{7}{2}VV_{,k}V_{,k}\Big]+O(\rho\epsilon^3), \tag{3.10a}$$

$$\Lambda^{0i}=16\Big[V_{,k}(V_{k,i}-V_{i,k})+\frac{3}{4}\dot{V}V_{,i}\Big]+O(\rho\epsilon^{5/2}), \tag{3.10b}$$

$$\Lambda^{ij}=4\Big(V_{,i}V_{,j}-\frac{1}{2}\delta_{ij}V_{,k}V_{,k}\Big)+16\Big[2V_{,(i}\dot{V}_{j)}-V_{k,i}V_{k,j}-V_{i,k}V_{j,k}+2V_{k,(i}V_{j),k}-\delta_{ij}\Big(\frac{3}{8}\dot{V}^2+V_{,k}\dot{V}_k-V_{m,k}V_{[m,k]}\Big)\Big]+O(\rho\epsilon^3), \tag{3.10c}$$

where overdot denotes $\partial/\partial t$. Notice the presence of the cubically nonlinear terms in Λ^{00}, involving either $V\times W$ or V^3.

IV. EVALUATION OF EPSTEIN-WAGONER MOMENTS

A. Basic strategy

The EW moments are integrals over a sphere of harmonic coordinate radius $\mathcal{R}$ about the center of mass of the system, with all variables entering the integrands to be evaluated at retarded time $u=t-r$. We substitute the matter stress-energy tensor $T^{\alpha\beta}$, and the second-iterated fields evaluated in the near-zone into Eqs. (2.19). We expand all quantities to the PN order needed to achieve a 2PN-accurate waveform. Each volume integral will be split into a ''compact'' (C) piece involving integration of the compact-support matter source, and a ''field'' (F) piece, involving integration of the nonlinear field contributions. In $I^{ij}_{\rm EW}$ and $I^{ijk}_{\rm EW}$ the two surface integrations at the boundary radius $\mathcal{R}$ will involve only the field contributions, and will require somewhat special treatment.

In integrating the field terms, we will frequently integrate by parts, but will carefully evaluate and save the surface terms, using the identity

$$\int_{\mathcal{M}}\partial_k F^{ij\cdots m}d^3x=\oint_{\partial\mathcal{M}}F^{ij\cdots m}|_{\mathcal{R}}\hat{n}^k\mathcal{R}^2d^2\Omega. \tag{4.1}$$

In order to simplify some of the integrations, we will frequently make a change of variables within integrals, in order to place one of the bodies at the origin of the new variables, for example $\mathbf{y}\equiv\mathbf{x}-\mathbf{x}_A$. Even though $d^3y=d^3x$, this shift has the consequence that the region of integration $\mathcal{M}_x=\{x^i||\mathbf{x}|\leqslant\mathcal{R}\}$ will now appear in the new coordinates to be a region bounded by $|\mathbf{y}|=|\mathcal{R}\hat{\mathbf{n}}-\mathbf{x}_A|$, i.e., not centered at $\mathbf{y}=0$. It is much easier in practice to integrate in y coordinates over a region $\mathcal{M}_y=\{y^i||\mathbf{y}|\leqslant\mathcal{R}\}$, which is shifted by $\mathbf{x}_A$ relative to the true region of integration. The two integrations can be related by taking into account the appropriate surface integrals, using the identity

$$\int_{\mathcal{M}_x}f(\mathbf{x})d^3x=\int_{\mathcal{M}_y}g(\mathbf{y})d^3y-\oint_{\partial\mathcal{M}_y}g(\mathbf{y})\hat{\mathbf{y}}\cdot\mathbf{x}_A\mathcal{R}^2d^2\Omega_y+\frac{1}{2}\oint_{\partial\mathcal{M}_y}\mathbf{x}_A\cdot\boldsymbol{\nabla}g(\mathbf{y})\hat{\mathbf{y}}\cdot\mathbf{x}_A\mathcal{R}^2d^2\Omega_y+\cdots, \tag{4.2}$$

where $g(\mathbf{y})\equiv f(\mathbf{y}+\mathbf{x}_A)$ and $\hat{\mathbf{y}}=\mathbf{y}/y$. Again, we evaluate and save the surface terms.

In the end, we will only be interested in the physically measurable, transverse-traceless (TT) components of the radiation-zone field h^{ij}. We will therefore make frequent use of the identities, which follow from the definition (2.29):

$$(\delta^{ij})_{\rm TT}=0,\quad(\hat{N}^iB^j)_{\rm TT}=0, \tag{4.3}$$

where $\mathbf{B}$ is arbitrary. These identities apply only to indices ''i'' and ''j'' appearing in the components of the final waveform; we do not apply them to fields which ultimately make up source terms.

In the field integrals, we will need explicit forms for the instantaneous potentials (3.7) evaluated inside the near zone. To the needed order, they are given by

$$U(u,\mathbf{x})=\sum_A\frac{m^*_A}{|\mathbf{x}-\mathbf{x}_A|}+O(\epsilon^3), \tag{4.4a}$$

$$X(u,\mathbf{x})=\sum_A m_A|\mathbf{x}-\mathbf{x}_A|[1+O(\epsilon)], \tag{4.4b}$$

$$U_i(u,\mathbf{x})=\sum_A\frac{m_Av^i_A}{|\mathbf{x}-\mathbf{x}_A|}+O(\epsilon^{5/2}), \tag{4.4c}$$

$$P(u,\mathbf{x})=\sum_A\frac{m_Av^2_A}{|\mathbf{x}-\mathbf{x}_A|}+\frac{1}{4}U^2-\frac{1}{2}\sum_{A\neq B}\frac{m_Am_B}{|\mathbf{x}-\mathbf{x}_A||\mathbf{x}_A-\mathbf{x}_B|}+O(\epsilon^3), \tag{4.4d}$$

where $P\equiv P_{ii}$, and where

$$m^*_A\equiv m_A\Big(1+\frac{3}{2}v^2_A-\sum_B m_B/|\mathbf{x}_A-\mathbf{x}_B|+O(\epsilon^2)\Big). \tag{4.5}$$

Equation (4.4d) can be easily obtained from Eq. (3.7d) (after contraction on ij) by integrating by parts, carefully checking the vanishing of all surface terms. Although the full potential P_{ij} appears (via W_{ij}) in Λ^{00}, we will not need its explicit form, as the integration of that particular term will be handled by a ''trick'' (see Appendix D). Note that the so-called ''superpotential'' $X(u,\mathbf{x})$ is needed only to lowest order because it always appears twice time-differentiated, e.g., in Eq. (3.6a), and so its contribution is already $O(\epsilon)$ relative to that of U.

B. The two-index moment $I^{ij}_{\rm EW}$

We write Eq. (2.19a) in the form

$$I^{ij}_{\rm EW}=I^{ij}_C+I^{ij}_F+I^{ij}_S\,, \qquad (4.6)$$

where the three terms represent the compact (C), field (F), and surface (S) contributions. Substituting Eqs. (3.3), (3.6), (3.8d), (3.9), and (4.4) into $(-g)T^{00}$ and expanding through $O(\rho\epsilon^2)$, we obtain

$$\begin{aligned}I^{ij}_C=&\sum_A m_A x^{ij}_A\left(1+\frac{1}{2}v^2_A+3\sum_B\frac{m_B}{r_{AB}}\right)+\frac{3}{8}\sum_A m_A x^{ij}_A v^4_A\\&+\sum_{AB} m_A m_B\frac{x^{ij}_A}{r_{AB}}\left(2v^2_B+\frac{7}{2}v^2_A-4\mathbf{v}_A\cdot\mathbf{v}_B-\frac{3}{2}(\mathbf{v}_B\cdot\hat{\mathbf{n}}_{AB})^2\right.\\&\left.-\sum_C\frac{m_C}{r_{BC}}+\frac{7}{2}\sum_C\frac{m_C}{r_{AC}}-\frac{3}{2}\mathbf{a}_B\cdot\mathbf{x}_{AB}\right)+O(\epsilon^3)\times m x^2_A\,,\end{aligned} \qquad (4.7)$$

where $\mathbf{x}_{AB}\equiv\mathbf{x}_A-\mathbf{x}_B$, $r_{AB}\equiv|\mathbf{x}_{AB}|$, $\hat{\mathbf{n}}_{AB}\equiv\mathbf{x}_{AB}/r_{AB}$, and $\mathbf{a}_A\equiv d^2\mathbf{x}_A/dt^2$. All sums are assumed to exclude cases where a denominator (e.g., r_{BC}) might vanish.

To the required order for calculating I^{ij}_F Λ^{00} can be written in terms of the instantaneous potentials,

$$\begin{aligned}\Lambda^{00}=&-14U_{,k}U_{,k}+16\left(-\frac{7}{8}U_{,k}\ddot{X}_{,k}-U\ddot{U}+U_{,k}\dot{U}_k-2U_k\dot{U}_{,k}\right.\\&+\frac{5}{8}\dot{U}^2+\frac{1}{2}U_{m,k}(U_{m,k}+3U_{k,m})+2P_{,k}U_{,k}-P_{km}U_{,km}\\&\left.-\frac{7}{2}UU_{,k}U_{,k}\right)+O(\rho\epsilon^3).\end{aligned} \qquad (4.8)$$

For the first term, the integral $-(14/16\pi)\int_{\mathcal M}U_{,k}U_{,k}x^{ij}d^3x$ is straighforward: integrating twice by parts and showing that the surface terms are proportional to $\mathcal{R}\delta^{ij}$, which has no TT part, we are left with the integral $(14/16\pi)\int_{\mathcal M}U\nabla^2Ux^{ij}d^3x$ $=-(7/2)\Sigma_{AB}m^*_Am^*_Bx^{ij}_A/r_{AB}$. This term is of 1PN and 2PN order via the PN contributions to m^*. The next term, $-(14/16\pi)\int_{\mathcal M}U_{,k}\ddot{X}_{,k}x^{ij}d^3x$ is already of 2PN order. We integrate once by parts to remove the derivative from U. Using the fact that $\nabla^2X=2U$, we find a surface integral $-(14/16\pi)\oint_{\partial\mathcal M}U\ddot{X}_{,k}x^{ij}\hat{n}^k\mathcal{R}^2d^2\Omega$, and the new integrals $(28/16\pi)\int_{\mathcal M}U\ddot{U}x^{ij}d^3x+(28/16\pi)\int_{\mathcal M}U\ddot{X}_{,k}\delta^{k(i}x^{j)}d^3x$. The first of these volume integrals can be combined with that arising from the third term in Eq. (4.8). We substitute Eqs. (4.4a) and (4.4b), including a δ-function term that arises in $\ddot{U}$ (see Appendix B). In the surface term, we expand the integrand in powers of r^{-1}, and obtain $-(7/15)\Sigma_{AB}m_Am_B\mathcal{R}(v^{ij}_A+x^{(i}_Aa^{j)}_A)+O(\mathcal{R}^{-1})$. We drop all terms that fall off with increasing $\mathcal{R}$. In the volume integrals, for each term in the sums $\ddot{U}=\Sigma_A\ddot{U}_A$ and $\ddot{X}_{,k}=\Sigma_A\ddot{X}_{A,k}$, we change integration variables from $\mathbf{x}$ to $\mathbf{y}=\mathbf{x}-\mathbf{x}_A$ so that, for a given A, the potentials $\ddot{U}_A$ and $\ddot{X}_{A,k}$ are centered at the origin of the new $\mathbf{y}$ coordinate, while U now takes the form $\Sigma_Bm_B/|\mathbf{y}+x_{AB}|$. We calculate the surface contributions that result from this change of variables using Eq. (4.2). For example the first integral then becomes

$$\begin{aligned}\int_{\mathcal M}U\ddot{U}x^{ij}d^3x=&\sum_{AB}m_Am_B\int_{\mathcal{M}_y}\frac{1}{|\mathbf{y}+\mathbf{x}_{AB}|}\\&\times\left[y\mathbf{a}_A\cdot\hat{\mathbf{y}}-v^2_A+3(\mathbf{v}_A\cdot\hat{\mathbf{y}})^2-\frac{4\pi}{3}v^2_Ay^3\delta^3(\mathbf{y})\right]\\&\times(y^2\hat{y}^{ij}+2y\hat{y}^{(i}x^{j)}_A+x^{ij}_A)y^{-3}d^3y.\end{aligned} \qquad (4.9)$$

We use the spherical harmonic expansion

$$\frac{1}{|\mathbf{y}+\mathbf{x}_{AB}|}\equiv\sum_{l,m}\frac{4\pi}{2l+1}\frac{(-r_<)^l}{r_>^{l+1}}Y^*_{lm}(\hat{\mathbf{n}}_{AB})Y_{lm}(\hat{\mathbf{y}}), \qquad (4.10)$$

where $r_{<(>)}$ denotes the lesser (greater) of r_{AB} and y, express all products of unit vectors $\hat{y}^k$ in terms of symmetric, trace-free (STF) products using Eqs. (A2), and integrate over directions $\hat{\mathbf{y}}$, using the identity

$$\sum_m\int Y^*_{lm}(\hat{\mathbf{n}})Y_{lm}(\hat{\mathbf{y}})\hat{y}^{\langle L'\rangle}d^2\Omega_y\equiv\hat{n}^{\langle L\rangle}\delta_{ll'}, \qquad (4.11)$$

(see Appendix A) where the superscript $\langle L\rangle$ over a unit vector denotes an l-dimensional STF product. We then integrate over y, using the formula

$$\int_0^{\mathcal R}\frac{r^l_<}{r^{l+1}_>}y^qdy=\frac{(2l+1)r^q_{AB}}{(l+q+1)(l-q)}\left[1-\frac{l+q+1}{2l+1}\left(\frac{\mathcal R}{r_{AB}}\right)^{q-l}\right]\qquad(q-l\neq0). \qquad (4.12)$$

The result is a series of terms of three types: those with nonvanishing TT part that are independent of $\mathcal{R}$ and linear in $\mathcal{R}$, which we keep; terms with vanishing TT part which we discard (regardless of their dependence on $\mathcal{R}$); and terms that fall off with increasing $\mathcal{R}$, which we also discard. An example of the second type of term would be a contribution to $I^{ij}_{\rm EW}$ proportional to $\mathcal{R}\delta^{ij}$. The contribution of such a term to h^{ij} has no TT part; equivalently, it can be eliminated to the necessary order by a finite gauge or coordinate transformation.

Many of the field integrations that we encounter in evaluating the EW moments are amenable to this general method: (i) integrate by parts to leave one potential undifferentiated, (ii) change variables to put the center of the differentiated potentials at the origin, (iii) expand the undifferentiated potential in spherical harmonics, (iv) express all unit vector products in STF terms, (v) integrate over d^3y using the identites (4.11) and (4.12), (vi) retain all relevant contributions from surface integrals that arise in steps (i) and (ii).

Terms 2–8 contributed by Λ^{00} [Eq. (4.8)] can be handled using this method, as can the compact contributions to P_{ij} and P (proportional to velocities) in terms 9 and 10. However the nonlinear field contributions to P_{ij} and P lead to additional complications, although the basic method still applies. These terms are discussed in Appendix D. Finally, the cubically nonlinear term 11 can be calculated easily by integrating by parts. Computation of these terms is straightforward but tedious. In evaluating 2PN terms, we make repeated use of the fact, valid to Newtonian order, that $\Sigma_A m_A \mathbf{x}_A = 0$.

We now turn to the surface term I_S^{ij} given by Eq. (2.20a). Because the surface lies outside the matter source, only the field contribution, Λ^{ij} is needed. The term can be rewritten in the form

$$(d/dt)^2 I_S^{ij} = (1/16\pi) \oint_{\partial\mathcal{M}} (2\Lambda^{k(i}\hat{n}^{j)k}\mathcal{R}^3 - \Lambda^{kl}{}_{,l}\hat{n}^{ijk}\mathcal{R}^4) d^2\Omega. \tag{4.13}$$

However, because I_S^{ij} is essentially two anti-time derivatives of the surface integral, reducing its order by ϵ, we need to know Λ^{ij} to $O(\rho\epsilon^3)$, i.e., to $O(\epsilon^2)$ beyond its leading order terms, at least in principle. This is in contrast to having to know Λ^{00} in the spatial integral I_F^{ij} only to $O(\epsilon)$ beyond its leading order. This would present considerable complications, except for the fact that we only need to calculate a surface integral, and retain terms that are either independent of or grow with $\mathcal{R}$. Consequently we only need to retain contributions to Λ^{ij} that vary as $\mathcal{R}^{-2}$ or $\mathcal{R}^{-3}$. To see what terms must be retained, we return to the definition of Λ^{ij}, Eq. (2.6). Far from the source, the fields $h^{\alpha\beta}$ have the leading ϵ and r dependences $h^{00} \sim \epsilon/r$, $h^{0i} \sim \epsilon^{3/2}/r^2$ (r^{-2} here because the net momentum of the system vanishes), and $h^{ij} \sim \epsilon^2/r$; Λ^{ij} has the schematic form $(h_{,\lambda})^2 + h(h_{,\lambda})^2 + h^2(h_{,\lambda})^2 + \cdots$. By combining the leading forms of $h^{\alpha\beta}$ with the knowledge that time derivatives increase the order by $\epsilon^{1/2}$, while spatial derivatives either increase the rate of fall off by one power of r^{-1} or increase the order by $\epsilon^{1/2}$ via the retarded time dependence, it can be shown by inspection that terms of order $h(h_{,\lambda})^2$ and higher are either of higher than 2PN order, or fall off faster than $\mathcal{R}^{-3}$, or generate angular dependence that leads to no TT parts. However, the purely quadratic terms proportional to $(h_{,\lambda})^2$ do contribute; their explicit contribution is given by the nonlinear terms of Eq. (2.6) with $g_{\mu\nu}$ replaced by $\eta_{\mu\nu}$. Again, inspection shows that, to the required order, we can write

$$\Lambda^{ij} = -h^{00}\ddot{h}^{ij} + \frac{1}{4}h^{00}{}_{,i}h^{00}{}_{,j} + 2h^{00,(i}\dot{h}^{j)0} - \delta^{ij}\left(\frac{1}{8}h^{00}{}_{,k}h^{00}{}_{,k} + h^{00,k}\dot{h}^{k0}\right). \tag{4.14}$$

Further inspection shows that knowing $h^{\alpha\beta}$ to the accuracy shown in Eq. (3.5) suffices; the higher-order terms not explicitly shown in those expressions contribute terms either at higher-than-2PN order, or at faster-than-$\mathcal{R}^{-3}$ falloff. We do need to evaluate V, V_i, and W_{ij} carefully, however. Expanding these functions in powers of $|\mathbf{x}-\mathbf{x}'|$ about $t=u$, but to higher orders than that shown in Eq. (3.6), using Eqs. (3.9) for σ, σ_i, and σ_{ij}, and displaying only terms that lead to the appropriate contributions in Λ^{ij}, we find, in the vicinity of $r=\mathcal{R}$,

$$V = \frac{\tilde{m}}{r} + \frac{1}{4r}\ddot{Q}^{kl}(3\delta^{kl} - n^k n^l) + O(\epsilon^3/r) - \frac{2}{3}\overset{(3)}{Q} + O(\epsilon^3 r^0) + \frac{r}{16}\overset{(4)}{Q}{}^{kl}(\delta^{kl} + n^k n^l) + O(\epsilon^4 r) + O(\epsilon^2/r^2) + O(\epsilon^2/r^3), \tag{4.15a}$$

$$V_i = -\frac{1}{2r^2}(\epsilon^{ijk}J^k - \dot{Q}^{ij})n^j + O(\epsilon^{5/2})/r^2 - \frac{1}{4}\overset{(3)}{Q}{}^{ij}n^j + O(\epsilon^3 r^0) + O(\epsilon^{3/2}/r^3), \tag{4.15b}$$

$$W^{ij} = \frac{1}{2r}\ddot{Q}^{ij} + O(\epsilon^2/r^2) + O(\epsilon^3/r), \tag{4.15c}$$

where we define here and for future use

$$\tilde{m} \equiv m + E, \tag{4.16a}$$

$$E \equiv \frac{1}{2}\sum_A \left(m_A v_A^2 - \sum_B m_A m_B / r_{AB} \right), \tag{4.16b}$$

$$\mathbf{X} \equiv \tilde{m}^{-1}\sum_A m_A \mathbf{x}_A \left(1 + \frac{1}{2}v_A^2 - \frac{1}{2}\sum_B m_B/r_{AB} \right) = 0, \tag{4.16c}$$

$$Q^{ij} \equiv \sum_A m_A x_A^{ij}, \tag{4.16d}$$

$$Q^{ijk} \equiv \sum_A m_A x_A^{ijk}, \tag{4.16e}$$

$$J^i \equiv \sum_A m_A \epsilon^{ilm} x_A^l v_A^m, \tag{4.16f}$$

$$J^{ij} \equiv \sum_A m_A \epsilon^{ilm} x_A^l v_A^m x_A^j, \tag{4.16g}$$

where $m = \Sigma_A m_A$, and $Q = Q^{ii}$. In Eq. (4.15) we show schematically the ϵ order and the r dependence of the terms neglected. Note that, by virtue of the Newtonian equations of motion, E and J^i are constant to leading order. Here $\tilde{m}$, Q^{ij}, and J^i are to be evaluated at $u = t - R$. Combining Eqs. (4.15), (3.5), (4.14), and (4.13), we find, to the required order that $I_S^{ij} = -(7/6) m\mathcal{R}\ddot{Q}^{ij}$.

Combining I_C^{ij}, I_F^{ij}, and I_S^{ij}, we obtain finally

$$I^{ij}_{\rm EW}=\sum_A m_A x_A^{ij}\left(1+\frac{1}{2}v_A^2-\frac{1}{2}\sum_B \frac{m_B}{r_{AB}}\right)+\frac{3}{8}\sum_A m_A x_A^{ij} v_A^4+\frac{1}{12}\sum_{AB} m_A x_A^{ij}\frac{m_B}{r_{AB}}$$
$$\times\left\{28v_A^2-11v_B^2-22\mathbf{v}_A\cdot\mathbf{v}_B-(\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB})^2+2(\mathbf{v}_B\cdot\hat{\mathbf{n}}_{AB})^2-2\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}\mathbf{v}_B\cdot\hat{\mathbf{n}}_{AB}+2(\mathbf{a}_A+\mathbf{a}_B)\cdot\mathbf{x}_{AB}+6\sum_C\frac{m_C}{r_{BC}}\right\}$$
$$-\frac{1}{12}\sum_{AB}\frac{m_A m_B}{r_{AB}}\left\{\frac{1}{2}\{(\mathbf{v}_A+\mathbf{v}_B)^2-[(\mathbf{v}_A+\mathbf{v}_B)\cdot\hat{\mathbf{n}}_{AB}]^2\}x_A^{(i}x_B^{j)}+2(\mathbf{v}_A+\mathbf{v}_B)\cdot\mathbf{x}_{AB}(10v_A^{(i}x_A^{j)}+11v_A^{(i}x_B^{j)})\right.$$
$$\left.-(26v_A^{ij}-49v_A^{(i}v_B^{j)})r_{AB}^2\right\}-\frac{1}{12}\sum_{AB} m_A m_B r_{AB}\{\mathbf{a}_A\cdot\hat{\mathbf{n}}_{AB}x_A^{(i}\hat{n}_{AB}^{j)}-a_A^{(i}x_A^{j)}+23(a_A+a_B)^{(i}x_A^{j)}\}$$
$$-3\sum_{AB} m_A^2 m_B\hat{n}_{AB}^{ij}+\mathcal{G}^{ij}_{(3)}-\frac{14}{5}m\mathcal{R}\ddot{Q}^{ij}+O(\epsilon^3)Q^{ij}, \tag{4.17}$$

where $\mathcal{G}^{ij}_{(3)}$ is a complicated three-body term arising from the $P_{km}U_{,km}$ term in Eq. (4.8), that vanishes identically for two-body systems. It is evaluated in Appendix D.

C. The three-index moment $I^{ijk}_{\rm EW}$

Since $I^{ijk}_{\rm EW}$ is dominantly of 1/2PN order, we need to calculate only the first post-Newtonian corrections to it, i.e., terms of 3/2PN order. We first note that $I^{ijk}_{\rm EW}$ [Eqs. (2.19b) and (2.20b)] can be written

$$I^{ijk}_{\rm EW}=\tilde{I}^{ijk}_{\rm EW}+\tilde{I}^{jik}_{\rm EW}-\tilde{I}^{kij}_{\rm EW}, \tag{4.18}$$

where we separate $\tilde{I}^{ijk}_{\rm EW}$ into compact, field, and surface contributions, given by

$$\tilde{I}^{ijk}_C+\tilde{I}^{ijk}_F=\int_{\mathcal{M}}\tau^{0i}x^j x^k d^3x,$$
$$(d/dt)\tilde{I}^{ijk}_S=(1/16\pi)\oint_{\partial\mathcal{M}}\Lambda^{li}\hat{n}^{jkl}\mathcal{R}^4 d^2\Omega. \tag{4.19}$$

Substituting Eqs. (3.3), (3.6a), (3.8d), (3.9), and (4.4) into $(-g)T^{0i}$ and expanding through $O(\rho\epsilon^{3/2})$, we obtain

$$\tilde{I}^{ijk}_C=\sum_A m_A v_A^i x_A^{jk}\left(1+\frac{1}{2}v_A^2+3\sum_B\frac{m_B}{r_{AB}}\right)+O(\epsilon^{5/2})\times Q^{ij}. \tag{4.20}$$

To the required order,

$$\Lambda^{0i}=16\left[U_{,k}(U_{k,i}-U_{i,k})+\frac{3}{4}\dot{U}U_{,i}\right]. \tag{4.21}$$

We then calculate $\tilde{I}^{ijk}_F$ following the method laid out in Sec. IV A. In the course of this calculation we find no TT terms dependent on positive powers of $\mathcal{R}$. Finally we evaluate the surface contribution using Eqs. (4.14) and (4.15) evaluated to lowest order, and find no contributions. The final result is

$$\tilde{I}^{ijk}_{\rm EW}=\sum_A m_A v_A^i x_A^{jk}\left(1+\frac{1}{2}v_A^2-\frac{1}{2}\sum_B\frac{m_B}{r_{AB}}\right)-\frac{1}{2}\sum_{AB}\frac{m_A m_B}{r_{AB}}\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}\hat{n}^i_{AB}x_A^{jk}-\frac{1}{12}\sum_{AB} m_A m_B r_{AB}[2\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}\hat{n}^{ijk}_{AB}$$
$$+11(2v_A^i\hat{n}^{jk}_{AB}-v_A^j\hat{n}^{ik}_{AB}-v_A^k\hat{n}^{ij}_{AB})]+\frac{1}{2}\sum_{AB} m_A m_B[\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}\hat{n}^{i(j}_{AB}x_A^{k)}+7v_A^i x_A^{(j}\hat{n}^{k)}_{AB}-7v_A^{(j}x_A^{k)}\hat{n}^i_{AB}]+O(\epsilon^{5/2})\times Q^{ij}. \tag{4.22}$$

The result agrees with Eq. (A52) of [35]. The full moment $I^{ijk}_{\rm EW}$ can be constructed from this using Eq. (4.18).

D. The four-index moment $I^{ijkl}_{\rm EW}$

Since this moment contributes to the waveform already at PN order, we only need to evaluate the integrands through their first PN corrections. We write Eq. (2.19c) in the form

$$I^{ijkl}_{\rm EW}=I^{ijkl}_C+I^{ijkl}_F \tag{4.23}$$

(there is no surface contribution). Expanding $(-g)T^{ij}$ through $O(\rho\epsilon^2)$, we obtain

$$I_C^{ijkl}=\sum_A m_A v_A^{ij} x_A^{kl}\left(1+\frac{1}{2}v_A^2+3\sum_B \frac{m_B}{r_{AB}}\right)+O(\epsilon^3)\times Q^{ij}. \tag{4.24}$$

To the required order, Λ^{ij} can be written in terms of the instantaneous potentials

$$\Lambda^{ij}=4\left(U_{,i}U_{,j}-\frac{1}{2}\delta_{ij}U_{,k}U_{,k}\right)+16\Bigg[\frac{1}{4}U_{,(i}\ddot{X}_{,j)}+2U_{,(i}\dot{U}_{j)}-U_{k,i}U_{k,j}-U_{i,k}U_{j,k}+2U_{k,(i}U_{j),k}$$
$$-\delta_{ij}\left(\frac{1}{8}U_{,k}\ddot{X}_{,k}+\frac{3}{8}\dot{U}^2+U_{,k}\dot{U}_k-U_{m,k}U_{[m,k]}\right)\Bigg]+O(\rho\epsilon^3). \tag{4.25}$$

The terms proportional to δ^{ij} produce no TT contributions to the waveform, so we drop them.

The method proceeds as in the previous cases, without the complications of cubic nonlinearities. The result is

$$I_{\rm EW}^{ijkl}=\sum_A m_A\left(v_A^{ij}-\frac{1}{2}\sum_B m_B\hat{n}_{AB}^{ij}/r_{AB}\right)x_A^{kl}+\frac{1}{12}\sum_{AB} m_Am_Br_{AB}\hat{n}_{AB}^{ij}(\hat{n}_{AB}^{kl}-\delta^{kl})+\frac{1}{2}\sum_A m_Av_A^2v_A^{ij}x_A^{kl}$$
$$-\frac{1}{4}\sum_{AB} m_Am_Bx_A^{kl}/r_{AB}\Bigg(2v_A^{ij}+(4v_{AB}^2-v_B^2)\hat{n}_{AB}^{ij}-3(\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB})^2\hat{n}_{AB}^{ij}-4(3\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}-4\mathbf{v}_B\cdot\hat{\mathbf{n}}_{AB})v_A^{(i}\hat{n}_{AB}^{j)}-16a_B^{(i}x_{AB}^{j)}-2a_A^{(i}x_{AB}^{j)}$$
$$+\mathbf{a}_A\cdot\mathbf{x}_{AB}\hat{n}_{AB}^{ij}-2\sum_C m_C(1/r_{AC}+1/r_{BC})\hat{n}_{AB}^{ij}\Bigg)-\frac{1}{24}\sum_{AB} m_Am_Br_{AB}\delta^{kl}$$
$$\times\Bigg((4v_{AB}^2-v_A^2)\hat{n}_{AB}^{ij}+2(8v_{AB}^{ij}-23v_B^{ij})-(\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB})^2\hat{n}_{AB}^{ij}-8\mathbf{v}_{AB}\cdot\hat{\mathbf{n}}_{AB}v_{AB}^{(i}\hat{n}_{AB}^{j)}+4\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}v_A^{(i}\hat{n}_{AB}^{j)}-14a_A^{(i}x_{AB}^{j)}+\mathbf{a}_A\cdot\mathbf{x}_{AB}\hat{n}_{AB}^{ij}$$
$$-4\sum_C (m_C/r_{AC})\hat{n}_{AB}^{ij}\Bigg)+\frac{1}{24}\sum_{AB} m_Am_Br_{AB}\hat{n}_{AB}^{ijkl}\left(4v_{AB}^2-5v_A^2+9(\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB})^2-4\sum_C m_C/r_{AC}-3\mathbf{a}_A\cdot\mathbf{x}_{AB}\right)$$
$$+\frac{1}{3}\sum_{AB} m_Am_B\hat{n}_{AB}^{(k}x_A^{l)}(v_A^2\hat{n}_{AB}^{ij}+10v_A^{ij}+4(\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}-3\mathbf{v}_{AB}\cdot\hat{\mathbf{n}}_{AB})v_A^{(i}\hat{n}_{AB}^{j)}-3(\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB})^2\hat{n}_{AB}^{ij}-14a_A^{(i}x_{AB}^{j)}+\mathbf{a}_A\cdot\mathbf{x}_{AB}\hat{n}_{AB}^{ij})$$
$$+\frac{1}{12}\sum_{AB} m_Am_Br_{AB}\hat{n}_{AB}^{ij}(v_A^{kl}+2\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}v_A^{(k}\hat{n}_{AB}^{l)}-a_A^{(k}x_{AB}^{l)}+2a_A^{(k}x_A^{l)})+\frac{1}{12}\sum_{AB} m_Am_Br_{AB}\hat{n}_{AB}^{kl}(4v_{AB}^{ij}-21v_A^{ij}$$
$$+8\mathbf{v}_{AB}\cdot\hat{\mathbf{n}}_{AB}v_{AB}^{(i}\hat{n}_{AB}^{j)}-6\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}v_A^{(i}\hat{n}_{AB}^{j)}+35a_A^{(i}x_{AB}^{j)})+\frac{1}{3}\sum_{AB} m_Am_Br_{AB}(2v_{AB}^{(i}\hat{n}_{AB}^{j)}v_{AB}^{(k}\hat{n}_{AB}^{l)}-v_A^{(i}\hat{n}_{AB}^{j)}v_A^{(k}\hat{n}_{AB}^{l)})$$
$$+\frac{1}{3}\sum_{AB} m_Am_B(2v_A^{(i}\hat{n}_{AB}^{j)}v_A^{(k}x_A^{l)}-\mathbf{v}_A\cdot\hat{\mathbf{n}}_{AB}\hat{n}_{AB}^{ij}v_A^{(k}x_A^{l)}-12v_A^{(i}\hat{n}_{AB}^{j)}v_{AB}^{(k}x_A^{l)})-\frac{8}{35}m\mathcal{R}\ddot{Q}^{ij}\delta^{kl}+O(\epsilon^3)\times Q^{ij}. \tag{4.26}$$

E. The five- and six-index moments $I_{\rm EW}^{ijklm}$ and $I_{\rm EW}^{ijklmn}$

These moments contribute to the waveform at 3/2PN and 2PN order, respectively, thus we only need to evaluate the dominant, Newtonian contributions to the integrands. Splitting the moments into a compact and a field piece, substituting the lowest-order contributions to τ^{ij} at $O(\rho\epsilon)$, into Eq. (2.19c), namely $(-g)T^{ij}=\Sigma_A m_A v_A^{ij}\delta^3(\mathbf{x}-\mathbf{x}_A)$, and $\Lambda^{ij}=4(U_{,i}U_{,j}-\frac{1}{2}\delta_{ij}U_{,k}U_{,k})$, and carrying out the integration procedures as above, we obtain

$$I_{\rm EW}^{ijklm}=\frac{1}{3}\frac{d}{dt}\left\{\sum_A m_Ax_A^{klm}\left(v_A^{ij}-\frac{1}{2}\sum_B\frac{m_B}{r_{AB}}\hat{n}_{AB}^{ij}\right)+\frac{1}{4}\sum_{AB} m_Am_Br_{AB}\hat{n}_{AB}^{ij}x_A^{(k}(\hat{n}_{AB}^{lm)}-\delta^{lm})+O(\epsilon^{5/2})\times Q^{ij},\right. \tag{4.27a}$$

$$I_{\rm EW}^{ijklmn}=\frac{1}{12}\frac{d^2}{dt^2}\Bigg\{\sum_A m_Ax_A^{klmn}\left(v_A^{ij}-\frac{1}{2}\sum_B\frac{m_B}{r_{AB}}\hat{n}_{AB}^{ij}\right)+\frac{1}{2}\sum_{AB} m_Am_Br_{AB}\Bigg[\hat{n}_{AB}^{ij}x_A^{(kl}(\hat{n}_{AB}^{mn)}-\delta^{mn)})-\frac{1}{10}x_{AB}^{ij}(2\hat{n}_{AB}^{klmn}-2\hat{n}_{AB}^{(kl}\delta^{mn)}$$
$$-\delta^{(kl}\delta^{mn)})\Bigg]-\frac{8}{105}m\mathcal{R}Q^{ij}\delta^{(kl}\delta^{mn)}\Bigg\}+O(\epsilon^3)\times Q^{ij}. \tag{4.27b}$$

Equation (4.27a) is equivalent to Eq. (A53d) of [35].

V. EVALUATION OF RADIATION-ZONE CONTRIBUTIONS

We now turn to the evaluation of the contribution $h^{ij}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})$ given by the integral over the remainder of the past light cone of the observer, Eq. (2.23). There is no material source now, so $\tau^{ij}=\Lambda^{ij}/16\pi$. On the other hand, the time dependence in the integrand of Eq. (2.23) is not the simple fixed retarded time $u=t-R$ of the EW moments. The $(u'+r')$ dependence of τ^{ij} in Eq. (2.23) reflects the variation in retarded time along each two-dimensional intersection of the past light cone of the event $(t,\mathbf{x})$ with the future light cone of the event $(u',0)$. However, τ^{ij} is a functional of retarded potentials, such as V. When evaluated at $u'+r'$, V has the form

$$V(u'+r',\mathbf{x}')=\int \frac{d^3x''}{|\mathbf{x}'-\mathbf{x}''|}\sigma(u'+r'-|\mathbf{x}'-\mathbf{x}''|,\mathbf{x}''), \tag{5.1}$$

Notice that, because $|\mathbf{x}''|\ll\mathcal{R}$, while $|\mathbf{x}'|>\mathcal{R}$, we can approximate

$$u'+r'-|\mathbf{x}'-\mathbf{x}''|\approx u'+\hat{\mathbf{n}}'\cdot\mathbf{x}''+(2r')^{-1}[(\hat{\mathbf{n}}'\cdot\mathbf{x}'')^2-r''^2]+\cdots, \tag{5.2}$$

where $\hat{\mathbf{n}}'=\mathbf{x}'/r'$, and then expand such retarded functions about u' in powers of the small quantity r''/r'. For a given u', the retarded fields that contribute to Λ^{ij} along the intersection between the two light cones in Fig. 5 all have their source in the near zone, on slices of the near-zone world tube that pass through the center of mass at time u'. The expansion (5.2) simply reflects the fact that, as one moves around the source in angle [integration over $d^2\Omega$ in Eq. (2.23)], the orientation of the slice of the near-zone world tube that generates the fields precesses (see Fig. 6).

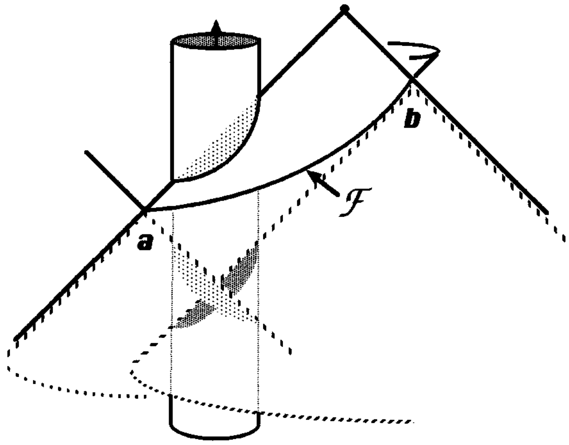

FIG. 6. Fields contributing to $\Lambda^{\alpha\beta}$ at two representative points a and b on F have sources near same event u' at $r=0$. Only orientation of near-zone source slice varies as angular integration moves around F.

Since the ingredients of Λ^{ij} are all fields evaluated in the radiation zone, we can use expansions in powers of $1/r'$, such as those of Eq. (2.14). The angular dependence of such expansions can always be expressed in terms of STF products of radial unit vectors $\hat{\mathbf{n}}'$ (analogues of spherical harmonics). Thus Λ^{ij} can be written, in the regime $r'\gg\mathcal{R}$, as a sequence of terms of the generic form $f^{ij}_{N,l}(u')\hat{n}'^{\langle L\rangle}r'^{-N}$. Then a change of variables

$$\zeta\equiv(t-u')/r=1+(u-u')/r \tag{5.3}$$

puts Eq. (2.26) into the form, for each (N,l) term,

$$h^{ij}_{\mathcal{C}-\mathcal{N}}(N,l)=\left(\frac{2}{r}\right)^{N-2}\int_1^{1+2\mathcal{R}/r}\frac{d\zeta}{(\zeta^2-1)^{N-2}}f_{N,l}[u-r(\zeta-1)](4\pi)^{-1}\int_0^{2\pi}d\phi'\int_{1-\alpha}^{1}\hat{n}'^{\langle L\rangle}(\zeta-\hat{\mathbf{n}}'\cdot\hat{\mathbf{n}})^{N-3}d\cos\theta'$$
$$+\left(\frac{2}{r}\right)^{N-2}\int_{1+2\mathcal{R}/r}^{\infty}\frac{d\zeta}{(\zeta^2-1)^{N-2}}f_{N,l}[u-r(\zeta-1)](4\pi)^{-1}\int_0^{2\pi}d\phi'\int_{-1}^{1}\hat{n}'^{\langle L\rangle}(\zeta-\hat{\mathbf{n}}'\cdot\hat{\mathbf{n}})^{N-3}d\cos\theta', \tag{5.4}$$

where [cf. Eq. (2.25)] $\alpha=(\zeta-1)(\zeta+1-2\mathcal{R}/r)(r/2\mathcal{R})$. We first carry out the angular integrals, which yield $\hat{n}^{\langle L\rangle}A_{N,l}(\zeta,\alpha)$, where $A_{N,l}$ can be computed from Legendre polynomials $P_l(z)$ by $A_{N,l}(\zeta,\alpha)=\frac{1}{2}\int_{1-\alpha}^{1}P_l(z)(\zeta-z)^{N-3}dz$ [see Appendix A, Eq. (A5); $\alpha=2$ corresponds to the full 4π angular integration]. Then, in the ζ integration from 1 to $1+2\mathcal{R}/r$, we expand the retarded time dependence of the $f_{N,l}$ about u, then integrate; this is valid since $\mathcal{R}<r$. In the integrals from $1+2\mathcal{R}/r$ to ∞, we integrate by parts numerous times, each time increasing the number of time derivatives of $f_{N,l}$, stopping when the result exceeds the PN order required. The boundary terms that arise are evaluated at $\zeta=1+2\mathcal{R}/r$ and $\zeta=\infty$, corresponding to retarded time $u-2\mathcal{R}$ and $-\infty$, respectively. At the former boundary, we again expand the functions about u; at the latter boundary the contributions are assumed to be zero, which is equivalent to making the usual and reasonable assumption that the source is not extraordinarily dynamical in the infinite past.

For the cases where the field point is inside the near zone, Eq. (5.4) still applies, except that now $r<R$, and the first ζ integral runs from $-1+2\mathcal{R}/r$ to $1+2\mathcal{R}/r$ (Fig. 5).

In working to 2PN order, just as in the case of the EW surface integrals, Eqs. (2.20), here we must also evaluate the integrand Λ^{ij} to $O(\rho\epsilon^3)$. Here, as before, it can be shown that only the twice-iterated fields are needed in practice. This can be seen as follows. We are only interested in the $1/r$ part of the waveform. Thus a contribution to Λ^{ij} that is already $O(\rho\epsilon^3)$ but that falls off faster than r'^{-3} $(N>3)$ can be dropped. This would apply to all terms that are quartically nonlinear and higher, such as terms of the form $h^2(h_{,\lambda})^2$, which fall off as r'^{-6}. Cubically nonlinear terms of the form $h(h_{,\lambda})^2$ can also be dropped; at leading order, they are $O(\rho\epsilon^2)$, but fall off as r'^{-5}. One might worry that by expanding $f^{ij}_{N,l}$ in Eq. (5.4) about u (the value of retarded time at which all contributions to the waveform are to be evaluated in the end), one could reduce the rate of fall off by one power of r for each retarded time derivative. But each time derivative either raises the order of the term by $\epsilon^{1/2}$ or kills it outright via a conservation law, such as for the Newtonian potential $h\sim m/r$. Thus the leading cubically nonlinear contribution turns out to be of order $\rho\epsilon^3/r'^5$, which can be dropped. Thus only quadratically nonlinear terms of the form $(h_{,\lambda})^2$ need to be considered. As before, a knowledge of $h^{\alpha\beta}$ to the accuracy shown in Eq. (3.5) suffices; higher-order terms contribute terms of order $\rho\epsilon^3/r'^4$. However, we must now be careful in evaluating the terms which *do* contribute. For example a term of $O(\rho\epsilon)$ that falls off as r'^{-6}, can, after three terms in the Taylor expansion of its retarded time dependence about u in powers of $r(\zeta-1)$, lead to a $1/r$ contribution to the waveform at $O(\Box^{-1}\rho\epsilon^{5/2})$, which is 3/2PN order beyond quadrupole order. A term of this form would arise from the cross term between the gradient of the Newtonian potential m/r and the Newtonian quadrupole potential $\sim Q^{ij}/r^3$. Similarly a $(\rho\epsilon)r'^{-7}$ term would contribute a 2PN contribution to the waveform. Such a term would arise from a cross term between the Newtonian potential and the Newtonian octupole potential $\sim Q^{ijk}/r^4$. A consequence of these considerations is that, in expanding the second-iterated fields $h^{\alpha\beta}$, we must use the general multipole expansions of Eq. (2.14), expanded through octupole order. This amounts to expanding h^{00} through $q=3$, h^{0i} through $q=2$, and h^{00} through $q=1$. Evaluating the integrals $M^{\alpha\beta k_1\cdots k_q}$ to the needed order, using the general method for integrating over the near-zone hypersurface $\mathcal{M}$ described in Sec. II D, and adding any contributions to $h^{\alpha\beta}$ from the radiation-zone integrations (primarily from W^{ij}; see Appendix C), we obtain

$$
\begin{aligned}
h^{00}&=4m/r'+7(m/r')^2+2\{r'^{-1}Q^{ij}(u')\}_{,ij}\\
&\quad-\frac{2}{3}\{r'^{-1}Q^{ijk}(u')\}_{,ijk},\\
h^{0i}&=-2\{r'^{-1}[\dot{Q}^{ij}(u')-\epsilon^{ija}J^a(u')]\}_{,j}\\
&\quad+\frac{2}{3}\{r'^{-1}[\dot{Q}^{ijk}(u')-2\epsilon^{ika}J^{aj}(u')]\}_{,jk},\\
h^{ij}&=(m/r')^2\hat{n}'^{ij}+2\ddot{Q}^{ij}(u')/r'\\
&\quad-\frac{2}{3}\{r'^{-1}[\ddot{Q}^{ijk}(u')-4\epsilon^{(i|ka}\dot{J}^{a|j)}(u')]\}_{,k},
\end{aligned}
\tag{5.5}
$$

where Q^{ij}, Q^{ijk}, J^a, and J^{aj} are defined in Eqs. (4.16d)–(4.16g), and where the superscript notation ${}^{(i|a\cdots k|j)}$ denotes symmetrization only on i and j.

To the required order, we then have

$$
\Lambda^{ij}=-h^{00}\ddot{h}^{ij}+\frac{1}{4}h^{00}{}_{,i}h^{00}{}_{,j}+\frac{1}{2}h^{00}{}_{,(i}h^{kk}{}_{,j)}+2h^{00,(i}\dot{h}^{j)0},
\tag{5.6}
$$

with the result

$$
\begin{aligned}
\Lambda^{ij}=\frac{4m}{r'^2}\Bigg[&\hat{n}'^{\langle ijkl\rangle}\left(\frac{15Q^{\langle kl\rangle}}{r'^4}+\frac{15\dot{Q}^{\langle kl\rangle}}{r'^3}+\frac{6\ddot{Q}^{\langle kl\rangle}}{r'^2}+\frac{\overset{(3)}{Q}{}^{\langle kl\rangle}}{r'}\right)+\hat{n}'^{\langle k(i\rangle}\left(\frac{18Q^{\langle j)k\rangle}}{7r'^4}+\frac{18\dot{Q}^{\langle j)k}}{7r'^3}-\frac{18\ddot{Q}^{\langle j)k}}{7r'^2}-\frac{24\overset{(3)}{Q}{}^{\langle j)k}}{7r'}\right)\\
&-\left(\frac{6\ddot{Q}^{\langle ij\rangle}}{5r'^2}+\frac{6\overset{(3)}{Q}{}^{\langle ij\rangle}}{5r'}+2\overset{(4)}{Q}{}^{\langle ij\rangle}\right)+\hat{n}'^{\langle ijklm\rangle}\left(\frac{35Q^{\langle klm\rangle}}{r'^5}+\frac{35\dot{Q}^{\langle klm\rangle}}{r'^4}+\frac{15\ddot{Q}^{\langle klm\rangle}}{r'^3}+\frac{10\overset{(3)}{Q}{}^{\langle klm\rangle}}{3r'^2}+\frac{\overset{(4)}{Q}{}^{\langle klm\rangle}}{3r'}\right)\\
&+\hat{n}'^{\langle kl(i\rangle}\left(\frac{25Q^{\langle j)kl\rangle}}{3r'^5}+\frac{25\dot{Q}^{\langle j)kl\rangle}}{3r'^4}-\frac{25\overset{(3)}{Q}{}^{\langle j)kl\rangle}}{9r'^2}-\frac{10\overset{(4)}{Q}{}^{\langle j)kl\rangle}}{9r'}\right)+\hat{n}'^{\langle kl(i\rangle}\epsilon^{j)ka}\left(\frac{8\dot{J}^{ak}}{r'^3}+\frac{8\ddot{J}^{ak}}{r'^2}+\frac{8\overset{(3)}{J}{}^{ak}}{3r'}\right)\\
&-\hat{n}'^{\langle ijl\rangle}\epsilon^{kla}\left(\frac{4\dot{J}^{ak}}{r'^3}+\frac{4\ddot{J}^{ak}}{r'^2}+\frac{4\overset{(3)}{J}{}^{ak}}{3r'}\right)-\hat{n}'^{k}\left(\frac{10\ddot{Q}^{\langle ijk\rangle}}{7r'^3}+\frac{10\overset{(3)}{Q}{}^{\langle ijk\rangle}}{7r'^2}-\frac{4\overset{(4)}{Q}{}^{\langle ijk\rangle}}{21r'}+\frac{4\overset{(4)}{Q}{}^{ijk}}{3r'}+\frac{2}{3}\overset{(5)}{Q}{}^{ijk}\right)\\
&+\hat{n}'^{k}\epsilon^{(i|ka}\left(\frac{8\dot{J}^{a|j)}}{5r'^3}+\frac{8\ddot{J}^{a|j)}}{5r'^2}+\frac{16\overset{(3)}{J}{}^{a|j)}}{5r'}+\frac{8}{3}\overset{(4)}{J}{}^{a|j)}\right)-\hat{n}'^{(i}\epsilon^{j)ka}\left(\frac{4\dot{J}^{ak}}{5r'^3}+\frac{4\ddot{J}^{ak}}{5r'^2}-\frac{16\overset{(3)}{J}{}^{ak}}{15r'}\right)\Bigg].
\end{aligned}
\tag{5.7}
$$

The terms in Eq. (5.7) are of the generic form $f_{N,l}(u')\hat{n}'^{\langle L\rangle}r'^{-N}$. We substitute each such term into Eq. (5.4), integrate using the procedure outlined above, and keep only terms through 2PN order that falloff as $1/r$. Evaluating at the detector distance R, we obtain, finally,

$$h^{ij}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x}) = \frac{4m}{R}\int_0^\infty ds \overset{(4)}{Q}{}^{ij}(u-s)\left[\ln\left(\frac{s}{2R+s}\right)+\frac{11}{12}\right] + \frac{4m}{3R}\hat{N}^k\int_0^\infty ds \overset{(5)}{Q}{}^{ijk}(u-s)\left[\ln\left(\frac{s}{2R+s}\right)+\frac{97}{60}\right] - \frac{16m}{3R}\epsilon^{(i|ka}\hat{N}^k\int_0^\infty ds \overset{(4)}{J}{}^{a|j)}(u-s) \times\left[\ln\left(\frac{s}{2R+s}\right)+\frac{7}{6}\right] + \frac{1912}{315}\frac{m}{R}\overset{(4)}{Q}{}^{ij}(u)\mathcal{R}. \quad (5.8)$$

As with the calculation of EW moments, we discard terms that fall off with increasing $\mathcal{R}$.

The integrals involving the logarithm of retarded time are the tail terms, and are in complete agreement with [33], including the constants (11/12, 97/60, 7/6) added to the logarithms. Their origin is the backscatter of the outgoing gravitational waves off the lowest-order, Schwarzschild-like, static background curvature of the spacetime surrounding the source. More precisely, the logarithmic integrals can be seen to arise directly from the term $-h^{00}\ddot{h}^{ij}$ in Eq. (5.6), which represents a modification of the flat spacetime characteristics by the potential $h^{00}\sim m/r$. The first tail term, arising from the $2d^4Q^{\langle ij\rangle}/du^4$ term in Eq. (5.7), is actually of 3/2PN order, while the second and third terms, arising from the $\frac{2}{3}d^5Q^{ijk}/du^5$ and $\frac{8}{3}d^4J^{aj}/du^4$ terms in Eq. (5.7), are of 2PN order. On the other hand, only the 3/2PN tail term contributes to the energy flux at 2PN order, resulting in the ''4π'' term for circular orbits in Eq. (1.4). Notice that the tail terms show no dependence on the near-zone boundary radius $\mathcal{R}$. In the BDI framework, the tail terms contain a scale b which is associated with a gauge transformation from harmonic coordinates to a set of radiative coordinates used in that framework; physical results in the end do not depend on b, and the tail effects in the two frameworks are in complete agreement.

It is easy to see that the final term in Eq. (5.8), which depends linearly on $\mathcal{R}$ *exactly cancels* the sum of the corresponding terms arising from the two-, four-, and six-index EW moments [Eqs. (4.17), (4.26), and (4.27b)].

Thus combining the contributions of the six EW moments to Eq. (2.18) with these contributions gives the gravitational waveform, valid to 2PN order, for a general N-body system. *The waveform is explicitly finite, with no divergent integrals or undefined terms.* Henceforth, we shall not display any $\mathcal{R}$-dependent terms.

VI. REDUCTION TO TWO-BODY SYSTEMS

A. Center of mass and equations of motion to 2PN order

We now specialize to the case of two bodies. Through 2PN order the dynamics of two-body systems are well known. The motion is governed by a Lagrangian that admits a conserved total energy and angular momentum, as well as a ''conserved'' center-of-mass definition. We define the system's center of mass $\mathbf{X}$ and the relative position $\mathbf{x}$ by

$$\mathbf{X}\equiv m^{-1}(m_1\mathbf{x}_1+m_2\mathbf{x}_2)+\mathbf{f}^{(1)}(\mathbf{x}_1,\mathbf{x}_2) + \mathbf{f}^{(2)}(\mathbf{x}_1,\mathbf{x}_2)+O(\epsilon^3)\times\mathbf{X}, \quad (6.1a)$$

$$\mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2, \quad (6.1b)$$

where $m=m_1+m_2$, and $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ denote 1PN and 2PN corrections to the center-of-mass definition. Inverting these expressions and setting $\mathbf{X}=0$, we obtain

$$\mathbf{x}_1=(m_2/m)\mathbf{x}-\mathbf{f}^{(1)}-\mathbf{f}^{(2)}+O(\epsilon^3)\times\mathbf{x}_1, \quad (6.2a)$$

$$\mathbf{x}_2=-(m_1/m)\mathbf{x}-\mathbf{f}^{(1)}-\mathbf{f}^{(2)}+O(\epsilon^3)\times\mathbf{x}_2. \quad (6.2b)$$

The only place the 2PN correction $\mathbf{f}^{(2)}$ could conceivably be needed is in the lowest-order quadrupole moment, but in this case it is straightforward to show that it is not, since

$$Q^{ij}=\sum_A m_Ax_A^ix_A^j=\mu x^ix^j+mf^{(1)i}f^{(1)j}+O(\epsilon^3)\times Q^{ij}, \quad (6.3)$$

where $f^{(1)i}=-\frac{1}{2}\eta(\delta m/m)(v^2-m/r)x^i$ (see, e.g., [59]), and where we define the two-body variables $\mu=m_1m_2/m$ (reduced mass), $\eta=\mu/m$, $\delta m=m_1-m_2$, $\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2$, and $r=|\mathbf{x}|$. The two-body equations of motion then take the effective one-body relative form, through 2PN order:

$$\mathbf{a}=\mathbf{a}_N+\mathbf{a}^{(1)}_{\rm PN}+\mathbf{a}^{(3/2)}_{\rm SO}+\mathbf{a}^{(2)}_{\rm 2PN}+\mathbf{a}^{(2)}_{\rm SS}+O(\mathbf{a}^{(5/2)}), \quad (6.4)$$

where the subscripts denote the nature of the term, post-Newtonian (PN), spin-orbit (SO), post-post-Newtonian (2PN), and spin-spin (SS); and the superscripts denote the order in ϵ. The individual terms (excluding spins) are given by

$$\mathbf{a}_N=-\frac{m}{r^2}\hat{\mathbf{n}}, \quad (6.5a)$$

$$\mathbf{a}^{(1)}_{\rm PN}=-\frac{m}{r^2}\left\{\hat{\mathbf{n}}\left[-2(2+\eta)\frac{m}{r}+(1+3\eta)v^2-\frac{3}{2}\eta\dot{r}^2\right]-2(2-\eta)\dot{r}\mathbf{v}\right\}, \quad (6.5b)$$

$$\mathbf{a}^{(2)}_{\rm 2PN}=-\frac{m}{r^2}\left\{\hat{\mathbf{n}}\left[\frac{3}{4}(12+29\eta)\left(\frac{m}{r}\right)^2+\eta(3-4\eta)v^4+\frac{15}{8}\eta(1-3\eta)\dot{r}^4-\frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2-\frac{1}{2}\eta(13-4\eta)\frac{m}{r}v^2 -(2+25\eta+2\eta^2)\frac{m}{r}\dot{r}^2\right]-\frac{1}{2}\dot{r}\mathbf{v}\left[\eta(15+4\eta)v^2-(4+41\eta+8\eta^2)\frac{m}{r}-3\eta(3+2\eta)\dot{r}^2\right]\right\}. \quad (6.5c)$$

B. Two-body Epstein-Wagoner moments

Restricting the summations in the EW moments to two bodies and substituting Eqs. (6.2), we obtain, through 2PN order,

$$I^{ij}_{\rm EW}=\mu x^{ij}\left[1+\frac{1}{2}(1-3\eta)v^2-\frac{1}{2}(1-2\eta)m/r\right]+\mu x^{ij}\left[\frac{3}{8}(1-7\eta+13\eta^2)v^4+\frac{1}{12}(28-79\eta-54\eta^2)v^2(m/r)\right.$$
$$\left.-\frac{1}{4}(5+27\eta-4\eta^2)(m/r)^2-\frac{1}{12}(1-13\eta+30\eta^2)\dot r^2(m/r)\right]+\mu mr\left[\frac{1}{6}(13+23\eta)v^{ij}-\frac{5}{3}(1-4\eta)\dot r v^{(i}\hat n^{j)}\right],\quad (6.6a)$$

$$I^{ijk}_{\rm EW}=\mu(\delta m/m)\left\{x^{ij}v^k-2v^{(i}x^{j)k}-v^{(i}x^{j)k}\left[(1-5\eta)v^2+\frac{1}{3}(7+12\eta)(m/r)\right]\right.$$
$$\left.+\frac{1}{2}x^{ij}v^k\left[(1-5\eta)v^2+\frac{1}{3}(17+12\eta)(m/r)\right]+\frac{1}{6}(1-6\eta)(m\dot r/r^2)x^{ijk}\right\},\quad (6.6b)$$

$$I^{ijkl}_{\rm EW}=\mu x^{kl}(1-3\eta)\left(v^{ij}-\frac{1}{3}\hat n^{ij}m/r\right)-\frac{1}{6}\mu mr\hat n^{ij}\delta^{kl}+\mu x^{kl}\left[\frac{1}{2}(1-9\eta+21\eta^2)v^2v^{ij}\right.$$
$$-\frac{1}{24}(13-46\eta+36\eta^2)v^2\hat n^{ij}m/r+\frac{1}{4}(7-10\eta-36\eta^2)v^{ij}m/r+\frac{1}{6}(7-12\eta-36\eta^2)\dot r v^{(i}\hat n^{j)}m/r$$
$$\left.+\frac{1}{8}(1-6\eta+12\eta^2)\dot r^2\hat n^{ij}m/r+\frac{1}{24}(37-122\eta+48\eta^2)\hat n^{ij}(m/r)^2\right]$$
$$+\mu mr\delta^{kl}\left[\frac{1}{12}(7-46\eta)v^{ij}-\frac{1}{24}(7+2\eta)v^2\hat n^{ij}+\frac{1}{6}(3+2\eta)\dot r v^{(i}\hat n^{j)}+\frac{1}{24}(1-2\eta)\dot r^2\hat n^{ij}-\frac{3}{8}\hat n^{ij}m/r\right]$$
$$+\mu mr\left[\frac{1}{12}(1-2\eta)\hat n^{ij}v^{kl}-\frac{1}{6}(1-4\eta)\dot r\hat n^{ij}v^{(k}\hat n^{l)}-\frac{1}{3}(7-20\eta)v^{(i}\hat n^{j)}v^{(k}\hat n^{l)}\right],\quad (6.6c)$$

$$I^{ijklm}_{\rm EW}=-\frac{1}{3}(d/dt)\mu(\delta m/m)\left[(1-2\eta)\left(v^{ij}-\frac{1}{4}\hat n^{ij}m/r\right)x^{klm}-\frac{1}{4}mr\hat n^{ij}x^{(k}\delta^{lm)}\right],\quad (6.6d)$$

$$I^{ijklmn}_{\rm EW}=\frac{1}{12}\mu(d/dt)^2\left[(1-5\eta+5\eta^2)\left(v^{ij}-\frac{1}{5}\hat n^{ij}m/r\right)x^{klmn}-\frac{1}{10}(3-10\eta)mr\hat n^{ij}x^{(kl}\delta^{mn)}+\frac{1}{10}mrx^{ij}\delta^{(kl}\delta^{mn)}\right].\quad (6.6e)$$

In addition, the moments that appear in the tail terms, Eq. (5.8), reduce, to the required order, to

$$Q^{ij}=\mu x^{ij},\quad (6.7)$$

$$Q^{ijk}=-\mu(\delta m/m)x^{ijk},\quad (6.8)$$

$$J^{aj}=-\mu(\delta m/m)(\mathbf{x}\times\mathbf{v})^a x^j.\quad (6.9)$$

C. Two-body gravitational waveform and energy loss

Substituting the EW two-body moments (6.6) into Eq. (2.18), calculating the time derivatives using the 2PN equations of motion (6.5) to the accuracy needed, and adding the tail terms from the radiation zone integral (5.8), we obtain the gravitational waveform. An alternative method is first to calculate the so-called ''symmetric trace-free'' (STF) moments defined by Thorne [34] and used by BDI, and then to calculate the waveform. The procedures and formulas needed to do this are given in Appendix E. The result for the waveform is

$$h^{ij}=\frac{2\mu}{R}[\tilde Q^{ij}+P^{1/2}Q^{ij}+PQ^{ij}+PQ^{ij}_{\rm SO}+P^{3/2}Q^{ij}+P^{3/2}Q^{ij}_{\rm tail}+P^{3/2}Q^{ij}_{\rm SO}+P^2Q^{ij}+P^2Q^{ij}_{\rm tail}+P^2Q^{ij}_{\rm SO}+P^2Q^{ij}_{\rm SS}+O(\epsilon^{5/2})]_{\rm TT},\quad (6.10)$$

where, as before, the superscripts denote the effective PN order, and subscripts label the nature of the term, and where the individual nonspin pieces are given by

$$\tilde Q^{ij}=2\left(v^iv^j-\frac{m}{r}\hat n^i\hat n^j\right),\quad (6.11a)$$

$$P^{1/2}Q^{ij}=\frac{\delta m}{m}\left\{3(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\frac{m}{r}[2\hat{n}^{(i}v^{j)}-\dot{r}\hat{n}^i\hat{n}^j]+(\hat{\mathbf{N}}\cdot\mathbf{v})\left[\frac{m}{r}\hat{n}^i\hat{n}^j-2v^iv^j\right]\right\}, \tag{6.11b}$$

$$\begin{aligned}PQ^{ij}=&\frac{1}{3}\Bigg\{(1-3\eta)\Bigg[(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2\frac{m}{r}\left[\left(3v^2-15\dot{r}^2+7\frac{m}{r}\right)\hat{n}^i\hat{n}^j+30\dot{r}\hat{n}^{(i}v^{j)}-14v^iv^j\right]\\&+(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})\frac{m}{r}[12\dot{r}\hat{n}^i\hat{n}^j-32\hat{n}^{(i}v^{j)}]+(\hat{\mathbf{N}}\cdot\mathbf{v})^2\left[6v^iv^j-2\frac{m}{r}\hat{n}^i\hat{n}^j\right]\Bigg]+\left[3(1-3\eta)v^2-2(2-3\eta)\frac{m}{r}\right]v^iv^j\\&+4\frac{m}{r}\dot{r}(5+3\eta)\hat{n}^{(i}v^{j)}+\frac{m}{r}\left[3(1-3\eta)\dot{r}^2-(10+3\eta)v^2+29\frac{m}{r}\right]\hat{n}^i\hat{n}^j\Bigg\},\end{aligned} \tag{6.11c}$$

$$\begin{aligned}P^{3/2}Q^{ij}=&\frac{\delta m}{m}(1-2\eta)\Bigg\{(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^3\frac{m}{r}\left[\frac{5}{4}\left(3v^2-7\dot{r}^2+6\frac{m}{r}\right)\dot{r}\hat{n}^i\hat{n}^j-\frac{17}{2}\dot{r}v^iv^j-\frac{1}{6}\left(21v^2-105\dot{r}^2+44\frac{m}{r}\right)\hat{n}^{(i}v^{j)}\right]\\&+\frac{1}{4}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2(\hat{\mathbf{N}}\cdot\mathbf{v})\frac{m}{r}\left[58v^iv^j+\left(45\dot{r}^2-9v^2-28\frac{m}{r}\right)\hat{n}^i\hat{n}^j-108\dot{r}\hat{n}^{(i}v^{j)}\right]+\frac{3}{2}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})^2\frac{m}{r}[10\hat{n}^{(i}v^{j)}-3\dot{r}\hat{n}^i\hat{n}^j]\\&+\frac{1}{2}(\hat{\mathbf{N}}\cdot\mathbf{v})^3\left[\frac{m}{r}\hat{n}^i\hat{n}^j-4v^iv^j\right]\Bigg\}+\frac{1}{12}\frac{\delta m}{m}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\frac{m}{r}\Bigg[2\hat{n}^{(i}v^{j)}\left(\dot{r}^2(63+54\eta)-\frac{m}{r}(128-36\eta)+v^2(33-18\eta)\right)\\&+\hat{n}^i\hat{n}^j\dot{r}\left(\dot{r}^2(15-90\eta)-v^2(63-54\eta)+\frac{m}{r}(242-24\eta)\right)-\dot{r}v^iv^j(186+24\eta)\Bigg]\\&+\frac{\delta m}{m}(\hat{\mathbf{N}}\cdot\mathbf{v})\Bigg[\frac{1}{2}v^iv^j\left(\frac{m}{r}(3-8\eta)-2v^2(1-5\eta)\right)-\hat{n}^{(i}v^{j)}\frac{m}{r}\dot{r}(7+4\eta)\\&-\hat{n}^i\hat{n}^j\frac{m}{r}\left(\frac{3}{4}(1-2\eta)\dot{r}^2+\frac{1}{3}(26-3\eta)\frac{m}{r}-\frac{1}{4}(7-2\eta)v^2\right)\Bigg],\end{aligned} \tag{6.11d}$$

$$P^{3/2}Q^{ij}_{\rm tail}=4m\int_0^\infty\left\{\frac{m}{r^3}\left[\left(3v^2+\frac{m}{r}-15\dot{r}^2\right)\hat{n}^i\hat{n}^j+18\dot{r}\hat{n}^{(i}v^{j)}-4v^iv^j\right]\right\}_{u-s}\left[\ln\left(\frac{s}{2R+s}\right)+\frac{11}{12}\right]ds, \tag{6.11e}$$

$$\begin{aligned}P^2Q^{ij}=&\frac{1}{60}(1-5\eta+5\eta^2)\Bigg\{24(\hat{\mathbf{N}}\cdot\mathbf{v})^4\left[5v^iv^j-\frac{m}{r}\hat{n}^i\hat{n}^j\right]+\frac{m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^4\Bigg[2\left(175\frac{m}{r}-465\dot{r}^2+93v^2\right)v^iv^j\\&+30\dot{r}\left(63\dot{r}^2-50\frac{m}{r}-27v^2\right)\hat{n}^{(i}v^{j)}+\left(1155\frac{m}{r}\dot{r}^2-172\left(\frac{m}{r}\right)^2-945\dot{r}^4-159\frac{m}{r}v^2+630\dot{r}^2v^2-45v^4\right)\hat{n}^i\hat{n}^j\Bigg]\\&+24\frac{m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^3(\hat{\mathbf{N}}\cdot\mathbf{v})\left[87\dot{r}v^iv^j+5\dot{r}\left(14\dot{r}^2-15\frac{m}{r}-6v^2\right)\hat{n}^i\hat{n}^j+16\left(5\frac{m}{r}-10\dot{r}^2+2v^2\right)\hat{n}^{(i}v^{j)}\right]\\&+288\frac{m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})^3[\dot{r}\hat{n}^i\hat{n}^j-4\hat{n}^{(i}v^{j)}]+24\frac{m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2(\hat{\mathbf{N}}\cdot\mathbf{v})^2\left[\left(35\frac{m}{r}-45\dot{r}^2+9v^2\right)\hat{n}^i\hat{n}^j-76v^iv^j+126\dot{r}\hat{n}^{(i}v^{j)}\right]\Bigg\}\\&+\frac{1}{15}(\hat{\mathbf{N}}\cdot\mathbf{v})^2\Bigg\{\left[5(25-78\eta+12\eta^2)\frac{m}{r}-(18-65\eta+45\eta^2)v^2+9(1-5\eta+5\eta^2)\dot{r}^2\right]\frac{m}{r}\hat{n}^i\hat{n}^j\\&+3\left[5(1-9\eta+21\eta^2)v^2-2(4-25\eta+45\eta^2)\frac{m}{r}\right]v^iv^j+18(6-15\eta-10\eta^2)\frac{m}{r}\dot{r}\hat{n}^{(i}v^{j)}\Bigg\}\\&+\frac{1}{15}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})\frac{m}{r}\Bigg\{\left[3(36-145\eta+150\eta^2)v^2-5(127-392\eta+36\eta^2)\frac{m}{r}-15(2-15\eta+30\eta^2)\dot{r}^2\right]\dot{r}\hat{n}^i\hat{n}^j\\&+6(98-295\eta-30\eta^2)\dot{r}v^iv^j+2\left[5(66-221\eta+96\eta^2)\frac{m}{r}-9(18-45\eta-40\eta^2)\dot{r}^2-(66-265\eta+360\eta^2)v^2\right]\hat{n}^{(i}v^{j)}\Bigg\}\\&+\frac{1}{60}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2\frac{m}{r}\Bigg\{\Bigg[3(33-130\eta+150\eta^2)v^4+105(1-10\eta+30\eta^2)\dot{r}^4+15(181-572\eta+84\eta^2)\frac{m}{r}\dot{r}^2\end{aligned}$$

$$-(131-770\eta+930\eta^2)\frac{m}{r}v^2-60(9-40\eta+60\eta^2)v^2\dot{r}^2-8(131-390\eta+30\eta^2)\left(\frac{m}{r}\right)^2\Bigg]\hat{n}^i\hat{n}^j$$

$$+4\left[(12+5\eta-315\eta^2)v^2-9(39-115\eta-35\eta^2)\dot{r}^2+5(29-104\eta+84\eta^2)\frac{m}{r}\right]v^iv^j$$

$$+4\left[15(18-40\eta-75\eta^2)\dot{r}^2-5(197-640\eta+180\eta^2)\frac{m}{r}+3(21-130\eta+375\eta^2)v^2\right]\dot{r}\hat{n}^{(i}v^{j)}\Bigg\}$$

$$+\frac{1}{60}\Bigg\{\Bigg[(467+780\eta-120\eta^2)\frac{m}{r}v^2-15(61-96\eta+48\eta^2)\frac{m}{r}\dot{r}^2-(144-265\eta-135\eta^2)v^4+6(24-95\eta+75\eta^2)v^2\dot{r}^2$$

$$-2(642+545\eta)\left(\frac{m}{r}\right)^2-45(1-5\eta+5\eta^2)\dot{r}^4\Bigg]\frac{m}{r}\hat{n}^i\hat{n}^j+\Bigg[4(69+10\eta-135\eta^2)\frac{m}{r}v^2-12(3+60\eta+25\eta^2)\frac{m}{r}\dot{r}^2$$

$$+45(1-7\eta+13\eta^2)v^4-10(56+165\eta-12\eta^2)\left(\frac{m}{r}\right)^2\Bigg]v^iv^j$$

$$+4\left[2(36+5\eta-75\eta^2)v^2-6(7-15\eta-15\eta^2)\dot{r}^2+5(35+45\eta+36\eta^2)\frac{m}{r}\right]\frac{m}{r}\dot{r}\hat{n}^{(i}v^{j)}\Bigg\}, \tag{6.11f}$$

$$P^2Q^{ij}_{\rm tail}=2\delta m\int_0^\infty\Bigg\{\frac{m}{r^3}\Bigg[15\left(3v^2+2\frac{m}{r}-7\dot{r}^2\right)\dot{r}\hat{n}^i\hat{n}^j\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}-\left(13v^2+\frac{22}{3}\frac{m}{r}-65\dot{r}^2\right)(\hat{n}^i\hat{n}^j\mathbf{v}\cdot\hat{\mathbf{N}}+2\hat{n}^{(i}v^{j)}\hat{\mathbf{n}}\cdot\hat{\mathbf{N}})$$

$$-40\dot{r}(v^iv^j\hat{\mathbf{n}}\cdot\hat{\mathbf{N}}+2\hat{n}^{(i}v^{j)}\mathbf{v}\cdot\hat{\mathbf{N}})+20v^iv^j\mathbf{v}\cdot\hat{\mathbf{N}}\Bigg]\Bigg\}_{u-s}\left[\ln\left(\frac{s}{2R+s}\right)+\frac{97}{60}\right]ds$$

$$+8\delta m\int_0^\infty\Bigg\{\frac{m}{r^3}\Bigg[\left(v^2-\frac{2}{3}\frac{m}{r}-5\dot{r}^2\right)(\hat{n}^i\hat{n}^j\mathbf{v}\cdot\hat{\mathbf{N}}-n^{(i}v^{j)}\hat{\mathbf{n}}\cdot\hat{\mathbf{N}})-2\dot{r}(v^iv^j\hat{\mathbf{n}}\cdot\hat{N}-\hat{n}^{(i}v^{j)}\mathbf{v}\cdot\hat{\mathbf{N}})\Bigg]\Bigg\}_{u-s}\left[\ln\left(\frac{s}{2R+s}\right)+\frac{7}{6}\right]ds. \tag{6.11g}$$

The leading PN and 3/2PN spin-orbit and the 2PN spin-spin contributions to the waveform can be found in Eqs. (3.22) of [41] and in Appendix F. There will also be in principle 2PN spin-orbit terms; these have not been calculated to date.

Although we have differentiated the moments appearing in the tail terms explicitly using the equations of motion in order to display the waveform contributions in a consistent manner, this is not the best form of the tail terms for explicit numerical evaluation in the case of general orbits. The reason is the slow falloff of the logarithmic term with increasing s. Instead, it is preferable to revert to the forms of the tail terms given in Eq. (5.8), split each integral over s into a finite part from 0 to s_0, where s_0 corresponds to several dynamical time scales of the source, and a remaining integral from s_0 to ∞. The first integral can be done using the expressions given in Eqs. (6.11). The remaining integral is integrated by parts twice. One can then show [38] that the latter integral falls off as $1/s_0$ generally, and for nearly periodic orbits, as $1/s_0^2$. By choosing s_0 sufficiently large (generally a few dynamical time scales or orbital periods), one then can obtain accurate numerical representations of the tail terms, without having to integrate over the entire past history of the source.

Differentiating h^{ij} with respect to time, using the 2PN equation of motion (6.5) where required, and substituting into Eq. (2.30); or equivalently, taking the appropriate time derivatives of the STF moments (Appendix E), and substituting into Eq. (E5b), one finds, for the energy flux,

$$\frac{dE}{dt}=\dot{E}_N+\dot{E}_{\rm PN}+\dot{E}_{\rm SO}+\dot{E}_{\rm tail}+\dot{E}_{\rm 2PN}+\dot{E}_{\rm SS}+O(\epsilon^{5/2})\dot{E}_N, \tag{6.12}$$

where the nonspin contributions are

$$\dot{E}_N=\frac{8}{15}\frac{m^2\mu^2}{r^4}\{12v^2-11\dot{r}^2\}, \tag{6.13a}$$

$$\dot{E}_{\rm PN}=\frac{8}{15}\frac{m^2\mu^2}{r^4}\Bigg\{\frac{1}{28}\Bigg[(785-852\eta)v^4-2(1487-1392\eta)v^2\dot{r}^2+3(687-620\eta)\dot{r}^4-160(17-\eta)\frac{m}{r}v^2$$

$$+8(367-15\eta)\frac{m}{r}\dot{r}^2+16(1-4\eta)\left(\frac{m}{r}\right)^2\Bigg]\Bigg\}, \tag{6.13b}$$

$$\dot{E}_{\rm tail}=-\frac{4m}{5}\overset{(4)}{Q}{}^{\langle ij\rangle}(u)\int_0^\infty \overset{(4)}{Q}{}^{\langle ij\rangle}(u-s)\ln[s/(2R+s)]ds, \tag{6.13c}$$

$$\begin{aligned}\dot{E}_{\rm 2PN}=\frac{8}{15}\frac{m^2\mu^2}{r^4}\Bigg\{\frac{1}{756}\Bigg[&18(1692-5497\eta+4430\eta^2)v^6-54(1719-10\,278\eta+6292\eta^2)v^4\dot{r}^2\\&+54(2018-15\,207\eta+7572\eta^2)v^2\dot{r}^4-18(2501-20\,234\eta+8404\eta^2)\dot{r}^6-12(33\,510-60\,971\eta+14\,290\eta^2)\frac{m}{r}\dot{r}^4\\&-36(4446-5237\eta+1393\eta^2)\frac{m}{r}v^4+108(4987-8513\eta+2165\eta^2)\frac{m}{r}\dot{r}^2v^2-3(106\,319+9798\eta+5376\eta^2)\left(\frac{m}{r}\right)^2\dot{r}^2\\&+(281\,473+81\,828\eta+4368\eta^2)\left(\frac{m}{r}\right)^2v^2-24(253-1026\eta+56\eta^2)\left(\frac{m}{r}\right)^3\Bigg]\Bigg\}.\end{aligned} \tag{6.13d}$$

The 3/2PN spin-orbit and 2PN spin-spin contributions can be found in Eqs. (3.25) of [41] and Appendix F. The tail contribution is formally of 3/2PN order, arising from a cross term involving $P^{3/2}Q^{ij}_{\rm tail}$ and $\widetilde{Q}^{ij}$; for simplicity, we do not write it out explicitly (for circular orbits we evaluate it below). The ''11/12'' term in Eq. (5.8) contributes a term of the schematic form $(d^4Q/du^4)(d^3Q/du^3)$, which can be written as a total time derivative and absorbed into a redefinition of the energy E at an order above that at which it is well defined as a conserved quantity (see, e.g., [60,61] for a discussion of this point). In the same way, the form of the tail term shown in Eq. (6.13c) has been achieved by integrating the tail contribution once by parts and moving the total time derivative over to the left-hand side. The 2PN tail terms in the waveform make no contribution to the energy flux to 2PN order because their cross product with the quadrupole piece contains an odd number of unit vectors $\hat{\mathbf{N}}$, and thus vanishes on integration over solid angle. They will, however, produce 5/2PN contributions to $\dot{E}$ via cross terms with the 1/2PN waveform terms $P^{1/2}Q^{ij}$.

Through first PN order, Eqs. (6.13) agree with [17,62].

VII. QUASICIRCULAR ORBITS

A. Orbit equations and gravitational waveforms

Because gravitational radiation reaction circularizes orbits, the late stage of inspiral of a compact binary, such as that of the binary pulsar PSR 1913+16, will be characterized by a quasicircular orbit, that is, an orbit which is circular apart from the slow inspiral caused by radiation damping. We define the Newtonian angular momentum $\mathbf{L}_N\equiv\mu\mathbf{x}\times\mathbf{v}$, the unit vector $\boldsymbol{\lambda}\equiv\hat{\mathbf{L}}_N\times\hat{\mathbf{n}}$, and the angular velocity $\omega\equiv|\mathbf{L}_N|/\mu r^2$. A circular orbit is given by the conditions $\ddot{r}=\dot{r}=0$. Solving the 2PN two-body equations of motion (6.5) under these conditions gives

$$\omega^2=\frac{m}{r^3}\left[1-\frac{m}{r}(3-\eta)+\left(\frac{m}{r}\right)^2\left(6+\frac{41}{4}\eta+\eta^2\right)\right]. \tag{7.1}$$

Then the orbital velocity is $\mathbf{v}=r\omega\boldsymbol{\lambda}$ and the orbital energy through 2PN order is

$$E=-\eta\frac{m^2}{2r}\left[1-\frac{1}{4}\frac{m}{r}(7-\eta)-\frac{1}{8}\left(\frac{m}{r}\right)^2(7-49\eta-\eta^2)\right]. \tag{7.2}$$

In order to calculate waveforms as observed by an Earth-bound detector, we must choose conventions for the direction and orientation of the orbit. The standard convention is to choose a triad of vectors composed of $\hat{\mathbf{N}}$, the radial direction to the observer, $\hat{\mathbf{p}}$, lying along the intersection of the orbital plane with the plane of the sky (line of nodes), and $\hat{\mathbf{q}}=\hat{\mathbf{N}}\times\hat{\mathbf{p}}$ (see Fig. 7). The normal to the orbit $\hat{\mathbf{L}}_N$ is inclined an angle i relative to $\hat{\mathbf{N}}$ $(0\leqslant i\leqslant\pi)$. The orbital phase $\phi=\omega u+\text{const}$ of body 1 is measured from the line of nodes in a positive (out of the plane) sense (orbits seen to be moving clockwise correspond to $i\geqslant\pi/2$). The two basic waveform polarizations h_+ and $h_\times$ are given by

$$h_+=\frac{1}{2}(\hat{p}_i\hat{p}_j-\hat{q}_i\hat{q}_j)h^{ij}, \tag{7.3a}$$

$$h_\times=\frac{1}{2}(\hat{p}_i\hat{q}_j+\hat{q}_i\hat{p}_j)h^{ij}. \tag{7.3b}$$

[There is no need to apply the TT projection in Eq. (6.10) before contracting on $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$.] From our conventions, we have that $\hat{\mathbf{n}}=\hat{\mathbf{p}}\cos\phi+(\hat{\mathbf{q}}\cos i+\hat{\mathbf{N}}\sin i)\sin\phi$ and $\boldsymbol{\lambda}=-\hat{\mathbf{p}}\sin\phi+(\hat{\mathbf{q}}\cos i+\hat{\mathbf{N}}\sin i)\cos\phi$. Since h^{ij} consists of terms of the

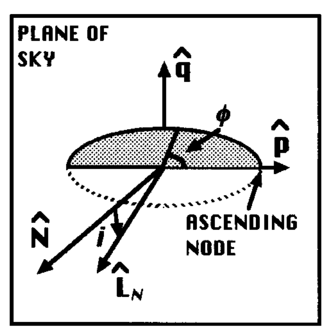

FIG. 7. Orientation of unit vectors defining + and × waveform polarizations. Direction of detector is $\hat{\mathbf{N}}$; $\hat{\mathbf{p}}$ lies along line of nodes and is the origin for orbital phase angle ϕ.

form $\hat{n}^i\hat{n}^j$, $\hat{\lambda}^i\hat{\lambda}^j$, or $\hat{n}^{(i}\hat{\lambda}^{j)}$, we find the following formulas to be useful in evaluating the polarizations:

$$(\hat{n}^i\hat{n}^j)_+ = \frac{1}{4}\sin^2 i + \frac{1}{4}(1+\cos^2 i)\cos 2\phi, \tag{7.4a}$$

$$(\hat{\lambda}^i\hat{\lambda}^j)_+ = \frac{1}{4}\sin^2 i - \frac{1}{4}(1+\cos^2 i)\cos 2\phi, \tag{7.4b}$$

$$(\hat{n}^{(i}\hat{\lambda}^{j)})_+ = -\frac{1}{4}(1+\cos^2 i)\sin 2\phi, \tag{7.4c}$$

$$(\hat{n}^i\hat{n}^j)_\times = \frac{1}{2}\cos i \sin 2\phi, \tag{7.4d}$$

$$(\hat{\lambda}^i\hat{\lambda}^j)_\times = -\frac{1}{2}\cos i \sin 2\phi, \tag{7.4e}$$

$$(\hat{n}^{(i}\hat{\lambda}^{j)})_\times = \frac{1}{2}\cos i \cos 2\phi, \tag{7.4f}$$

$$\hat{\mathbf{N}}\cdot\hat{\mathbf{n}} = \sin i \sin\phi, \tag{7.4g}$$

$$\hat{\mathbf{N}}\cdot\boldsymbol{\lambda} = \sin i \cos\phi. \tag{7.4h}$$

Substituting $\dot{r}=0$ and Eq. (7.1) into Eqs. (6.11) [keeping PN and 2PN corrections in Eq. (7.1) as needed], and using Eqs. (7.4), we can evaluate h_+ and $h_\times$ explicitly as functions of orbital phase and orbital orientation. The waveforms can be expressed in terms of powers of m/r, but it is observationally more useful to express them in terms of $m\omega \approx (m/r)^{3/2}$, since ω is directly related to the observed gravitational-wave frequency. Instead of showing the result here, we refer the reader to [48] where the complete, ''ready-to-use'' pair of 2PN waveform polarizations are displayed and discussed. Similar substitution into Eqs. (6.13) results in Eq. (1.4).

B. Tail terms

Because they involve integration over the past history of the source, the tail contributions to the waveform and energy flux require additional discussion. For circular orbits, the $+$ and $\times$ polarizations of the quantity $P^{3/2}Q^{ij}_{\rm tail}$ are given by

$$(P^{3/2}Q_{\rm tail})_+ = 8m(1+\cos^2 i)\int_0^\infty \left(\frac{m^2}{r^4}\cos 2\phi\right)_{u-s} \times\left[\ln\left(\frac{s}{2R+s}\right)+\frac{11}{12}\right]ds, \tag{7.5a}$$

$$(P^{3/2}Q_{\rm tail})_\times = 16m\cos i\int_0^\infty \left(\frac{m^2}{r^4}\sin 2\phi\right)_{u-s} \times\left[\ln\left(\frac{s}{2R+s}\right)+\frac{11}{12}\right]ds. \tag{7.5b}$$

Because r and ω evolve on a radiation-reaction timescale τ_{RR} which is long compared to an orbital period, we can approximate them to be constant in the above integrals; the results will be valid up to corrections of order $(\omega\tau_{RR})^{-1}\ln(\omega\tau_{RR})\ll 1$ [45]. Notice that the integrals converge as $s\to\infty$, even if we approximate $m^2/r^4\approx$const (in fact, $r\to\infty$ in the infinite past [63], so the integrals truly converge). Thus we can substitute $\omega(u-s)$ for ϕ with $\omega=$const in the tail integrals, pull out the m^2/r^4 factor, and use the fact that, for any integer n,

$$\mathcal{P}_S^{(n)} \equiv \int_0^\infty \sin(n\omega s)\ln\left(\frac{s}{2R+s}\right)ds = -\frac{1}{n\omega}\{\gamma+\ln(2n\omega R)+O[(2n\omega R)^{-2}]\}, \tag{7.6a}$$

$$\mathcal{P}_C^{(n)} \equiv \int_0^\infty \cos(n\omega s)\ln\left(\frac{s}{2R+s}\right)ds = -\frac{1}{n\omega}\left(\frac{\pi}{2}+O[(2n\omega R)^{-1}]\right), \tag{7.6b}$$

where γ is Euler's constant. The result is

$$(P^{3/2}Q_{\rm tail})_+ = -4(1+\cos^2 i)\left(\frac{m}{r}\right)^{5/2} \times\left\{\frac{\pi}{2}\cos 2\phi+\left(\gamma+\ln(4\omega R)-\frac{11}{12}\right)\sin 2\phi\right\}, \tag{7.7a}$$

$$(P^{3/2}Q_{\rm tail})_\times = -8\cos i\left(\frac{m}{r}\right)^{5/2} \times\left\{\frac{\pi}{2}\sin 2\phi-\left(\gamma+\ln(4\omega R)-\frac{11}{12}\right)\cos 2\phi\right\}, \tag{7.7b}$$

It is useful to combine these tail terms with the lowest-order quadrupole terms, given from Eq. (6.11a) by $\widetilde{Q}_+ = -(m/r)(1+\cos^2 i)\cos 2\phi$ and $\widetilde{Q}_\times = -2(m/r)\cos i \sin 2\phi$, into the forms

$$\widetilde{Q}_+ \approx -\frac{m}{r}(1+\cos^2 i)\left[1+2\pi\left(\frac{m}{r}\right)^{3/2}\right]\cos 2\psi, \tag{7.8a}$$

$$\widetilde{Q}_\times \approx -2\frac{m}{r}\cos i\left[1+2\pi\left(\frac{m}{r}\right)^{3/2}\right]\sin 2\psi, \tag{7.8b}$$

where

$$\psi = \phi - 2(m/r)^{3/2}[\gamma+\ln(4\omega R e^{-11/12})] = \omega\{u-2m\ln R-2m[\gamma+\ln(4\omega e^{-11/12})]\}. \tag{7.9}$$

We first note that one effect of the tail term is to shift the phase of the quadrupole piece by an irrelevant constant, and by a term which varies logarithmically with ω as the inspiral proceeds. This slowly varying phase shift was studied in [38].

We also recognize that $u-2m\ln R = t-R-2m\ln R$ is retarded time with respect to the ''true'' null cone that intersects the observation point at (t,R). This can be seen by noting that, in the asymptotic, Schwarzschild-like spacetime of the source, in harmonic coordinates, outgoing radial null geodesics obey $t-r-2m\ln r+O(1/r)=\text{const}$. An identical R dependence in the phase shows up at the next 1/2PN order, when one combines the two polarization states of $P^{1/2}Q^{ij}$ with those of $P^2Q^{ij}_{\text{tail}}$. We thus conclude that, at least through the 2PN order considered, our procedure for calculating the tail terms yields gravitational waves that asymptotically propagate along the true harmonic null cones, toward true future null infinity, despite the use of a flat-spacetime wave equation for $h^{\alpha\beta}$. This avoids the need for further matching or other devices to connect our solutions to true null infinity, and answers another long-standing criticism of the EW framework [7]. It is useful to note also that, in the BDI approach, a similar logarithmic term appears in the phase shift (7.9), but there the term depends on the parameter b used in the transformation from harmonic to radiative coordinates. The appearance of such a parameter can be shown to have no physical consequences, as expected [38,64]. Our method is explicitly free of such arbitrary parameters, all effects of $\mathcal{R}$ having cancelled. The only external radius which appears is that of the observer.

The tail contribution to the energy flux, given by Eq. (6.13c) can also be calculated in closed form using the above assumptions together with Eq. (7.6b). The result is the ''4π'' term in Eq. (1.4).

C. Display of the waveforms

We now display our results explicitly by plotting the waveform for an inspiralling binary as a function of time. We will assume that the binary is in a quasicircular orbit in its last few moments before the final plunge to coalescence. The time evolution of the orbital phase velocity in this regime can be obtained by integrating the equation

$$\frac{d\omega}{dt}=\frac{\dot{E}}{dE/d\omega}, \tag{7.10}$$

where $\dot{E}$ is given by Eqs. (1.4) and $dE/d\omega$ can be obtained from Eqs. (7.1) and (7.2). The orbital phase angle ϕ can, in turn, be obtained by integrating the orbital phase velocity. The results are

$$\begin{aligned}\omega(t)=&\frac{1}{8m}(T_c-T)^{-3/8}\bigg\{1+\left[\frac{743}{2688}+\frac{11}{32}\eta\right](T_c-T)^{-1/4}\\&-\frac{3\pi}{10}(T_c-T)^{-3/8}+\left[\frac{1\,855\,099}{14\,450\,688}+\frac{56\,975}{258\,048}\eta\right.\\&\left.+\frac{371}{2048}\eta^2\right](T_c-T)^{-1/2}\bigg\},\end{aligned} \tag{7.11a}$$

$$\begin{aligned}\phi(t)=&\phi_c-\frac{1}{\eta}(T_c-T)^{5/8}\bigg\{1+\left[\frac{3715}{8064}+\frac{55}{96}\eta\right](T_c-T)^{-1/4}\\&-\frac{3\pi}{4}(T_c-T)^{-3/8}+\left[\frac{9\,275\,495}{14\,450\,688}+\frac{284\,875}{258\,048}\eta\right.\\&\left.+\frac{1855}{2048}\eta^2\right](T_c-T)^{-1/2}\bigg\},\end{aligned} \tag{7.11b}$$

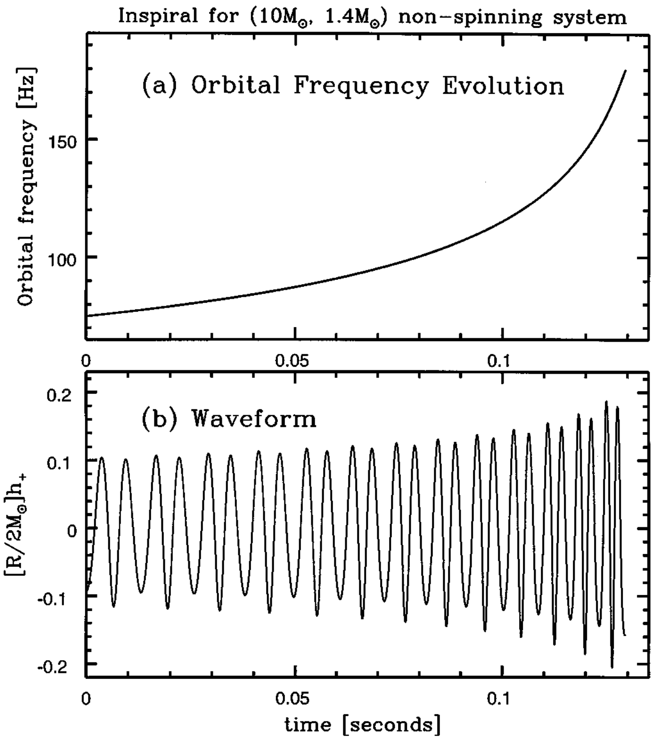

FIG. 8. (a) Orbital frequency and (b) waveform for a $1.4M_\odot$ neutron star spiralling into a $10M_\odot$ black hole plotted vs time in sec. Orbit is viewed edge-on, therefore only ''+'' polarization is present.

where T is a dimensionless time variable related to the coordinate retarded time u by $T=\eta(u/5m)$, and ϕ_c and T_c are constants of integration. The constant T_c is the dimensionless retarded time at coalescence (the time at which the frequency in Eq. (7.11) formally becomes infinite), and ϕ_c is the orbital phase at coalescence.

We can now use the orbital phase evolution along with Eqs. (7.3), (7.4), and (7.1) to write h_+ and $h_\times$ as explicit functions of time. We will not display the result here (there are enough large equations in this paper already), but rather refer the reader to Eqs. (2)–(4) in [48] for ''ready-to-use'' waveforms. The ''ready-to-use'' waveforms are essentially Eqs. (6.11) boiled down to the circular orbit case.

For the case of a 1.4 $M_\odot$ neutron star spiralling into a 10 $M_\odot$ black hole the resulting frequency sweep and waveform are shown in Fig. 8. The observer is viewing the orbital motion edge on, so that $i=\pi/2$ in Eqs. (7.4). In this case the gravitational radiation is linearly polarized (only h_+ is present). The upper cut-off frequency in Fig. 8 is chosen to be 180 Hz; this is approximately the orbital frequency at the innermost stable circular orbit [65,66] for this type of system. For the initial LIGO detector, Finn [67] has shown that a substantial fraction of the signal-to-noise ratio available is accumulated when integrating a matched filter against the signal in the frequency range we have displayed. In other words, the segment of the waveform shown in Fig. 8(b), sweeping from 75 Hz to 180 Hz, is the portion of the waveform which is actually most *detectable* for the initial LIGO detector.

As energy is extracted from the system by the radiation, the orbital radius shrinks and the orbital frequency increases.

This gives rise to the dominant ''chirp'' feature of the waveform in Fig. 8(b): the growing amplitude and the bunching of peaks at late time. However, because the coordinate velocity rises to about $0.5c$, this system is quite relativistic, and thus the inclusion of higher multipoles of the radiation causes the waveform to differ considerably from the simple *cosine* chirp that one would compute just using quadrupole radiation. The pairing of wave crests (alternately closer together and farther apart) signifies the onset of the gravitational analogue of synchrotron spikes. Just as in electricity and magnetism this feature comes from the inclusion of many harmonics of the radiation. In our analysis we have included multipoles through the six-index multipole $I_{\rm EW}^{ijklmn}$. This allows us consistently to include components of the radiation in our waveform at multiples of the orbital phase $n\phi_{\rm orbital}$ where n ranges from 1 to 6, $n=2$ being the dominant quadrupole contribution.

Another interesting feature of Fig. 8(b) is that adjacent troughs are not the same depth, but adjacent crests are essentially the same height. This effect also has a discernable physical origin. The deeper troughs arise when the lighter mass is coming toward the observer; thus the observer is in the forward synchrotron beam pattern of the lighter, faster-moving mass. The shallower troughs arise when the lighter mass is receding from the observer. [At the left-hand side of the figure, the phase is arbitrarily set to zero, i.e., the heavier mass (chosen to be m_1) is passing through the ascending node coming toward the observer and the lighter mass is receding. The waveform is clearly in the not-so-deep trough at this leftmost point.] The crests are essentially the same height because the radiation is virtually the same when the masses are moving transverse to the line of sight of the observer regardless of which mass is closer to the observer (see [35] for further discussion of the asymmetric radiation emission). The extent to which the harmonic structure might be measurable by a gravitational-wave detector is currently under investigation [68]. Preliminary analysis shows that neglecting the harmonic structure (i.e., just using the quadrupole amplitude to describe the wave) results in approximately a 4% loss in signal-to-noise ratio. In Appendix F we show how the effects of spin modify the waveform and frequency evolution.

VIII. DISCUSSION

We have extended the Epstein-Wagoner framework for calculating gravitational radiation from slow-motion systems to produce a method that is free of divergences or undefined integrals. The extension involved adding to the original framework the integral of the effective source over that part of the past null cone of the field point that is *exterior* to the near zone. When expressed in appropriate variables, that integral can be shown to be convergent, and can be evaluated in a straightforward way, to any chosen PN order. The exterior integral yielded (a) terms that explicitly cancel terms from the EW framework previously thought to be divergent (b) tail terms, in agreement with other methods based on matching, and (c) phasing terms that verify that the radiation asymptotically propagates along true null cones of the curved spacetime.

This new, well-defined framework, provides a basis for extending the calculation of gravitational radiation to higher PN orders. An extension to 5/2PN order in the BDI framework has been achieved by Blanchet [69]; such an extension in the improved EW framework is in progress. Extension to 3PN order will be a bigger challenge, simply because of the complexity of the terms, including quadratically nonlinear integrals, and the rapidly increasing number of computations. However, we foresee no obstacle in principle to such an extension in the improved framework.

This improved framework will also allow derivation of near-zone gravitational fields in a form that will yield equations of motion for the sources to high PN orders. It should be possible to derive radiation-reaction terms in the two-body equations of motion, at order $\epsilon^{5/2}$ and $\epsilon^{7/2}$ beyond Newtonian gravity [60,61,70], without the presence of ill-defined or divergent terms, and without the need for matching between zones. One goal would be to derive the nondissipative, 3PN terms in the equations of motion. This would improve the accuracy of estimates, using a hybrid Schwarzschild-PN equation of motion, of the transition point between inspiral and unstable plunge in the late stage of compact binary inspiral [65,66]. Calculation of the near-zone fields will also be important in developing interfaces between the post-Newtonian approach, which works well for most of the inspiral, and numerical relativity methods which must be used for the final few orbits and the coalescence. Work on this latter subject is in progress.

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APPENDIX A: STF TENSORS AND THEIR PROPERTIES

In calculating field integrals we make frequent use of the properties of symmetric, trace-free (STF) products of unit vectors. The general formula for such STF products is

$$\hat{n}^{\langle L\rangle}\equiv\sum_{p=0}^{[l/2]}(-1)^p\frac{(2l-l-2p)!!}{(2l-1)!!}[\hat{n}^{L-2P}\delta^P+\mathrm{sym}(q)], \tag{A1}$$

where $[l/2]$ denotes the integer just less than or equal to $l/2$, the capitalized superscripts denote the dimensionality, $l-2p$ or p, of products of $\hat{n}^i$ or δ^{ij} respectively, and ''sym(q)'' denotes all distinct terms arising from permutations of indices, where $q=l!/[(2^p p!(l-2p)!]$ is the total number of such terms (see [34,28] for compendia of formulas). For convenience, we display the first several examples explicitly:

$$\hat{n}^{\langle ij\rangle}=\hat{n}^{ij}-\frac{1}{3}\delta^{ij}, \tag{A2a}$$

$$\hat{n}^{\langle ijk\rangle}=\hat{n}^{ijk}-\frac{1}{5}(\hat{n}^i\delta^{jk}+\hat{n}^j\delta^{ik}+\hat{n}^k\delta^{ij}), \tag{A2b}$$

$$\hat{n}^{\langle ijkl\rangle}=\hat{n}^{ijkl}-\frac{1}{7}[\hat{n}^{ij}\delta^{kl}+\mathrm{sym}(6)]+\frac{1}{35}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}), \tag{A2c}$$

$$\hat{n}^{\langle ijklm\rangle}=\hat{n}^{ijklm}-\frac{1}{9}[\hat{n}^{ijk}\delta^{kl}+\mathrm{sym}(10)]+\frac{1}{63}[\hat{n}^{i}\delta^{jk}\delta^{lm}+\mathrm{sym}(15)], \tag{A2d}$$

$$\hat{n}^{\langle ijklmn\rangle}=\hat{n}^{ijklmn}-\frac{1}{11}[\hat{n}^{ijkl}\delta^{mn}+\mathrm{sym}(15)]+\frac{1}{99}[\hat{n}^{ij}\delta^{kl}\delta^{mn}+\mathrm{sym}(45)]-\frac{1}{693}[\delta^{ij}\delta^{kl}\delta^{mn}+\mathrm{sym}(15)]. \tag{A2e}$$

There is a close connection between these STF tensors and spherical harmonics. For example, it is straightforward to show that, for any unit vector $\hat{\mathbf{N}}$, the contraction of $\hat{N}^L$ with $\hat{n}^{\langle L\rangle}$ is given by

$$\hat{N}^L\hat{n}^{\langle L\rangle}=\frac{l!}{(2l-1)!!}P_l(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}), \tag{A3}$$

where P_l is a Legendre polynomial. This latter property can be used to establish the identity (4.11):

$$\sum_m \int Y^*_{lm}(\hat{\mathbf{n}})Y_{lm}(\hat{\mathbf{y}})\hat{y}^{\langle L'\rangle}d^2\Omega_y\equiv\hat{n}^{\langle L\rangle}\delta_{ll'}. \tag{A4}$$

Since the left-hand side is STF, and depends only on the unit vector $\hat{\mathbf{n}}$, then it must be proportional to the STF combination $\hat{n}^{\langle L'\rangle}$. To establish the normalization, contract both sides with the L'-dimensional non-STF product $\hat{N}^{L'}$, where $\hat{\mathbf{N}}$ represents the z direction. Using Eq. (A3), and recalling that $P_{l'}=\mathcal{N}_{l'}Y_{l'0}$, where $\mathcal{N}_{l'}$ is a normalization coefficient, we find that the integral yields $[l!/(2l-1)!!]P_{l'}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\delta_{ll'}$, establishing the unit coefficient in Eqs. (4.11) and (A4).

In calculating the radiation-zone contributions to h^{ij}, we must also evaluate the integrals $(4\pi)^{-1}\int_0^{2\pi}d\phi'\int_{1-\alpha}^1\hat{n}'^{\langle L\rangle}(\zeta-\hat{\mathbf{n}}'\cdot\hat{\mathbf{n}})^{N-3}d\cos\theta'$, where $\alpha=(\zeta-1)(\zeta+1-2\mathcal{R}/r)(r/2\mathcal{R})$. The result must be an l-dimensional STF tensor, dependent on the only vector in the problem, $\hat{\mathbf{n}}$, and thus must be proportional to $\hat{n}^{\langle L\rangle}$. To determine the proportionality factor, which will be a function of ζ and α, we contract with $\hat{n}^L$, choose $\hat{\mathbf{n}}$ to be in the z direction, and substitute Eq. (A3). The result is

$$(4\pi)^{-1}\int_0^{2\pi}d\phi'\int_{1-\alpha}^1\hat{n}'^{\langle L\rangle}(\zeta-\hat{\mathbf{n}}'\cdot\hat{\mathbf{n}})^{N-3}d\cos\theta'=A_{N,l}(\zeta,\alpha)\hat{n}^{\langle L\rangle}, \tag{A5a}$$

$$A_{N,l}(\zeta,\alpha)=\frac{1}{2}\int_{1-\alpha}^1P_l(z)(\zeta-z)^{N-3}dz. \tag{A5b}$$

APPENDIX B: DERIVATIVES OF GRAVITATIONAL POTENTIALS

In evaluating the ''field'' parts of EW moments, we have repeated occasion to integrate expressions involving two derivatives, spatial, time, and mixed, of the potential U and two spatial derivatives of $\ddot{X}$. For a field point external to the bodies, such derivatives can be calculated easily from the expressions (4.4a) and (4.4b). However, because the integrations run over the locations $\mathbf{x}_A$ of the bodies themselves, we must carefully evaluate the singular behavior of such double derivatives at $\mathbf{x}=\mathbf{x}_A$. Consider, for example, the expression for $\ddot{U}$, written in terms of a smooth density distribution:

$$\ddot{U}=\int\rho'\left[\frac{\mathbf{a}'\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}+\frac{3v'^{ij}(x-x')^{\langle ij\rangle}}{|\mathbf{x}-\mathbf{x}'|^5}\right]d^3x', \tag{B1}$$

where $\mathbf{a}'=d\mathbf{v}'/dt$. For a field point outside the bodies, shrinking the density distribution to a point yields a result equivalent to that obtained by differentiating Eq. (4.4a). For a point inside, say, body A, we find that the integral $\int_{\text{body }A}\ddot{U}d^3x\rightarrow-(4\pi/3)m_Av_A^2$ as the size of body A shrinks to a point. Consequently we must add a δ-function term to all double derivatives of U and X found using Eqs. (4.4a) and (4.4b). The results are

$$U_{,ij}=U^\dagger_{,ij}-(4\pi/3)\sum_A m_A\delta^{ij}\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{B2a}$$

$$\dot{U}_{,i}=\dot{U}^\dagger_{,i}+(4\pi/3)\sum_A m_Av_A^i\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{B2b}$$

$$\ddot{U}=\ddot{U}^\dagger-(4\pi/3)\sum_A m_Av_A^2\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{B2c}$$

$$\ddot{X}_{,ij}=\ddot{X}^\dagger_{,ij}-(8\pi/15)\sum_A m_A(v_A^2\delta^{ij}+2v_A^{ij})\delta^3(\mathbf{x}-\mathbf{x}_A), \tag{B2d}$$

where † denotes derivatives computed from Eqs. (4.4a) and (4.4b).

APPENDIX C: THE SECOND-ITERATED FIELDS

In Sec. III B, we wrote down the second-iterated solution for $h^{\alpha\beta}$ in terms of the potentials V, V_i, and W_{ij}. Here we discuss the solutions for these potentials, Eqs. (3.4), in more detail, especially the potential W_{ij}, whose source is noncompact.

We first consider field points in the radiation zone. Since their sources have compact support, the potentials V and V_i do not have to be divided into contributions from integrals over the near zone and over the radiation zone. They can be expanded using the analogue of Eq. (2.14), and written to the needed order in the form

$$V(t,\mathbf{x})=\widetilde{m}/r+\frac{1}{2}\{r^{-1}Q^{ij}(u)\}_{,ij}-\frac{1}{6}\{r^{-1}Q^{ijk}(u)\}_{,ijk}+\frac{1}{2}\ddot{Q}/r-\{r^{-1}F^i(u)\}_{,i}+O(\epsilon^3), \quad \text{(C1a)}$$

$$V_i(t,\mathbf{x})=-\frac{1}{2}\{r^{-1}[\dot{Q}^{ij}(u)-\epsilon^{ija}J^a(u)]\}_{,j}+\frac{1}{6}\{r^{-1}[\dot{Q}^{ijk}(u)-2\epsilon^{ika}J^{aj}(u)]\}_{,jk}+O(\epsilon^{5/2}), \quad \text{(C1b)}$$

where $\ddot{Q}$ and $F^i\equiv\Sigma_A m_A x_A^i(v_A^2-\Sigma_B m_B/2r_{AB})$, respectively, represent the difference between the monopole and dipole moments of the potential V, and the 1PN accurate, constant total mass $\widetilde{m}$, Eq. (4.16a), and the vanishing center of mass $\mathbf{X}$, Eq. (4.16c). In constructing h^{00} using Eq. (3.5a) these two terms are cancelled by terms from $W=W^{ii}$.

The potential W_{ij} must first be divided into near-zone and radiation-zone contributions, $W_{ij}=(W_{ij})_{\mathcal{N}}+(W_{ij})_{\mathcal{C}-\mathcal{N}}$. To the $O(\epsilon^2)$ needed for the use of W_{ij} in source terms for higher iterations [see Eqs. (3.5)], we can approximate the integrand in Eq. (3.4c) by $\sigma_{ij}+(4\pi)^{-1}(U_{,i}U_{,j}-\frac{1}{2}\delta_{ij}U_{,k}U_{,k})\equiv\tau_{(1)}^{ij}/4$, with $\sigma_{ij}=\Sigma_A m_A v_A^i v_A^j\delta^3(\mathbf{x}-\mathbf{x}_A)$. Here $\tau_{(1)}^{ij}$ denotes the first-iterated effective stress energy. Then $(W_{ij})_{\mathcal{N}}$ can be expanded using the analogue of Eq. (2.14), with the result

$$(W_{ij})_{\mathcal{N}}=\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\left(\frac{1}{r}M^{ijk_1\cdots k_q}\right)_{,k_1\cdots k_q}, \quad \text{(C2)}$$

where

$$M^{ijk_1\cdots k_q}(u)=\int_{\mathcal{M}}\tau_{(1)}^{ij}(u,\mathbf{x})x^{k_1\cdots k_q}d^3x. \quad \text{(C3)}$$

Using the expression above for $\tau_{(1)}^{ij}$ in each of the moments in Eq. (C3), and using the strategy for evaluating field integrals described in Secs. IV A and IV B, we find, to the needed accuracy,

$$M^{ij}=\frac{1}{2}\ddot{Q}^{ij}, \quad \text{(C4a)}$$

$$M^{ijk}=\frac{1}{6}\ddot{Q}^{ijk}-\frac{2}{3}\epsilon^{(i|ka}\dot{J}^{a|j)}, \quad \text{(C4b)}$$

$$M^{ijkl}=\frac{1}{15}m^2\mathcal{R}\left(2\delta^{i(k}\delta^{l)j}-\frac{3}{2}\delta^{ij}\delta^{kl}\right)+\text{term independent of }\mathcal{R}, \quad \text{(C4c)}$$

where we discard terms that fall off with increasing $\mathcal{R}$, but retain all other terms. Although we never actually need the contribution from the moment M^{ijkl}, we show the $\mathcal{R}$-dependent term to illustrate its ultimate cancellation.

To evaluate $(W_{ij})_{\mathcal{C}-\mathcal{N}}$, we use the fact that, to the required order, in the radiation zone, $\tau_{(1)}^{ij}=(4\pi)^{-1}(m^2/r'^4)(\hat{n}'^{\langle ij\rangle}-\frac{1}{6}\delta^{ij})$. Using Eq. (5.4), and remembering the factor of 4 difference between h^{ij} and W_{ij}, we obtain

$$(W_{ij})_{\mathcal{C}-\mathcal{N}}=\frac{1}{4}\frac{m^2}{r^2}\hat{n}^{ij}-\frac{1}{5}\frac{m^2}{r^3}\mathcal{R}\hat{n}^{\langle ij\rangle}, \quad \text{(C5)}$$

where we again discard terms that fall off with $\mathcal{R}$. It is easy to see that the $\mathcal{R}$-dependent term in Eq. (C5) exactly cancels the corresponding term in $(W_{ij})_{\mathcal{N}}$ resulting from Eqs. (C4c) and (C2). Combining the contributions to W_{ij} through octupole order, and substituting them along with Eqs. (C1) for V and V_i into Eqs. (3.5) yields the second-iterated radiation-zone fields $h^{\alpha\beta}$, Eqs. (5.5). It is interesting to note that the $(m^2/4r^2)\hat{n}^{ij}$ term in $(W_{ij})_{\mathcal{C}-\mathcal{N}}$ is required in order that the far-zone field correctly approximate the Schwarzschild geometry in harmonic coordinates in the static limit: namely,

$$h^{00}=4m/r+7(m/r)^2, \quad \text{(C6a)}$$

$$h^{0i}=0, \quad \text{(C6b)}$$

$$h^{ij}=(m/r)^2\hat{n}^{ij} \quad \text{(C6c)}$$

[compare Eq. (5.5)]. This contribution could not have been found using the EW approach without our new formulation of the radiation-zone integrals.

We next consider field points within the near zone. Expanding the retardation about $t=u$ with $|\mathbf{x}-\mathbf{x}'|$ as the small parameter, we obtain Eqs. (3.6) and (3.7). The compact contributions to U, X, U_i, and P_{ij} can be evaluated directly; the noncompact part of P_{ij} is left unevaluated until it is incorporated into an EW moment (see Appendix D). It remains to evaluate the radiation-zone contribution $(W_{ij})_{\mathcal{C}-\mathcal{N}}$ with a near-zone field point. Using the form of $\tau_{(1)}^{ij}$ above, and using the near-zone field-point version of Eq. (5.4), we find only contributions proportional to $m^2r^2/\mathcal{R}^4$ and $m^2/\mathcal{R}^2$. Thus we can discard such terms.

APPENDIX D: CUBIC NONLINEARITIES IN I_{EW}^{ij}

At 2PN order, the nonlinear field source Λ^{00} Eq. (4.8) contains terms that are cubically nonlinear, i.e., that depend on effective products of three gravitational potentials. The contribution of the final such term in Eq. (4.8), proportional to $UU_{,k}U_{,k}$, to the integral $\int_{\mathcal{M}}\Lambda^{00}x^ix^jd^3x$ can be evaluated straightforwardly by integrating by parts. However, the two terms $2P_{,k}U_{,k}-P_{km}U_{,km}$ are more difficult because P_{ij} itself [Eq. (3.7d)] is a potential, one of whose pieces is produced by a nonlinear source. The contribution of the compact source σ_{ij} can be handled easily by the methods of Sec. IV A. Here we focus on the nonlinear piece. We define the nonlinear potential

$$p_{ij}(u,\mathbf{x})\equiv\frac{1}{4\pi}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left(U_{,i}U_{,j}-\frac{1}{2}\delta_{ij}U_{,k}U_{,k}\right)(u,\mathbf{x}'). \tag{D1}$$

We then need to evaluate the integral

$$(1/\pi)\int_{\mathcal{M}}(2p_{,k}U_{,k}-p_{km}U_{,km})x^ix^jd^3x. \tag{D2}$$

We integrate the first term by parts, show that the surface terms fall off with $\mathcal{R}$, and obtain $8\Sigma_A m_A p(\mathbf{x}_A)x_A^{ij}-(4/\pi)\int pU^{,(i}x^{j)}d^3x$. The first of these terms may be evaluated using the nonlinear pieces of Eq. (4.4d). The second term may be written in the form

$$\frac{1}{2\pi^2}\int_{\mathcal{M}}|\nabla' U'|^2d^3x'\sum_A m_A\int_{\mathcal{M}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{(x-x_A)^{(i}x^{j)}}{|\mathbf{x}-\mathbf{x}_A|^3}d^3x. \tag{D3}$$

In the x integration, we change variables to $\mathbf{y}=\mathbf{x}-\mathbf{x}_A$ and integrate using the general method described in Sec. IV A. The result is the integral $(1/2\pi)\Sigma_A m_A\int_{\mathcal{M}}|\nabla' U'|^2d^3x'(x'^{ij}-x_A^{ij})/|\mathbf{x}_A-\mathbf{x}'|$, which can be easily evaluated by integrating by parts.

The second term in Eq. (D2) can be written

$$-\frac{1}{4\pi^2}\int_{\mathcal{M}}\left(U'_{,k}U'_{,m}-\frac{1}{2}\delta_{km}|\nabla' U'|^2\right)d^3x'\sum_A m_A \times\int_{\mathcal{M}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left(\frac{3(x-x_A)^{\langle km\rangle}}{|\mathbf{x}-\mathbf{x}_A|^5}-\frac{4\pi}{3}\delta^{km}\delta^3(\mathbf{x}-\mathbf{x}_A)\right)x^{ij}d^3x. \tag{D4}$$

Again we do the x integration by changing variables to $\mathbf{y}=\mathbf{x}-\mathbf{x}_A$, and using the method of Sec. IV A. Integration of the δ-function term is straightforward. The remaining x' integration takes the form

$$\frac{1}{\pi}\int_{\mathcal{M}}\left(U'_{,k}U'_{,m}-\frac{1}{2}\delta_{km}|\nabla' U'|^2\right)d^3x'\sum_A m_A\left(\frac{1}{6}\Phi^A_{,ijkm}-\Psi^A_{,k(i}\delta_{j)m}+\frac{1}{2}\Psi^A_{,km(i}x_A^{j)}-2X^A_{,k}\delta_{m(i}x_A^{j)}+\frac{1}{2}X^A_{,km}x_A^{ij}-\frac{1}{3}\frac{\delta_{km}x_A^{ij}}{|\mathbf{x}'-\mathbf{x}_A|}+X^A\delta_{k(i}\delta_{j)m}-\frac{1}{5}\mathcal{R}\delta_{k(i}\delta_{j)m}\right), \tag{D5}$$

where $\Phi^A\equiv|\mathbf{x}'-\mathbf{x}_A|^5/15$, $\Psi^A\equiv|\mathbf{x}'-\mathbf{x}_A|^3/3$, and $X^A\equiv|\mathbf{x}'-\mathbf{x}_A|$. The first five terms in Eq. (D5) can be evaluated simply by integrating by parts. The sixth term is equivalent to the cubically nonlinear term in Λ^{00} proportional to $UU_{,k}U_{,k}$ [Eq. (4.8)]. The final term proportional to $\mathcal{R}$ is straightforward.

The seventh term requires extra work. Dropping contributions with no TT part, we find that the integral to be evaluated is $\pi^{-1}\int_{\mathcal{M}}U_{,i}U_{,j}Xd^3x$. Defining U_A and X_A to be the contribution to U and X from body A, respectively, we write

$$\int_{\mathcal{M}}U_{,i}U_{,j}Xd^3x=\sum_A\int_{\mathcal{M}}U_{A,i}U_{A,j}X_Ad^3x+\sum_{A\neq B}\int_{\mathcal{M}}U_{A,i}U_{A,j}X_Bd^3x+2\sum_{A\neq B}\int_{\mathcal{M}}U_{A,(i}U_{B,j)}X_Ad^3x+\sum_{A\neq B\neq C}\int_{\mathcal{M}}U_{A,(i}U_{B,j)}X_Cd^3x. \tag{D6}$$

The first term has no TT part, while the second two terms can be evaluated using the standard methods of Sec. IV A, and lead to the term $-3\Sigma_{AB}m_A^2m_B\hat{n}_{AB}^{ij}$ in Eq. (4.17). We define the third term to be $\mathcal{G}_{(3)}^{ij}$, change variables to $\mathbf{u}=\mathbf{x}-\mathbf{x}_C$, $\mathbf{y}=\mathbf{x}_A-\mathbf{x}_C$, and $\mathbf{z}=\mathbf{x}_B-\mathbf{x}_C$, verify that no surface contributions at $\mathcal{R}$ are so generated, and show that $\mathcal{G}$ can be written $\mathcal{G}_{(3)}^{ij}=\Sigma_{ABC}m_Am_Bm_C\nabla_y^i\nabla_z^jF(\mathbf{y},\mathbf{z})$, where $F(\mathbf{y},\mathbf{z})\equiv\int_{\mathcal{M}}|\mathbf{u}-\mathbf{y}|^{-1}|\mathbf{u}-\mathbf{z}|^{-1}ud^3u$. The latter step involves ensuring that the piece of F that diverges with $\mathcal{R}$ contributes no TT part to $\mathcal{G}$, so that the integration can effectively be commuted with the y and z derivatives. Note that F has units of (distance)^2, is symmetric on y and z, is a function only of $|\mathbf{y}|$, $|\mathbf{z}|$ and $w\equiv|\mathbf{y}-\mathbf{z}|$, and has the property that $\nabla_y^2F=-4\pi y/w$, $\nabla_z^2F=-4\pi z/w$. It is then straightforward to show that the function with these properties is given by $F(\mathbf{y},\mathbf{z})=-(2\pi/3)[(y+z)w-yz+(y^2+z^2-w^2)\ln(y+z+w)]$, modulo terms that give no TT contribution to $\mathcal{G}$. Thus the solution for $\mathcal{G}_{(3)}^{ij}$ in Eq. (4.17) is

$$\mathcal{G}_{(3)}^{ij}=\sum_{A\neq B\neq C}m_Am_Bm_C\nabla_A^i\nabla_B^jF(\mathbf{x}_{AC},\mathbf{x}_{BC}), \tag{D7a}$$

$$F(\mathbf{x}_{AC},\mathbf{x}_{BC})=-\frac{2}{3}[(r_{AC}+r_{BC})r_{AB}-r_{AC}r_{BC}+2\mathbf{x}_{AC}\cdot\mathbf{x}_{BC}\ln(r_{AC}+r_{BC}+r_{AB})]. \tag{D7b}$$

Note that, because $\nabla_A^i\nabla_B^j(\mathbf{x}_{AC}\cdot\mathbf{x}_{BC})=\delta_{ij}$, no logarithmic dependence on source variables actually survives in $h_{\rm TT}^{ij}$. For two-body systems, this term does not enter the formula for the EW moment.

APPENDIX E: STF-MULTIPOLE DECOMPOSITION

Although the Epstein-Wagoner multipoles arose very naturally in our retarded-time expansion of the relaxed Einstein equation, these are not the only multipoles for displaying the answer. An alternative set are the symmetric trace-free (STF) multipoles, which arise naturally in angular decompositions of the waveform (see, e.g., [34]), and are multipoles of choice in the BDI framework. Thus it is useful to obtain a transformation between the EW multipoles and the STF multipoles.

If the waveform is known then the STF multipoles can be projected out. This is exactly analogous to projecting the coefficients of spherical harmonics from a scalar function. The STF multipoles can be projected from the TT waveform by integrating over the sphere [see [34], Eq. (4.11)]:

$$\frac{d^m}{du^m}\mathcal{I}^{a_1a_2\cdots a_m}_{\rm STF}=\left[\frac{m(m-1)(2m+1)!!}{2(m-1)(m+2)}\frac{R}{4\pi}\times\int h^{a_1a_2}_{\rm TT}N^{a_3}\cdots N^{a_m}d\Omega\right], \tag{E1a}$$

$$\frac{d^m}{du^m}\mathcal{J}^{a_1a_2\cdots a_m}_{\rm STF}=\left[\frac{(m-1)(2m-1)!!}{4(m+2)}\frac{R}{4\pi}\times\int \epsilon^{a_1jk}N^jh^{ka_2}_{\rm TT}N^{a_2}\cdots N^{a_m}d\Omega\right], \tag{E1b}$$

where $\mathcal{I}^{a_1a_2\cdots a_m}_{\rm STF}$ are called ''mass'' multipole moments and $\mathcal{J}^{a_1a_2\cdots a_m}_{\rm STF}$ are called ''current'' or ''spin'' multipole moments. Substituting the expansion of $h^{a_1a_2}_{\rm TT}$ in terms of EW moments, Eq. (2.18), and adding the radiation-zone tail terms, Eq. (5.8), we obtain the transformations, correct to 2PN order:

$$\mathcal{I}^{ij}_{\rm STF}=\left[I^{ij}_{\rm EW}+\frac{1}{21}(11I^{ijkk}_{\rm EW}-12I^{k(ij)k}_{\rm EW}+4I^{kkij}_{\rm EW})+\frac{1}{63}(23I^{ijaabb}_{\rm EW}-32I^{a(ij)abb}_{\rm EW}+10I^{aaijbb}_{\rm EW}+2I^{abijab}_{\rm EW})\right]_{\rm STF}+\mathcal{I}^{ij}_{\rm tail}, \tag{E2a}$$

$$\dot{\mathcal{I}}^{ijk}_{\rm STF}=[3I^{ijk}_{\rm EW}+(3I^{ijkaa}_{\rm EW}-3I^{iaakj}_{\rm EW}+I^{aaijk}_{\rm EW})]_{\rm STF}+\dot{\mathcal{I}}^{ijk}_{\rm tail}, \tag{E2b}$$

$$\ddot{\mathcal{I}}^{ijkl}_{\rm STF}=\left[12I^{ijkl}_{\rm EW}+\frac{72}{55}(13I^{ijklmm}_{\rm EW}-12I^{immjkl}_{\rm EW}+4I^{mmijkl}_{\rm EW})\right]_{\rm STF}, \tag{E2c}$$

$$\overset{(3)}{\mathcal{I}}{}^{ijklm}_{\rm STF}=[60I^{ijklm}_{\rm EW}]_{\rm STF}, \tag{E2d}$$

$$\overset{(4)}{\mathcal{I}}{}^{ijklmn}_{\rm STF}=[360I^{ijklmn}_{\rm EW}]_{\rm STF}, \tag{E2e}$$

$$\mathcal{J}^{ij}_{\rm STF}=\left[\frac{1}{2}\epsilon_{ipq}I^{jqp}_{\rm EW}+\frac{1}{28}\epsilon_{ipq}(9I^{jqpmm}_{\rm EW}-3I^{qmmjp}_{\rm EW})\right]_{\rm STF}+\mathcal{J}^{ij}_{\rm tail}, \tag{E2f}$$

$$\dot{\mathcal{J}}^{ijk}_{\rm STF}=\left[2\epsilon_{ipq}I^{jqpk}_{\rm EW}+\frac{4}{15}\epsilon_{ipq}(7I^{jqpkmm}_{\rm EW}-2I^{qmmpjk}_{\rm EW})\right]_{\rm STF}, \tag{E2g}$$

$$\ddot{\mathcal{J}}^{ijkl}_{\rm STF}=[9\epsilon_{ipq}I^{jqpkl}_{\rm EW}]_{\rm STF}, \tag{E2h}$$

$$\overset{(3)}{\mathcal{I}}{}^{ijklm}_{\rm STF}=[48\epsilon_{ipq}I^{jqpklm}_{\rm EW}]_{\rm STF}, \tag{E2i}$$

where the STF notation on the right-hand side means symmetrize and remove all traces (note that the STF tensors are symmetric on all indices, while the EW moments are symmetric only on selected pairs). These transformations can also be established using Eqs. (5.23) and (5.24) of [34].

For two-body systems in general orbits, the resulting STF moments are given by

$$\mathcal{I}^{ij}_{\rm STF}=\mu r^2\Bigg\{\hat n^i\hat n^j+\frac{1}{42}\left[\hat n^i\hat n^j\left(29(1-3\eta)v^2-6(5-8\eta)\frac{m}{r}\right)-24(1-3\eta)\dot r\hat n^{(i}v^{j)}+22(1-3\eta)v^iv^j\right]$$
$$+\frac{1}{1512}\hat n^i\hat n^j\left[3(253-1835\eta+3545\eta^2)v^4-6(355+1906\eta-337\eta^2)\left(\frac{m}{r}\right)^2+2(2021-5947\eta-4883\eta^2)\frac{m}{r}v^2\right.$$
$$\left.-2(131-907\eta+1273\eta^2)\frac{m}{r}\dot r^2\right]+\frac{1}{378}v^iv^j\left[2(742-335\eta-985\eta^2)\frac{m}{r}+3(41-337\eta+733\eta^2)v^2\right.$$
$$\left.\left.+30(1-5\eta+5\eta^2)\dot r^2\right]-\frac{1}{378}\hat n^{(i}v^{j)}\dot r\left[(1085-4057\eta-1463\eta^2)\frac{m}{r}+12(13-101\eta+209\eta^2)v^2\right]\Bigg\}_{\rm STF}+\mathcal{I}^{ij}_{\rm tail}, \tag{E3a}$$

$$\mathcal{I}^{ijk}_{\rm STF}=-\mu\frac{\delta m}{m}r^3\left\{\hat n^i\hat n^j\hat n^k\left[1+\frac{1}{6}(5-19\eta)v^2-\frac{1}{6}(5-13\eta)\frac{m}{r}\right]+(1-2\eta)(v^iv^j\hat n^k-\dot r v^i\hat n^j\hat n^k)\right\}_{\rm STF}+\mathcal{I}^{ijk}_{\rm tail}, \tag{E3b}$$

$$\mathcal{I}^{ijkl}_{\rm STF}=\mu r^4\left\{\hat n^i\hat n^j\hat n^k\hat n^l\left[(1-3\eta)+\frac{1}{110}(103-735\eta+1395\eta^2)v^2-\frac{1}{11}(10-61\eta+105\eta^2)\frac{m}{r}\right]+\frac{6}{55}(1-5\eta+5\eta^2)(13v^iv^j\hat n^k\hat n^l-12\dot r v^i\hat n^j\hat n^k\hat n^l)\right\}_{\rm STF}, \tag{E3c}$$

$$\mathcal{I}^{ijklm}_{\rm STF}=-\mu\frac{\delta m}{m}r^5\{(1-2\eta)\hat n^i\hat n^j\hat n^k\hat n^l\hat n^m\}_{\rm STF}, \tag{E3d}$$

$$\mathcal{I}^{ijklmn}_{\rm STF}=\mu r^6\{(1-5\eta+5\eta^2)\hat{n}^i\hat{n}^j\hat{n}^k\hat{n}^l\hat{n}^m\hat{n}^n\}_{\rm STF}, \tag{E3e}$$

$$\mathcal{J}^{ij}_{\rm STF}=-\mu\frac{\delta m}{m}r\left\{(\mathbf{x}\times\mathbf{v})^i\left[\hat{n}^j\left(1+\frac{1}{28}(13-68\eta)v^2+\frac{3}{14}(9+10\eta)\frac{m}{r}\right)+\frac{5}{28}(1-2\eta)\dot{r}v^j\right]\right\}_{\rm STF}+\mathcal{J}^{ij}_{\rm tail}, \tag{E3f}$$

$$\mathcal{J}^{ijk}_{\rm STF}=\mu r^2\left\{(\mathbf{x}\times\mathbf{v})^i\left[\hat{n}^j\hat{n}^k\left(1-3\eta+\frac{1}{90}(41-385\eta+925\eta^2)v^2+\frac{2}{9}(7-8\eta-43\eta^2)\frac{m}{r}\right)\right.\right.$$
$$\left.\left.+\frac{1}{45}(1-5\eta+5\eta^2)(10\dot{r}v^j\hat{n}^k+7v^jv^k)\right]\right\}_{\rm STF}, \tag{E3g}$$

$$\mathcal{J}^{ijkl}_{\rm STF}=-\mu\frac{\delta m}{m}r^3(1-2\eta)\{(\mathbf{x}\times\mathbf{v})^i\hat{n}^j\hat{n}^k\hat{n}^l\}_{\rm STF}, \tag{E3h}$$

$$\mathcal{J}^{ijklm}_{\rm STF}=\mu r^4(1-5\eta+5\eta^2)\{(\mathbf{x}\times\mathbf{v})^i\hat{n}^j\hat{n}^k\hat{n}^l\hat{n}^m\}_{\rm STF}, \tag{E3i}$$

where the ''tail'' STF moments are given by

$$\ddot{\mathcal{I}}^{ij}_{\rm tail}=2m\int_0^\infty ds\,\overset{(4)}{Q}{}^{ij}(u-s)\left[\ln\left(\frac{s}{2R+s}\right)+\frac{11}{12}\right]_{\rm STF}, \tag{E4a}$$

$$\dddot{\mathcal{I}}^{ijk}_{\rm tail}=2m\int_0^\infty ds\,\overset{(5)}{Q}{}^{ijk}(u-s)\left[\ln\left(\frac{s}{2R+s}\right)+\frac{97}{60}\right]_{\rm STF}, \tag{E4b}$$

$$\ddot{\mathcal{J}}^{ij}_{\rm tail}=2m\int_0^\infty ds\,\overset{(4)}{J}{}^{ij}(u-s)\left[\ln\left(\frac{s}{2R+s}\right)+\frac{7}{6}\right]_{\rm STF}. \tag{E4c}$$

Through 3/2PN order, these moments agree with [35], and in the circular orbit limit, through 2PN order, they agree with BDI [39].

In terms of STF moments, the waveform and energy flux may be written [34]

$$h^{ij}_{\rm TT}=\frac{1}{R}\sum_{l=2}^{\infty}\left[\frac{4}{l!}\overset{(l)}{\mathcal{I}}{}^{ija_1\cdots a_{l-2}}_{\rm STF}(u)\hat{N}^{a_1\cdots a_{l-2}}+\frac{8l}{(l+1)!}\epsilon_{pq(i}\overset{(l)}{\mathcal{J}}{}^{j)pa_1\cdots a_{l-2}}_{\rm STF}(u)\hat{N}^{qa_1\cdots a_{l-2}}\right]_{\rm TT}, \tag{E5a}$$

$$\frac{dE}{dt}=\sum_{l=2}^{\infty}\left[\frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!}\overset{(l+1)}{\mathcal{I}}{}^{a_1\cdots a_l}_{\rm STF}(u)\overset{(l+1)}{\mathcal{I}}{}^{a_1\cdots a_l}_{\rm STF}(u)+\frac{4l(l+2)}{(l-1)(l+1)!(2l+1)!!}\overset{(l+1)}{\mathcal{J}}{}^{a_1\cdots a_l}_{\rm STF}(u)\overset{(l+1)}{\mathcal{J}}{}^{a_1\cdots a_l}_{\rm STF}(u)\right]. \tag{E5b}$$

Substitution of Eqs. (E3) into Eqs. (E5a) and (E5b), using 2PN equations of motions in any acceleration terms generated by time derivatives, and keeping terms through 2PN order, yields Eqs. (6.10), (6.11), (6.12) and (6.13).

APPENDIX F: SPIN EFFECTS

In this paper, we have used our augmented Epstein-Wagoner formalism to give a complete description of the gravitational radiation for inspiralling ''point-mass'' binaries through $O(\epsilon^2)$ beyond the lowest-order quadrupole contribution. In this appendix we demonstrate that our formalism is also adequate for computing contributions to the radiation which arise from the finite spatial extent of the bodies. Our primary goal will be to compute the contributions to the radiation from the bodies' spin angular momenta, but in the process we will show how other extended-body effects, such as those due to a body's intrinsic quadrupole moment, could be computed with our formalism. The results will be presented in such a way that the spin contributions computed here can just be *added* to results already presented here and elsewhere. In particular we give the spin-orbit ($PQ^{ij}_{\rm SO}$ and $P^{3/2}Q^{ij}_{\rm SO}$) and spin-spin ($P^2Q^{ij}_{\rm SS}$) contributions to the waveform Eq. (6.10) for general orbits. We also give a restricted circular-orbit version of the results which can be added to the ''ready-to-use'' waveforms in [48].

In order to derive the spin corrections to the waveform, we relax our ''point-mass'' assumption and allow the bodies to have spatial extent *small* compared to the interbody distances. We further assume that the bodies are uniformly spinning fluid balls, approximately spherical in harmonic coordinates. (A full discussion of this ''fluid sphere'' formalism is given in Appendix A of [35], where it is used to derive the waveform produced by nonspinning bodies through $O[\epsilon^{3/2}]$.) Although formally, our PN approach restricts us to weak internal gravity, we anticipate applying the results to neutron stars and black holes, as in the nonspinning case, by relying upon the strong equivalence principle (see Sec. II B for discussion of this point). It is now conventional, in treating spinning compact bodies, to view the spin **S** of each body as a quantity measured in units of its $(\text{mass})^2$, as is the case for black holes. Given that, formally, $S \sim md\overline{v}$, where d is the size of the body, and $\overline{v}$ is its rotational velocity, our convention implies that $S_{\rm compact}/S_{\rm formal} \sim m/d\overline{v} \sim \epsilon^{1/2}$, with the result that spin effects are viewed as 1/2PN order smaller per factor of spin than would be the case formally (see [40,41] for further discussion).

The leading-order spin corrections to the waveform arise solely from terms in the source [Eq. (2.5)] directly dependent upon fluid velocities. Since these terms have compact support, they generate no contributions to the waveform from surface terms or from far-zone integrals, at the order we are considering in this appendix. Thus the spin corrections can all be obtained from the compact support pieces of the EW moments Eq. (2.19). We illustrate the procedure for computing the spin contributions by examining the four-index EW multipole Eq. (2.19c):

$$I^{ijkl}_{\rm EW} = \int_{\mathcal{M}} \tau^{ij} x^k x^l d^3x. \tag{F1}$$

Using Eq. (2.9) and Eq. (2.5) we can write

$$I^{ijkl}_{\rm EW} = \int_{\mathcal{M}} [\rho v^i v^j + (\text{terms independent of velocity}) + O(\rho\epsilon^2)] x^k x^l d^3x. \tag{F2}$$

Terms which are independent of the fluid velocity will not contribute to the spin terms that we are computing here; they give nonspin terms which we have already calculated. Any spin terms that might result from the $O(\rho\epsilon^2)$ contributions will, in our convention, be at least $O(\epsilon^{1/2})$ smaller, beyond the 2PN order at which we are working. We now write the source-point position and velocity as

$$x^i \equiv x^i_A + \overline{x^i_A}, \tag{F3a}$$

$$v^i \equiv v^i_A + \overline{v^i_A}, \tag{F3b}$$

where x^i_A is a suitably defined, PN-order, coordinate "center of mass" of body A and $\overline{x^i_A}$ is a coordinate displacement vector from the center of mass to the fluid element within the body. Similarly $v^i_A = dx^i_A/dt$ is the coordinate velocity of the center of mass. (See, e.g., [71,40,41] for the definition of the center of mass.)

Substituting Eq. (F3) into Eq. (F2) and integrating we obtain

$$I^{ijkl}_{\rm EW} = \sum_A m_A v^{ij}_A x^{kl}_A + 2 v^{(i}_A \epsilon^{j)m(k} x^{l)}_A S^m_A, \tag{F4}$$

where we have defined the spin vector by the formula

$$\int_A \rho \overline{x^i_A v^j_A} d^3x \equiv \frac{1}{2}\epsilon^{kij} S^k_A, \tag{F5}$$

having assumed that $\int_A \rho \overline{x^{(i}_A v^{j)}_A} d^3x = (1/2) dI^{ij}_A/dt = 0$, where I^{ij}_A is the body's intrinsic moment-of-inertia tensor. The first term in Eq. (F4) is the leading-order velocity-dependent term in Eq. (4.26), and the second term is the spin-orbit correction to this multipole, of order $\epsilon^{1/2}$ smaller. In obtaining Eq. (F4) from Eq. (F2) we have neglected a number of terms because (1) they vanish because of our assumption of spherical symmetry, (2) they have vanishing transverse-traceless projection, or (3) they are higher order in the bodies' *small* dimension $(\sim m)$, and therefore effectively of higher PN order. Such higher order moments can in principle be retained and incorporated into the framework.

Keeping terms up to $O(\rho\epsilon)$ in the source τ^{ij}, and proceeding in precisely the same manner we can compute the spin-orbit contributions to the other EW multipoles

$$I^{ij}_{\rm EW} = \sum_A [m_A x^{ij}_A + x^{(i}_A (\mathbf{v}_A \times \mathbf{S}_A)^{j)}], \tag{F6}$$

$$\tilde{I}^{ijk}_{\rm EW} = \sum_A (m_A v^i_A x^{jk}_A + x^{(j}_A \epsilon^{k)li} S^l_A). \tag{F7}$$

Here again, the first terms are the leading-order non-spin contributions to the multipoles, Eqs. (4.17) and (4.22). The spin-orbit correction terms are, respectively, of order $\epsilon^{3/2}$ and $\epsilon^{1/2}$ smaller than the leading terms.

In generating the expressions for the multipoles and waveforms, we must include *spin* corrections to the equations of motion. However, in the case of spinning bodies there is a delicate point to be considered in this procedure. The center of mass of body A, denoted by $\mathbf{x}_A$ used in our derivation of the multipole expressions turns out not to be precisely the same as the definition of the body's position used in the derivation of the conventional spin-orbit equations of motion, as given, say, by Damour [44], or Eq. (F14) below. The difference is related to the use of different so-called "spin supplementary conditions" which fix the center of mass of spinning bodies (see [40,41] for a thorough discussion). We have previously shown [40] that, to bring our center-of-mass definition into accord with that used in the equations of motion we need to shift the position of body A in the following manner:

$$x^i_A \rightarrow x^i_A + \frac{1}{2m_A}(\mathbf{v_A} \times \mathbf{S_A})^i. \tag{F8}$$

Performing this transformation replaces Eq. (F6) with

$$I^{ij}_{\rm EW} = \sum_A [m_A x^{ij}_A + 2 x^{(i}_A (\mathbf{v}_A \times \mathbf{S}_A)^{j)}]. \tag{F9}$$

Since we are working only to 3/2PN order in the spin-orbit correction, the transformation Eq. (F8) has no effect on the other multipoles. However if one were deriving the 2PN spin-orbit correction to the waveform [i.e., $P^2 Q^{ij}_{\rm SO}$ in Eq. (6.10)] it would be necessary to use the tranformation on Eq. (F7) as well.

The spin pieces of Eqs. (F9), (F7) and (F4) can just be added to their N-body point-mass counterparts in Sec. IV, Eqs. (4.17), (4.22), and (4.26), respectively.

We now wish to restrict our attention to the two-body case and express our multipoles in terms of relative coordinates. The reduction parallels the two-body (nonspin) reduction given in Sec. VI. We introduce the spin corrections to the definition of the system center of mass, Eq. (4.16c) (see [40,41]), find the relation between the coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$ and the relative coordinate $\mathbf{x}$ corresponding to Eqs. (6.2), and substitute into the two-body EW moments. It is useful to define two relative spin quantities

$$\boldsymbol{\chi}_s=\frac{1}{2}\left(\frac{\mathbf{S_1}}{m_1^2}+\frac{\mathbf{S_2}}{m_2^2}\right), \tag{F10a}$$

$$\boldsymbol{\chi}_a=\frac{1}{2}\left(\frac{\mathbf{S_1}}{m_1^2}-\frac{\mathbf{S_2}}{m_2^2}\right). \tag{F10b}$$

With the spins normalized by the individual $(\text{masses})^2$, these vectors are essentially the vectorial sum and difference of the dimensionless angular-momentum (Kerr) parameters of the individual bodies. For orbital systems composed of two Kerr black holes or neutron stars these vectors will have a maximum magnitude of unity. Stability studies of rotating neutron stars show that the dimensionless angular momentum parameter is bounded above by 0.63–0.74 [72] depending on the equation of state. Defining the vector spin quantities in this way also has the advantage that they are comparable in maximum magnitude to the other vectors that are used to form the terms in the waveform, namely $\hat{\mathbf{n}}$, $\hat{\mathbf{N}}$, and $\mathbf{v}$. As one computes the two-body multipoles, the waveform, the energy flux, and the orbital phase evolution, the spins appear in many combinations with the masses. With the spin-quantity definitions as above, the reduced mass parameter η never appears in any denominators, so that the extreme mass ratio limit ($\eta\rightarrow 0$) is always transparent in all expressions below [73]. This may seem like a minor aesthetic point, but it also means that the equations in the form we present them are suitable for stable numerical implementation with mass parameters free to roam from the equal mass case to the test mass case, and spin parameters free to roam independently of the mass choice from magnitude zero to unity.

The spin corrections to the relation between $\mathbf{x}_1$, $\mathbf{x}_2$ and the relative coordinate $\mathbf{x}$ [41] take the form

$$\mathbf{x_1}=\frac{m_2}{m}\mathbf{x}-m\mathbf{v}\times[\boldsymbol{\chi}_s(\delta m/m)+\boldsymbol{\chi}_a], \tag{F11a}$$

$$\mathbf{x_2}=-\frac{m_1}{m}\mathbf{x}-m\mathbf{v}\times[\boldsymbol{\chi}_s(\delta m/m)+\boldsymbol{\chi}_a]. \tag{F11b}$$

Substituting these transformations into the leading order term in Eq. (F9), we find that these spin-orbit corrections cancel, to the required order [compare Eq. (6.3)]. Substituting these definitions into the N-body multipoles gives the spin-orbit corrections to the two-body Epstein-Wagoner multipoles

$$I^{ij}_{\mathrm{EW(SO)}}=4m^2\eta^2(\mathbf{v}\times\boldsymbol{\chi}_s)^{(i}x^{j)}, \tag{F12a}$$

$$I^{ijk}_{\mathrm{EW(SO)}}=2m^2\eta x^{(i}\epsilon^{j)lk}[(\delta m/m)\boldsymbol{\chi}_s+\boldsymbol{\chi}_a]^l, \tag{F12b}$$

$$I^{ijkl}_{\mathrm{EW(SO)}}=4m^2\eta^2 v^{(i}\epsilon^{j)m(k}s^{l)}\chi_s^m. \tag{F12c}$$

These corrections can be added to the two-body multipoles given in Sec. VI. STF multipoles can be projected from the EW multipoles using the formulas given in Appendix E. The results are

$$\mathcal{I}^{ij}_{\mathrm{STF\,(SO)}}=\frac{8}{3}m^2\eta^2[2x^i(\mathbf{v}\times\boldsymbol{\chi}_s)^j-v^i(\mathbf{x}\times\boldsymbol{\chi}_s)^j]_{\mathrm{STF}}, \tag{F13a}$$

$$\mathcal{J}^{ij}_{\mathrm{STF\,(SO)}}=\frac{3}{2}m^2\eta[[(\delta m/m)\boldsymbol{\chi}_s+\boldsymbol{\chi}_a]^i x^j]_{\mathrm{STF}}, \tag{F13b}$$

$$\mathcal{J}^{ijk}_{\mathrm{STF\,(SO)}}=4m^2\eta^2[x^ix^j\chi_s^k]_{\mathrm{STF}}. \tag{F13c}$$

Equations (F13) are in agreement with [40,41]. These spin-orbit contributions can be added to the STF multipoles given in Appendix E. It is interesting to note that the four-index EW multipole I^{ijkl}_{EW} is needed to describe spin dependence of the radiation, but there is no spin contribution from the four-index STF multipole $\mathcal{I}^{ijkl}_{\mathrm{STF}}$. The multipole I^{ijab}_{EW} does contribute to the multipole $\mathcal{I}^{ij}_{\mathrm{STF}}$ and $\mathcal{J}^{ijk}_{\mathrm{STF}}$ through Eqs. (E2a) and (E2g).

In order to derive the spin contributions to the waveform from the multipoles we must also augment the equations of motion [Eq. (6.4)] with spin-orbit and spin-spin contributions. These can be found in [40,41], and in our notation are given by

$$\begin{aligned}\mathbf{a}_{\mathrm{SO}}=\frac{m^2}{r^3}\{&6\hat{\mathbf{n}}(\hat{\mathbf{n}}\times\mathbf{v})\cdot[\boldsymbol{\chi}_s+(\delta m/m)\boldsymbol{\chi}_a]\\&-2\mathbf{v}\times[(2-\eta)\boldsymbol{\chi}_s+2(\delta m/m)\boldsymbol{\chi}_a]\\&+6\dot{r}\hat{\mathbf{n}}\times[(1-\eta)\boldsymbol{\chi}_s+(\delta m/m)\boldsymbol{\chi}_a]\},\end{aligned} \tag{F14a}$$

$$\begin{aligned}\mathbf{a}_{\mathrm{SS}}=-\frac{m^3}{r^4}\{&\hat{\mathbf{n}}[|\boldsymbol{\chi}_s|^2-|\boldsymbol{\chi}_a|^2-5(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s)^2+5(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a)^2]\\&+2[\boldsymbol{\chi}_s(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s)-\boldsymbol{\chi}_a(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a)]\}.\end{aligned} \tag{F14b}$$

We now substitute our EW multipoles into Eq. (2.18) and use the equations of motion to eliminate acceleration terms to obtain the final spin contributions to the waveform

$$PQ^{ij}_{\mathrm{SO}}=2\left(\frac{m}{r}\right)^2\{\hat{\mathbf{N}}\times[(\delta m/m)\boldsymbol{\chi}_s+\boldsymbol{\chi}_a]\}^{(i}n^{j)}, \tag{F15a}$$

$$\begin{aligned}P^{3/2}Q^{ij}_{\mathrm{SO}}=4\left(\frac{m}{r}\right)^2\{&3(\hat{\mathbf{n}}\times\mathbf{v})\cdot[\boldsymbol{\chi}_s+(\delta m/m)\boldsymbol{\chi}_a]n^in^j\\&-[\mathbf{v}\times[(2+\eta)\boldsymbol{\chi}_s+2(\delta m/m)\boldsymbol{\chi}_a]]^{(i}n^{j)}\\&+3\dot{r}[\hat{\mathbf{n}}\times[\boldsymbol{\chi}_s+(\delta m/m)\boldsymbol{\chi}_a]]^{(i}n^{j)}-2\eta(\hat{\mathbf{n}}\times\boldsymbol{\chi}_s)^{(i}v^{j)}\\&+\eta[2(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\mathbf{v}+2(\hat{\mathbf{N}}\cdot\mathbf{v})\hat{\mathbf{n}}\\&-3\dot{r}(\hat{\mathbf{N}}\cdot\mathbf{n})\hat{\mathbf{n}}]^{(i}(\hat{\mathbf{N}}\times\boldsymbol{\chi}_s)^{j)}\},\end{aligned} \tag{F15b}$$

$$\begin{aligned}P^2Q^{ij}_{\mathrm{SS}}=-6\left(\frac{m}{r}\right)^3\eta\{&[|\boldsymbol{\chi}_s|^2-|\boldsymbol{\chi}_a|^2-5(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s)^2\\&+5(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a)^2]n^in^j\\&+2[\boldsymbol{\chi}_s(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s)-\boldsymbol{\chi}_a(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a)]^{(i}n^{j)}\}.\end{aligned} \tag{F15c}$$

Note that the spin-spin term comes entirely from the effects of the equations of motion. Thus we have computed the complete waveform, including leading-order spin effects, using our augmented EW formalism. The formalism can be extended to compute additional spin terms and other finite-size effects, such as the 2PN spin-orbit contribution to the waveform.

Either by a direct computation starting with the waveform or by using the STF multipoles in Eq. (E5b) we can compute the spin contributions to the rate of energy loss, Eq. (6.12),

$$\dot{E}_{\rm SO}=\frac{8}{15}\frac{m^3\mu^2}{r^5}[\hat{\mathbf{n}}\times\mathbf{v}]\cdot\Bigg\{[\boldsymbol{\chi}_s+(\delta m/m)\boldsymbol{\chi}_a]\times\left(27\dot{r}^2-37v^2-12\frac{m}{r}\right)+4\eta\boldsymbol{\chi}_s\left(12\dot{r}^2-3v^2+8\frac{m}{r}\right)\Bigg\}, \tag{F16}$$

$$\dot{E}_{\rm SS}=\frac{8}{15}\frac{m^4\mu^2}{r^6}\eta\{3[|\boldsymbol{\chi}_s|^2-|\boldsymbol{\chi}_a|^2](47v^2-55\dot{r}^2)-3[(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s)^2+(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a)^2](168v^2-269\dot{r}^2)+71[(\mathbf{v}\cdot\boldsymbol{\chi}_s)^2+(\mathbf{v}\cdot\boldsymbol{\chi}_a)^2]-342\dot{r}[(\mathbf{v}\cdot\boldsymbol{\chi}_s)(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s)-(\mathbf{v}\cdot\boldsymbol{\chi}_a)(\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a)]\}. \tag{F17}$$

Although they are not needed in our discussion, for completeness we include expressions for the precession of our spin vectors [40,41]

$$m\dot{\boldsymbol{\chi}}_s=\boldsymbol{\Pi}_1\times\boldsymbol{\chi}_s+\boldsymbol{\Pi}_2\times\boldsymbol{\chi}_a-2(\delta m/m)\boldsymbol{\chi}_a\times\boldsymbol{\chi}_s, \tag{F18a}$$

$$m\dot{\boldsymbol{\chi}}_a=\boldsymbol{\Pi}_2\times\boldsymbol{\chi}_a+\boldsymbol{\Pi}_1\times\boldsymbol{\chi}_s-2(1-2\eta)\boldsymbol{\chi}_s\times\boldsymbol{\chi}_a. \tag{F18b}$$

The precession vectors are given by

$$\boldsymbol{\Pi}_1=\frac{3}{4}\left(\frac{m}{r}\right)^2\left[(1+2\eta/3)(\hat{\mathbf{n}}\times\mathbf{v})+2\frac{m}{r}[(1-2\eta)\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s+(\delta m/m)\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a]\hat{\mathbf{n}}\right], \tag{F19a}$$

$$\boldsymbol{\Pi}_2=-\frac{3}{4}\left(\frac{m}{r}\right)^2\left[(\delta m/m)(\hat{\mathbf{n}}\times\mathbf{v})+2\frac{m}{r}[(\delta m/m)\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_s+(1-2\eta)\hat{\mathbf{n}}\cdot\boldsymbol{\chi}_a]\hat{\mathbf{n}}\right]. \tag{F19b}$$

When spinning bodies are involved, the full gravitational-wave signal can become quite complicated; the orbital plane and the spin vectors of the individual bodies can precess, giving rise to a complicated modulation of the signal [41,74]. However in the special case when the spins are aligned (or antialigned) with the orbital angular momentum axis, the spin vectors and the orbital angular momentum vector do not precess [Eqs. (F14), (F18), (F19)]. In this special case there is a simple circular orbit solution to the equation of motion and it is straightforward to compute the spin contributions to the phase evolution. The spin contributions to orbital frequency can obtained from Eq. (F14),

$$\omega^2=\frac{m}{r^3}\left\{1-2\left(\frac{m}{r}\right)^{3/2}[(1+\eta)\chi_s+(\delta m/m)\chi_a]-3\eta\left(\frac{m}{r}\right)^2[(\chi_s)^2-(\chi_a)^2]\right\}, \tag{F20}$$

where χ_s and χ_a now represent the projections of $\boldsymbol{\chi}_s$ and $\boldsymbol{\chi}_a$ onto the angular momentum axis. These quantities are positive when the spins are aligned in the same direction as the angular momentum axis and negative when they are antialigned. The orbital energy and energy flux take the simple form in the case of aligned spins and circular motion:

$$E=-\eta\frac{m^2}{2r}\left\{1+2\left(\frac{m}{r}\right)^{3/2}[(1-\eta)\chi_s+(\delta m/m)\chi_a]+\left(\frac{m}{r}\right)^2[(\chi_s)^2-(\chi_a)^2]\right\}, \tag{F21a}$$

$$\dot{E}=\frac{32\eta^2}{5}\left(\frac{m}{r}\right)^5\left\{1-\left(\frac{m}{r}\right)^{3/2}\left[\frac{73}{12}[\chi_s+(\delta m/m)\chi_a]-\frac{\eta\chi_s}{2}\right]-\frac{71\eta}{8}\left(\frac{m}{r}\right)^2[(\chi_s)^2-(\chi_a)^2]\right\}. \tag{F21b}$$

These spin corrections can be added to the nonspin formulas Eq. (7.2) and Eq. (1.4). With these we can proceed as in Sec. VI to obtain the orbital angular velocity and orbital phase as explicit functions of time:

$$\omega(t)=\frac{1}{8m}(T_c-T)^{-3/8}\left\{1+\left[\frac{113}{160}[\chi_s+(\delta m/m)\chi_a]-\frac{19}{40}\eta\chi_s\right](T_c-T)^{-3/8}-\frac{237}{512}\eta[(\chi_s)^2-(\chi_a)^2](T_c-T)^{-1/2}\right\}, \tag{F22a}$$

$$\phi(t)=\phi_c-\frac{1}{\eta}(T_c-T)^{5/8}\left\{1+\left[\frac{113}{64}[\chi_s+(\delta m/m)\chi_a]-\frac{19}{16}\eta\chi_s\right](T_c-T)^{-3/8}-\frac{1185}{512}\eta[(\chi_s)^2-(\chi_a)^2](T_c-T)^{-1/2}\right\}. \tag{F22b}$$

Again, the spin contributions can be inserted directly into Eqs. (7.11). [The definition of the dimensionless time $T=\eta(u/5m)$ is unchanged.] The explicit contributions to the $+$ and $\times$ polarizations for this specialized circular orbit case can be obtained from Eq. (F15). In the notation of [48] they are given by

$$h_{+,\times}=\frac{2m\eta}{R}x\{H^0_{+,\times}+\cdots+xH^{(1,{\rm SO})}_{+,\times}+x^{3/2}H^{(3/2,{\rm SO})}_{+,\times}+x^2H^{(2,{\rm SS})}_{+,\times}\}, \tag{F23}$$

where $x\equiv m\omega$ and where the ellipsis represents the nonspin contributions given in [48]. In keeping with the notation used

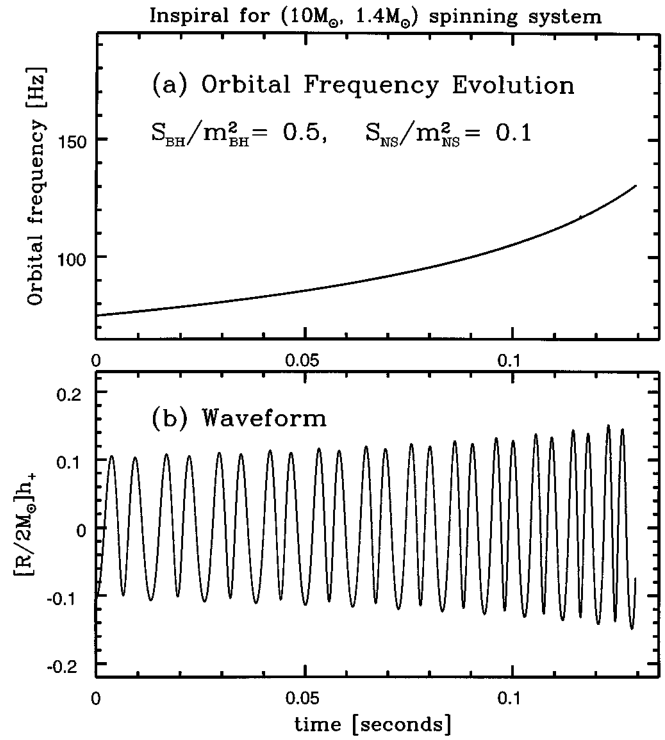

FIG. 9. Same configuration as Fig. 8, but bodies are spinning. Both spins are aligned with orbital angular momentum axis. Angular momentum of black hole is $S_{\rm BH}=0.5m_{\rm BH}^2$ and of neutron star is $S_{\rm NS}=0.5m_{\rm NS}^2$. Note frequency does not sweep as fast as nonspinning case because of dragging of inertial frames.

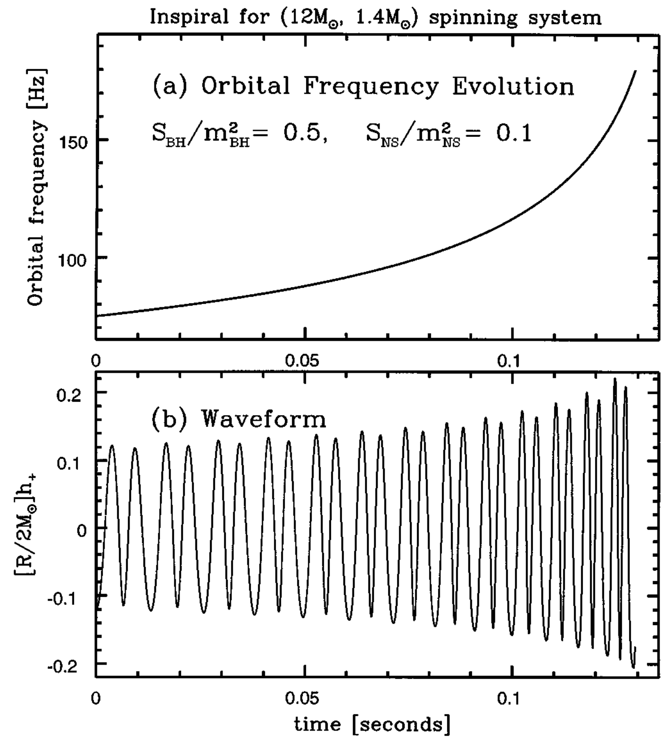

FIG. 10. Spins are same as in Fig. 9, but heavier mass is now 12 $M_\odot$. The frequency evolution is the same as the nonspinning case. Comparing this with Fig. 8 is an explicit demonstration of degeneracy in mass and spin parameters.

in [48] the superscripts represent the post-Newtonian order and the physical nature of each term. The plus polarization spin-orbit and spin-spin contributions are

$$H_+^{(1,\rm SO)}=-\sin i[(\delta m/m)\chi_s+\chi_a]\cos\phi, \tag{F24a}$$

$$H_+^{(3/2,\rm SO)}=\frac{4}{3}[(1+\cos^2 i)[\chi_s+(\delta m/m)\chi_a] + \eta(1-5\cos^2 i)\chi_s]\cos 2\phi, \tag{F24b}$$

$$H_+^{(2,\rm SS)}=-2\eta(1+\cos^2 i)[(\chi_s)^2-(\chi_a)^2]\cos 2\phi, \tag{F24c}$$

and the cross polarization contributions are

$$H_\times^{(1,\rm SO)}=-\sin i\cos i[(\delta m/m)\chi_s+\chi_a]\sin\phi, \tag{F25a}$$

$$H_\times^{(3/2,\rm SO)}=\frac{4}{3}\cos i[2[\chi_s+(\delta m/m)\chi_a] - \eta(1+3\cos^2 i)\chi_s]\sin 2\phi, \tag{F25b}$$

$$H_\times^{(2,\rm SS)}=4\eta\cos i[(\chi_s)^2-(\chi_a)^2]\sin 2\phi. \tag{F25c}$$

We emphasize that these are only valid for quasicircular orbits in the case where the the spins are aligned (or anti-aligned) with the orbital angular momentum vector. These restrictive assumptions about the configuration of the system suppress many of the intricate features of the waveform produced by spinning bodies [41,74].

Figure 9 shows an inspiral waveform for the same system as in Fig. 8 (10 $M_\odot$ black hole and a 1.4 $M_\odot$ neutron star spiralling to coalescence), but in this case the objects are spinning. The spins are aligned with the orbital angular momentum axis. The spin contributions to both the waveform Eq. (F23) and the frequency evolution Eq. (F22) have been incorporated into the plot. The black hole has been given a spin of $S_{\rm BH}/m_{\rm BH}^2=0.5$ and the neutron star has $S_{\rm NS}/m_{\rm NS}^2=0.1$ (i.e., $\chi_s=0.3$ and $\chi_a=0.2$). Notice the significant change in the frequency evolution; the system only sweeps to about 130 Hz in the same time it took for the nonspinning system to sweep to 180 Hz. Consequently, the peaks are not as closely bunched as they are in the nonspinning case. This slower orbital decay and frequency evolution is due to the dragging of inertial frames, which is inherent in the equations of motion and thus in our phase evolution equation (F22). At the left side of Figs. 8 and 9, the waveforms are clearly in phase with each other, but after a few cycles they are out of phase. Since the phase evolution of the system is crucial in analyzing gravitational waves from an inspiral, it might seem that this sensitivity to spin in the phase evolution could be exploited and the spins of the bodies be determined with great accuracy. However, by leaving the spins the same but adjusting the masses slightly, we can recover the basic structure of the nonspinning case almost exactly. This is depicted in Fig. 10, in which the frequency sweep and the waveform itself are virtually identical to the nonspinning waveform in Fig. 8. This signal degeneracy in the spin and mass parameters has been previously noted in [24,25]. It is also interesting to notice that the inclusion of the spins virtually removes the jagged features from the troughs of the waves.

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