Constant mean curvature slices and trapped surfaces in asymptotically flat spherical spacetimes

Mirta Iriondo
Royal Institute of Technology, Mathematics Department, Stockholm, Sweden

Edward Malec
Institute of Physics, Jagellonian University, 30-059 Cracow, Reymonta 4, Poland

Niall Ó Murchadha
Physics Department, University College, Cork, Ireland

(Received 21 February 1995)

We investigate trapped surfaces in asymptotically flat spherical spacetimes using constant mean curvature slicing. Precise necessary and sufficient conditions for the formation of such surfaces are derived. We write down an explicit expression for the constant mean curvature foliation of the Reissner-Nordström spacetime. A set of criteria describing the formation of horizons in arbitrary slicings of asymptotically flat and spherically symmetric spacetimes is given in the Appendix. [S0556-2821(96)05620-2]

PACS number(s): 98.80.Hw, 04.20.Gz, 95.30.Sf, 97.60.Lf

I. INTRODUCTION

In the analysis of general relativity as a Hamiltonian system [1] one chooses a time function and considers the foliation of the spacetime by the slices of constant time. Two natural geometrical quantities arise on such three-slices. One is the intrinsic three-metric, usually g_{ab}, and the other is the extrinsic curvature K^{ab}, the time derivative of g_{ab}. These are not independent: They are related by the constraints

$$\mathcal{R}^{(3)} - K^{ab}K_{ab} + (\mathrm{tr}K)^2 = 16\pi\rho,$$

$$\nabla_a K^{ab} - g^{ab}\nabla_a \mathrm{tr}K = -8\pi j^a,$$

where $\mathcal{R}^{(3)}$ is the three scalar curvature, ρ is the energy density and j^a is the current density of the sources.

It is often useful to specify the foliation, and thus the time, by placing a condition on the extrinsic curvature, The most common choice in asymptotically flat spacetimes is the maximal slicing condition $\mathrm{tr}K=0$. In cosmologies, the favored slicing is the constant mean curvature (CMC) foliation with $\mathrm{tr}K=$ const.

Such CMC slices have also been used in an asymptotically flat context [3]. They are everywhere spacelike, but at infinity they approach null infinity. Thus they are very useful in investigating the relationship between spatial and null infinity. A standard model of CMC hypersurfaces is the mass hyperboloids in Minkowski space [4]. In general asymptotically flat spacetimes, since the CMC slices approach null infinity, they are not good Cauchy slices. In the spherically symmetric case, however, if the matter has compact support, the exterior geometry is fixed and so good initial data can be prescribed on them.

In this paper we investigate a very special class of CMC, those which are spherically symmetric. Because of the absence of gravitational radiation, spherical spacetimes are particularly simple, yet realistic, models of general solutions to the Einstein equations.

If we have a spherically symmetric three-surface, the intrinsic metric can be written as

$$ds^2 = dl^2 + R^2 d\Omega^2,$$

where l is the proper distance in the radial direction and R is the Schwarzschild or areal radius. The geometry is encoded into the relationship between R and l and a useful object to use is the mean curvature of the spherical two surfaces, given by

$$p = \frac{2}{R}\frac{dR}{dl}.$$

The constraints now can be written as

$$\partial_l p = -8\pi\rho - \frac{3}{4}(K_r^r)^2 - \frac{3}{4}p^2 + \frac{1}{R^2} + \frac{1}{2}\mathrm{tr}KK_r^r + \frac{1}{4}(\mathrm{tr}K)^2,$$

$$\partial_l(K_r^r - \mathrm{tr}K) = -\frac{3}{2}pK_r^r + \frac{1}{2}p\,\mathrm{tr}K - 8\pi j_l. \tag{1}$$

It has been recently shown [2] that the constraints of general relativity in the spherically symmetric case can be expressed very simply by using the null expansions as subsidiary variables and the constraints can be expressed as a system of quasilinear first order ordinary differential equations (ODE's). We apply this new formulation in the CMC case to investigate a number of interesting problems.

Much work has been carried out in recent years on how concentrations of matter may gravitationally collapse [2,5,6]. One of the motivations for repeating the calculation in various slicings of asymptotically flat spacetimes is due to the fact that no covariant formulation of the question has been found. This article, in which we derive both necessary and sufficient conditions for the formation of trapped surfaces, can be regarded as an attempt to see how the criteria we obtain are more or less independent of the details of the

0556-2821/96/54(8)/4792(7)/$10.00

slicing used. Let us emphasize that the appearance of trapped surfaces indicates that irreversible gravitational collapse has commenced.

We derive, for the sake of completeness, the general line element for the Reissner-Nordström spacetime in the slicing by constant mean curvature hypersurfaces, that is, a generalization of the corresponding solution in the maximal slicing [7]. This makes concrete our earlier claim that the exterior geometry is completely fixed. This can be regarded as a consequence of the Birkoff theorem.

In the Appendix we derive general criteria, valid for arbitrary slicings of spherically symmetric geometries, for the formation of trapped surfaces.

II. CMC HYPERSURFACES IN MINKOWSKI SPACE

Let us consider spherically symmetric CMC hypersurfaces in Minkowski space. We write the four-metric as

$$ds^2 = -d\tau^2 + \tau^2[dr^2 + \sinh^2 r d\Omega^2], \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard round two-metric. The scalar curvature $\mathcal{R}^{(3)}$ of the three-space defined by $\tau =$ const is $\mathcal{R}^{(3)} = -6/\tau^2$.

The extrinsic curvature of this slice is pure trace, $K_{ab} \equiv \frac{1}{2}\partial_\tau g_{ab} = (1/\tau) g_{ab}$, which implies tr$K = K = 3/\tau$. The proper radial distance l along the slice is related to the coordinate radius r by $\tau dr = dl$ which yields

$$r = \frac{l}{\tau} = \frac{Kl}{3}. \tag{3}$$

The Schwarzschild radius R is given by

$$R = \tau \sinh r = \frac{3}{K} \sinh \frac{Kl}{3}, \tag{4}$$

and its derivative reads

$$R' = \frac{dR}{dl} = \cosh \frac{Kl}{3}. \tag{5}$$

The primary objects we deal with are the optical scalars, the expansion θ of the outgoing null rays, and the convergence θ' of the ingoing light rays. These are given by

$$R\theta = 2R' + \tfrac{2}{3}KR = 2 \cosh \frac{Kl}{3} + 2 \sinh \frac{Kl}{3} = 2e^{Kl/3} \tag{6}$$

and, similarly,

$$R\theta' = 2e^{-Kl/3}; \tag{7}$$

therefore, the product of $R\theta$ and $R\theta'$ remains constant, $R\theta R\theta' = 4$. We also have

$$R\theta = \frac{4RK}{3} + 2e^{-Kl/3}. \tag{8}$$

Thus at the origin we have

$$R\theta' = R\theta = 2, \tag{9}$$

and at infinity one of the scalars is divergent while the other vanishes,

$$R\theta \to \tfrac{4}{3}KR, \quad R\theta' \to 0. \tag{10}$$

An alternative form of the metric (2) can be written in terms of the Schwarzschild radius

$$ds^2 = \frac{-\tau^2}{\tau^2 + R^2} d\tau^2 - \frac{2R\tau}{\tau^2 + R^2} dR d\tau + \frac{\tau^2}{\tau^2 + R^2} dR^2 + R^2 d\Omega^2. \tag{11}$$

III. GENERAL STRUCTURE OF THE SPHERICALLY SYMMETRIC CONSTRAINTS

The two divergences of outgoing null rays are given by

$$\omega_+ = R\theta = Rp - RK_r^r + RK, \tag{12}$$

$$\omega_- = R\theta' = Rp + RK_r^r - RK, \tag{13}$$

where

$$p = \frac{2}{R}\frac{dR}{dl} \tag{14}$$

is the mean curvature of a surface of constant R in the slice where R is the Schwarzschild radius and l is the proper distance. The constraints now can be written as

$$\partial_l(\omega_+) = -8\pi R(\rho - j) - \frac{1}{4R}(2\omega_+^2 - \omega_+\omega_- - 4 - 4\omega_+ RK), \tag{15}$$

$$\partial_l(\omega_-) = -8\pi R(\rho + j) - \frac{1}{4R}(2\omega_-^2 - \omega_+\omega_- - 4 + 4\omega_- RK), \tag{16}$$

$$\partial_l R = R' = \frac{1}{4}(\omega_+ + \omega_-). \tag{17}$$

We assume we are given ρ (the energy density), $j = \vec{j} \cdot \hat{n}$ (the current density), where $\hat{n}$ is the outgoing radial normal and RK as functions of l and then solve the triplet of ODE's (15), (16), and (17) for (R, ω_+, ω_-). The only conditions we assume are regularity at the origin ($R = 0, \omega_+ = \omega_- = 2$), asymptotic flatness and that the sources satisfy the dominant energy condition, $\rho \geqslant |j|$.

Combining Eqs. (15) and (16) we can write

$$\partial_l(\omega_-\omega_+) = -8\pi R[\rho(\omega_+ + \omega_-) + j(\omega_+ - \omega_-)] - \frac{1}{4R}(\omega_-\omega_+ - 4)(\omega_+ + \omega_-), \tag{18}$$

and by regularity and asymptotic flatness we have that $\lim_{R\to 0}\omega_-\omega_+=4$ also $\lim_{R\to\infty}\omega_-\omega_+=4$.

Suppose that $\omega_-\omega_+>4$; if both are positive, we have that the right-hand side of Eq. (18) is strictly negative and if both are negative the right-hand side is positive. Thus we have

$$\omega_-\omega_+\leqslant 4. \tag{19}$$

IV. CONSTRAINTS ON ASYMPTOTICALLY FLAT CMC HYPERSURFACES

When we consider asymptotically flat CMC hypersurfaces, it is useful to use variables that are finite at the origin and infinity. From the Minkowski analysis, it is clear that we need as boundary conditions that $\omega_+\to 2e^{Kl/3}$ and $\omega_-\to 2e^{-Kl/3}$. Thus the natural variables to use are $A=\omega_+e^{-Kl/3}$ and $B=\omega_-e^{Kl/3}$. Using these Eqs. (15) and (16) become

$$\partial_l A=-8\pi Re^{-Kl/3}(\rho-j)-\frac{e^{Kl/3}}{4R}[2A^2-\tfrac{8}{3}KRe^{-Kl/3}A -ABe^{-2Kl/3}-4e^{-2Kl/3}], \tag{20}$$

$$\partial_l B=-8\pi Re^{Kl/3}(\rho+j)-\frac{e^{-Kl/3}}{4R}[2B^2+\tfrac{8}{3}KRe^{Kl/3}B -ABe^{2Kl/3}-4e^{2Kl/3}]. \tag{21}$$

We know, from the previous section [inequality (19)] that $AB=\omega_+\omega_-$ is bounded above by 4 . Let us write the expression which does not depend on the sources in Eq. (20) as

$$-\frac{e^{Kl/3}}{2R}[A^2-\tfrac{4}{3}KRe^{-Kl/3}A-4e^{-2Kl/3}]-\frac{e^{-Kl/3}}{4R}[4-AB]. \tag{22}$$

Consider

$$\alpha=2\sqrt{K^2R^2/9+1}+\frac{2}{3}KR, \tag{23}$$

$$\beta=2\sqrt{K^2R^2/9+1}-\frac{2}{3}KR;$$

these are essentially the roots of the quadratic equation in A in Eq. (22). If A lies outside the range

$$[-\beta e^{-Kl/3},\alpha e^{-Kl/3}], \tag{24}$$

then every term on the right-hand side of Eq. (20) is negative and therefore $\partial_l A<0$.

It is clear that $\alpha>\frac{4}{3}KR$ and also $\alpha\geqslant 2$, and these are the limiting values of A at infinity and at the origin, respectively. At any maximum of A we have that $\partial_l A=0$ which implies that $A\leqslant\alpha e^{-Kl/3}$ at that point. We can show that this is a global upper bound, or, equivalently,

$$\omega_+\leqslant 2\sqrt{\frac{K^2R^2}{9}+1}+\frac{2}{3}KR. \tag{25}$$

Consider the function

$$f(l)=2\sqrt{\frac{K^2R^2}{9}+1}+\frac{2}{3}KR-\omega_+, \tag{26}$$

$f(l)$ is zero at the origin and by asymptotic flatness it is positive at infinity. We can show that it is always positive. Let us assume, to the contrary, that $f(l)$ is negative somewhere. This means that there must exist a negative minimum, i.e., a point where

(i) $\omega_+>2\sqrt{K^2R^2/9+1}+\frac{2}{3}KR$

and

(ii) $f'(l)=0$.

However, we can show that if (i) holds, then $f'(l)>0$. Using Eqs. (15) and (17), we can calculate

$$f'(l)=\frac{1}{4R}\left[\frac{KR}{3}\frac{\alpha(\omega_++\omega_-)}{\sqrt{K^2R^2/9+1}}+2\omega_+^2-\omega_+\omega_- -4\omega_+KR-4\right]+8\pi R(\rho-j). \tag{27}$$

The coefficient of ω_- is

$$\frac{1}{4R}\left[-\omega_++\frac{KR\alpha}{3\sqrt{K^2R^2/9+1}}\right]. \tag{28}$$

This is obviously negative since $\omega_+\geqslant\alpha$. Therefore we minimize $f'(l)$ by choosing the maximum value of ω_-. There is a condition, Eq. (19), that constrains the product of both optical scalars, $\omega_+\omega_-\leqslant 4$. Therefore the maximum of ω_- is $4/\omega_+$.

Now consider the function

$$\tilde{f}(l)=\frac{1}{4R}\left[\frac{KR}{3}\frac{\alpha(\omega_++4/\psi_+)}{\sqrt{K^2R^2/9+1}}+2\omega_+^2-8-4\omega_+KR\right]. \tag{29}$$

The derivative of this function with respect to ω_+ is positive (assuming $\omega_+\geqslant\alpha$) and therefore its minimum is achieved at the minimum of ω_+, i.e., at $\omega_+=\alpha$.

Hence

$$f'(l)\geqslant\frac{1}{4R}\left[\frac{KR}{3}\frac{\alpha^2+4}{\sqrt{K^2R^2/9+1}}+2\alpha^2-8-4\alpha KR\right] +8\pi R(\rho-j). \tag{30}$$

It is easy to see that this expression is positive. Hence we get a contradiction. We could show, in a similar vein, the existence of the upper bound on B as well of the lower bounds. Therefore we have the following global bounds on the optical scalars:

$$-\beta\leqslant\omega_+\leqslant\alpha, \tag{31}$$

$$-\alpha\leqslant\omega_-\leqslant\beta. \tag{32}$$

Those bounds are valid for both signs, positive and negative, of the trace of the extrinsic curvature K.

V. SUFFICIENT CONDITIONS FOR TRAPPED SURFACES IN CMC HYPERSURFACES

We can use the formulae (15) and (17) in Sec. II to derive

$$\partial_l(\omega_+ R) = -8\pi R^2(\rho - j) + 1 + \tfrac{1}{4}(2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK), \tag{33}$$

and we have the bounds on ω_+ and ω_- from the previous section. It is easy to show that the maximum value of $(2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK)$ occurs when (i) $\omega_+ = \alpha$ and $\omega_- = \beta$ if $K > 0$, and (ii) $\omega_+ = -\beta$ and $\omega_- = -\alpha$ if $K < 0$. Hence we get in both cases

$$2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK \leqslant 4 + \frac{16}{3}K^2R^2 + \frac{16}{3}|K|R\sqrt{\frac{K^2R^2}{9} + 1}. \tag{34}$$

We can complete the square in the square root to finally get

$$\tfrac{1}{4}(2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK) \leqslant 1 + \tfrac{8}{9}K^2R^2 + \tfrac{4}{3}|K|R; \tag{35}$$

thus, we get

$$\partial_l(\omega_+ R) \leqslant -8\pi R^2(\rho - j) + 2 + \tfrac{8}{9}K^2R^2 + \tfrac{4}{3}|K|R. \tag{36}$$

If we integrate this equation out to some surface S we get

$$\omega_+ R|_S \leqslant -2(M-P) + 2L + \frac{2K^2}{9\pi}V + \frac{4}{3}|K|\int_0^l R\,dl, \tag{37}$$

where $M = \int 4\pi R^2 \rho\, dl$ is the total amount of matter inside S, $P = \int 4\pi R^2 j\, dl$ is the total outward radial momentum of the matter, L is the proper radius, and V is the volume inside S (notice that $4\pi R^2 dl = dV$ is the proper volume element). Therefore, we have that if

$$(M-P)(S) \geqslant L + \frac{K^2}{9\pi}V + \frac{2}{3}|K|\int R\,dl,$$

we must have that $\omega_+ R|_S$ is negative and so the surface at S is a outer future-trapped surface. We can estimate $\int R\,dl$ as

$$\int R\,dl \leqslant \left[\int R^2 dl\right]^{1/2}\left[\int dl\right]^{1/2} = \left(\frac{VL}{4\pi}\right)^{1/2}. \tag{38}$$

Therefore a sufficient condition for the appearance of a outer future-trapped surface on a slice with constant trace of the extrinsic curvature is that

$$(M-P)(S) \geqslant L + \frac{K^2}{9\pi}V + \frac{|K|}{3}\left(\frac{VL}{\pi}\right)^{1/2}. \tag{39}$$

VI. NECESSARY CONDITIONS FOR A TRAPPED SURFACE IN A CMC HYPERSURFACE

Let us return to the equality (33) we derived in Sec. V and again integrate it out to some surface S:

$$\omega_+ R|_S = -2(M-P) + L(S) + \tfrac{1}{4}\int_0^L (2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK)\,dl, \tag{40}$$

but now we wish to minimize the integral rather than maximize it. We assume that no outer future trapped surface exists within S, i.e., $\omega_+ \geqslant 0$. We also assume that no outer past-trapped surface exists in S. Not only that, but that ω_- is strongly bounded away from zero, i.e., $\omega_- \geqslant C > 0$; that means that all radially ingoing null rays are converging.

In other words, we want to minimize the function $f(\omega_+, \omega_-) = 2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK$ in the region given by $0 \leqslant \omega_+ \leqslant \alpha$ and $C \leqslant \omega_- \leqslant \beta$. A simple calculation gives $f_{\min} = \min(f(\alpha, C), 0)$.

Because α is a function of R, we need to study the function

$$\tilde{f}(R) = f(\alpha(R), C) = 2\alpha C - \alpha^2 + 4\alpha RK$$

in order to find $f_{\min}$.

We will consider separately two cases, with the positive and negative traces of the extrinsic curvature.

(i) Let $K > 0$. By inspection we obtain that this function is an increasing function in the variable R, therefore,

$$\min(\tilde{f}) = f(\alpha(0), C) = 4C - 4.$$

Clearly, when $C \geqslant 1$, $f_{\min} = 0$; otherwise, $f_{\min} \geqslant 4C - 4$. Inserting this into Eq. (40) we obtain two estimates

$$\omega_+ R|_S \geqslant \begin{cases} -2(M-P)(S) + L(S) & \text{for } C \geqslant 1, \\ -2(M-P)(S) + CL(S) & \text{for } C \leqslant 1; \end{cases} \tag{41}$$

that is, since $\theta(S) = 0$,

$$M(S) - P(S) \geqslant \begin{cases} \dfrac{L}{2} & \text{for } C \geqslant 1, \\ \dfrac{CL}{2} & \text{for } C \leqslant 1 \end{cases}$$

is the necessary condition for the existence of a future trapped surface.

The above result obviously applies to maximal slices. In connection with that, two of us have to admit that theorem 2 in [2] should be stated as above; the actual statement of [2], that the necessary condition for future-trapped surfaces is $M(S) - P(S) \geqslant CL/2$ can be wrong.

(ii) Let $K < 0$. In this case one easily estimates $\tilde{f}(R)$ from below by

$$4(C-1) + \frac{4KR(4+C)}{3} \quad \text{for } C \leqslant 1,$$

$$\frac{4KR(4+C)}{3} \quad \text{for } C\geqslant 1.$$

That leads to a necessary condition for S to be future trapped

$$M(S)-P(S)+\frac{(4+C)|K|}{6}\int_0^{L(S)} dlR(l)\geqslant\begin{cases}\dfrac{L}{2} & \text{for } C\geqslant 1,\\[2ex] \dfrac{CL}{2} & \text{for } C\leqslant 1.\end{cases}$$

Using relations $dl=(2/Rp)\,dR$ and $pR=\frac{1}{2}(\omega_+ + \omega_-)\geqslant C/2$ one obtains

$$\int_0^{L(S)} dlR(l)\leqslant\frac{S}{2\pi C}$$

and the necessary condition

$$M(S)-P(S)+\frac{(4+C)|K|S}{12\pi C}\geqslant\begin{cases}\dfrac{L}{2} & \text{for } C\geqslant 1,\\[2ex] \dfrac{CL}{2} & \text{for } C\leqslant 1.\end{cases}$$

A similar necessary condition, under a somewhat stronger condition, has been obtained by Zannias [6]. Thus negative values of the trace of the extrinsic curvature can help to form trapped surfaces.

Let us recall that yet another necessary result has been derived in [8], where the following equation has been proved:

$$\frac{R^3}{8}\theta(S)\theta'(S)+m-\left(\frac{S}{16\pi}\right)^{1/2}=\pi\int_r^\infty \sqrt{a}R^3[\rho_0(\theta+\theta') +j(\theta-\theta')]. \tag{42}$$

This equation has been derived on maximal slices, but it holds true on any slicing, assuming a quick enough falloff of matter fields. Under the dominant energy condition one concludes that an outermost trapped surface S (future or past) must have a Schwarzschild radius R not greater than $2m$. This conclusion is slicing independent.

VII. REISSNER-NORDSTRÖM GEOMETRY USING CMC FOLIATIONS

In this section we will present an explicit line element for electrovacuum in constant mean curvature foliations. The most general spherically symmetric line element can be put

$$ds^2=-N^2dt^2+adr^2+R^2d\Omega^2. \tag{43}$$

We assume that the trace of the extrinsic curvature

$$K=\frac{\partial_t(\sqrt{a}R^2)}{2N\sqrt{a}R^2} \tag{44}$$

is constant on a particular slice and, moreover, is time independent. The three nonzero components of the extrinsic curvature are

$$K_r^r=\frac{\partial_t(\sqrt{a})}{2N\sqrt{a}},\quad K_\phi^\phi=K_\theta^\theta=\frac{\partial_t R}{NR}=\frac{1}{2}(K-K_r^r). \tag{45}$$

The spherically symmetric Einstein equations consist of constraint equations (1), the evolution equation

$$\partial_t(K_r^r-trK)=-\frac{p^3R^2}{2N}\frac{\partial_r}{\sqrt{a}}\left(\frac{N}{pR}\right)^2+\frac{3N}{2}(K_r^r)^2+8\pi(T_r^r+\rho) +\frac{N}{2}K^2-2NKK_r^r, \tag{46}$$

and the lapse equation

$$\Delta^{(3)}N=N[\tfrac{3}{2}(K_r^r)^2+\tfrac{1}{2}K^2-KK_r^r+4\pi(T_i^i+\rho)]. \tag{47}$$

In electrovacuum we have $q^2/8\pi R^4=\rho=T_i^i=-T_r^r$ where q is the electric charge. The mean curvature p of nested two spheres and the extrinsic curvatures are easily found from the constraints (1) and they read

$$pR=2\sqrt{1+\frac{C}{R}+\frac{q^2}{R^2}+\left(\frac{KR}{3}+\frac{C_1}{2R^2}\right)^2}, \tag{48}$$

$$K_r^r=\frac{K}{3}+\frac{C_1}{R^3}.$$

The lapse equation becomes now

$$\Delta^{(3)}N=N\left[\frac{3C_1^2}{2R^6}+\frac{q^2}{R^4}+\frac{K^2}{3}\right], \tag{49}$$

and one easily finds out that it is solved by

$$N=\gamma\frac{pR}{2}, \tag{50}$$

where γ is given by

$$\gamma(r,t)=1+\int_{R(r)}^\infty d\tilde{R}\frac{C_2(\tilde{R},t)}{(\tilde{R})^2(p\tilde{R})^3}. \tag{51}$$

Inserting the whole information into Eq. (46) and using the relation $\partial_r/\sqrt{a}=(Rp/2)\partial_R$ one obtains that the constant C_2 depends on the rate of change of the radial-radial component of the extrinsic curvature,

$$C_2(R,t)=4\partial_t[R^3(K_r^r-K)]. \tag{52}$$

The change of the coordinate variable r into the areal radius R transforms the line element (43) into

$$ds^2=dt^2\left[-N^2+\frac{\gamma^2}{4}\left(\frac{C_1}{R^2}-\frac{2KR}{3}\right)^2\right]+2\gamma\frac{C_1/R^3-2K/3}{p}dtdR+\frac{4}{(pR)^2}dR^2+R^2d\Omega^2$$
$$=-\frac{\gamma^2\omega_+\omega_-}{4}dt^2+2\gamma\frac{\omega_--\omega_+}{\omega_-+\omega_+}dRdt+\frac{16}{(\omega_-+\omega_+)^2}dR^2+R^2d\Omega^2, \qquad (53)$$

with N and p defined above. Let us point out that the parameter C that appears in the expression for p should be identified with $-2m_B$, where m_B is the Bondi mass. The spatial part Eq. (53) agrees, for $q=0=C_1$, with the constant mean curvature foliation of the Schwarzschild geometry given in [9].

VIII. "CMC SURFACES AVOID SINGULARITIES"

In the article [2] an argument was advanced as to how foliations with bounded trace of the extrinsic curvature might avoid singularities. In this section we wish to produce a different (and sharper) argument to the same end. This argument works for essentially any slicing, but we present it here specifically for CMC slices. Let us consider a model of a collapsing system where the support of the matter becomes ever smaller as the collapse continues so that eventually the star is confined to a region much smaller than that enclosed by the apparent horizon. If the star were to be compressed inside a boundary which satisfies $R \ll m$, where m is the conserved Arnowitt-Deser-Misner (ADM) mass of the star, before any singularity appears then regular CMC foliations will be excluded from this part of the spacetime.

From the inequalities (31) and (32) we can show that, on any regular CMC slice,

$$\omega_+\omega_- \geq -4\left(\frac{2K^2R^2}{9}+1+\frac{2|K|R}{3}\sqrt{\frac{1+K^2R^2}{9}}\right) \geq -4\left(\frac{2|K|R}{3}+1\right)^2. \qquad (54)$$

However, both the Schwarzschild radius R, and the product $\omega_+\omega_-$ are four-scalars and we have

$$\omega_+\omega_-=4\left(1-\frac{2m_H}{R}\right), \qquad (55)$$

where m_H is the so-called Hawking mass, which equals the constant ADM mass outside the support of the matter. Inequality (54) and equality (55) can be combined to give the inequality

$$m_H \leq \frac{2K^2R^3}{9}+\frac{2|K|R}{3}+R. \qquad (56)$$

This means that for a fixed positive m_H we have a lower bound for R. Let us assume that we are considering a spherical collapse and viewing it using a CMC foliation. Let us also assume that during this collapse a two-surface appears which violates this inequality (56) before the CMC slices become singular. This means that the CMC foliation cannot progress past this point, and the lapse collapses. Since the lapse equation is elliptical, not only does the lapse go to zero at this point; it becomes small on the whole interior, and the CMC slicing freezes.

This means that if we wish to find a solution where CMC slices run right up to the singularity, we cannot allow a large accumulation of matter near the center before the singularity appears. A possible way for this to happen is that if the collapse were such that in addition to the infall of matter, one also had an explosion that pushed significant amounts of the star outwards, away from the horizon.

This bound is valid for solutions which have the spatial topology $R^1\times S^2$ as in the extended Schwarzschild solution as well as topology R^3. Maximal slicing can be viewed as a special case of CMC slicing and it was observed many years ago (see [7]) that the regular maximal slicing of the Schwarzschild solution saturates at $R=3m/2$, in agreement with the bound stated above.

ACKNOWLEDGMENTS

This work was initiated during the ESI Summer School in Mathematical Relativity, Vienna. It has been partially supported by the Forbairt grant SC/94/225 and the KBN project 2 PO3B 090 08.

APPENDIX

In what follows we prove two results that hold true on any spherically symmetric slices with an arbitrary slicing condition $K=K^j_j$ albeit we need that $B=\sup_{0\leq R<\infty}KR<\infty$.

Theorem 1. Assume the energy condition $\rho \geq j$. In notation of Sec. V, if

$$M(S)-P(S)>\frac{3}{2}L+\frac{B^2}{2}L,$$

then a two-sphere S must be future trapped.

Proof. From Eq. (15) one obtains, after multiplying by R and integration by parts,

$$\omega_+R=-2[M(S)-P(S)]+\int_0^L dl\left(1-\frac{\omega_+^2}{4}+\omega_+KR+\frac{\omega_+\omega_-}{2}\right). \qquad (A1)$$

From that one gets, using the obvious estimate $-\omega_+^2/4+\omega_+KR\leq(KR)^2\leq B^2$, that

$$\omega_+R\leq-2[M(S)-P(S)]+\int_0^L dl\left(1+B^2+\frac{\omega_+\omega_-}{2}\right). \qquad (A2)$$

From [2] we have that $\omega_+\omega_-\leq 4$. Inserting that into Eq. (A2), one arrives at

$$\omega_+R=-2[M(S)-P(S)]+3L+B^2L,$$

which immediately yields theorem 1.

That result is not exact, as opposed to the sufficiency condition proved in the main text.

Theorem 2 (the Penrose inequality). Let Σ be a Cauchy hypersurface with an arbitrary slice K and the asymptotic mass m. Assume that matter is of compact support and that it satisfies the energy condition $\rho \geq j$ outside a sphere S. Let $A = 4\pi R^2$ denote the area of S. Then S cannot be trapped if $R > 2m$.

Proof. Multiplying Eq. (18) by R and integrating it, one gets the equation

$$\omega_+ \omega_- R - 4R = -8\pi \int_0^L dl R^2 [\rho(\omega_+ + \omega_-) + j(\omega_+ + \omega_-)]. \tag{A3}$$

The integral $4\pi \int_0^\infty dl R^2 [\rho(\omega_+ + \omega_-) + j(\omega_+ + \omega_-)]$ is conserved in time, as one can easily check; from the asymptotic flatness, taking into account the asymptotic behavior of optical scalars, one obtains that $4m$, where m is the asymptotic mass, is actually equal to the integral. Thus Eq. (A3) can be written as

$$\omega_+ \omega_- R - 4R = -8m + 8\pi \int_L^\infty dl R^2 [\rho(\omega_+ + \omega_-) + j(\omega_+ + \omega_-)]. \tag{A4}$$

At spatial infinity both optical scalars are positive. Let S be the last such sphere that one of the scalars vanishes. Then outside S both scalars are positive and one easily shows that the integrand $\rho(\omega_+ + \omega_-) + j(\omega_+ + \omega_-)$ is non-negative outside S if the energy condition of theorem 2 holds true. But that means that the sphere S must be placed within the area having areal radius $R \leq 2m$.

Let us point out that theorem 2 generalizes a result hitherto proven in [8] for the case of maximal slices (see also [10] for the proof of the Penrose inequality for Tolman-Bondi-Sharp-Misner-Podurets class of metrics). Theorem 2 excludes the existence of any trapped surfaces, both future and past, outside the Schwarzschild radius.

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[2] E. Malec and N. Ó Murchadha, Phys. Rev. D **50**, R6033 (1994).

[3] M. Iriondo, Ph.D. thesis, Royal Institute of Technology, Stockholm, Sweden, 1994.

[4] S. Y. Cheng and S. T. Yau, Ann. Math. **104**, 407 (1976); A. Treibergs, Invent. Math. **66**, 39 (1982).

[5] P. Bizoń, E. Malec, and N. Ó Murchadha, Phys. Rev. Lett. **61**, 1147 (1988); P. Bizoń, E. Malec, and N. Ó Murchadha, Class. Quantum Grav. **6**, 961 (1989); P. Koc, *ibid.* **11**, 1315 (1994).

[6] T. Zannias, Phys. Rev. D **45**, 2998 (1992).

[7] The Schwarzschild line element in maximal slicing is given by F. Estabrook, H. Wahlquist, S. Christensen, B. deWitt, L. Smarr, and E. Tsiang, Phys. Rev. D **7**, 2814 (1973); B. Reinhart, J. Math. Phys. **14**, 719 (1973); for the Reissner-Nordström solution see M. J. Duncan, Phys. Rev. D **31**, 1267 (1985); E. Malec, *ibid.* **49**, 6475 (1994).

[8] E. Malec and N. Ó Murchadha, Phys. Rev. D **49**, 6931 (1994).

[9] D. R. Brill, J. M. Cavallo, and J. A. Isenberg, J. Math. Phys. **21**, 2789 (1980).

[10] S. Hayward, Class. Quantum Grav. **11**, 3037 (1994).