Trapped surfaces in cosmological spacetimes

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(Received 21 February 1995)

We investigate the formation of trapped surfaces in cosmological spacetimes, using constant mean curvature slicing. Quantitative criteria for the formation of trapped surfaces demonstrate that cosmological regions enclosed by trapped surfaces may have matter density exceeding significantly the background matter density of the flat and homogeneous cosmological model. Cosmological trapped surfaces existing at the epoch of recombination would become seeds of the galaxy formation and would be hidden in their centers.
[S0556-2821(96)05720-7]

PACS number(s): 98.80.Hw, 04.20.Gz, 95.30.Sf, 97.60.Lf

I. INTRODUCTION

In our previous work [1] we investigated the formation of trapped surfaces in various cosmological models. Recently we have found a particularly useful formulation [2] of the spherically symmetric Einstein constraint equations that allowed us to improve our earlier estimates [3] for conditions determining the appearance of trapped surfaces. In the present paper we apply the new formalism to spherically symmetric cosmologies. As a result we find stronger criteria in spacetimes than were investigated previously and, more importantly, we are able to deal with hyperbolic universes where our previous attempts had failed.

The order of the article is as follows. The first section presents the formalism. In Sec. II we deal with the main results. Section III shows that regions enclosed by trapped surfaces must be invisible to external observers, although they might be detected indirectly through the observation of gravitational lensing. The last section contains our conclusions.

There exist three classes of homogeneous spherically symmetric cosmologies. The three are (i) the closed ($k=1$) cosmology with the metric

$$ds^2=-d\tau^2+a^2(\tau)[dr^2+\sin^2 r d\Omega^2], \tag{1}$$

(ii) the open flat ($k=0$) cosmology with the metric

$$ds^2=-d\tau^2+a^2(\tau)[dr^2+r^2 d\Omega^2], \tag{2}$$

(iii) the open ($k=-1$) cosmology with the metric

$$ds^2=-d\tau^2+a^2(\tau)[dr^2+\sinh^2 r d\Omega^2], \tag{3}$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ is the standard line element on the unit sphere with the angle variables $0\leqslant\phi<2\pi$ and $0\leqslant\theta\leqslant\pi$.

The geometric part of the initial data set of the Einstein equations consists of the intrinsic three-geometry and the extrinsic curvature K_{ab} which is essentially the first time derivative of the metric, all given at some time (say, $\tau=0$). The intrinsic geometries are, respectively,

$$a^2(\tau)[dr^2+\sin^2 r d\Omega^2], \tag{4}$$

$$a^2(\tau)[dr^2+r^2 d\Omega^2], \tag{5}$$

$$a^2(\tau)[dr^2+\sinh^2 r d\Omega^2], \tag{6}$$

and in each case the extrinsic curvature is pure trace,

$$K_{ab}=Hg_{ab} \tag{7}$$

where H is a time-dependent function that is constant on each slice $\tau=$const. It is called the Hubble constant and it is given by $H=\partial_\tau a/a$.

In the general case initial data consist of the quartet (g_{ij},K_{ij},ρ,J_i) where g_{ij} is the intrinsic metric, K_{ij} is the extrinsic curvature, ρ is the matter energy density, and J_i is the matter current density. These cannot be given arbitrarily but must satisfy the constraints

$$^{(3)}\mathcal{R}-K_{ij}K^{ij}+(\mathrm{tr}K)^2=16\pi\rho, \tag{8}$$

$$\nabla_i K^{ij}-\nabla^j \mathrm{tr}K=-8\pi J^j, \tag{9}$$

where $^{(3)}\mathcal{R}$ is the scalar curvature of the intrinsic metric.

The momentum constraint, Eq. (9), is identically satisfied in the case of homogeneous cosmologies (with $J_i=0$) and the Hamiltonian constraint, Eq. (8), reduces to

$$16\pi\rho=\frac{6k}{a^2}+6H^2, \tag{10}$$

where k is 1, 0, and -1 in the closed, flat, and hyperbolic cosmologies, respectively. Thus we can conclude that all slices of the constant coordinate time have a uniform energy density ρ_0 which is at rest.

In this article we wish to consider data for spherically symmetric cosmologies which either in the large approximate the standard cosmologies or asymptotically approach them. In all cases we will make the assumption that the initial slice is chosen so that the trace of the extrinsic curvature

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is constant on the slice. In order to retain the link with homogeneous cosmologies we define tr$K=3H$.

The initial data we consider is a spherically symmetric set consisting of a three-metric

$$ds^2_{(3)}=a^2dr^2+b^2(r)f^2(r)d\Omega^2, \tag{11}$$

an extrinsic curvature

$$K^r_r=H+K(r), \quad K^\theta_\theta=H-K/2, \quad K^\phi_\phi=H-K/2, \tag{12}$$

an energy density $\rho(r)$, and a current density $j_i(r)$. The function $f(r)$ will be one of the set (sin(r),r,sinh(r)), depending on the type of cosmology.

There are some useful geometric quantities that can be defined. One of them is the proper distance from the center of symmetry given by $dl=adr$. The Schwarzschild (also called ''areal'' or ''curvature'') radius R is given by $R=bf$.

The mean curvature of a centered two-sphere as embedded in a spacelike three-dimensional hypersuface is

$$p=\frac{2\partial_l R}{R}. \tag{13}$$

In a general spacetime we may investigate the geometry by considering the propagation of various beams of lightrays through a space-time. These beams in general will shear and either expand or contract; a number of (optical) functions will be required to describe their propagation. In a spherically symmetric spacetime we focus our attention on light rays moving orthogonally to two-spheres centered around a center of symmetry. We need only two functions. These are the divergence of future-directed outgoing light rays

$$\theta=\frac{2}{R}\frac{d}{d\tau_{\rm out}}R \tag{14}$$

and the divergence of past-directed outgoing light rays

$$\theta'=\frac{-2}{R}\frac{d}{d\tau_{\rm in}}R, \tag{15}$$

where $d/d\tau_{\rm out}$ is the derivative along future-pointing outgoing radial null rays and $d/d\tau_{\rm in}$ is the derivative along future-pointing ingoing radial null rays. One interesting property of θ and θ' is that they can be expressed purely in terms of initial data on a spacelike slice. In the spherically symmetric case we have

$$\theta=p-K^r_r+{\rm tr}K=p-K+2H \tag{16}$$

and

$$\theta'=p+K^r_r-{\rm tr}K=p+K-2H. \tag{17}$$

This means that θ and θ' are three-dimensional scalars. They are not four-scalars since they depend on a choice of affine parameters along the null rays. However, the product $\theta\theta'$ is a four-scalar.

In the homogeneous universes we find that $pR=2$, $pR=2\cos(r)$, and $pR=2\cosh(r)$ in the $k=0$, 1, and -1, cases, respectively, and

$$R\theta=2+2RH=2+2\sqrt{\frac{8\pi\rho_0}{3}}R, \quad R\theta'=2-2RH \tag{18}$$

for $k=0$,

$$\begin{aligned}R\theta&=2\cos(r)+2RH\\&=2\cos(r)+2aH\sin(r)\\&=2\cos(r)+2\sqrt{\left(\frac{8\pi\rho_0a^2}{3}-1\right)}\sin(r),\\R\theta'&=2\cos(r)-2\sqrt{\left(\frac{8\pi\rho_0a^2}{3}-1\right)}\sin(r)\end{aligned} \tag{19}$$

for $k=1$, and

$$\begin{aligned}R\theta&=2\cosh(r)+2RH\\&=2\cosh(r)+2\sqrt{\left(\frac{8\pi\rho_0a^2}{3}+1\right)}\sinh(r),\\R\theta'&=2\cosh(r)-2\sqrt{\left(\frac{8\pi\rho_0a^2}{3}+1\right)}\sinh(r)\end{aligned} \tag{20}$$

for $k=-1$.

A surface on which θ is negative is called, after Penrose [4], a outer future-trapped surface; a surface on which θ' is negative is called a outer past-trapped surface; if both are negative, we have an outer trapped surface, and if θ is negative and θ' positive, we have a future-trapped surface. The occurrence of such surfaces in a spacetime is an indication of the fact that gravitational collapse is well advanced. In the case of homogeneous closed cosmologies outer future-trapped surfaces exist for any $r>\mathrm{arccot}(aH)$. In neither $k=0$ nor $k=-1$ is $R\theta$ ever negative if $H>0$.

In this article we consider a universe that is homogeneous in the large but that it is dotted with numerous spherical inhomogeneities, far from each the metric approaches the background metric of a homogeneous universe. If we center our coordinate system at a particular lump, we expect that optical scalars approach the values given in Eqs. (18), (19), and (20) far away from the lump. In the case of closed cosmologies this limiting value is expected to be met for values of the coordinate radius r much less than $\pi/2$.

We assume local flatness at the origin, i.e., $\lim_{R\to0}R\theta=\lim_{R\to0}R\theta'=2$, although this condition can be relaxed to allow for a conical singularity there, i.e., $0<\lim_{R\to0}R\theta,\lim_{R\to0}R\theta'\leq2$.

II. MAIN CALCULATIONS

The spherical initial data must satisfy the constraints, which read, in terms of the functions θ and θ',

$$\partial_l(\theta R)=-8\pi R(\rho-j) -\frac{1}{4R}[2(\theta R)^2-\theta R\theta' R-4-12\theta RHR], \tag{21}$$

$$\partial_l(\theta' R) = -8\pi R(\rho + j) - \frac{1}{4R}[2(\theta' R)^2 - \theta R \theta' R - 4 + 12\theta RHR], \quad (22)$$

where $j = j_l$ is the radial component of the matter current density normalized so that $j^2 = j^k j_k$. We can manipulate Eqs. (21) and (22) to obtain

$$\partial_l(\theta' R \theta R) = -8\pi[\rho(\theta' R + \theta R) + j(\theta R - \theta' R)] - \frac{1}{2R}[(\theta R \theta' R - 4)(\theta' R + \theta R)]. \quad (23)$$

Let us now assume that the total matter satisfy the dominant energy condition, i.e., $\rho \geqslant |j|$. Assume that $\theta R \theta' R > 4$ at a particular point. Consider first the situation where both θR and $\theta' R$ are positive. Then $(\theta' R + \theta R) > |(-\theta' R + \theta R)|$ and $\rho(\theta' R + \theta R) + j(-\theta' R + \theta R) \geqslant 0$. This means that both terms of Eq. (23) are nonpositive and the derivative of the product $\theta R \theta' R$ is negative. On the other hand, when both θR and $\theta' R$ are negative and their product is greater than 4, then $\rho(\theta' R + \theta R) + j(-\theta' R + \theta R) < 0$ and the first term in Eq. (23) is positive. The second term becomes also positive, so that $\partial_l(\theta R \theta' R) > 0$. Thus in both cases if $\theta R \theta' R > 4$ then $\partial_l(\theta R \theta' R) \neq 0$.

Let us now consider the expressions for the product of the two scalars $\theta R \theta' R$ in each of the three homogeneous cosmologies. We get

$$R\theta R \theta' = 4 - 4R^2 H^2 \quad (24)$$

for $k = 0$,

$$R\theta R \theta' = 4\cos^2(r) - 4R^2 H^2 \quad (25)$$

for $k = 1$, and

$$R\theta R \theta' = 4\cosh^2(r) - 4\left(\frac{8\pi\rho_0 a^2}{3} + 1\right)\sinh^2(r) = 4 - \frac{32\pi\rho_0 a^2}{3}\sinh^2(r) \quad (26)$$

for $k = -1$.

In each of these cases we have $R\theta R \theta' = 4$ at the origin and never more than 4. We are considering initial geometries that have regular origins and asymptotically approach the homogeneous cosmologies, so that both at the origin and far from the center the product $R\theta R \theta'$ does not exceed 4. If it were to achieve a maximal value greater than 4 somewhere in between, then its derivative would have to vanish, but that is excluded in the preceding analysis. Therefore we have proved the following.

Lemma 1. Assume that one is given spherical cosmological data which are locally flat at the center, which are asymptotic to any of standard homogeneous cosmological models, and that the matter satisfies the dominant energy condition. Then

$$R\theta R \theta' \leqslant 4.$$

Remarks. (i) The above statement is true for any regular slice, with arbitrary (i.e., nonconstant on a part of a slice) tr$K$, assuming that the slice is asymptotic to a homogeneous constant mean curvature slice. (ii) It implies the positivity of the Hawking mass on a sphere centered around a symmetry center; $2M_H = R(1 - R\theta R \theta'/4)$ cannot become negative on a fixed slice.

Lemma 1 holds true for all three cosmological models.

The main issue that we will address in this paper is the question of the formation of trapped surfaces due to concentration of matter. The result will be obtained through a careful analysis of Eq. (21). What we do is multiply Eq. (21) by R, use Eq. (13), and write the resulting equation in the form

$$\partial_l(\theta R^2) = -8\pi R^2(\rho - j) + 1 + \tfrac{1}{2}\theta R \theta' R - \tfrac{1}{4}(\theta R)^2 + 3\theta RHR. \quad (27)$$

The substitution of Eqs. (16) and (17) into Eq. (27) gives

$$\partial_l(\theta R^2) = -8\pi R^2\left(\rho - \frac{3H^2}{8\pi} - j\right) + 1 + \frac{1}{4}(pR + KR)^2 - R^2K^2 + 2R^2Hp \quad (28)$$

or

$$\partial_l(\theta R^2) = -8\pi R^2\left(\rho - \rho_0 + \frac{3k}{8\pi a^2} - j\right) + 2 - \left(1 - \frac{1}{4}(pR + KR)^2\right) - R^2K^2 + 4RH\partial_l R, \quad (29)$$

where we used the relation (10) to eliminate the H^2 term and use the definition of the mean curvature p.

Let us integrate Eq. (29) from the origin out to a surface S. We identify

$$\Delta M = 4\pi\int_0^{L(S)} R^2(\rho - \rho_0)dl = \int_{V(S)} dV(\rho - \rho_0), \quad (30)$$

as the excess matter inside a volume $V(S)$ bounded by S and

$$P = 4\pi\int_0^{L(S)} R^2 j dl = \int_{V(S)} dVj \quad (31)$$

as the total radial momentum of the matter inside S. In this notation, the aforementioned integration yields

$$\theta R^2|_S = -2(\Delta M - P) - \frac{3k}{4\pi a^2}V + 2L + \frac{HA}{2\pi} - \int_{V(S)} dV\left(1 - \frac{1}{4}(pR + KR)^2 + R^2K^2\right), \quad (32)$$

where A is the area of the surface S and L is the geodesic distance of S from the center. Below we will prove, in a series of lemmas, that under some conditions we can control the sign of the last integral.

Lemma 2. Given spherical data which are locally flat at the origin and approach either the $k = 0$ or the $k = 1$ cosmology. If the energy condition $\rho-\rho_0-|j|\geq -3k/4\pi a^2$ is satisfied out to an asymptotic region, then

$$2\geq|pR+KR|,\quad 2\geq|pR-KR|.$$

Lemma 3. Given spherical data which are locally flat at tha origin and approach the $k=-1$ cosmology, if the energy condition $\rho-\rho_0-|j|\geq 3/4\pi a^2$ is satisfied inside a sphere S, then

$$2>(pR+KR),\ \ 2>(pR-KR)$$

inside the sphere.

Remark. The energy condition $\rho-\rho_0-|j|\geq 3/4\pi a^2$, using Eq. (10), can be written as ${}^{(3)}\mathcal{R}\geq 16\pi|j|+\frac{3}{2}K^2$, independent of k. In both the $k=1$ and the $k=0$ cases this condition will be satisfied in the external region, whereas in the $k=-1$ cosmology ${}^{(3)}\mathcal{R}$ approaches $-6/a^2$. This is the principal reason why lemma 3 differs from lemma 2.

Before proving the two lemmas, let us formulate two main results that give sufficient conditions for the formation of trapped surfaces.

Theorem 1. Given data which approach either the $k=0$ or the $k=1$ cosmology and is locally flat at the origin, if the energy condition $\rho-\rho_0-|j|\geq -3k/4\pi a^2$ is satisfied out to an asymptotic region and if

$$\Delta M-P\geq -3k/8\pi a^2V+L+\frac{HA}{4\pi} \tag{33}$$

at a surface S, then S is outer future trapped.

Proof. The result follows directly from Eq. (32) and the estimate of lemma 2.

Theorem 2. Assume that normally ingoing light light rays are everywhere convergent inside a volume V bounded by a surface S, $\theta'>0$. Given data which approach the $k=-1$ locally flat cosmology, if the energy condition $\rho-\rho_0-|j|\geq 3/4\pi a^2$ is satisfied inside the volume V and if

$$\Delta M-P\geq\frac{3}{8\pi a^2}V+L+\frac{HA}{4\pi} \tag{34}$$

at the surface S, then there exists a surface inside S that is future trapped.

Proof. Assume that there is no future-trapped surface inside S, i.e., $\theta=p+2H-K>0$. Since we also assume that there is no past-trapped surface, we may conclude that inside S $p-K>-2H, p+K>2H$. We know that p is positive inside S because we have that $p=(\theta+\theta')/2$ and each of θ and θ' is positive. We also have $2>pR-KR>-2HR$ and $2>pR+KR>2HR$; the last inequalities follow from lemma 3. If $H>0$, we have that $pR+KR$ is positive and thus $(p+K)^2R^2\leq 4$ and the last integral of Eq. (32) is strictly negative. On the other hand, if $H<0$, we must have that $pR-KR$ is positive and $(pR-KR)^2\leq 4$ but we could have that $pR+KR$ be negative. This can only happen while K is negative since we know that p is positive. In this case we write the integrand of Eq. (32) as $1-\frac{1}{4}(pR-KR)^2-pKR^2+K^2R^2$. This is clearly nonnegative. Thus we also have in this case that the last term in Eq. (32) is negative. This contradicts the assumption that there is no trapped surface. Hence, under the assumptions of theorem 2, there must exist a trapped surface inside S.

In order to prove lemmas 2 and 3 we shall return to Eqs. (21) and (22) and write them in terms of Rp, RK, and RH. Equation (21) can be written as

$$\partial_l(pR-KR)=-8\pi R\left(\rho-j-\frac{3H^2}{8\pi}\right)-\frac{1}{2R}(Rp-RK)^2+\frac{1}{4R}(Rp-RK)(Rp+RK)+\frac{1}{R} \tag{35}$$

and Eq. (22) as

$$\partial_l(pR+KR)=-8\pi R\left(\rho+j-\frac{3H^2}{8\pi}\right)-\frac{1}{2R}(Rp+RK)^2+\frac{1}{4R}(Rp-RK)(Rp+RK)+\frac{1}{R}. \tag{36}$$

We will prove first the upper bound of $pR+KR$, and $pR-KR$, simultaneously for both lemmas 2 and 3; this part of the proof does not depend on the type of a cosmological spacetime. Also, as will become clear, the energy condition need be imposed only inside a sphere S if we are interested in finding the upper bound inside S (as opposed to the estimation from below that requires the global assumption made in lemma 2). According to the conditions made in the lemmas, the first term of either Eq. (35) or (36) is nonpositive. We show that in the situation of interest the remainders of each of the equations are also nonpositive.

At the center of symmetry the quantities $pR+KR$ and $pR-KR$ are equal to 2 for all types of cosmology. This means that right-hand sides of either Eq. (35) or (36) must be nonpositive and that the quantities in question start from the origin with the value 2 and start to decrease as soon as they meet either positive $\rho+j-3H^2/8\pi$ or $\rho-j-3H^2/8\pi$.

Let us assume that further out one of the two, say, $pR+KR$, rises up to 2 with $pR-KR$ lagging behind. In this case we can write the nonmaterial part of the right-hand side of Eq. (36) as

$$-\tfrac{1}{2}(Rp+RK)^2+\tfrac{1}{4}(Rp-RK)(Rp+RK)+1=-1+\tfrac{1}{2}(Rp-RK)\leq 0. \tag{37}$$

Because the material part of Eq. (36) is nonpositive, we get that $\partial_l(Rp+RK)\leq 0$ so that $pR+KR$ cannot exceed 2. A similar argument can be made for $pR-KR$. Thus lemma 3 and the upper bound of lemma 2 are proved; as is clear from the above derivation, in order to have a bound that is valid inside a sphere S we need that the energy condition be imposed only inside S.

The same reasoning can be applied to complete the proof of lemma 2. We will show that if one of the two quantities in question reaches the value -2, then at least one of them must be less than -2, thus breaking either the demand of geometries being asymptotic to a homogeneous cosmology in the sense expressed in Eqs. (18) and (19).

In order to show this we need the global energy condition of lemma 2. Let us assume that there exists a point where $pR+KR=-2$, with $pR-KR \geqslant pR+KR$. Then the nonmaterial part of Eq. (36) reads

$$-\tfrac{1}{2}(Rp+RK)^2+\tfrac{1}{4}(Rp-RK)(Rp+RK)+1 \leqslant 0. \quad (38)$$

Equation (36) implies now (assuming the energy condition) that $pR+KR$ has to become more negative if $pR+KR<pR-KR$ and may stay at -2 in the case of equality only if the matter contribution vanishes. However, if we can impose an outer boundary condition such that $pR+KR \geqslant -2$, then we get a contradiction. A similar argument works for $pR-KR$. The outer boundary condition is guaranteed in the cases of interest. Cosmological spacetime dotted with inhomogeneities have the property that asymptotically $pR+KR$ and $pR-KR$ approach values given by Eqs. (18) and (19) which must be strictly larger than -2. That ends the proof of lemma 2.

It is interesting that we obtain an exact criterion with the constant 1; this suggests that the above theorem constitutes a part of a more complex true statement that can be formulated for general nonspherical spacetimes. It suggests also that $M(S)$ is a sensible measure of the energy of a gravitational system that might appear as a part of a quasilocal energy measure in nonspherical systems.

It is clear that the analysis performed here can include cases where the sources are distributions rather than classical functions; in particular, we have no difficulty with shells of matter. All we get on crossing the shell is a downward step in θ and θ'. More interestingly, we can extend the analysis to include conical singularities at the origin [6], in a way analogous to that described in [2].

III. CONFINING PROPERTY OF TRAPPED SURFACES

In this section we show that a region enclosed by trapped surfaces cannot be seen by external observers. This fact has been proven (without referring to the cosmic censor hypothesis) by Israel [5]. Here we will present a different version of the proof that is based on a 1+3 decomposition of a spacetime (as opposed to the proof of Israel who used a 2+2 decomposition).

We need the evolution part of the Einstein equations and the lapse equation. These are

$$\partial_t[\delta K_r^r-2H(t)]=\frac{3\alpha}{4}(\delta K_r^r)^2-\frac{\alpha p^2}{4}-\frac{p}{\sqrt{a}}\partial_r\alpha+\frac{\alpha}{R^2}+8\pi\alpha T_r^r+3\alpha H^2-3H\delta K_r^r \quad (39)$$

and

$$\nabla_i\partial^i\alpha=\alpha\left(\frac{3}{2}(\delta K_r^r)^2 4\pi(\rho+T_i^i)+3H^2\right)+3\partial_t H. \quad (40)$$

In addition we need the evolution equation of the mean curvature p of centered spheres:

$$\partial_t p=\frac{\partial_r\alpha}{\sqrt{a}}(-\delta K_r^r+2H)+8\pi\alpha\frac{j_r}{\sqrt{a}}+\frac{p\alpha}{2}(\delta K_r^r-2H). \quad (41)$$

Using these equations we can find the full time derivative of θ along a trajectory of null geodesics normal to centered spheres:

$$\left(\partial_t+\frac{\alpha}{\sqrt{a}}\right)\theta=\frac{\partial_r\alpha}{\sqrt{a}}\theta'\theta-8\pi\alpha\left(-2\frac{j_r}{\sqrt{a}}+\rho+T_r^r\right)-\alpha\theta^2+3\alpha H\theta. \quad (42)$$

Take now an apparent horizon, i.e., a centered sphere S of vanishing $\theta(S)$. Assume that there exists a foliation with a "good" lapse $\alpha>0$ in an external region extending outward from the horizon. Equation (42) implies that photons that start from S will forever remain inside an apparent horizon, if the strong energy condition $-2j_r/\sqrt{a}+\rho+T_r^r \geqslant 0$ is assumed. Hence apparent horizons move faster than light in cosmological spacetimes (in contrast with asymptotically flat spacetimes, where they can eventually stabilize to the speed of light); they act as one-way membranes for nontachyonic matter. This means that outside observers cannot detect any information from any inside region that is enclosed by a trapped surface. The only way to draw any conclusions about a piece of a spacetime that is enclosed by a trapped surface is through the observation of "long-wave" effects, through the attractive force that large massive objects exert on their surrounding. One such possibility would be the observation of gravitational lensing.

IV. DISCUSSION

For a fixed value of H, for all three kinds of cosmologies, we know that we need a minimum density of matter before we expect trapped surfaces to appear. What we need is $\rho \approx \rho_F$ where $\rho_F=3H^2/8\pi$ is the density in the flat cosmology. In other words, we need that the scalar curvature ${}^{(3)}\mathcal{R}$ be, at least on average, positive in the interior. The specific results that we prove capture this very well. For example, in lemmas 2 and 3, the energy condition required is that $\rho-\rho_F \geqslant |j|$. Further, if we define

$$\Delta\tilde{M}=4\pi\int_0^{L(S)}R^2(\rho-\rho_F)dl=\int_{V(S)}dV(\rho-\rho_F), \quad (43)$$

then the sufficient conditions for the appearance of trapped surfaces in both theorems 1 and 2 can be written as

$$\Delta\tilde{M}-P \geqslant L+\frac{HA}{4\pi}. \quad (44)$$

This means that it is much harder to form trapped surfaces in a hyperbolic universe than in the flat or closed case because we need a much larger enhancement over the background.

Cosmological trapped surfaces that we discuss in the first three sections can, if they exist, accumulate an enormous amount of energy. Typically, as we have shown, the matter content of a trapped surface having a geodesic radius L is of the order L plus the background energy $M_H=3H^2V/8\pi$ (we neglect here the effects related to the possibility of a nonzero curvature of the spacelike slice and the surface term $HA/2\pi$). Assume that there exists a trapped surface with a proper radius of the order of 1000 Mpc. Then its excess energy is of the order of 1000 (in units of Mpc). The present value of the Hubble constant is about 50 km s Mpc or (in units in which the speed of light $c=1$) 1/6000 Mpc. Therefore the expected value of the background energy inside the above ball is of the order $0.5\times 10^6 \times (1/6000)^2$ Mpc=14 Mpc, which is about 10^2 times less than the energy content that is needed in order to form, say, a spherical massive shell that creates a trapped surface. We include this crude calculation to point out that the formalism of general relativity does allow for cosmological regions with high concentrations of matter that can be detected only indirectly by external observers.

On the other hand, the observed mass density contrast is of the order of 10^{-2} [7] on scales of 1000 Mpc, that is almost four orders less than required in order to have a trapped surface. A trapped surface of the size 1000 Mpc, that is, of the mass of order of 10^{22} solar masses, would form a powerful gravitational lense and should locally deform the cosmic background radiation. Therefore the existence of cosmological trapped surfaces in the present universe seems to be ruled out.

They could have, however, existed during earlier epochs, when the universe was more dense. The evolution of spherical inhomogeneities formed by pressureless dust can be exactly solved in Lemaitre-Tolman coordinates and it probably gives a good fit to the evolution of spherical inhomogeneities in cosmological models after recombination when the pressure became negligible. A simple calculation [8] shows that a trapped surface existing at the epoch of recombination and having initial radius of the order of 10^{-6} pc and initial mass of 10^7 solar masses would become a seed of a galaxy of a final mass 10^{10} solar masses. The size of the galaxy would be nowadays of the order of 0.15 Mpc, while the size of a trapped surface inside it would have to be much less than 10^{-3} pc. Accretion of surrounding matter onto the trapped surface would ignite a powerful source of radiation at an early phase of the evolution. That calculation suggests that *cosmological trapped surfaces* that formed in the past can exist as huge black holes at centers of some galaxies, notably quasars and their remnants.

ACKNOWLEDGMENTS

This work was partially supported by Forbairt Grant No. SC/94/225 and the KBN Grant No. 2 PO3B 090 08.

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